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GROWTH OF SOLUTIONS OF CERTAIN NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS

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ABSTRACT. In this paper, we investigate the growth of solutions of the differential equation $f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F$, where $A_0(z), \ldots, A_{k-1}(z), F(z) \not\equiv 0$ are entire functions, and we obtain general estimates of the hyper-exponent of convergence of distinct zeros and the hyper-order of solutions for the above equation.

Key words and phrases: Differential equations, Hyper-order, Hyper-exponent of convergence of distinct zeros, Wiman-Valiron theory.

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1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper, we will use the standard notations of the Nevanlinna value distribution theory (see [8]). In addition, we use the notations $\sigma(f)$ and $\mu(f)$ to denote respectively the order and the lower order of growth of $f(z)$. Recalling the following definitions of hyper-order and hyper-exponent of convergence of distinct zeros.

Definition 1.1. ([3] – [6], [12]). Let f be an entire function. Then the hyper-order $\sigma_2(f)$ of $f(z)$ is defined by

$$\sigma_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log T(r,f)}{\log r} = \overline{\lim_{r\to+\infty}} \frac{\log\log\log M(r,f)}{\log r}, \tag{1.1}$$

where $T(r, f)$ is the Nevanlinna characteristic function of f (see [8]), and $M(r,f) = \max_{|z|=r} |f(z)|$.

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Definition 1.2. ([5]). Let f be an entire function. Then the hyper-exponent of convergence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log \overline{N}\left(r, \frac{1}{f}\right)}{\log r}, \tag{1.2}$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{|z| < r\}$. We define the linear measure of a set $E \subset [0, +\infty[$ by $m(E) = \int_0^{+\infty} \chi_E(t)\, dt$ and the logarithmic measure of a set $F \subset [1, +\infty[$ by $lm(F) = \int_1^{+\infty} \frac{\chi_F(t)dt}{t}$, where χ_H is the characteristic function of a set H. The upper and the lower densities of E are defined by

$$\overline{dens}E = \overline{\lim_{r\to+\infty}} \frac{m(E \cap [0, r])}{r}, \ \underline{dens}E = \underline{\lim}_{r\to+\infty} \frac{m(E \cap [0, r])}{r}. \tag{1.3}$$

The upper and the lower logarithmic densities of F are defined by

$$\overline{\log dens}(F) = \overline{\lim_{r\to+\infty}} \frac{lm(F \cap [1, r])}{\log r}, \ \underline{\log dens}(F) = \underline{\lim}_{r\to+\infty} \frac{lm(F \cap [1, r])}{\log r}. \tag{1.4}$$

In the study of the solutions of complex differential equations, the growth of a solution is a very important property. Recently, Z. X. Chen and C. C. Yang have investigated the growth of solutions of the non-homogeneous linear differential equation of second order

$$f'' + A_1(z) f' + A_0(z) f = F, \tag{1.5}$$

and have obtained the following two results:

Theorem A. [5, p. 276]. *Let E be a set of complex numbers satisfying $\overline{dens}\{|z| : z \in E\} > 0$, and let $A_0(z)$, $A_1(z)$ be entire functions, with $\sigma(A_1) \le \sigma(A_0) = \sigma < +\infty$ such that for a real constant $C\,(> 0)$ and for any given $\varepsilon > 0$,*

$$|A_1(z)| \le \exp\left(o(1)|z|^{\sigma-\varepsilon}\right) \tag{1.6}$$

and

$$|A_0(z)| \ge \exp\left((1 + o(1))\, C\, |z|^{\sigma-\varepsilon}\right) \tag{1.7}$$

as $z \to \infty$ for $z \in E$, and let $F \not\equiv 0$ be an entire function with $\sigma(F) < +\infty$. Then every entire solution $f(z)$ of the equation (1.5) satisfies $\overline{\lambda_2}(f) = \sigma_2(f) = \sigma$, with at most one exceptional solution f_0 satisfying $\sigma(f_0) < \sigma$.

