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A VARIANT OF JENSEN-STEFFENSEN'S INEQUALITY FOR CONVEX AND SUPERQUADRATIC FUNCTIONS

S. ABRAMOVICH, M. KLARIČIĆ BAKULA, AND S. BANIĆ

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HAIFA
HAIFA, 31905, ISRAEL
abramos@math.haifa.ac.il

DEPARTMENT OF MATHEMATICS
FACULTY OF NATURAL SCIENCES, MATHEMATICS AND EDUCATION
UNIVERSITY OF SPLIT
TESLINA 12, 21000 SPLIT
CROATIA
milica@pmfst.hr

FACULTY OF CIVIL ENGINEERING AND ARCHITECTURE
UNIVERSITY OF SPLIT
MATICE HRVATSKE 15, 21000 SPLIT
CROATIA
Senka.Banic@gradst.hr

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ABSTRACT. A variant of Jensen-Steffensen's inequality is considered for convex and for superquadratic functions. Consequently, inequalities for power means involving not only positive weights have been established.

Key words and phrases: Jensen-Steffensen's inequality, Monotonicity, Superquadraticity, Power means.

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1. INTRODUCTION.

Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ a convex function on I. If $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m)$ is any m-tuple in I^m and $\boldsymbol{p} = (p_1, \dots, p_m)$ any nonnegative m-tuple such that $\sum_{i=1}^m p_i > 0$, then the well known Jensen's inequality (see for example [7, p. 43])

$$f\left(\frac{1}{P_m}\sum_{i=1}^m p_i\xi_i\right) \le \frac{1}{P_m}\sum_{i=1}^m p_i f(\xi_i) \tag{1.1}$$

holds, where $P_m = \sum_{i=1}^m p_i$.

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If f is strictly convex, then (1.1) is strict unless $\xi_i = c$ for all $i \in \{j : p_j > 0\}$.

It is well known that the assumption "$\boldsymbol{p}$ is a nonnegative m-tuple" can be relaxed at the expense of more restrictions on the m-tuple $\boldsymbol{\xi}$.

If $\boldsymbol{p}$ is a real m-tuple such that

$$0 \leq P_j \leq P_m \ , \ j = 1, \ldots, m, \quad P_m > 0, \tag{1.2}$$

where $P_j = \sum_{i=1}^{j} p_i$, then for any monotonic m-tuple $\boldsymbol{\xi}$ (increasing or decreasing) in I^m we get

$$\overline{\xi} = \frac{1}{P_m} \sum_{i=1}^{m} p_i \xi_i \in I,$$

and for any function f convex on I (1.1) still holds. Inequality (1.1) considered under conditions (1.2) is known as the Jensen-Steffensen's inequality [7, p. 57] for convex functions.

In his paper [5] A. McD. Mercer considered some monotonicity properties of power means. He proved the following theorem:

Theorem A. *Suppose that $0 < a < b$ and $a \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq b$ hold with at least one of the x_k satisfying $a < x_k < b$. If $\boldsymbol{w} = (w_1, \ldots, w_n)$ is a positive n-tuple with $\sum_{i=1}^{n} w_i = 1$ and $-\infty < r < s < +\infty$, then*

$$a < Q_r\left(a, b; \boldsymbol{x}\right) < Q_s\left(a, b; \boldsymbol{x}\right) < b \ ,$$

where

$$Q_t\left(a, b; \boldsymbol{x}\right) \equiv \left(a^t + b^t - \sum_{i=1}^{n} w_i x_i^t\right)^{\frac{1}{t}}$$

for all real $t \neq 0$, and

$$Q_0\left(a, b; \boldsymbol{x}\right) \equiv \frac{ab}{G}, \quad G = \prod_{i=1}^{n} x_i^{w_i}.$$

In his next paper [6], Mercer gave a variant of Jensen's inequality for which Witkowski presented in [8] a shorter proof. This is stated in the following theorem:

Theorem B. *If f is a convex function on an interval containing an n-tuple $\boldsymbol{x} = (x_1, \ldots, x_n)$ such that $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ and $\boldsymbol{w} = (w_1, \ldots, w_n)$ is a positive n-tuple with $\sum_{i=1}^{n} w_i = 1$, then*

$$f\left(x_1 + x_n - \sum_{i=1}^{n} w_i x_i\right) \leq f\left(x_1\right) + f\left(x_n\right) - \sum_{i=1}^{n} w_i f\left(x_i\right).$$

This theorem is a special case of the following theorem proved in [4] by Abramovich, Klaričić Bakula, Matić and Pečarić:

Theorem C ([4, Th. 2])**.** *Let $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$ and let $[a, b] \subseteq I$, $a < b$. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be a monotonic n-tuple in $[a, b]^n$ and $\boldsymbol{v} = (v_1, \ldots, v_n)$ a real $n-$tuple such that $v_i \neq 0$, $i = 1, \ldots, n$, and $0 \leq V_j \leq V_n$, $j = 1, \ldots, n$, $V_n > 0$, where $V_j = \sum_{i=1}^{j} v_i$. If f is convex on I, then*

$$f\left(a + b - \frac{1}{V_n} \sum_{i=1}^{n} v_i x_i\right) \leq f\left(a\right) + f\left(b\right) - \frac{1}{V_n} \sum_{i=1}^{n} v_i f\left(x_i\right). \tag{1.3}$$

