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# ON THE HYERS-ULAM STABILITY OF QUADRATIC FUNCTIONAL EQUATIONS

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ABSTRACT. In this paper, we obtain the general solution and the generalized Hyers-Ulam stability for quadratic functional equations $f(2x+y)+f(2x-y) = f(x+y)+f(x-y)+6f(x)$ and $f(2x + y) + f(x + 2y) = 4f(x + y) + f(x) + f(y)$.

*Key words and phrases:* Hyers-Ulam-Rassias stability; Quadratic function.

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## 1. INTRODUCTION

In 1940, S.M. Ulam [20] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of homomorphisms:

*Let $G_1$ be a group and let $G_2$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?*

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. If we turn our attention to the case of functional equations, we can ask the question: When the solutions of an equation differing slightly from a given one must be close to the true solution of the given equation.

The case of approximately additive functions was solved by D. H. Hyers [9] under the assumption that $G_1$ and $G_2$ are Banach spaces. In 1978, a generalized version of the theorem of Hyers for approximately linear mappings was given by Th. M. Rassias [17]. During the last

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decades, the stability problems of several functional equations have been extensively investigated by a number of authors [2, 6, 11, 15]. The terminology generalized Hyers-Ulam stability originates from these historical backgrounds. These terminologies are also applied to the case of other functional equations. For more detailed definitions of such terminologies, we can refer to [10, 12, 18].

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is related to a symmetric biadditive function ([1], [16]). It is natural that each equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function $f$ between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function $B$ such that $f(x)=B(x,x)$ for all $x$ (see [1], [16]). The biadditive function $B$ is given by

$$B(x,y)=\frac{1}{4}(f(x+y)-f(x-y)). \tag{1.2}$$

A Hyers-Ulam stability problem for the quadratic functional equation (1.1) was proved by F. Skof for functions $f:E_1\to E_2$ , where $E_1$ is a normed space and $E_2$ a Banach space (see [19]). P. W. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain $E_1$ is replaced by an abelian group. In the paper [4], S. Czerwik proved the Hyers-Ulam-Rassias stability of the quadratic functional equation (1.1). A. Grabiec [8] has generalized these results mentioned above. K. W. Jun and Y. H. Lee [13] proved the Hyers-Ulam-Rassias stability of the pexiderized quadratic equation (1.1).

Now, we introduce the following functional equations, which are somewhat different from (1.1),

$$f(2x+y)+f(2x-y) = f(x+y)+f(x-y)+6f(x), \tag{1.3}$$
$$f(2x+y)+f(x+2y) = 4f(x+y)+f(x)+f(y). \tag{1.4}$$

In this paper, we establish the general solution and the generalized Hyers-Ulam stability problem for the equations (1.3), (1.4), which are equivalent to (1.1). It is significant for us to decrease the possible estimator of the stability problem for the functional equations. This work is possible if we consider the stability problem in the sense of Hyers-Ulam-Rassias for the functional equations (1.3), (1.4). As a result, we have much better possible upper bounds for the equations (1.3), (1.4) than those of Czerwik [4] and Skof-Cholewa [3].

## 2. SOLUTION OF (1.3), (1.4)

Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers and let both $E_1$ and $E_2$ be real vector spaces. We here present the general solution of (1.3), (1.4).

**Theorem 2.1.** *A function $f:E_1\to E_2$ satisfies the functional equation (1.1) if and only if $f:E_1\to E_2$ satisfies the functional equation (1.4) if and only if $f:E_1\to E_2$ satisfies the functional equation (1.3). Therefore, every solution of functional equations (1.3) and (1.4) is also a quadratic function.*

*Proof.* Let $f:E_1\to E_2$ satisfy the functional equation (1.1). Putting $x=0=y$ in (1.1), we get $f(0)=0$. Set $x=0$ in (1.1) to get $f(y)=f(-y)$. Letting $y=x$ and $y=2x$ in (1.1), respectively, we obtain that $f(2x)=4f(x)$ and $f(3x)=9f(x)$ for all $x\in E_1$. By induction, we lead to $f(kx)=k^2f(x)$ for all positive integer $k$. Replacing $x$ and $y$ by $2x+y$ and $x+2y$

in (1.1), respectively, we have

$$f(2x+y)+f(x+2y) = \frac{1}{2}[f(3x+3y)+f(x-y)] = 4f(x+y)+\frac{1}{2}[f(x+y)+f(x-y)] = 4f(x+y)+f(x)+f(y) \tag{2.1}$$

for all $x,\ y \in E_1$.

