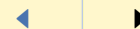
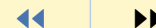




Application Of Differential
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Waggas Galib Atshan
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ON APPLICATION OF DIFFERENTIAL SUBORDINATION FOR CERTAIN SUBCLASS OF MEROMORPHICALLY p -VALENT FUNCTIONS WITH POSITIVE COEFFICIENTS DEFINED BY LINEAR OPERATOR

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Abstract:

This paper is mainly concerned with the application of differential subordinations for the class of meromorphic multivalent functions with positive coefficients defined by a linear operator satisfying the following:

$$-\frac{z^{p+1}(L^n f(z))'}{p} \prec \frac{1 + Az}{1 + Bz} \quad (n \in \mathbb{N}_0; z \in U).$$

In the present paper, we study the coefficient bounds, δ -neighborhoods and integral representations. We also obtain linear combinations, weighted and arithmetic means and convolution properties.

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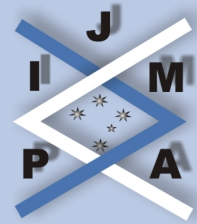
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1. Introduction

Let $L(p, m)$ be a class of all meromorphic functions $f(z)$ of the form:

$$(1.1) \quad f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k \quad \text{for any } m \geq p, \quad p \in \mathbb{N} = \{1, 2, \dots\}, \quad a_k \geq 0,$$

which are p -valent in the punctured unit disk

$$U^* = \{z : z \in \mathbb{C}, 0 < |z| < 1\} = U \setminus \{0\}.$$

Definition 1.1. Let f, g be analytic in U . Then g is said to be subordinate to f , written $g \prec f$, if there exists a Schwarz function $w(z)$, which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ ($z \in U$) such that $g(z) = f(w(z))$ ($z \in U$). Hence $g(z) \prec f(z)$ ($z \in U$), then $g(0) = f(0)$ and $g(U) \subset f(U)$. In particular, if the function $f(z)$ is univalent in U , we have the following (e.g. [6]; [7]):

$$g(z) \prec f(z) (z \in U) \quad \text{if and only if} \quad g(0) = f(0) \quad \text{and} \quad g(U) \subset f(U).$$

Definition 1.2. For functions $f(z) \in L(p, m)$ given by (1.1) and $g(z) \in L(p, m)$ defined by

$$(1.2) \quad g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k, \quad (b_k \geq 0, p \in \mathbb{N}, m \geq p),$$

we define the convolution (or Hadamard product) of $f(z)$ and $g(z)$ by

$$(1.3) \quad (f * g)(z) = z^{-p} + \sum_{k=m}^{\infty} a_k b_k z^k, \quad (p \in \mathbb{N}, m \geq p, z \in U).$$

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Definition 1.3 ([9]). Let $f(z)$ be a function in the class $L(p, m)$ given by (1.1). We define a linear operator L^n by

$$L^0 f(z) = f(z),$$
$$L^1 f(z) = z^{-p} + \sum_{k=m}^{\infty} (p+k+1)a_k z^k = \frac{(z^{p+1} f(z))'}{z^p}$$

and in general

$$(1.4) \quad L^n f(z) = L(L^{n-1} f(z))$$
$$= z^{-p} + \sum_{k=m}^{\infty} (p+k+1)^n a_k z^k$$
$$= \frac{(z^{p+1} L^{n-1} f(z))'}{z^p}, \quad (n \in \mathbb{N}).$$

It is easily verified from (1.4) that

$$(1.5) \quad z(L^n f(z))' = L^{n+1} f(z) - (p+1)L^n f(z),$$
$$(f \in L(p, m), \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

1. Liu and Srivastava [4] introduced recently the linear operator when $m = 0$, investigating several inclusion relationships involving various subclasses of meromorphically p -valent functions, which they defined by means of the linear operator L^n (see [4]).
2. Uralegaddi and Somanatha [10] introduced the linear operator L^n when $p = 1$ and $m = 0$.
3. Aouf and Hossen [2] obtained several results involving the linear operator L^n when $m = 0$ and $p \in \mathbb{N}$.