In case f is strictly convex, the equality holds in (1.3) iff one of the following two cases occurs:

(1) *either $\overline{x} = a$ or $\overline{x} = b$,*

(2) *there exists* $l \in \{2, \dots, n-1\}$ *such that* $\overline{x} = x_1 + x_n - x_l$ *and*

$$\left\{ \begin{array}{l} x_1 = a,\ x_n = b \text{ or } x_1 = b,\ x_n = a, \\ \overline{V}_j \left(x_{j-1} - x_j\right) = 0,\ j = 2, \dots, l, \\ V_j \ \left(x_j - x_{j+1}\right) = 0,\ j = l, \dots, n-1, \end{array} \right. \tag{1.4}$$

where $\overline{V}_j = \sum_{i=j}^{n} v_i$, $j = 1, \dots, n$ *and* $\overline{x} = (1/V_n) \sum_{i=1}^{n} v_i x_i$.

In the special case where $\boldsymbol{v} > 0$ *and* f *is strictly convex, the equality holds in (1.4) iff* $x_i = a,\ i = 1, \dots, n$, *or* $x_i = b$, $i = 1, \dots, n$.

Here, as in the rest of the paper, when we say that an n-tuple $\boldsymbol{\xi}$ is increasing (decreasing) we mean that $\xi_1 \le \xi_2 \le \cdots \le \xi_n$ ($\xi_1 \ge \xi_2 \ge \cdots \ge \xi_n$). Similarly, when we say that a function $f : I \to \mathbb{R}$ is increasing (decreasing) on I we mean that for all $u, v \in I$ we have $u < v \Rightarrow f(u) \le f(v)$ ($u < v \Rightarrow f(u) \ge f(v)$).

In Section 2 we refine Theorems A, B, and C. These refinements are achieved by superquadratic functions which were introduced in [1] and [2].

As Jensen's inequality for convex functions is a generalization of Hölder's inequality for $f(x) = x^p,\ p \ge 1$, so the inequalities satisfied by superquadratic functions are generalizations of the inequalities satisfied by the superquadratic functions $f(x) = x^p,\ p \ge 2$ (see [1], [2]).

First we quote some definitions and state a list of basic properties of superquadratic functions.

Definition 1.1. A function $f : [0, \infty) \to \mathbb{R}$ is superquadratic provided that for all $x \ge 0$ there exists a constant $C(x) \in \mathbb{R}$ such that

$$f(y) - f(x) - f(|y - x|) \ge C(x)(y - x) \tag{1.5}$$

for all $y \ge 0$.

Definition 1.2. A function $f : [0, \infty) \to \mathbb{R}$ is said to be strictly superquadratic if (1.5) is strict for all $x \ne y$ where $xy \ne 0$.

Lemma A ([2, Lemma 2.3]). *Suppose that* f *is superquadratic. Let* $\xi_i \ge 0$, $i = 1, \dots, m$, *and let* $\overline{\xi} = \sum_{i=1}^{m} p_i \xi_i$, *where* $p_i \ge 0$, $i = 1, \dots, m$, *and* $\sum_{i=1}^{m} p_i = 1$. *Then*

$$\sum_{i=1}^{m} p_i f(\xi_i) - f\left(\overline{\xi}\right) \ge \sum_{i=1}^{m} p_i f\left(\left|\xi_i - \overline{\xi}\right|\right).$$

Lemma B ([1, Lemma 2.2]). *Let* f *be superquadratic function with* $C(x)$ *as in Definition 1.1. Then:*

(i) $f(0) \le 0$,
(ii) *if* $f(0) = f'(0) = 0$ *then* $C(x) = f'(x)$ *whenever* f *is differentiable at* $x > 0$,
(iii) *if* $f \ge 0$, *then* f *is convex and* $f(0) = f'(0) = 0$.

In [3] the following refinement of Jensen's Steffensen's type inequality for nonnegative superquadratic functions was proved:

Theorem D ([3, Th. 1]). *Let* $f : [0, \infty) \to [0, \infty)$ *be a differentiable and superquadratic function, let* $\boldsymbol{\xi}$ *be a nonnegative monotonic* m*-tuple in* $\mathbb{R}^m$ *and* $\boldsymbol{p}$ *a real* m*-tuple,* $m \ge 3$, *satisfying*

$$0 \le P_j \le P_m,\ j = 1, \dots, m, \quad P_m > 0.$$

Let $\overline{\xi}$ *be defined as*

$$\overline{\xi} = \frac{1}{P_m} \sum_{i=1}^{m} p_i \xi_i.$$

Then

$$\sum_{i=1}^{m} p_i f(\xi_i) - P_m f(\overline{\xi}) \geq \sum_{i=1}^{k-1} P_i f(\xi_{i+1} - \xi_i) + P_k f(\overline{\xi} - \xi_k) + \overline{P}_{k+1} f(\xi_{k+1} - \overline{\xi}) + \sum_{i=k+2}^{m} \overline{P}_i f(\xi_i - \xi_{i-1}) \geq \left[\sum_{i=1}^{k} P_i + \sum_{i=k+1}^{m} \overline{P}_i\right] f\left(\frac{\sum_{i=1}^{m} p_i |\overline{\xi} - \xi_i|}{\sum_{i=1}^{k} P_i + \sum_{i=k+1}^{m} \overline{P}_i}\right) \geq (m-1) P_m f\left(\frac{\sum_{i=1}^{m} p_i |\overline{\xi} - \xi_i|}{(m-1) P_m}\right), \tag{1.6}$$

