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# ON HADAMARD'S INEQUALITY ON A DISK

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ABSTRACT. In this paper an inequality of Hadamard type for convex functions defined on a disk in the plane is proved. Some mappings naturally connected with this inequality and related results are also obtained.

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## 1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known in the literature as Hadamard's inequality for convex mappings. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping $f$.

In the paper [4] (see also [6] and [7]) the following mapping naturally connected with Hadamard's result is considered

$$H : [0,1] \to \mathbb{R}, \quad H(t) := \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx.$$

The following properties are also proved:

$(i)$ $H$ is convex and monotonic nondecreasing.
$(ii)$ One has the bounds

$$\sup_{t\in[0,1]} H(t) = H(1) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

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and

$$\inf_{t\in[0,1]} H(t) = H(0) = f\left(\frac{a+b}{2}\right).$$

Another mapping also closely connected with Hadamard's inequality is the following one [6] (see also [7])

$$F : [0,1] \to \mathbb{R}, \quad F(t) := \frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx + (1-t)y)\, dxdy.$$

The properties of this mapping are itemized below:

$(i)$ $F$ is convex on $[0,1]$ and monotonic nonincreasing on $\left[0, \frac{1}{2}\right]$ and nondecreasing on $\left[\frac{1}{2}, 1\right]$.

$(ii)$ $F$ is symmetric about $\frac{1}{2}$. That is,

$$F(t) = F(1-t), \quad \text{for all } t \in [0,1].$$

$(iii)$ One has the bounds

$$\sup_{t\in[0,1]} F(t) = F(0) = F(1) = \frac{1}{b-a}\int_a^b f(x)\, dx$$

and

$$\inf_{t\in[0,1]} F(t) = F\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2}\int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dxdy \geq f\left(\frac{a+b}{2}\right).$$

$(iv)$ The following inequality holds

$$F(t) \geq \max\{H(t), H(1-t)\}, \quad \text{for all } t \in [0,1].$$

In this paper we will point out a similar inequality to Hadamard's that applies to convex mappings defined on a disk embedded in the plane $\mathbb{R}^2$. We will also consider some mappings similar in a sense to the mappings $H$ and $F$ and establish their main properties.

For recent refinements, counterparts, generalizations and new Hadamard's type inequalities, see the papers [1]-[11] and [14]-[15] and the book [13].

## 2. HADAMARD'S INEQUALITY ON THE DISK

Let us consider a point $C = (a,b) \in \mathbb{R}^2$ and the disk $D(C,R)$ centered at the point $C$ and having the radius $R > 0$. The following inequality of Hadamard type holds.

**Theorem 2.1.** *If the mapping $f : D(C,R) \to \mathbb{R}$ is convex on $D(C,R)$, then one has the inequality*

$$f(C) \leq \frac{1}{\pi R^2}\iint_{D(C,R)} f(x,y)\, dxdy \leq \frac{1}{2\pi R}\int_{\mathfrak{S}(C,R)} f(\gamma)\, dl(\gamma) \tag{2.1}$$

*where $\mathfrak{S}(C,R)$ is the circle centered at the point $C$ with radius $R$. The above inequalities are sharp.*

*Proof.* Consider the transformation of the plane $\mathbb{R}^2$ in itself given by

$$h : \mathbb{R}^2 \to \mathbb{R}^2, \quad h = (h_1, h_2) \text{ and } h_1(x,y) = -x + 2a, \;\; h_2(x,y) = -y + 2b.$$

Then $h(D(C,R)) = D(C,R)$ and since

$$\frac{\partial(h_1,h_2)}{\partial(x,y)} = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1,$$

we have the change of variable

$$\begin{aligned}\iint_{D(C,R)} f(x,y)\,dxdy &= \iint_{D(C,R)} f(h_1(x,y), h_2(x,y)) \left|\frac{\partial(h_1,h_2)}{\partial(x,y)}\right| dxdy \\ &= \iint_{D(C,R)} f(-x+2a,-y+2b)\,dxdy.\end{aligned}$$