Theorem B. [5, p. 276]. *Let $A_1(z)$, $A_0(z) \not\equiv 0$ be entire functions such that $\sigma(A_0) < \sigma(A_1) < \frac{1}{2}$ (or A_1 is transcendental, $\sigma(A_1) = 0$, A_0 is a polynomial), and let $F \not\equiv 0$ be an entire function. Consider a solution f of the equation (1.5), we have*

(i) *If $\sigma(F) < \sigma(A_1)$ (or F is a polynomial when A_1 is transcendental, $\sigma(A_1) = 0$, A_0 is a polynomial), then every entire solution $f(z)$ of (1.5) satisfies $\overline{\lambda_2}(f) = \sigma_2(f) = \sigma(A_1)$.*

(ii) *If $\sigma(A_1) \le \sigma(F) < +\infty$, then every entire solution $f(z)$ of (1.5) satisfies $\overline{\lambda_2}(f) = \sigma_2(f) = \sigma(A_1)$, with at most one exceptional solution f_0 satisfying $\sigma(f_0) < \sigma(A_1)$.*

For $k \ge 2$, we consider the non-homogeneous linear differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F, \tag{1.8}$$

where $A_0(z), \ldots, A_{k-1}(z)$ and $F(z) \not\equiv 0$ are entire functions. It is well-known that all solutions of equation (1.8) are entire functions.

Recently, the concepts of hyper-order [3] – [6] and iterated order [10] were used to further investigate the growth of infinite order solutions of complex differential equations. The main purposes of this paper are to investigate the hyper-exponent of convergence of distinct zeros and the hyper-order of infinite order solutions for the above equation. We will prove the following two theorems:

Theorem 1.1. *Let E be a set of complex numbers satisfying $\overline{dens}\{|z| : z \in E\} > 0$, and let $A_0(z), \ldots, A_{k-1}(z)$ be entire functions, with $\max\{\sigma(A_j) : j = 1, \ldots, k\} \le \sigma(A_0) = \sigma < +\infty$ such that for real constants $0 \le \beta < \alpha$ and for any given $\varepsilon > 0$,*

$$|A_j(z)| \le \exp\left(\beta |z|^{\sigma-\varepsilon}\right) \quad (j = 1, \ldots, k-1) \tag{1.9}$$

and

$$|A_0(z)| \ge \exp\left(\alpha |z|^{\sigma-\varepsilon}\right) \tag{1.10}$$

as $z \to \infty$ for $z \in E$, and let $F \not\equiv 0$ be an entire function with $\sigma(F) < +\infty$. Then every entire solution $f(z)$ of the equation (1.8) satisfies $\overline{\lambda_2}(f) = \sigma_2(f) = \sigma$, with at most one exceptional solution f_0 satisfying $\sigma(f_0) < \sigma$.

Theorem 1.2. *Let $A_0(z), \ldots, A_{k-1}(z)$ be entire functions with $A_0(z) \not\equiv 0$ such that $\max\{\sigma(A_j) : j = 0, 2, \ldots, k-1\} < \sigma(A_1) < \frac{1}{2}$ (or A_1 is transcendental, $\sigma(A_1) = 0$, $A_0, A_2, \ldots, A_{k-1}$ are polynomials), and let $F \not\equiv 0$ be an entire function. Consider a solution f of the equation (1.8), we have*

(i) *If $\sigma(F) < \sigma(A_1)$ (or F is a polynomial when A_1 is transcendental, $\sigma(A_1) = 0$, $A_0, A_2, \ldots, A_{k-1}$ are polynomials), then every entire solution $f(z)$ of (1.8) satisfies $\overline{\lambda_2}(f) = \sigma_2(f) = \sigma(A_1)$.*

(ii) *If $\sigma(A_1) \le \sigma(F) < +\infty$, then every entire solution $f(z)$ of (1.8) satisfies $\overline{\lambda_2}(f) = \sigma_2(f) = \sigma(A_1)$, with at most one exceptional solution f_0 satisfying $\sigma(f_0) < \sigma(A_1)$.*

2. PRELIMINARY LEMMAS

Our proofs depend mainly upon the following lemmas.