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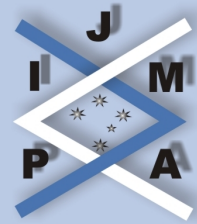
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We introduce a subclass of the function class $L(p, m)$ by making use of the principle of differential subordination as well as the linear operator L^n .

Definition 1.4. Let A and B ($-1 \leq B < A \leq 1$) be fixed parameters. We say that a function $f(z) \in L(p, m)$ is in the class $L(p, m, n, A, B)$, if it satisfies the following subordination condition:

$$(1.6) \quad \frac{z^{p+1}(L^n f(z))'}{p} \prec \frac{1 + Az}{1 + Bz} \quad (n \in \mathbb{N}_0; z \in U).$$

By the definition of differential subordination, (1.6) is equivalent to the following condition:

$$\left| \frac{z^{p+1}(L^n f(z))' + p}{Bz^{p+1}(L^n f(z))' + pA} \right| < 1, \quad (z \in U).$$

We can write

$$L\left(p, m, n, 1 - \frac{2\beta}{p}, -1\right) = L(p, m, n, \beta),$$

where $L(p, m, n, \beta)$ denotes the class of functions in $L(p, m)$ satisfying the following:

$$\operatorname{Re}\{-z^{p+1}(L^n f(z))'\} > \beta \quad (0 \leq \beta < p; z \in U).$$

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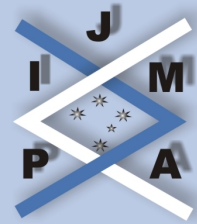
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2. Coefficient Bounds

Theorem 2.1. Let the function $f(z)$ of the form (1.1), be in $L(p, m)$. Then the function $f(z)$ belongs to the class $L(p, m, n, A, B)$ if and only if

$$(2.1) \quad \sum_{k=m}^{\infty} k(1-B)(p+k+1)^n a_k < (A-B)p,$$

where $-1 \leq B < A \leq 1$, $p \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m \geq p$.

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \frac{(A-B)p}{k(1-B)(p+k+1)^n} z^m, \quad m \geq p.$$

Proof. Assume that the condition (2.1) is true. We must show that $f \in L(p, m, n, A, B)$, or equivalently prove that

$$(2.2) \quad \left| \frac{z^{p+1}(L^n f(z))' + p}{Bz^{p+1}(L^n f(z))' + Ap} \right| < 1.$$

We have

$$\begin{aligned} \left| \frac{z^{p+1}(L^n f(z))' + p}{Bz^{p+1}(L^n f(z))' + Ap} \right| &= \left| \frac{z^{p+1}(-pz^{-(p+1)}) + \sum_{k=m}^{\infty} k(p+k+1)^n a_k z^{k-1} + p}{Bz^{p+1}(-pz^{-(p+1)}) + \sum_{k=m}^{\infty} k(p+k+1)^n a_k z^{k-1} + Ap} \right| \\ &= \left| \frac{\sum_{k=m}^{\infty} k(p+k+1)^n a_k z^{k+p}}{(A-B)p + B \sum_{k=m}^{\infty} k(p+k+1)^n a_k z^{k+p}} \right| \end{aligned}$$

$$\leq \left\{ \frac{\sum_{k=m}^{\infty} k(p+k+1)^n a_k}{(A-B)p + B \sum_{k=m}^{\infty} k(k+p+1)^n a_k} \right\} < 1.$$

The last inequality by (2.1) is true.