Now, by the convexity of $f$ on $D(C,R)$ we also have

$$\frac{1}{2}\left[f(x,y)+f(-x+2a,-y+2b)\right] \geq f(a,b)$$

which gives, by integration on the disk $D(C,R)$, that

$$\begin{aligned}\frac{1}{2}\left[\iint_{D(C,R)} f(x,y)\,dxdy + \iint_{D(C,R)} f(-x+2a,-y+2b)\,dxdy\right] \\ \geq f(a,b)\iint_{D(C,R)} dxdy = \pi R^2 f(a,b).\end{aligned} \tag{2.2}$$

In addition, as

$$\iint_{D(C,R)} f(x,y)\,dxdy = \iint_{D(C,R)} f(-x+2a,-y+2b)\,dxdy,$$

then by the inequality (2.2) we obtain the first part of (2.1).

Now, consider the transformation

$$g=(g_1,g_2):\ [0,R]\times[0,2\pi] \to D(C,R)$$

given by

$$g:\begin{cases} g_1(r,\theta)=r\cos\theta+a, \\ g_2(r,\theta)=r\sin\theta+b, \end{cases} \qquad r\in[0,R],\ \theta\in[0,2\pi].$$

Then we have

$$\frac{\partial(g_1,g_2)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r.$$

Thus, we have the change of variable

$$\begin{aligned}\iint_{D(C,R)} f(x,y)\,dxdy &= \int_0^R\int_0^{2\pi} f(g_1(r,\theta), g_2(r,\theta))\left|\frac{\partial(g_1,g_2)}{\partial(r,\theta)}\right| drd\theta \\ &= \int_0^R\int_0^{2\pi} f(r\cos\theta+a, r\sin\theta+b)\,rdrd\theta.\end{aligned}$$

Note that, by the convexity of $f$ on $D(C,R)$, we have

$$\begin{aligned}f(r\cos\theta+a,r\sin\theta+b) &= f\left(\frac{r}{R}(R\cos\theta+a,R\sin\theta+b)+\left(1-\frac{r}{R}\right)(a,b)\right) \\ &\leq \frac{r}{R}f(R\cos\theta+a,R\sin\theta+b)+\left(1-\frac{r}{R}\right)f(a,b),\end{aligned}$$

which yields that

$$f(r\cos\theta+a,r\sin\theta+b)\,r \leq \frac{r^2}{R}f(R\cos\theta+a,R\sin\theta+b)+r\left(1-\frac{r}{R}\right)f(a,b)$$

for all $(r,\theta)\in[0,R]\times[0,2\pi]$.

Integrating on $[0,R]\times[0,2\pi]$ we get

$$\begin{aligned}\iint_{D(C,R)} f(x,y)\,dxdy &\leq \int_0^R \frac{r^2}{R}dr\int_0^{2\pi} f(R\cos\theta+a,R\sin\theta+b)\,d\theta \\ &\quad + f(a,b)\int_0^{2\pi}d\theta\int_0^R r\left(1-\frac{r}{R}\right)dr \\ &= \frac{R^2}{3}\int_0^{2\pi} f(R\cos\theta+a,R\sin\theta+b)\,d\theta+\frac{\pi R^2}{3}f(a,b).\end{aligned} \tag{2.3}$$

Now, consider the curve $\gamma:[0,2\pi]\to\mathbb{R}^2$ given by

$$\gamma:\begin{cases} x(\theta):=R\cos\theta+a, \\ y(\theta):=R\sin\theta+b,\end{cases}\qquad \theta\in[0,2\pi].$$

Then $\gamma([0,2\pi])=\mathfrak{S}(C,R)$ and we write (integrating with respect to arc length)

$$\begin{aligned}\int_{\mathfrak{S}(C,R)} f(\gamma)\,dl(\gamma) &= \int_0^{2\pi} f(x(\theta),y(\theta))\left([\dot{x}(\theta)]^2+[\dot{y}(\theta)]^2\right)^{\frac{1}{2}}d\theta \\ &= R\int_0^{2\pi} f(R\cos\theta+a,R\sin\theta+b)\,d\theta.\end{aligned}$$