where $\overline{P}_i = \sum_{j=i}^{m} p_j$ *and* $k \in \{1, \ldots, m-1\}$ *satisfies*

$$\xi_k \leq \overline{\xi} \leq \xi_{k+1}.$$

In case f *is also strictly superquadratic, inequality*

$$\sum_{i=1}^{m} p_i f(\xi_i) - P_m f(\overline{\xi}) > (m-1) P_m f\left(\frac{\sum_{i=1}^{m} p_i \left(|\xi_i - \overline{\xi}|\right)}{(m-1) P_m}\right)$$

holds for $\boldsymbol{\xi} > \mathbf{0}$ *unless one of the following two cases occurs:*

(1) *either* $\overline{\xi} = \xi_1$ *or* $\overline{\xi} = \xi_m$,
(2) *there exists* $k \in \{3, \ldots, m-2\}$ *such that* $\overline{\xi} = \xi_k$ *and*

$$\begin{cases} P_j(\xi_j - \xi_{j+1}) = 0, & j = 1, \ldots, k-1 \\ \overline{P}_j(\xi_j - \xi_{j-1}) = 0, & j = k+1, \ldots, m. \end{cases} \tag{1.7}$$

In these two cases

$$\sum_{i=1}^{m} p_i f(\xi_i) - P_m f(\overline{\xi}) = 0.$$

In Section 2 we refine Theorem B and Theorem C for functions which are superquadratic and positive. One of the refinements is derived easily from Theorem D.

We use in Section 3 the following theorem [7, p. 323] to give an alternative proof of Theorem B.

Theorem E. *Let* I *be an interval in* $\mathbb{R}$, *and* $\boldsymbol{\xi}, \boldsymbol{\eta}$ *two decreasing* m*-tuples such that* $\boldsymbol{\xi}, \boldsymbol{\eta} \in I^m$. *Let* $\mathbf{p}$ *be a real* m*-tuple such that*

$$\sum_{i=1}^{k} p_i \xi_i \leq \sum_{i=1}^{k} p_i \eta_i \tag{1.8}$$

for $k = 1, 2, \ldots, m-1$, *and*

$$\sum_{i=1}^{m} p_i \xi_i = \sum_{i=1}^{m} p_i \eta_i\,. \tag{1.9}$$

Then for every continuous convex function $f : I \to \mathbb{R}$ *we have*

$$\sum_{i=1}^{m} p_i f(\xi_i) \leq \sum_{i=1}^{m} p_i f(\eta_i)\,. \tag{1.10}$$

2. VARIANTS OF JENSEN-STEFFENSEN'S INEQUALITY FOR POSITIVE SUPERQUADRATIC FUNCTIONS

In this section we refine in two ways Theorem C for functions which are superquadratic and positive. The refinement in Theorem 2.1 follows by showing that it is a special case of Theorem D for specific $\boldsymbol{p}$. The refinement in Theorem 2.2 follows the steps in the proof of Theorem B given by Witkowski in [8]. Therefore the second refinement is confined only to the specific $\boldsymbol{p}$ given in Theorem B, which means that what we get is a variant of Jensen's inequality and not of the more general Jensen-Steffensen's inequality.

Theorem 2.1. *Let $f : [0,\infty) \to [0,\infty)$ and let $[a,b] \subseteq [0,\infty)$. Let $\boldsymbol{x} = (x_1,\ldots,x_n)$ be a monotonic $n-$tuple in $[a,b]^n$ and $\boldsymbol{v} = (v_1,\ldots,v_n)$ a real n-tuple such that $v_i \neq 0$, $i = 1,\ldots,n$, $0 \le V_j \le V_n$, $j = 1,\ldots,n$, and $V_n > 0$, where $V_j = \sum_{i=1}^{j} v_i$. If f is differentiable and superquadratic, then*

$$f(a) + f(b) - \frac{1}{V_n}\sum_{i=1}^{n} v_i f(x_i) - f\left(a + b - \frac{1}{V_n}\sum_{i=1}^{n} v_i x_i\right) \ge (n+1)\, f\left(\frac{b - a - \frac{1}{V_n}\sum_{i=1}^{n} v_i \left|a + b - x_i - \frac{1}{V_n}\sum_{j=1}^{n} v_j x_j\right|}{n+1}\right). \tag{2.1}$$

In case f is also strictly superquadratic and $a > 0$, inequality (2.1) is strict unless one of the following two cases occurs:

(1) *either $\overline{x} = a$ or $\overline{x} = b$,*
(2) *there exists $l \in \{2,\ldots,n-1\}$ such that $\overline{x} = x_1 + x_n - x_l$ and*

$$\begin{cases} x_1 = a,\ x_n = b \ \text{ or } \ x_1 = b,\ x_n = a \\ \overline{V}_j\,(x_{j-1} - x_j) = 0\ ,\ j = 2,\ldots,l, \\ V_j\,(x_j - x_{j+1}) = 0\ ,\ j = l,\ldots,n-1, \end{cases} \tag{2.2}$$

where $\overline{V}_j = \sum_{i=j}^{n} v_i$, $j = 1,\ldots,n$, and $\overline{x} = \frac{1}{V_n}\sum_{i=1}^{n} v_i x_i$.
In these two cases we have

$$f(a) + f(b) - \frac{1}{V_n}\sum_{i=1}^{n} v_i f(x_i) - f\left(a + b - \frac{1}{V_n}\sum_{i=1}^{n} v_i x_i\right) = 0.$$

In the special case where $\boldsymbol{v} > \boldsymbol{0}$ and f is also strictly superquadratic, the equality holds in (2.1) iff $x_i = a,\ i = 1,\ldots,n$, or $x_i = b$, $i = 1,\ldots,n$.

Proof. Suppose that $\boldsymbol{x}$ is an increasing n-tuple in $[a,b]^n$. The proof of the theorem is an immediate result of Theorem D, by defining the $(n+2)$-tuples $\boldsymbol{\xi}$ and $\boldsymbol{p}$ as

$$\begin{aligned} &\xi_1 = a, \quad \xi_{i+1} = x_i,\ i = 1,\ldots,n, \quad \xi_{n+2} = b \\ &p_1 = 1, \quad p_{i+1} = -v_i/V_n,\ i = 1,\ldots,n, \quad p_{n+2} = 1. \end{aligned}$$

Then we get (2.1) from the last inequality in (1.6) and from the fact that in our special case we have

$$\sum_{i=1}^{k} P_i + \sum_{i=k+1}^{n+2} \overline{P}_i \le n + 1,$$

for $P_i = \sum_{j=1}^{i} p_j$ and $\overline{P}_i = \sum_{j=i}^{n+2} p_j$, and

$$\overline{\xi} = \frac{1}{P_m} \sum_{i=1}^{m} p_i \xi_i = a + b - \frac{1}{V_n} \sum_{i=1}^{n} v_i x_i = a + b - \overline{x}.$$

The proof of the equality case and the special case where $\boldsymbol{v} > \mathbf{0}$ follows also from Theorem D.

We have

$$P_j = \frac{\overline{V}_j}{V_n},\ \ j = 1, \dots, n,\ \ P_{n+1} = 0,\ \ P_{n+2} = 1,$$
$$\overline{P}_1 = 1,\ \ \overline{P}_2 = 0,\ \ \overline{P}_j = \frac{V_{j-2}}{V_n},\ \ j = 3, \dots, n+2.$$

Obviously, $\overline{\xi} = \xi_1$ is equivalent to $\overline{x} = b$ and $\overline{\xi} = \xi_{n+2}$ is equivalent to $\overline{x} = a$. Also, the existence of some $k \in \{3, \dots, m-2\}$ such that $\overline{\xi} = \xi_k$ and that (1.7) holds is equivalent to the existence of some $l \in \{2, \dots, n-1\}$ such that $\overline{x} = x_1 + x_n - x_l = a + b - x_l$ and that (2.2) holds. Therefore, applying Theorem D we get the desired conclusions. In the case when $\boldsymbol{x}$ is decreasing we simply replace $\boldsymbol{x}$ and $\boldsymbol{v}$ with $\widetilde{\boldsymbol{x}} = (x_n, \dots, x_1)$ and $\widetilde{\boldsymbol{v}} = (v_n, \dots, v_1)$, respectively, and then argue in the same manner.

In the special case that $\boldsymbol{v} > \mathbf{0}$ also, $V_i > 0$ and $\overline{V}_j > 0$, $i = 1, \dots, n$, and therefore according to (2.2) equality holds in (2.1) only when either $x_1 = \cdots = x_n = a$ or $x_1 = \cdots = x_n = b$. □

In the following theorem we will prove a refinement of Theorem B. Without loss of generality we assume that $\sum_{i=1}^{n} v_i = 1$.

Theorem 2.2. *Let $f : [0, \infty) \to [0, \infty)$ and let $[a, b] \subseteq [0, \infty)$, $a < b$. Let $\boldsymbol{x} = (x_1, \dots, x_n)$ be an n-tuple in $[a, b]^n$ and $\boldsymbol{v} = (v_1, \dots, v_n)$ a real n-tuple such that $\boldsymbol{v} \geq \mathbf{0}$ and $\sum_{i=1}^{n} v_i = 1$. If f is superquadratic we have*

$$\begin{aligned} f(a) + f(b) - \sum_{i=1}^{n} v_i f(x_i) - f\left(a + b - \sum_{i=1}^{n} v_i x_i\right) \\ \geq \sum_{i=1}^{n} v_i f\left(\left|\sum_{j=1}^{n} v_j x_j - x_i\right|\right) + 2\sum_{i=1}^{n} v_i \left[\frac{x_i - a}{b - a} f(b - x_i) + \frac{b - x_i}{b - a} f(x_i - a)\right] \\ \geq \sum_{i=1}^{n} v_i f\left(\left|\sum_{j=1}^{n} v_j x_j - x_i\right|\right) + 2\sum_{i=1}^{n} v_i f\left(\frac{2(x_i - a)(b - x_i)}{b - a}\right). \end{aligned} \tag{2.3}$$

If f is strictly superquadratic and $\boldsymbol{v} > \mathbf{0}$ equality holds in (2.3) iff $x_i = a$, $i = 1, \dots, n$, or $x_i = b$, $i = 1, \dots, n$.