Conversely, suppose that $f(z) \in L(p, m, n, A, B)$. We must show that the condition (2.1) holds true. We have

$$\left| \frac{z^{p+1}(L^n f(z))' + p}{Bz^{p+1}(L^n f(z))' + Ap} \right| < 1,$$

hence we get

$$\left| \frac{\sum_{k=m}^{\infty} k(p+k+1)^n a_k z^{k+p}}{(A-B)p + B \sum_{k=m}^{\infty} k(p+k+1)^n a_k z^{k+p}} \right| < 1.$$

Since $\operatorname{Re}(z) < |z|$, so we have

$$\operatorname{Re} \left\{ \frac{\sum_{k=m}^{\infty} k(p+k+1)^n a_k z^{k+p}}{(A-B)p + B \sum_{k=m}^{\infty} k(p+k+1)^n a_k z^{k+p}} \right\} < 1.$$

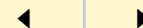
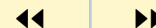
We choose the values of z on the real axis and letting $z \rightarrow 1^-$, then we obtain

$$\left\{ \frac{\sum_{k=m}^{\infty} k(p+k+1)^n a_k}{(A-B)p + B \sum_{k=m}^{\infty} k(p+k+1)^n a_k} \right\} < 1,$$



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then

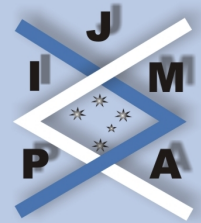
$$\sum_{k=m}^{\infty} k(1-B)(p+k+1)^n a_k < (A-B)p$$

and the proof is complete. □

Corollary 2.2. *Let $f(z) \in L(p, m, n, A, B)$, then we have*

$$a_k \leq \frac{(A-B)p}{k(1-B)(p+k+1)^n}, \quad k \geq m.$$

Corollary 2.3. *Let $0 \leq n_2 < n_1$, then $L(p, m, n_2, A, B) \subseteq L(p, m, n_1, A, B)$.*



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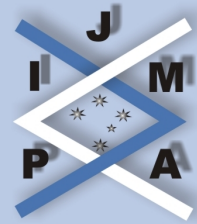


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3. Neighbourhoods and Partial Sums

Definition 3.1. Let $-1 \leq B < A \leq 1$, $m \geq p$, $n \in \mathbb{N}_0$, $p \in \mathbb{N}$ and $\delta \geq 0$. We define the δ -neighbourhood of a function $f \in L(p, m)$ and denote $N_\delta(f)$ such that

$$(3.1) \quad N_\delta(f) = \left\{ g \in L(p, m) : g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k, \text{ and } \sum_{k=m}^{\infty} \frac{k(1-B)(p+k+1)^n}{(A-B)p} |a_k - b_k| \leq \delta \right\}.$$

Goodman [3], Ruscheweyh [8] and Altintas and Owa [1] have investigated neighbourhoods for analytic univalent functions, we consider this concept for the class $L(p, m, n, A, B)$.

Theorem 3.2. Let the function $f(z)$ defined by (1.1) be in $L(p, m, n, A, B)$. For every complex number μ with $|\mu| < \delta$, $\delta \geq 0$, let $\frac{f(z)+\mu z^{-p}}{1+\mu} \in L(p, m, n, A, B)$, then $N_\delta(f) \subset L(p, m, n, A, B)$, $\delta \geq 0$.

Proof. Since $f \in L(p, m, n, A, B)$, f satisfies (2.1) and we can write for $\gamma \in \mathbb{C}$, $|\gamma| = 1$, that

$$(3.2) \quad \left[\frac{z^{p+1}(L^n f(z))' + p}{Bz^{p+1}(L^n f(z))' + pA} \right] \neq \gamma.$$

Equivalently, we must have

$$(3.3) \quad \frac{(f * Q)(z)}{z^{-p}} \neq 0, \quad z \in U^*,$$

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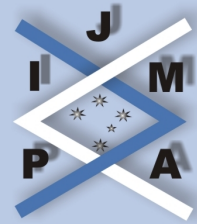
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where

$$Q(z) = z^{-p} + \sum_{k=m}^{\infty} e_k z^k,$$

such that $e_k = \frac{\gamma k(1-B)(p+k+1)^n}{(A-B)^p}$, satisfying $|e_k| \leq \frac{k(1-B)(p+k+1)^n}{(A-B)^p}$ and $k \geq m, p \in \mathbb{N}, n \in \mathbb{N}_0$.