By the inequality (2.3) we obtain

$$\iint_{D(C,R)} f(x,y)\,dxdy \leq \frac{R}{3}\int_{\mathfrak{S}(C,R)} f(\gamma)\,dl(\gamma)+\frac{\pi R^2}{3}f(a,b)$$

which gives the following inequality which is interesting in itself

$$\frac{1}{\pi R^2}\iint_{D(C,R)} f(x,y)\,dxdy \leq \frac{2}{3}\cdot\frac{1}{2\pi R}\int_{\mathfrak{S}(C,R)} f(\gamma)\,dl(\gamma)+\frac{1}{3}f(a,b). \tag{2.4}$$

As we proved that

$$f(C)\leq\frac{1}{\pi R^2}\iint_{D(C,R)} f(x,y)\,dxdy,$$

then by the inequality (2.4) we deduce the inequality

$$f(C)\leq\frac{1}{2\pi R}\int_{\mathfrak{S}(C,R)} f(\gamma)\,dl(\gamma). \tag{2.5}$$

Finally, by (2.5) and (2.4) we have

$$\begin{aligned}\frac{1}{\pi R^2}\iint_{D(C,R)} f(x,y)\,dxdy &\leq \frac{2}{3}\cdot\frac{1}{2\pi R}\int_{\mathfrak{S}(C,R)} f(\gamma)\,dl(\gamma)+\frac{1}{3}f(C) \\ &\leq \frac{1}{2\pi R}\int_{\mathfrak{S}(C,R)} f(\gamma)\,dl(\gamma)\end{aligned}$$

and the second part of (2.1) is proved.

Now, consider the map $f_0:D(C,R)\to\mathbb{R},\ \ f_0(x,y)=1$. Thus

$$\begin{aligned}1 &= f_0(\lambda(x,y)+(1-\lambda)(u,z)) \\ &= \lambda f_0(x,y)+(1-\lambda)f_0(u,z)=1.\end{aligned}$$

Therefore $f_0$ is convex on $D(C,R)\to\mathbb{R}$. We also have

$$f_0(C)=1,\ \ \frac{1}{\pi R^2}\iint_{D(C,R)} f_0(x,y)\,dxdy=1 \text{ and } \frac{1}{2\pi R}\int_{\mathfrak{S}(C,R)} f_0(\gamma)\,dl(\gamma)=1,$$

which shows us the inequalities (2.1) are sharp. □

## 3. SOME MAPPINGS CONNECTED TO HADAMARD'S INEQUALITY ON THE DISK

As above, assume that the mapping $f : D(C,R) \to \mathbb{R}$ is a convex mapping on the disk centered at the point $C = (a,b) \in \mathbb{R}^2$ and having the radius $R > 0$. Consider the mapping $H : [0,1] \to \mathbb{R}$ associated with the function $f$ and given by

$$H(t) := \frac{1}{\pi R^2} \iint_{D(C,R)} f(t(x,y) + (1-t)C)\, dxdy,$$

which is well-defined for all $t \in [0,1]$.

The following theorem contains the main properties of this mapping.

**Theorem 3.1.** *With the above assumption, we have:*

$(i)$ *The mapping $H$ is convex on* $[0,1]$.

$(ii)$ *One has the bounds*

$$\inf_{t\in[0,1]} H(t) = H(0) = f(C) \tag{3.1}$$

*and*

$$\sup_{t\in[0,1]} H(t) = H(1) = \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\, dxdy. \tag{3.2}$$

$(iii)$ *The mapping $H$ is monotonic nondecreasing on* $[0,1]$.