Lemma 2.1. *([3]). Let E be a set of complex numbers satisfying $\overline{dens}\,\{|z| : z \in E\} > 0$, and let $A_0(z), \ldots, A_{k-1}(z)$ be entire functions, with $\max\{\sigma(A_j) : j = 1, \ldots, k\} \le \sigma(A_0) = \sigma < +\infty$ such that for some real constants $0 \le \beta < \alpha$ and for any given $\varepsilon > 0$,*

$$|A_j(z)| \le \exp\left(\beta |z|^{\sigma-\varepsilon}\right) \quad (j = 1, \ldots, k-1) \tag{2.1}$$

and

$$|A_0(z)| \ge \exp\left(\alpha |z|^{\sigma-\varepsilon}\right) \tag{2.2}$$

as $z \to \infty$ for $z \in E$. Then every entire solution $f \not\equiv 0$ of the equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0 \tag{2.3}$$

satisfies $\sigma(f) = +\infty$ and $\sigma_2(f) = \sigma(A_0)$.

Lemma 2.2. *([7]). Let $f(z)$ be a nontrivial entire function, and let $\alpha > 1$ and $\varepsilon > 0$ be given constants. Then there exist a constant $c > 0$ and a set $E \subset [0, +\infty)$ having finite linear measure such that for all z satisfying $|z| = r \notin E$, we have*

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le c\left[T(\alpha r, f)\, r^{\varepsilon} \log T(\alpha r, f)\right]^j \quad (j \in \mathbf{N}). \tag{2.4}$$

Lemma 2.3. *([7]). Let $f(z)$ be a transcendental meromorphic function, and let $\alpha > 1$ be a given constant. Then there exists a set $E \subset (1, +\infty)$ of finite logarithmic measure and a constant $B > 0$ that depends only on α and (m, n) (m, n positive integers with $m < n$) such that for all z satisfying $|z| = r \notin [0, 1] \cup E$, we have*

$$\left| \frac{f^{(n)}(z)}{f^{(m)}(z)} \right| \leq B \left[\frac{T(\alpha r, f)}{r} (\log^{\alpha} r) \log T(\alpha r, f) \right]^{n-m}. \tag{2.5}$$

Lemma 2.4. *([5]). Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of infinite order with the hyper-order $\sigma_2(f) = \sigma$, $\mu(r)$ be the maximum term, i.e $\mu(r) = \max\{|a_n| r^n; n = 0, 1, \ldots\}$ and let $\nu_f(r)$ be the central index of f, i.e $\nu_f(r) = \max\{m, \mu(r) = |a_m| r^m\}$. Then*

$$\overline{\lim_{r \to +\infty}} \frac{\log \log \nu_f(r)}{\log r} = \sigma. \tag{2.6}$$

Lemma 2.5. *(Wiman-Valiron, [9, 11]). Let $f(z)$ be a transcendental entire function and let z be a point with $|z| = r$ at which $|f(z)| = M(r, f)$. Then for all $|z|$ outside a set E of r of finite logarithmic measure, we have*

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z} \right)^j (1 + o(1)) \;\; (j \text{ is an integer, } r \notin E). \tag{2.7}$$

Lemma 2.6. *([1]). Let $f(z)$ be an entire function of order $\sigma(f) = \sigma < \frac{1}{2}$, and denote $A(r) = \inf_{|z|=r} \log |f(z)|$, $B(r) = \sup_{|z|=r} \log |f(z)|$. If $\sigma < \alpha < 1$, then*

$$\underline{\log dens} \{r : \; A(r) > (\cos \pi\alpha) B(r)\} \geq 1 - \frac{\sigma}{\alpha}. \tag{2.8}$$