Since $\frac{f(z) + \mu z^{-p}}{1 + \mu} \in L(p, m, n, A, B)$, by (3.3),

$$\frac{1}{z^{-p}} \left(\frac{f(z) + \mu z^{-p}}{1 + \mu} * Q(z) \right) \neq 0,$$

and then

$$(3.4) \quad \frac{1}{z^{-p}} \left(\frac{(f * Q)(z) + \mu z^{-p}}{1 + \mu} \right) \neq 0.$$

Now assume that $\left| \frac{(f * Q)(z)}{z^{-p}} \right| < \delta$. Then, by (3.4), we have

$$\left| \frac{1}{1 + \mu} \frac{f * Q}{z^{-p}} + \frac{\mu}{1 + \mu} \right| \geq \frac{|\mu|}{|1 + \mu|} - \frac{1}{|1 + \mu|} \left| \frac{(f * Q)(z)}{z^{-p}} \right| > \frac{|\mu| - \delta}{|1 + \mu|} \geq 0.$$

This is a contradiction as $|\mu| < \delta$. Therefore $\left| \frac{(f * Q)(z)}{z^{-p}} \right| \geq \delta$.

Letting

$$g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k \in N_{\delta}(f),$$

then

$$\begin{aligned}
 \delta - \left| \frac{(g * Q)(z)}{z^{-p}} \right| &\leq \left| \frac{((f - g) * Q)(z)}{z^{-p}} \right| \\
 &\leq \left| \sum_{k=m}^{\infty} (a_k - b_k) e_k z^k \right| \\
 &\leq \sum_{k=m}^{\infty} |a_k - b_k| |e_k| |z|^k \\
 &< |z|^m \sum_{k=m}^{\infty} \left[\frac{k(1-B)(p+k+1)^n}{(A-B)p} \right] |a_k - b_k| \\
 &\leq \delta,
 \end{aligned}$$

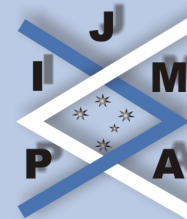
therefore $\frac{(g * Q)(z)}{z^{-p}} \neq 0$, and we get $g(z) \in L(p, m, n, A, B)$, so $N_\delta(f) \subset L(p, m, n, A, B)$. \square

Theorem 3.3. Let $f(z)$ be defined by (1.1) and the partial sums $S_1(z)$ and $S_q(z)$ be defined by $S_1(z) = z^{-p}$ and

$$S_q(z) = z^{-p} + \sum_{k=m}^{m+q-2} a_k z^k, \quad q > m, \quad m \geq p, \quad p \in \mathbb{N}.$$

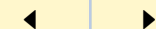
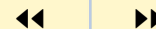
Also suppose that $\sum_{k=m}^{\infty} C_k a_k \leq 1$, where

$$C_k = \frac{k(1-B)(p+k+1)^n}{(A-B)p}.$$



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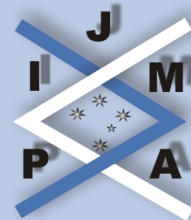


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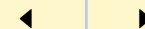
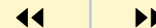
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Then

$$(3.5) \quad \begin{aligned} & \text{(i)} \quad f \in L(p, m, n, A, B) \\ & \text{(ii)} \quad \operatorname{Re} \left\{ \frac{f(z)}{S_q(z)} \right\} > 1 - \frac{1}{C_q}, \end{aligned}$$

$$(3.6) \quad \operatorname{Re} \left\{ \frac{S_q(z)}{f(z)} \right\} > \frac{C_q}{1 + C_q}, \quad z \in U, q > m.$$

Proof.