*Proof.* $(i)$ Let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Then we have

$$\begin{aligned} H(\alpha t_1 + \beta t_2) &= \frac{1}{\pi R^2} \iint_{D(C,R)} f(\alpha (t_1 (x,y) + (1-t_1) C) \\ &\quad + \beta (t_2 (x,y) + (1-t_2) C))\, dxdy \\ &\leq \alpha \cdot \frac{1}{\pi R^2} \iint_{D(C,R)} f(t_1 (x,y) + (1-t_1) C)\, dxdy \\ &\quad + \beta \cdot \frac{1}{\pi R^2} \iint_{D(C,R)} f(t_2 (x,y) + (1-t_2) C)\, dxdy \\ &= \alpha H(t_1) + \beta H(t_2), \end{aligned}$$

which proves the convexity of $H$ on $[0,1]$.

$(ii)$ We will prove the following identity

$$H(t) = \frac{1}{\pi t^2 R^2} \iint_{D(C,tR)} f(x,y)\, dxdy \tag{3.3}$$

for all $t \in (0,1]$.

Fix $t$ in $(0,1]$ and consider the transformation $g = (\psi, \eta) : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$g : \begin{cases} \psi(x,y) := tx + (1-t)a, \\ \eta(x,y) := ty + (1-t)b, \end{cases} \qquad (x,y) \in \mathbb{R}^2;$$

then $g(D(C,R)) = D(C,tR)$.

Indeed, for all $(x,y) \in D(C,R)$ we have

$$(\psi - a)^2 + (\eta - b)^2 = t^2 \left[(x-a)^2 + (y-b)^2\right] \leq (tR)^2$$

which shows that $(\psi, \eta) \in D(C,tR)$, and conversely, for all $(\psi, \eta) \in D(C,tR)$, it is easy to see that there exists $(x,y) \in D(C,R)$ so that $g(x,y) = (\psi, \eta)$.

We have the change of variable

$$\begin{aligned}\iint_{D(C,tR)} f(\psi,\eta)\, d\psi d\eta &= \iint_{D(C,R)} f(\psi(x,y),\eta(x,y)) \left|\frac{\partial(\psi,\eta)}{\partial(x,y)}\right| dxdy \\ &= \iint_{D(C,R)} f(t(x,y)+(1-t)(a,b))\, t^2 dxdy \\ &= \pi R^2 t^2 H(t)\end{aligned}$$

since $\left|\frac{\partial(\psi,\eta)}{\partial(x,y)}\right| = t^2$, which gives us the equality (3.3).

Now, by the inequality (2.1), we have

$$\frac{1}{\pi t^2 R^2}\iint_{D(C,tR)} f(x,y)\, dxdy \geq f(C)$$

which gives us $H(t) \geq f(C)$ for all $t \in [0,1]$ and since $H(0) = f(C)$, we obtain the bound (3.1).

By the convexity of $f$ on the disk $D(C,R)$ we have

$$\begin{aligned} H(t) &\leq \frac{1}{\pi R^2}\iint_{D(C,R)} [tf(x,y)+(1-t)f(C)]\, dxdy \\ &= \frac{t}{\pi R^2}\iint_{D(C,R)} f(x,y)\, dxdy + (1-t) f(C) \\ &\leq \frac{t}{\pi R^2}\iint_{D(C,R)} f(x,y)\, dxdy + \frac{1-t}{\pi R^2}\iint_{D(C,R)} f(x,y)\, dxdy \\ &= \frac{1}{\pi R^2}\iint_{D(C,R)} f(x,y)\, dxdy. \end{aligned}$$

As we have

$$H(1) = \frac{1}{\pi R^2}\iint_{D(C,R)} f(x,y)\, dxdy,$$

then the bound (3.2) holds.

$(iii)$ Let $0 \leq t_1 < t_2 \leq 1$. Then, by the convexity of the mapping $H$ we have

$$\frac{H(t_2)-H(t_1)}{t_2-t_1} \geq \frac{H(t_1)-H(0)}{t_1} \geq 0$$

as $H(t_1) \geq H(0)$ for all $t_1 \in [0,1]$. This proves the monotonicity of the mapping $H$ in the interval $[0,1]$.