Lemma 2.7. *([2]). Let $f(z)$ be an entire function with $\mu(f) = \mu < \frac{1}{2}$ and $\mu < \sigma(f) = \sigma$. If $\mu \leq \delta < \min\left(\sigma, \frac{1}{2}\right)$ and $\delta < \alpha < \frac{1}{2}$, then*

$$\overline{\log dens} \left\{r : \; A(r) > (\cos \pi\alpha) B(r) > r^{\delta}\right\} > C(\sigma, \delta, \alpha), \tag{2.9}$$

where $C(\sigma, \delta, \alpha)$ is a positive constant depending only on σ, δ and α.

Lemma 2.8. *Suppose that $A_0(z), \ldots, A_{k-1}(z)$ are entire functions such that $A_0(z) \not\equiv 0$ and*

$$\max\{\sigma(A_j) \; : j = 0, 2, \ldots, k-1\} < \sigma(A_1) < \frac{1}{2}. \tag{2.10}$$

Then every transcendental solution $f \not\equiv 0$ of (2.3) is of infinite order.

Proof. Using the same argument as in the proof of Theorem 4 in [6, p. 222], we conclude that $\sigma(f) = +\infty$. □

3. PROOF OF THEOREM 1.1

We affirm that (1.8) can only possess at most one exceptional solution f_0 such that $\sigma(f_0) < \sigma$. In fact, if f^* is a second solution with $\sigma(f^*) < \sigma$, then $\sigma(f_0 - f^*) < \sigma$. But $f_0 - f^*$ is a solution of the corresponding homogeneous equation (2.3) of (1.8). This contradicts Lemma 2.1. We assume that f is a solution of (1.8) with $\sigma(f) = +\infty$ and $f_1, \ldots, f_k$ are k entire solutions of the corresponding homogeneous equation (2.3). Then by Lemma 2.1, we have $\sigma_2(f_j) = \sigma(A_0) = \sigma \; (j = 1, \ldots, k)$. By variation of parameters, f can be expressed in the form

$$f(z) = B_1(z) f_1(z) + \cdots + B_k(z) f_k(z), \tag{3.1}$$

where $B_1(z), \dots, B_k(z)$ are determined by

$$B_1'(z) f_1(z) + \cdots + B_k'(z) f_k(z) = 0$$
$$B_1'(z) f_1'(z) + \cdots + B_k'(z) f_k'(z) = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$B_1'(z) f_1^{(k-1)}(z) + \cdots + B_k'(z) f_k^{(k-1)}(z) = F. \tag{3.2}$$

Noting that the Wronskian $W(f_1, f_2, \dots, f_k)$ is a differential polynomial in $f_1, f_2, \dots, f_k$ with constant coefficients, it easy to deduce that $\sigma_2(W) \le \sigma_2(f_j) = \sigma(A_0) = \sigma$. Set

$$W_i = \begin{vmatrix} f_1, \dots, \overset{(i)}{0}, \dots, f_k \\ \cdots \\ \cdots \\ f_1^{(k-1)}, \dots, F, \dots, f_k^{(k-1)} \end{vmatrix} = F \cdot g_i \ \ (i = 1, \dots, k), \tag{3.3}$$

where g_i are differential polynomials in $f_1, f_2, \dots, f_k$ with constant coefficients. So

$$\sigma_2(g_i) \le \sigma_2(f_j) = \sigma(A_0), \ B_i' = \frac{W_i}{W} = \frac{F \cdot g_i}{W} \ \ (i = 1, \dots, k) \tag{3.4}$$

and

$$\sigma_2(B_i) = \sigma_2\left(B_i'\right) \le \max\left(\sigma_2(F), \ \sigma(A_0)\right) = \sigma(A_0) \ \ (i = 1, \dots, k), \tag{3.5}$$

because $\sigma_2(F) = 0 \ (\sigma(F) < +\infty)$. Then from (3.1) and (3.5), we get

$$\sigma_2(f) \le \max\left(\sigma_2(f_j), \ \sigma_2(B_i)\right) = \sigma(A_0). \tag{3.6}$$