(i) Since $\frac{z^{-p} + \mu z^{-p}}{1 + \mu} = z^{-p} \in L(p, m, n, A, B)$, $|\mu| < 1$, then by Theorem 3.2, we have $N_1(z^{-p}) \subset L(p, m, n, A, B)$, $p \in \mathbb{N}(N_1(z^{-p}))$ denoting the 1-neighbourhood). Now since

$$\sum_{k=m}^{\infty} C_k a_k \leq 1,$$

then $f \in N_1(z^{-p})$ and $f \in L(p, m, n, A, B)$.

(ii) Since $\{C_k\}$ is an increasing sequence, we obtain

$$(3.7) \quad \sum_{k=m}^{m+q-2} a_k + C_q \sum_{k=q+m-1}^{\infty} a_k \leq \sum_{k=m}^{\infty} C_k a_k \leq 1.$$

Setting

$$G_1(z) = C_q \left(\frac{f(z)}{S_q(z)} - \left(1 - \frac{1}{C_q} \right) \right) = \frac{C_q \sum_{k=q+m-1}^{\infty} a_k z^{k+p}}{1 + \sum_{k=m}^{m+q-2} a_k z^{k+p}} + 1,$$

from (3.7) we get

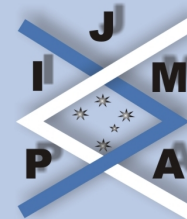
$$\left| \frac{G_1(z) - 1}{G_1(z) + 1} \right| = \left| \frac{C_q \sum_{k=q+m-1}^{\infty} a_k z^{k+p}}{2 + 2 \sum_{k=m}^{m+q-2} a_k z^{k+p} + C_q \sum_{k=q+m-1}^{\infty} a_k z^{k+p}} \right|$$

$$\leq \frac{C_q \sum_{k=q+m-1}^{\infty} a_k}{2 - 2 \sum_{k=m}^{m+q-2} a_k - C_q \sum_{k=q+m-1}^{\infty} a_k} \leq 1.$$

This proves (3.5). Therefore, $\operatorname{Re}(G_1(z)) > 0$ and we obtain $\operatorname{Re} \left\{ \frac{f(z)}{S_q(z)} \right\} > 1 - \frac{1}{C_q}$.
 Now, in the same manner, we can prove the assertion (3.6), by setting

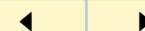
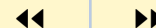
$$G_2(z) = (1 + C_q) \left(\frac{S_q(z)}{f(z)} - \frac{C_q}{1 + C_q} \right).$$

This completes the proof. □



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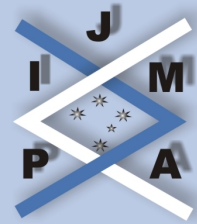


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4. Integral Representation

In the next theorem we obtain an integral representation for $L^n f(z)$.

Theorem 4.1. Let $f \in L(p, m, n, A, B)$, then

$$L^n f(z) = \int_0^z \frac{p(A\psi(t) - 1)}{t^{p+1}(1 - B\psi(t))} dt,$$

where $|\psi(z)| < 1, z \in U^*$.

Proof. Let $f(z) \in L(p, m, n, A, B)$. Letting $-\frac{z^{p+1}(L^n f(z))'}{p} = y(z)$, we have

$$y(z) \prec \frac{1 + Az}{1 + Bz}$$

or we can write $\left| \frac{y(z)-1}{By(z)-A} \right| < 1$, so that consequently we have

$$\frac{y(z) - 1}{By(z) - A} = \psi(z), \quad |\psi(z)| < 1, \quad z \in U.$$

We can write

$$\frac{-z^{p+1}(L^n f(z))'}{p} = \frac{1 - A\psi(z)}{1 - B\psi(z)},$$

which gives

$$(L^n f(z))' = \frac{p(A\psi(z) - 1)}{z^{p+1}(1 - B\psi(z))}.$$

Hence

$$L^n f(z) = \int_0^z \frac{p(A\psi(t) - 1)}{t^{p+1}(1 - B\psi(t))} dt,$$

and this gives the required result. □



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5. Linear Combination

In the theorem below, we prove a linear combination for the class $L(p, m, n, A, B)$.