Proof. The proof follows the technique in [8] and refines the result to positive superquadratic functions. From Lemma A we know that for any $\lambda \in [0, 1]$ the following holds:

$$\begin{aligned} &\lambda f(a) + (1-\lambda) f(b) - f(\lambda a + (1-\lambda) b) \\ &\geq \lambda f(|a - \lambda a - (1-\lambda) b|) + (1-\lambda) f(|b - \lambda a - (1-\lambda) b|) \\ &= \lambda f(|(1-\lambda)(a-b)|) + (1-\lambda) f(|\lambda(b-a)|) \\ &= \lambda f((1-\lambda)(b-a)) + (1-\lambda) f(\lambda(b-a)). \end{aligned} \tag{2.4}$$

Also, for any $x_i \in [a, b]$ there exists a unique $\lambda_i \in [0, 1]$ such that $x_i = \lambda_i a + (1 - \lambda_i) b$. We have

$$f(a) + f(b) - \sum_{i=1}^{n} v_i f(x_i) = f(a) + f(b) - \sum_{i=1}^{n} v_i f(\lambda_i a + (1-\lambda_i) b). \tag{2.5}$$

Applying (2.4) on every $x_i = \lambda_i a + (1-\lambda_i) b$ in (2.5) we obtain

$$\begin{aligned} f(a) + f(b) - \sum_{i=1}^{n} v_i f(x_i) &\geq f(a) + f(b) + \sum_{i=1}^{n} v_i \big[-\lambda_i f(a) - (1-\lambda_i) f(b) \\ &\quad + \lambda_i f((1-\lambda_i)(b-a)) + (1-\lambda_i) f(\lambda_i (b-a)) \big] \\ &= \sum_{i=1}^{n} v_i [(1-\lambda_i) f(a) + \lambda_i f(b)] \\ &\quad + \sum_{i=1}^{n} v_i [\lambda_i f((1-\lambda_i)(b-a)) + (1-\lambda_i) f(\lambda_i (b-a))] . \end{aligned} \tag{2.6}$$

Applying again (2.4) on (2.6) we get

$$\begin{aligned} f(a) + f(b) - \sum_{i=1}^{n} v_i f(x_i) &\geq \sum_{i=1}^{n} v_i f((1-\lambda_i) a + \lambda_i b) \\ &\quad + 2\sum_{i=1}^{n} v_i [\lambda_i f((1-\lambda_i)(b-a)) + (1-\lambda_i) f(\lambda_i (b-a))] . \end{aligned} \tag{2.7}$$

Applying again Lemma A on (2.7) we obtain

$$\begin{aligned} f(a) + f(b) - \sum_{i=1}^{n} v_i f(x_i) &\geq f\left(\sum_{i=1}^{n} v_i [(1-\lambda_i) a + \lambda_i b] \right) \\ &\quad + \sum_{i=1}^{n} v_i f\left(\left| (1-\lambda_i) a + \lambda_i b - \sum_{j=1}^{n} v_j [(1-\lambda_j) a + \lambda_j b] \right| \right) \\ &\quad + 2\sum_{i=1}^{n} v_i [(1-\lambda_i) f(\lambda_i (b-a)) + \lambda_i f((1-\lambda_i)(b-a))] \\ &= f\left(a + b - \sum_{i=1}^{n} v_i x_i \right) + \sum_{i=1}^{n} v_i f\left(\left| \sum_{j=1}^{n} v_j x_j - x_i \right| \right) \\ &\quad + 2\sum_{i=1}^{n} v_i \left[\frac{x_i - a}{b-a} f(b - x_i) + \frac{b - x_i}{b-a} f(x_i - a) \right] , \end{aligned} \tag{2.8}$$

and this is the first inequality in (2.3).

Since f is a nonnegative superquadratic function, from Lemma B we know that it is also convex, so from (2.8) we have

$$\sum_{i=1}^{n} v_i \left[\frac{x_i - a}{b-a} f(b - x_i) + \frac{b - x_i}{b-a} f(x_i - a) \right] \geq \sum_{i=1}^{n} v_i f\left(\frac{2(b-x_i)(x_i - a)}{b-a} \right) ,$$

hence, the second inequality in (2.3) is proved.

For the case when f is strictly superquadratic and $\boldsymbol{v} > \mathbf{0}$ we may deduce that inequalities (2.6) and (2.7) become equalities iff each of the λ_i, $i = 1, \ldots, n$, is either equal to 1 or equal to 0, which means that $x_i \in \{a, b\}$, $i = 1, \ldots, n$. However, since we also have

$$\sum_{j=1}^{n} v_j x_j - x_i = 0, \quad i = 1, \ldots, n,$$

we deduce that $x_i = a$, $i = 1, \ldots, n$, or $x_i = b$, $i = 1, \ldots, n$.