□

Further on, we shall introduce another mapping connected to Hadamard's inequality

$$h:[0,1]\to\mathbb{R}, \quad h(t) := \begin{cases} \dfrac{1}{2\pi tR}\displaystyle\int_{\mathfrak{S}(C,tR)} f(\gamma)\, dl(\gamma(t)), & t\in(0,1], \\ f(C), & t=0, \end{cases}$$

where $f : D(C,R) \to \mathbb{R}$ is a convex mapping on the disk $D(C,R)$ centered at the point $C = (a,b) \in \mathbb{R}^2$ and having the same radius $R$.

The main properties of this mapping are embodied in the following theorem.

**Theorem 3.2.** *With the above assumptions one has:*

$(i)$ *The mapping* $h:[0,1]\to\mathbb{R}$ *is convex on* $[0,1]$.

$(ii)$ *One has the bounds*

$$\inf_{t\in[0,1]} h(t) = h(0) = f(C) \tag{3.4}$$

*and*

$$\sup_{t\in[0,1]} h(t) = h(1) = \frac{1}{2\pi R}\int_{\mathfrak{S}(C,R)} f(\gamma)\, dl(\gamma). \tag{3.5}$$

$(iii)$ *The mapping* $h$ *is monotonic nondecreasing on* $[0,1]$.

$(iv)$ *We have the inequality*

$$H(t) \le h(t) \quad \textit{for all } t \in [0,1].$$

*Proof.* For a fixed $t$ in $[0,1]$ consider the curve

$$\gamma : \begin{cases} x(\theta) = tR\cos\theta + a, \\ y(\theta) = tR\sin\theta + b, \end{cases} \qquad \theta \in [0, 2\pi].$$

Then $\gamma([0,2\pi]) = \mathfrak{S}(C, tR)$ and

$$\begin{aligned}
&\frac{1}{2\pi tR}\int_{\mathfrak{S}(C,tR)} f(\gamma)\, dl(\gamma) \\
&\qquad = \frac{1}{2\pi tR}\int_0^{2\pi} f(tR\cos\theta + a, tR\sin\theta + b)\sqrt{(\dot{x}(\theta))^2 + (\dot{y}(\theta))^2}\, d\theta \\
&\qquad = \frac{1}{2\pi}\int_0^{2\pi} f(tR\cos\theta + a, tR\sin\theta + b)\, d\theta.
\end{aligned}$$

We note that, then

$$\begin{aligned}
h(t) &= \frac{1}{2\pi}\int_0^{2\pi} f(tR\cos\theta + a, tR\sin\theta + b)\, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(t(R\cos\theta, R\sin\theta) + (a,b))\, d\theta
\end{aligned}$$

for all $t \in [0,1]$.

$(i)$ Let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$. Then, by the convexity of $f$ we have that

$$\begin{aligned}
h(\alpha t_1 + \beta t_2) &= \frac{1}{2\pi}\int_0^{2\pi} f(\alpha [t_1 (R\cos\theta, R\sin\theta) + (a,b)] \\
&\quad + \beta [t_2 (R\cos\theta, R\sin\theta) + (a,b)])\, d\theta \\
&\le \alpha \cdot \frac{1}{2\pi}\int_0^{2\pi} f(t_1 (R\cos\theta, R\sin\theta) + (a,b))\, d\theta \\
&\quad + \beta \cdot \frac{1}{2\pi}\int_0^{2\pi} f(t_2 (R\cos\theta, R\sin\theta) + (a,b))\, d\theta \\
&= \alpha h(t_1) + \beta h(t_2)
\end{aligned}$$

which proves the convexity of $h$ on $[0,1]$.

$(iv)$ In the above theorem we showed that

$$H(t) = \frac{1}{\pi t^2 R^2}\iint_{D(C,tR)} f(x,y)\, dxdy \text{ for all } t \in (0,1].$$

By Hadamard's inequality (2.1) we can state that

$$\frac{1}{\pi t^2 R^2}\iint_{D(C,tR)} f(x,y)\,dxdy \leq \frac{1}{2\pi t R}\int_{\mathfrak{S}(C,tR)} f(\gamma)\,dl(\gamma)$$

which gives us that

$$H(t) \leq h(t) \quad \text{for all } t \in (0,1].$$

As it is easy to see that $H(0) = h(0) = f(C)$, then the inequality embodied in $(iv)$ is proved.