Theorem 5.1. *Let*

$$f_i(z) = z^{-p} + \sum_{k=m}^{\infty} a_{k,i} z^k, \quad (a_{k,i} \geq 0, i = 1, 2, \dots, \ell, k \geq m, m \geq p)$$

belong to $L(p, m, n, A, B)$, then

$$F(z) = \sum_{i=1}^{\ell} c_i f_i(z) \in L(p, m, n, A, B),$$

where $\sum_{i=1}^{\ell} c_i = 1$.

Proof. By Theorem 2.1, we can write for every $i \in \{1, 2, \dots, \ell\}$

$$\sum_{k=m}^{\infty} \frac{k(1-B)(p+k+1)^n}{(A-B)^p} a_{k,i} < 1,$$

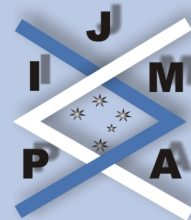
therefore

$$F(z) = \sum_{i=1}^{\ell} c_i \left(z^{-p} + \sum_{k=m}^{\infty} a_{k,i} z^k \right) = z^{-p} + \sum_{k=m}^{\infty} \left(\sum_{i=1}^{\ell} c_i a_{k,i} \right) z^k.$$

However,

$$\sum_{k=m}^{\infty} \frac{k(1-B)(p+k+1)^n}{(A-B)^p} \left(\sum_{i=1}^{\ell} c_i a_{k,i} \right) = \sum_{i=1}^{\ell} \left[\sum_{k=m}^{\infty} \frac{k(1-B)(p+k+1)^n}{(A-B)^p} a_{k,i} \right] c_i \leq 1,$$

then $F(z) \in L(p, m, n, A, B)$, so the proof is complete. \square



6. Weighted Mean and Arithmetic Mean

Definition 6.1. Let $f(z)$ and $g(z)$ belong to $L(p, m)$, then the weighted mean $h_j(z)$ of $f(z)$ and $g(z)$ is given by

$$h_j(z) = \frac{1}{2}[(1 - j)f(z) + (1 + j)g(z)].$$

In the theorem below we will show the weighted mean for this class.

Theorem 6.2. If $f(z)$ and $g(z)$ are in the class $L(p, m, n, A, B)$, then the weighted mean of $f(z)$ and $g(z)$ is also in $L(p, m, n, A, B)$.

Proof. We have for $h_j(z)$ by Definition 6.1,

$$\begin{aligned} h_j(z) &= \frac{1}{2} \left[(1 - j) \left(z^{-p} + \sum_{k=m}^{\infty} a_k z^k \right) + (1 + j) \left(z^{-p} + \sum_{k=m}^{\infty} b_k z^k \right) \right] \\ &= z^{-p} + \sum_{k=m}^{\infty} \frac{1}{2} ((1 - j)a_k + (1 + j)b_k) z^k. \end{aligned}$$

Since $f(z)$ and $g(z)$ are in the class $L(p, m, n, A, B)$ so by Theorem 2.1 we must prove that

$$\begin{aligned} & \sum_{k=m}^{\infty} k(1 - B)(p + k + 1)^n \left[\frac{1}{2}(1 - j)a_k + \frac{1}{2}(1 + j)b_k \right] \\ &= \frac{1}{2}(1 - j) \sum_{k=m}^{\infty} k(1 - B)(p + k + 1)^n a_k + \frac{1}{2}(1 + j) \sum_{k=m}^{\infty} k(1 - B)(p + k + 1)^n b_k \\ &\leq \frac{1}{2}(1 - j)(A - B)p + \frac{1}{2}(1 + j)(A - B)p. \end{aligned}$$

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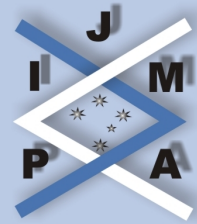
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The proof is complete. □

Theorem 6.3. Let $f_1(z), f_2(z), \dots, f_\ell(z)$ defined by

$$(6.1) \quad f_i(z) = z^{-p} + \sum_{k=m}^{\infty} a_{k,i} z^k, \quad (a_{k,i} \geq 0, i = 1, 2, \dots, \ell, k \geq m, m \geq p)$$

be in the class $L(p, m, n, A, B)$, then the arithmetic mean of $f_i(z)$ ($i = 1, 2, \dots, \ell$) defined by

$$(6.2) \quad h(z) = \frac{1}{\ell} \sum_{i=1}^{\ell} f_i(z)$$

is also in the class $L(p, m, n, A, B)$.