This completes the proof of the theorem. □

Corollary 2.3. *Let $\boldsymbol{v} = (v_1, \ldots, v_n)$be a real n-tuple such that $\boldsymbol{v} \geq \mathbf{0}$, $\sum_{i=1}^{n} v_i = 1$ and let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be an n-tuple in $[a, b]^n$, $0 < a < b$. Then for any real numbers r and s such that $\frac{s}{r} \geq 2$ we have*

$$\begin{aligned}\left(\frac{Q_s(a,b;\boldsymbol{x})}{Q_r(a,b;\boldsymbol{x})}\right)^s - 1 &\geq \frac{1}{Q_r(a,b;\boldsymbol{x})^s}\left[\sum_{i=1}^{n} v_i \left|\sum_{j=1}^{n} v_j x_j^r - x_i^r\right|^{\frac{s}{r}}\right.\\ &\qquad \left.+2\sum_{i=1}^{n} v_i \left(\frac{x_i^r - a^r}{b^r - a^r}(b^r - x_i^r)^{\frac{s}{r}} + \frac{b^r - x_i^r}{b^r - a^r}(x_i^r - a^r)^{\frac{s}{r}}\right)\right] \\ &\geq \frac{1}{Q_r(a,b;\boldsymbol{x})^s}\left[\sum_{i=1}^{n} v_i \left|\sum_{j=1}^{n} v_j x_j^r - x_i^r\right|^{\frac{s}{r}} + 2\sum_{i=1}^{n} v_i \left(\frac{2(x_i^r - a^r)(b^r - x_i^r)}{b^r - a^r}\right)^{\frac{s}{r}}\right],\end{aligned} \tag{2.9}$$

where $Q_p(a,b;\boldsymbol{x}) = (a^p + b^p - \sum_{i=1}^{n} v_i x_i)^{\frac{1}{p}}$, $p \in \mathbb{R} \setminus \{0\}$.
If $\frac{s}{r} > 2$ and $\boldsymbol{v} > \mathbf{0}$, the equalities hold in (2.9) iff $x_i = a,\ i = 1, \ldots, n$ or $x_i = b,\ i = 1, \ldots, n$.

Proof. We define a function $f : (0, \infty) \to (0, \infty)$ as $f(x) = x^{\frac{s}{r}}$. It can be easily checked that for any real numbers r and s such that $\frac{s}{r} \geq 2$ the function f is superquadratic. We define a new positive n-tuple $\boldsymbol{\xi}$ in $[a^r, b^r]$ as $\xi_i = x_i^r$, $i = 1, \ldots, n$. From Theorem 2.2 we have

$$\begin{aligned}a^s + b^s - \sum_{i=1}^{n} v_i x_i^s - \left(a^r + b^r - \sum_{i=1}^{n} v_i x_i^r\right)^{\frac{s}{r}} &\geq \sum_{i=1}^{n} v_i \left|\sum_{j=1}^{n} v_j x_j^r - x_i^r\right|^{\frac{s}{r}} + 2\sum_{i=1}^{n} v_i \left[\frac{x_i^r - a^r}{b^r - a^r}(b^r - x_i^r)^{\frac{s}{r}} + \frac{b^r - x_i^r}{b^r - a^r}(x_i^r - a^r)^{\frac{s}{r}}\right] \\ &\geq \sum_{i=1}^{n} v_i \left|\sum_{j=1}^{n} v_j x_j^r - x_i^r\right|^{\frac{s}{r}} + 2\sum_{i=1}^{n} v_i \left(\frac{2(x_i^r - a^r)(b^r - x_i^r)}{b^r - a^r}\right)^{\frac{s}{r}} \geq 0.\end{aligned} \tag{2.10}$$

We have

$$a^s + b^s - \sum_{i=1}^{n} v_i x_i^s - \left(a^r + b^r - \sum_{i=1}^{n} v_i x_i^r\right)^{\frac{s}{r}} = Q_s(a,b;\boldsymbol{x})^s - Q_r(a,b;\boldsymbol{x})^s$$

so from (2.10) the inequalities in (2.9) follow.

The equality case follows from the equality case in Theorem 2.2, as the function $f(x) = x^{\frac{s}{r}}$ is strictly superquadratic for $\frac{s}{r} > 2$. □

Remark 2.4. It is an immediate result of Corollary 2.3 that if $\frac{s}{r} > 2$ and there is at least one $j \in \{1, \dots, n\}$ such that

$$v_j \left(x_j^r - a^r\right)\left(b^r - x_j^r\right) > 0,$$

then for this j we have

$$\left(\frac{Q_s(a,b;\boldsymbol{x})}{Q_r(a,b;\boldsymbol{x})}\right)^s - 1 > \frac{2v_j}{Q_r(a,b;\boldsymbol{x})^s}\left(\frac{2\left(x_j^r - a^r\right)\left(b^r - x_j^r\right)}{b^r - a^r}\right)^{\frac{s}{r}} > 0.$$

3. AN ALTERNATIVE PROOF OF THEOREM B

In this section we give an interesting alternative proof of Theorem B based on Theorem E. To carry out that proof we need the following technical lemma.