Now from (1.8), it follows that

$$|A_0(z)| \le \left|\frac{f^{(k)}}{f}\right| + |A_{k-1}(z)| \left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1(z)| \left|\frac{f'}{f}\right| + \left|\frac{F}{f}\right|. \tag{3.7}$$

Then by Lemma 2.2, there exists a set $E_1 \subset [0, +\infty)$ with a finite linear measure such that for all z satisfying $|z| = r \notin E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le r\left[T(2r, f)\right]^{k+1} \ \ (j = 1, \dots, \ k). \tag{3.8}$$

Also, by the hypothesis of Theorem 1.1, there exists a set E_2 with $\overline{dens}\ \{|z| : \ z \in E_2\} > 0$ such that for all z satisfying $z \in E_2$, we have

$$|A_j(z)| \le \exp\left(\beta |z|^{\sigma - \varepsilon}\right) \ \ (\ j = 1, \dots, \ k-1) \tag{3.9}$$

and

$$|A_0(z)| \ge \exp\left(\alpha |z|^{\sigma - \varepsilon}\right) \tag{3.10}$$

as $z \to \infty$. Since $\sigma(f) = +\infty$, then for a given arbitrary large $\rho > \sigma(F)$,

$$M(r, f) \ge \exp(r^{\rho}) \tag{3.11}$$

holds for sufficiently large r. On the other hand, for a given ε with $0 < \varepsilon < \rho - \sigma(F)$, we have

$$|F(z)| \le \exp\left(r^{\sigma(F)+\varepsilon}\right), \ \ \left|\frac{F(z)}{f(z)}\right| \le \exp\left(r^{\sigma(F)+\varepsilon} - r^{\rho}\right) \to 0 \ \ (r \to +\infty), \tag{3.12}$$

where $|f(z)| = M(r, f)$ and $|z| = r$. Hence from (3.7) – (3.10) and (3.12), it follows that for all z satisfying $z \in E_2$, $|z| = r \notin E_1$ and $|f(z)| = M(r, f)$

$$\exp\left(\alpha |z|^{\sigma - \varepsilon}\right) \le |z| \left[T(2|z|, f)\right]^{k+1} \left[1 + (k-1)\exp\left(\beta |z|^{\sigma - \varepsilon}\right)\right] + o(1) \tag{3.13}$$

as $z \to \infty$. Thus there exists a set $E \subset [0, +\infty)$ with a positive upper density such that

$$\exp\left(\alpha r^{\sigma-\varepsilon}\right) \le dr \exp\left(\beta r^{\sigma-\varepsilon}\right) \left[T\left(2r, f\right)\right]^{k+1} \tag{3.14}$$

as $r \to +\infty$ in E, where $d\,(> 0)$ is some constant. Therefore

$$\sigma_2\left(f\right) = \overline{\lim_{r\to+\infty}} \frac{\log\log T\left(r,f\right)}{\log r} \ge \sigma - \varepsilon. \tag{3.15}$$

Since ε is arbitrary, then by (3.15) we get $\sigma_2\left(f\right) \ge \sigma\left(A_0\right) = \sigma$. This and the fact that $\sigma_2\left(f\right) \le \sigma$ yield $\sigma_2\left(f\right) = \sigma\left(A_0\right) = \sigma$.