Let $f : E_1 \to E_2$ satisfy the functional equation (1.4). Putting $x = 0 = y$ in (1.4), we get $f(0) = 0$. Set $y = 0$ in (1.4) to get $f(2x) = 4f(x)$. Letting $y = x$ and $y = -2x$ in (1.4), we obtain that $f(3x) = 9f(x)$ and $f(x) = f(-x)$ for all $x \in E_1$. Putting $x$ and $y$ by $x + y$ and $x + y$ in (1.4), respectively, we obtain

$$f(2x+3y)+f(x+3y) = 4f(x+2y)+f(x+y)+f(y), \tag{2.2}$$

$$f(3x+y)+f(3x+2y) = 4f(2x+y)+f(x)+f(x+y). \tag{2.3}$$

Adding (2.2) to (2.3) and using (1.4), we obtain

$$f(2x+3y)+f(3x+2y)+f(x+3y)+f(3x+y) = 18f(x+y)+5f(x)+5f(y) \tag{2.4}$$

for all $x,\ y \in E_1$. Replacing $y$ by $2y$ and $x$ by $2x$ in (1.4), respectively, we have

$$4f(x+y)+f(x+4y) = 4f(x+2y)+f(x)+4f(y), \tag{2.5}$$

$$4f(x+y)+f(4x+y) = 4f(2x+y)+4f(x)+f(y) \tag{2.6}$$

for all $x,\ y \in E_1$. Adding (2.5) to (2.6) and using (1.4), we get

$$f(x+4y)+f(4x+y) = 8f(x+y)+9f(x)+9f(y) \tag{2.7}$$

for all $x,\ y \in E_1$.

On the other hand, using (1.4), we get

$$f(x+4y)+f(4x+y) = f(6x+9y)+f(9x+6y)-4f(5x+5y) = 9f(2x+3y)+9f(3x+2y)-100f(x+y), \tag{2.8}$$

which yields the relation by virtue of (2.7)

$$f(2x+3y)+f(3x+2y) = 12f(x+y)+f(x)+f(y) \tag{2.9}$$

for all $x,\ y \in E_1$. Combining the last equation with (2.4), we get

$$f(x+3y)+f(3x+y) = 6f(x+y)+4f(x)+4f(y). \tag{2.10}$$

Replacing $x$ and $y$ by $\frac{x+y}{2}$ and $\frac{x-y}{2}$ in (2.10), respectively, we have the desired result (1.3).

Now, let $f : E_1 \to E_2$ satisfy the functional equation (1.3). Putting $x = 0 = y$ in (1.3), we get $f(0) = 0$. Letting $y = 0$ and $y = x$ in (1.3), respectively, we obtain that $f(2x) = 4f(x)$ and $f(3x) = 9f(x)$ for all $x \in E_1$. Putting $y = 2x$ in (1.3), we get $f(x) = f(-x)$. Replacing $x$ and $y$ by $x + y$ and $x - y$, respectively, in (1.3), we have

$$f(3x+y)+f(x+3y) = 6f(x+y)+4f(x)+4f(y) \tag{2.11}$$

for all $x,\ y \in E_1$. Replacing $y$ by $x + y$ in (1.3), we obtain

$$f(3x+y)+f(x-y) = 6f(x)+f(2x+y)+f(y). \tag{2.12}$$

Interchange $x$ with $y$ in (2.12) to get the relation

$$f(3y+x)+f(x-y) = 6f(y)+f(2y+x)+f(x). \tag{2.13}$$

Adding (2.12) to (2.13), we obtain

$$6f(x+y)+2f(x-y) = f(2x+y)+f(x+2y)+3f(x)+3f(y) \tag{2.14}$$

for all $x,\ y \in E_1$. Setting $-y$ instead of $y$ in (2.14) and using the evenness of $f$, we get the relation

$$6f(x-y)+2f(x+y)=f(2x-y)+f(2y-x)+3f(x)+3f(y). \tag{2.15}$$

Adding (2.14) to (2.15), we obtain the result (1.1). □

## 3. STABILITY OF (1.3)

From now on, let $X$ be a real vector space and let $Y$ be a Banach space unless we give any specific reference. We will investigate the Hyers-Ulam-Rassias stability problem for the functional equation (1.3). Thus we find the condition that there exists a true quadratic function near an approximately quadratic function.