Lemma 3.1. *Let $\boldsymbol{y} = (y_1, \dots, y_m)$ be a decreasing real m-tuple and $\boldsymbol{p} = (p_1, \dots, p_m)$ a nonnegative real m-tuple with $\sum_{i=1}^m p_i = 1$. We define*

$$\overline{y} = \sum_{i=1}^m p_i y_i$$

and the m-tuple

$$\overline{\boldsymbol{y}} = (\overline{y}, \overline{y}, \dots, \overline{y}).$$

Then the m-tuples $\boldsymbol{\eta} = \boldsymbol{y}$, $\boldsymbol{\xi} = \overline{\boldsymbol{y}}$ and $\boldsymbol{p}$ satisfy conditions (1.8) and (1.9).

Proof. Note that $\overline{y}$ is a convex combination of $y_1, y_2, \dots, y_m$, so we know that

$$y_m \le \overline{y} \le y_1.$$

From the definitions of the m-tuples $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ we have

$$\sum_{i=1}^m p_i\xi_i = \overline{y}\sum_{i=1}^m p_i = \overline{y} = \sum_{i=1}^m p_i y_i = \sum_{i=1}^m p_i\eta_i.$$

Hence, condition (1.9) is satisfied. Furthermore, for $k = 1, 2, \dots, m-1$ we have

$$\begin{aligned}\sum_{i=1}^k p_i\eta_i - \sum_{i=1}^k p_i\xi_i &= \sum_{i=1}^k p_i y_i - \overline{y}\sum_{i=1}^k p_i \\ &= \sum_{i=1}^k p_i y_i - \sum_{j=1}^m p_j y_j \sum_{i=1}^k p_i\,.\end{aligned}$$

Since $\sum_{i=1}^m p_i = 1$, we can write

$$\begin{aligned}\sum_{i=1}^k p_i\eta_i - \sum_{i=1}^k p_i\xi_i &= \left(\sum_{j=1}^k p_j + \sum_{j=k+1}^m p_j\right)\sum_{i=1}^k p_i y_i - \left(\sum_{j=1}^k p_j y_j + \sum_{j=k+1}^m p_j y_j\right)\sum_{i=1}^k p_i \\ &= \sum_{j=k+1}^m p_j \sum_{i=1}^k p_i y_i - \sum_{i=1}^k p_i \sum_{j=k+1}^m p_j y_j \\ &= \sum_{i=1}^k p_i\left(\sum_{j=k+1}^m p_j y_i - \sum_{j=k+1}^m p_j y_j\right) \\ &= \sum_{i=1}^k p_i \sum_{j=k+1}^m p_j\left(y_i - y_j\right).\end{aligned}$$

Since $\boldsymbol{p}$ is nonnegative and $\boldsymbol{y}$ is decreasing, we obtain

$$\sum_{i=1}^{k} p_i\eta_i - \sum_{i=1}^{k} p_i\xi_i \geq 0, \quad k = 1, 2, \ldots, m-1,$$

which means that condition (1.8) is satisfied as well. □

Now we can give an alternative proof of Theorem B which is mainly based on Theorem E.

Proof of Theorem B. Since $\overline{x} = \sum_{i=1}^{n} w_i x_i$ is a convex combination of $x_1, x_2, \ldots, x_n$ it is clear that there is an $s \in \{1, 2, \ldots, n-1\}$ such that

$$x_1 \leq \cdots \leq x_s \leq \overline{x} \leq x_{s+1} \leq \cdots \leq x_n,$$

that is,

$$-x_1 \geq \cdots \geq -x_s \geq -\overline{x} \geq -x_{s+1} \geq \cdots \geq -x_n. \tag{3.1}$$

Adding $x_1 + x_n$ to all the inequalities in (3.1) we obtain

$$x_n \geq \cdots \geq x_1 + x_n - x_s \geq x_1 + x_n - \overline{x} \geq x_1 + x_n - x_{s+1} \geq \cdots \geq x_1,$$

which gives us

$$x_1 + x_n - \overline{x} = x_1 + x_n - \sum_{i=1}^{n} w_i x_i \in [x_1, x_n]. \tag{3.2}$$

We use (1.10) to prove the theorem. For this, we define the $(n+2)$-tuples $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ and $\boldsymbol{p}$ as follows:

$$\eta_1 = x_n, \quad \eta_2 = x_n, \quad \eta_3 = x_{n-1}, \quad \ldots, \quad \eta_n = x_2, \quad \eta_{n+1} = x_1, \quad \eta_{n+2} = x_1,$$
$$p_1 = 1, \quad p_2 = -w_n, \quad p_3 = -w_{n-1}, \quad \ldots, \quad p_n = -w_2, \quad p_{n+1} = -w_1, \quad p_{n+2} = 1,$$
$$\xi_1 = \xi_2 = \cdots = \xi_{n+2} = \overline{\eta}, \quad \overline{\eta} = \sum_{i=1}^{n+2} p_i\eta_i = x_1 + x_n - \sum_{j=1}^{n} w_j x_j.$$

It is easily verified that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are decreasing and that $\sum_{i=1}^{n+2} p_i = 1$. It remains to see that $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ and $\boldsymbol{p}$ satisfy conditions (1.8) and (1.9).