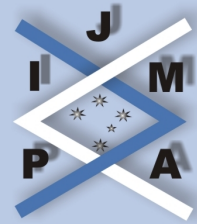
Proof. By (6.1), (6.2) we can write

$$h(z) = \frac{1}{\ell} \sum_{i=1}^{\ell} \left(z^{-p} + \sum_{k=m}^{\infty} a_{k,i} z^k \right) = z^{-p} + \sum_{k=m}^{\infty} \left(\frac{1}{\ell} \sum_{i=1}^{\ell} a_{k,i} \right) z^k.$$

Since $f_i(z) \in L(p, m, n, A, B)$ for every $i = 1, 2, \dots, \ell$, so by using Theorem 2.1, we prove that

$$\begin{aligned} & \sum_{k=m}^{\infty} k(1-B)(p+k+1)^n \left(\frac{1}{\ell} \sum_{i=1}^{\ell} a_{k,i} \right) \\ &= \frac{1}{\ell} \sum_{i=1}^{\ell} \left(\sum_{k=m}^{\infty} k(1-B)(p+k+1)^n a_{k,i} \right) \leq \frac{1}{\ell} \sum_{i=1}^{\ell} (A-B)p. \end{aligned}$$

The proof is complete. □



7. Convolution Properties

Theorem 7.1. If $f(z)$ and $g(z)$ belong to $L(p, m, n, A, B)$ such that

$$(7.1) \quad f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k, \quad g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k,$$

then

$$T(z) = z^{-p} + \sum_{k=m}^{\infty} (a_k^2 + b_k^2) z^k$$

is in the class $L(p, m, n, A_1, B_1)$ such that $A_1 \geq (1 - B_1)\mu^2 + B_1$, where

$$\mu = \frac{\sqrt{2}(A - B)}{\sqrt{m(m + 2)^n(1 - B)}}.$$

Proof. Since $f, g \in L(p, m, n, A, B)$, Theorem 2.1 yields

$$\sum_{k=m}^{\infty} \left(\left[\frac{k(1 - B)(p + k + 1)^n}{(A - B)p} \right] a_k \right)^2 \leq 1$$

and

$$\sum_{k=m}^{\infty} \left(\left[\frac{k(1 - B)(p + k + 1)^n}{(A - B)p} \right] b_k \right)^2 \leq 1.$$

We obtain from the last two inequalities

$$(7.2) \quad \sum_{k=m}^{\infty} \frac{1}{2} \left[\frac{k(1 - B)(p + k + 1)^n}{(A - B)p} \right]^2 (a_k^2 + b_k^2) \leq 1.$$

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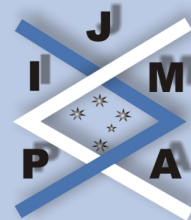
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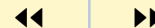
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However, $T(z) \in L(p, m, n, A_1, B_1)$ if and only if

$$(7.3) \quad \sum_{k=m}^{\infty} \left[\frac{k(1 - B_1)(p + k + 1)^n}{(A_1 - B_1)p} \right] (a_k^2 + b_k^2) \leq 1,$$

where $-1 \leq B_1 < A_1 \leq 1$, but (7.2) implies (7.3) if

$$\frac{k(1 - B_1)(p + k + 1)^n}{(A_1 - B_1)p} < \frac{1}{2} \left[\frac{k(1 - B)(p + k + 1)^n}{(A - B)p} \right]^2.$$