$(ii)$ The bound (3.4) follows by the above considerations and we shall omit the details.

By the convexity of $f$ on the disk $D(C,R)$ we have

$$\begin{aligned}
h(t) &= \frac{1}{2\pi}\int_0^{2\pi} f(t[(R\cos\theta, R\sin\theta)+(a,b)]+(1-t)(a,b))\,d\theta \\
&\leq t\cdot\frac{1}{2\pi}\int_0^{2\pi} f(R\cos\theta+a, R\sin\theta+b)\,d\theta + (1-t) f(a,b)\frac{1}{2\pi}\int_0^{2\pi} d\theta \\
&\leq t\cdot\frac{1}{2\pi}\int_0^{2\pi} f(R\cos\theta+a, R\sin\theta+b)\,d\theta \\
&\quad + (1-t)\cdot\frac{1}{2\pi}\int_0^{2\pi} f(R\cos\theta+a, R\sin\theta+b)\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(R\cos\theta+a, R\sin\theta+b)\,d\theta = h(1),
\end{aligned}$$

for all $t \in [0,1]$, which proves the bound (3.5).

$(iii)$ Follows by the above considerations as in the Theorem 3.1. We shall omit the details.

□

For a convex mapping $f$ defined on the disk $D(C,R)$ we can also consider the mapping

$$g(t,(x,y)) := \frac{1}{\pi R^2}\iint_{D(C,R)} f(t(x,y)+(1-t)(z,u))\,dzdu$$

which is well-defined for all $t \in [0,1]$ and $(x,y) \in D(C,R)$.

The main properties of the mapping $g$ are enclosed in the following proposition.

**Proposition 3.3.** *With the above assumptions on the mapping $f$ one has:*

$(i)$ *For all $(x,y) \in D(C,R)$, the map $g(\cdot,(x,y))$ is convex on $[0,1]$.*

$(ii)$ *For all $t \in [0,1]$, the map $g(t,\cdot)$ is convex on $D(C,R)$.*

*Proof.* $(i)$ Let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geq 0$ with $\alpha+\beta = 1$. By the convexity of $f$ we have

$$\begin{aligned}
g(\alpha t_1+\beta t_2,(x,y)) &= \frac{1}{\pi R^2}\iint_{D(C,R)} f(\alpha[t_1(x,y)+(1-t_1)(z,u)] \\
&\quad + \beta[t_2(x,y)+(1-t_2)(z,u)])\,dzdu \\
&\leq \alpha\cdot\frac{1}{\pi R^2}\iint_{D(C,R)} f(t_1(x,y)+(1-t_1)(z,u))\,dzdu \\
&\quad +\beta\cdot\frac{1}{\pi R^2}\iint_{D(C,R)} f(t_2(x,y)+(1-t_2)(z,u))\,dzdu \\
&= \alpha g(t_1,(x,y))+\beta g(t_2,(x,y)),
\end{aligned}$$

for all $(x,y) \in D(C,R)$, and the statement is proved.

$(ii)$ Let $(x_1, y_1), (x_2, y_2) \in D(C, R)$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Then

$$\begin{aligned} g(t, \alpha(x_1, y_1) + \beta(x_2, y_2)) &= \frac{1}{\pi R^2} \iint_{D(C,R)} f[\alpha(t(x_1, y_1) + (1-t)(z, u)) \\ &\quad + \beta(t(x_2, y_2) + (1-t)(z, u))] \, dzdu \\ &\leq \alpha \frac{1}{\pi R^2} \iint_{D(C,R)} f(t(x_1, y_1) + (1-t)(z, u)) \, dzdu \\ &\quad + \beta \frac{1}{\pi R^2} \iint_{D(C,R)} f(t(x_2, y_2) + (1-t)(z, u)) \, dzdu \\ &= \alpha g(t, (x_1, y_1)) + \beta g(t, (x_2, y_2)), \end{aligned}$$

for all $t \in [0, 1]$, and the statement is proved.