By (1.8), it is easy to see that if f has a zero at z_0 of order $\alpha\,(> k)$, then F must have a zero at z_0 of order $\alpha - k$. Hence,

$$n\left(r, \frac{1}{f}\right) \le k\,\overline{n}\left(r, \frac{1}{f}\right) + n\left(r, \frac{1}{F}\right) \tag{3.16}$$

and

$$N\left(r, \frac{1}{f}\right) \le k\,\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{F}\right). \tag{3.17}$$

Now (1.8) can be rewritten as

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_1\frac{f'}{f} + A_0\right). \tag{3.18}$$

By (3.18), we have

$$m\left(r, \frac{1}{f}\right) \le \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + \sum_{j=1}^{k} m\left(r, A_{k-j}\right) + m\left(r, \frac{1}{F}\right) + O\left(1\right). \tag{3.19}$$

By (3.17) and (3.19), we get for $|z| = r$ outside a set E_3 of finite linear measure,

$$\begin{aligned} T\left(r,f\right) &= T\left(r, \frac{1}{f}\right) + O\left(1\right) \\ &\le k\overline{N}\left(r, \frac{1}{f}\right) + \sum_{j=1}^{k} T\left(r, A_{k-j}\right) + T\left(r, F\right) + O\left(\log\left(rT\left(r,f\right)\right)\right). \end{aligned} \tag{3.20}$$

For sufficiently large r, we have

$$O\left(\log r + \log T\left(r,f\right)\right) \le \frac{1}{2}T\left(r,f\right) \tag{3.21}$$

$$T\left(r, A_0\right) + \cdots + T\left(r, A_{k-1}\right) \le k\,r^{\sigma+\varepsilon} \tag{3.22}$$

$$T\left(r, F\right) \le r^{\sigma(F)+\varepsilon}. \tag{3.23}$$

Thus, by (3.20) – (3.23), we have

$$T\left(r,f\right) \le 2k\,\overline{N}\left(r, \frac{1}{f}\right) + 2k\,r^{\sigma+\varepsilon} + 2r^{\sigma(F)+\varepsilon} \quad \left(|z| = r \notin E_3\right). \tag{3.24}$$

Hence for any f with $\sigma_2\left(f\right) = \sigma$, by (3.24), we have $\sigma_2\left(f\right) \le \overline{\lambda_2}\left(f\right)$. Therefore, $\overline{\lambda_2}\left(f\right) = \sigma_2\left(f\right) = \sigma$.

4. PROOF OF THEOREM 1.2

Assume that $f(z)$ is an entire solution of (1.8). For case (i), we assume $\sigma(A_1) > 0$ (when $\sigma(A_1) = 0$, Theorem 1.2 clearly holds). By (1.8) we get

$$A_1 = \frac{F}{f'} - \frac{f^{(k)}}{f'} - A_{k-1}\frac{f^{(k-1)}}{f'} - \cdots - A_2\frac{f''}{f'} - A_0\frac{f}{f'} = \frac{F}{f}\frac{f}{f'} - \frac{f^{(k)}}{f'} - A_{k-1}\frac{f^{(k-1)}}{f'} - \cdots - A_2\frac{f''}{f'} - A_0\frac{f}{f'}. \tag{4.1}$$

By Lemma 2.3, we see that there exists a set $E_4 \subset (1, +\infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_4$, we have

$$\left|\frac{f^{(j)}(z)}{f'(z)}\right| \leq Br\,[T(2r,\ f)]^k \quad (j = 2, \ldots,\ k). \tag{4.2}$$

Now set $b = \max\{\sigma(A_j) : j = 0,\ 2, \ldots,\ k-1;\ \sigma(F)\}$, and we choose real numbers $\alpha,\ \beta$ such that

$$b < \alpha < \beta < \sigma(A_1). \tag{4.3}$$

Then for sufficiently large r, we have

$$|A_j(z)| \leq \exp(r^\alpha) \quad (j = 0, 2, \ldots, k-1), \tag{4.4}$$

$$|F(z)| \leq \exp(r^\alpha). \tag{4.5}$$

By Lemma 2.6 (if $\mu(A_1) = \sigma(A_1)$) or Lemma 2.7 (if $\mu(A_1) < \sigma(A_1)$) there exists a subset $E_5 \subset (1, +\infty)$ with logarithmic measure $lm(E_5) = \infty$ such that for all z satisfying $|z| = r \in E_5$, we have