Hence, if

$$\frac{1 - B_1}{A_1 - B_1} < \frac{k(p + k + 1)^n}{2p} \alpha^2, \quad \text{where } \alpha = \frac{1 - B}{A - B}.$$

In other words,

$$\frac{1 - B_1}{A_1 - B_1} < \frac{k(k + 2)^n}{2} \alpha^2.$$

This is equivalent to

$$\frac{A_1 - B_1}{1 - B_1} > \frac{2}{k(k + 2)^n \alpha^2}.$$

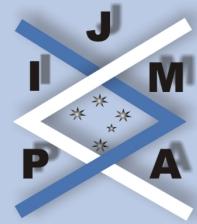
So we can write

$$(7.4) \quad \frac{A_1 - B_1}{1 - B_1} > \frac{2(A - B)^2}{m(m + 2)^n(1 - B)^2} = \mu^2.$$

Hence we get $A_1 \geq (1 - B_1)\mu^2 + B_1$. □

Theorem 7.2. Let $f(z)$ and $g(z)$ of the form (7.1) belong to $L(p, m, n, A, B)$. Then the convolution (or Hadamard product) of two functions f and g belong to the class, that is, $(f * g)(z) \in L(p, m, n, A_1, B_1)$, where $A_1 \geq (1 - B_1)v + B_1$ and

$$v = \frac{(A - B)^2}{m(1 - B)^2(m + 2)^n}.$$



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Proof. Since $f, g \in L(p, m, n, A, B)$, by using the Cauchy-Schwarz inequality and Theorem 2.1, we obtain

$$(7.5) \quad \sum_{k=m}^{\infty} \frac{k(1-B)(p+k+1)^n}{(A-B)p} \sqrt{a_k b_k} \\ \leq \left(\sum_{k=m}^{\infty} \frac{k(1-B)(p+k+1)^n}{(A-B)p} a_k \right)^{\frac{1}{2}} \left(\sum_{k=m}^{\infty} \frac{k(1-B)(p+k+1)^n}{(A-B)p} b_k \right)^{\frac{1}{2}} \leq 1.$$

We must find the values of A_1, B_1 so that

$$(7.6) \quad \sum_{k=m}^{\infty} \frac{k(1-B_1)(p+k+1)^n}{(A_1-B_1)p} a_k b_k < 1.$$

Therefore, by (7.5), (7.6) holds true if

$$(7.7) \quad \sqrt{a_k b_k} \leq \frac{(1-B)(A_1-B_1)}{(1-B_1)(A-B)}, \quad k \geq m, \quad m \geq p, \quad a_k \neq 0, \quad b_k \neq 0.$$

By (7.5), we have $\sqrt{a_k b_k} < \frac{(A-B)p}{k(1-B)(p+k+1)^n}$, therefore (7.7) holds true if

$$\frac{k(1-B_1)(p+k+1)^n}{(A_1-B_1)p} \leq \left[\frac{k(1-B)(p+k+1)^n}{(A-B)p} \right]^2,$$

which is equivalent to

$$\frac{(1-B_1)}{(A_1-B_1)} < \frac{k(1-B)^2(p+k+1)^n}{(A-B)^2p}.$$

Alternatively, we can write

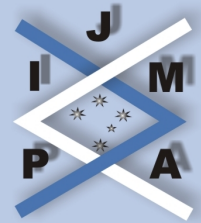
$$\frac{(1-B_1)}{(A_1-B_1)} < \frac{k(1-B)^2(k+2)^n}{(A-B)^2},$$

to obtain

$$\frac{A_1 - B_1}{1 - B_1} > \frac{(A - B)^2}{m(1 - B)^2(m + 2)^n} = v.$$

Hence we get $A_1 > v(1 - B_1) + B_1$.

□



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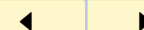
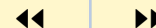
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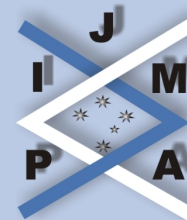
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