Condition (1.9) is trivially fulfilled since

$$\sum_{i=1}^{n+2} p_i\xi_i = \overline{\eta}\sum_{i=1}^{n+2} p_i = \overline{\eta} = \sum_{i=1}^{n+2} p_i\eta_i.$$

Further, we have $\xi_i = \overline{\eta}$, $i = 1, 2, \ldots, n+2$. To prove (1.8), we need to demonstrate that

$$\overline{\eta}\sum_{i=1}^{k} p_i \leq \sum_{i=1}^{k} p_i\eta_i, \quad k = 1, 2, \ldots, n+1. \tag{3.3}$$

For $k = 1$, (3.3) becomes $\overline{\eta} \leq x_n$, and this holds because of (3.2). On the other hand, for $k = n+1$, (3.3) becomes

$$\overline{\eta}\left(1 - \sum_{i=1}^{n} w_i\right) \leq x_n - \sum_{i=1}^{n} w_i x_i,$$

that is,

$$0 \leq x_n - \overline{x},$$

and this holds because of (3.2).

If $k \in \{2, \dots, n\}$, (3.3) can be rewritten and in its stead we have to prove that

$$\overline{\eta}\left(1-\sum_{i=n+2-k}^{n} w_i\right) \leq x_n - \sum_{i=n+2-k}^{n} w_i x_i. \tag{3.4}$$

Let us consider the decreasing n-tuple $\boldsymbol{y}$, where

$$y_i = x_1 + x_n - x_i, \quad i = 1, 2, \dots, n.$$

We have

$$\begin{aligned}\overline{y} &= \sum_{i=1}^{n} w_i y_i \\ &= \sum_{i=1}^{n} w_i \left(x_1 + x_n - x_i\right) \\ &= x_1 + x_n - \sum_{i=1}^{n} w_i x_i = x_1 + x_n - \overline{x} = \overline{\eta}.\end{aligned}$$

If we apply Lemma 3.1 to the n-tuple $\boldsymbol{y}$ and to the weights $\boldsymbol{w}$, then $m = n$ and for all $l \in \{1, 2, \dots, n-1\}$ the inequality

$$\overline{y}\sum_{i=1}^{l} w_i \leq \sum_{i=1}^{l} w_i \left(x_1 + x_n - x_i\right)$$

holds. Taking into consideration that $\overline{y} = \overline{\eta}$, $\sum_{i=1}^{l} w_i = 1 - \sum_{i=l+1}^{n} w_i$ and changing indices as $l = n + 1 - k$, we deduce that

$$\overline{\eta}\left(1-\sum_{i=n+2-k}^{n} w_i\right) \leq \sum_{i=1}^{n+1-k} w_i \left(x_1 + x_n - x_i\right), \tag{3.5}$$

for all $k \in \{2, \dots, n\}$. The difference between the right side of (3.4) and the right side of (3.5) is

$$\begin{aligned}&x_n - \sum_{i=n+2-k}^{n} w_i x_i - \sum_{i=1}^{n+1-k} w_i \left(x_1 + x_n - x_i\right) \\ &\quad = x_n - \sum_{i=n+2-k}^{n} w_i x_i - x_n \sum_{i=1}^{n+1-k} w_i - \sum_{i=1}^{n+1-k} w_i \left(x_1 - x_i\right) \\ &\quad = x_n \left(1 - \sum_{i=1}^{n+1-k} w_i\right) - \sum_{i=n+2-k}^{n} w_i x_i - \sum_{i=1}^{n+1-k} w_i \left(x_1 - x_i\right) \\ &\quad = x_n \sum_{i=n+2-k}^{n} w_i - \sum_{i=n+2-k}^{n} w_i x_i - \sum_{i=1}^{n+1-k} w_i \left(x_1 - x_i\right) \\ &\quad = \sum_{i=n+2-k}^{n} w_i \left(x_n - x_i\right) + \sum_{i=1}^{n+1-k} w_i \left(x_i - x_1\right) \geq 0,\end{aligned}$$

since $\boldsymbol{w}$ is nonnegative and $\boldsymbol{x}$ is increasing. Therefore, the inequality

$$\sum_{i=1}^{n+1-k} w_i \left(x_1 + x_n - x_i\right) \leq x_n - \sum_{i=n+2-k}^{n} w_i x_i \tag{3.6}$$

holds for all $k \in \{2, \ldots, n\}$. From (3.5) and (3.6) we obtain (3.4) . This completes the proof that the m-tuples $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ and $\boldsymbol{p}$ satisfy conditions (1.8) and (1.9) and we can apply Theorem E to obtain

$$\sum_{i=1}^{n+2} p_i f\left(\overline{\eta}\right) \leq f\left(x_n\right) - \sum_{i=1}^{n} w_i f\left(x_i\right) + f\left(x_1\right).$$

Taking into consideration that $\sum_{i=1}^{n+2} p_i = 1$ and $\overline{\eta} = x_1 + x_n - \sum_{j=1}^{n} w_j x_j$ we finally get

$$f\left(x_1 + x_n - \sum_{i=1}^{n} w_i x_i\right) \leq f\left(x_1\right) + f\left(x_n\right) - \sum_{i=1}^{n} w_i f\left(x_i\right).$$

□

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