$$|A_1(z)| > \exp\left(r^\beta\right). \tag{4.6}$$

Since $M(r, f) > 1$ for sufficiently large r, we have by (4.5)

$$\frac{|F(z)|}{M(r,f)} \leq \exp(r^\alpha). \tag{4.7}$$

On the other hand, by Lemma 2.5, there exists a set $E_6 \subset (1, +\infty)$ of finite logarithmic measure such that (2.7) holds for some point z satisfying $|z| = r \notin [0,1] \cup E_6$ and $|f(z)| = M(r,f)$. By (2.7), we get

$$\left|\frac{f'(z)}{f(z)}\right| \geq \frac{1}{2}\left|\frac{\nu_f(r)}{z}\right| > \frac{1}{2r}$$

or

$$\left|\frac{f(z)}{f'(z)}\right| < 2r. \tag{4.8}$$

Now by (4.1), (4.2), (4.4), and (4.6) – (4.8), we get

$$\exp\left(r^\beta\right) \leq Lr\,[T(2r,\ f)]^k\ 2\exp(r^\alpha)\,2r$$

for $|z| = r \in E_5 \setminus ([0,1] \cup E_4 \cup E_6)$ and $|f(z)| = M(r,f)$, where $L(>0)$ is some constant. From this and since β is arbitrary, we get $\sigma(f) = +\infty$ and $\sigma_2(f) \geq \sigma(A_1)$.

On the other hand, for any given $\varepsilon > 0$, if r is sufficiently large, we have

$$|A_j(z)| \leq \exp\left(r^{\sigma(A_1)+\varepsilon}\right) \quad (j = 0,\ 1, \ldots,\ k-1), \tag{4.9}$$

$$|F(z)| \leq \exp\left(r^{\sigma(A_1)+\varepsilon}\right). \tag{4.10}$$

Since $M(r,f) > 1$ for sufficiently large r, we have by (4.10)

$$\frac{|F(z)|}{M(r,f)} \leq \exp\left(r^{\sigma(A_1)+\varepsilon}\right). \tag{4.11}$$

Substituting (2.7), (4.9) and (4.11) into (1.8), we obtain

$$\begin{aligned}\left(\frac{\nu_f(r)}{|z|}\right)^k |1+o(1)| &\leq \exp\left(r^{\sigma(A_1)+\varepsilon}\right)\left(\frac{\nu_f(r)}{|z|}\right)^{k-1}|1+o(1)| \\ &\quad + \exp\left(r^{\sigma(A_1)+\varepsilon}\right)\left(\frac{\nu_f(r)}{|z|}\right)^{k-2}|1+o(1)| + \cdots \\ &\quad + \exp\left(r^{\sigma(A_1)+\varepsilon}\right)\left(\frac{\nu_f(r)}{|z|}\right)|1+o(1)| + 2\exp\left(r^{\sigma(A_1)+\varepsilon}\right),\end{aligned} \tag{4.12}$$

where z satisfies $|z| = r \notin [0,1] \cup E_6$ and $|f(z)| = M(r,f)$. By (4.12), we get

$$\overline{\lim_{r\to+\infty}} \frac{\log\log \nu_f(r)}{\log r} \leq \sigma(A_1) + \varepsilon. \tag{4.13}$$

Since ε is arbitrary, by (4.13) and Lemma 2.4 we have $\sigma_2(f) \leq \sigma(A_1)$. This and the fact that $\sigma_2(f) \geq \sigma(A_1)$ yield $\sigma_2(f) = \sigma(A_1)$.

By a similar argument to that used in the proof of Theorem 1.1, we can get $\overline{\lambda_2}(f) = \sigma_2(f) = \sigma(A_1)$.

Finally, case (ii) can also be obtained by using Lemma 2.8 and an argument similar to that in the proof of Theorem 1.1.

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