□

By the use of this mapping we can introduce the following application as well

$$G : [0, 1] \to \mathbb{R}, \quad G(t) := \frac{1}{\pi R^2} \iint_{D(C,R)} g(t, (x, y)) \, dxdy$$

where $g$ is as above.

The main properties of this mapping are embodied in the following theorem.

**Theorem 3.4.** *With the above assumptions we have:*

$(i)$ *For all* $s \in \left[0, \frac{1}{2}\right]$

$$G\left(s + \frac{1}{2}\right) = G\left(\frac{1}{2} - s\right),$$

*and for all* $t \in [0, 1]$ *one has*

$$G(1-t) = G(t).$$

$(ii)$ *The mapping* $G$ *is convex on the interval* $[0, 1]$.

$(iii)$ *One has the bounds*

$$\begin{aligned} \inf_{t \in [0,1]} G(t) &= G\left(\frac{1}{2}\right) \\ &= \frac{1}{(\pi R^2)^2} \iiiint_{D(C,R) \times D(C,R)} f\left(\frac{x+z}{2}, \frac{y+u}{2}\right) dxdydzdu \geq f(C) \end{aligned}$$

*and*

$$\sup_{t \in [0,1]} G(t) = G(0) = G(1) = \frac{1}{\pi R^2} \iint_{D(C,R)} f(x, y) \, dxdy.$$

$(iv)$ *The mapping* $G$ *is monotonic nonincreasing on* $\left[0, \frac{1}{2}\right]$ *and nondecreasing on* $\left[\frac{1}{2}, 1\right]$.

$(v)$ *We have the inequality*

$$G(t) \geq \max\{H(t), H(1-t)\}, \quad \text{for all } t \in [0, 1]. \tag{3.6}$$

*Proof.* The statements $(i)$ and $(ii)$ are obvious by the properties of the mapping $g$ defined above and we shall omit the details.

$(iii)$ By $(i)$ and $(ii)$ we have

$$G(t) = \frac{G(t) + G(1-t)}{2} \geq G\left(\frac{1}{2}\right), \quad \text{for all } t \in [0, 1]$$

which proves the first bound in $(iii)$.

Note that the inequality

$$G\left(\frac{1}{2}\right) \geq f(C)$$

follows by (3.6) for $t = \frac{1}{2}$ and taking into account that $H\left(\frac{1}{2}\right) \geq f(C)$.

We also have

$$\begin{aligned} G(t) &= \frac{1}{(\pi R^2)^2} \iint_{D(C,R)} \left( \iint_{D(C,R)} f(t(x,y) + (1-t)(z,u))\, dz du \right) dx dy \\ &\leq \frac{1}{(\pi R^2)^2} \\ &\quad \times \iint_{D(C,R)} \left[ t f(x,y)\, \pi R^2 + (1-t) \iint_{D(C,R)} f(z,u)\, dz du \right] dx dy \\ &= \frac{1}{(\pi R^2)^2} \\ &\quad \times \left[ t\pi R^2 \iint_{D(C,R)} f(x,y)\, dx dy + (1-t)\, \pi R^2 \iint_{D(C,R)} f(x,y)\, dx dy \right] \\ &= \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\, dx dy \end{aligned}$$

for all $t \in [0,1]$, and the second bound in $(iii)$ is also proved.

$(iv)$ The argument is similar to the proof of Theorem 3.1 $(iii)$ (see also [6]) and we shall omit the details.

$(v)$ By Theorem 2.1 we have that

$$\begin{aligned} G(t) &= \frac{1}{\pi R^2} \iint_{D(C,R)} g(t,(x,y))\, dx dy \\ &\geq g(t,(a,b)) = \frac{1}{\pi R^2} \iint_{D(C,R)} f(t(x,y) + (1-t)(a,b))\, dx dy = H(t) \end{aligned}$$

for all $t \in [0,1]$.

As $G(t) = G(1-t) \geq H(1-t)$, we obtain the desired inequality (3.6).

The theorem is thus proved.

□

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