**Theorem 3.1.** *Let $\phi : X^2 \to \mathbb{R}^+$ be a function such that*

$$\sum_{i=0}^{\infty} \frac{\phi(2^i x, 0)}{4^i} \qquad \left(\sum_{i=1}^{\infty} 4^i \phi(\frac{x}{2^i}, 0),\ \textit{respectively}\right) \tag{3.1}$$

*converges and*

$$\lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{4^n} = 0 \qquad \left(\lim_{n\to\infty} 4^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0\right) \tag{3.2}$$

*for all $x,\ y \in X$. Suppose that a function $f : X \to Y$ satisfies*

$$\|f(2x+y)+f(2x-y)-f(x+y)-f(x-y)-6f(x)\| \le \phi(x,y) \tag{3.3}$$

*for all $x, y \in X$. Then there exists a unique quadratic function $T : X \to Y$ which satisfies the equation (1.3) and the inequality*

$$\|f(x)-T(x)\| \le \frac{1}{8}\sum_{i=0}^{\infty} \frac{\phi(2^i x, 0)}{4^i} \tag{3.4}$$

$$\left(\|f(x)-T(x)\| \le \frac{1}{8}\sum_{i=1}^{\infty} 4^i \phi\left(\frac{x}{2^i}, 0\right)\right)$$

*for all $x \in X$. The function $T$ is given by*

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n} \qquad \left(T(x) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)\right) \tag{3.5}$$

*for all $x \in X$.*

*Proof.* Putting $y = 0$ in (3.3) and dividing by $8$, we have

$$\left\|\frac{f(2x)}{4} - f(x)\right\| \le \frac{1}{8}\phi(x,0) \tag{3.6}$$

for all $x \in X$. Replacing $x$ by $2x$ in (3.6) and dividing by $4$ and summing the resulting inequality with (3.6), we get

$$\left\|\frac{f(2^2 x)}{4^2} - f(x)\right\| \le \frac{1}{8}\left[\phi(x,0) + \frac{\phi(2x,0)}{4}\right] \tag{3.7}$$

for all $x \in X$. Using the induction on a positive integer $n$, we obtain that

$$\left\|\frac{f(2^n x)}{4^n} - f(x)\right\| \leq \frac{1}{8}\sum_{i=0}^{n-1}\frac{\phi(2^i x, 0)}{4^i} \leq \frac{1}{8}\sum_{i=0}^{\infty}\frac{\phi(2^i x, 0)}{4^i} \tag{3.8}$$

for all $x \in X$. In order to prove convergence of the sequence $\left\{\frac{f(2^n x)}{4^n}\right\}$, we divide inequality (3.8) by $4^m$ and also replace $x$ by $2^m x$ to find that for $n,\ m > 0$,

$$\begin{aligned} \left\|\frac{f(2^n 2^m x)}{4^{n+m}} - \frac{f(2^m x)}{4^m}\right\| &= \frac{1}{4^m}\left\|\frac{f(2^n 2^m x)}{4^n} - f(2^m x)\right\| \\ &\leq \frac{1}{8\cdot 4^m}\sum_{i=0}^{n-1}\frac{\phi(2^i 2^m x, 0)}{4^i} \\ &\leq \frac{1}{8}\sum_{i=0}^{\infty}\frac{\phi(2^i 2^m x, 0)}{4^{m+i}}. \end{aligned} \tag{3.9}$$

Since the right hand side of the inequality tends to $0$ as $m$ tends to infinity, the sequence $\left\{\frac{f(2^n x)}{4^n}\right\}$ is a Cauchy sequence. Therefore, we may define $T(x) = \lim_{n\to\infty} 2^{-2n} f(2^n x)$ for all $x \in X$. By letting $n \to \infty$ in (3.8), we arrive at the formula (3.4). To show that $T$ satisfies the equation (1.3), replace $x, y$ by $2^n x, 2^n y$, respectively, in (3.3) and divide by $4^n$, then it follows that

$$\begin{aligned} 4^{-n}\,\| f(2^n(2x+y)) + f(2^n(2x-y)) - f(2^n(x+y)) \\ - f(2^n(x-y)) - 6f(2^n x)\| \leq 4^{-n}\phi(2^n x, 2^n y). \end{aligned}$$

Taking the limit as $n \to \infty$, we find that $T$ satisfies (1.3) for all $x, y \in X$.

To prove the uniqueness of the quadratic function $T$ subject to (3.4), let us assume that there exists a quadratic function $S : X \to Y$ which satisfies (1.3) and the inequality (3.4). Obviously, we have $S(2^n x) = 4^n S(x)$ and $T(2^n x) = 4^n T(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Hence it follows from (3.4) that

$$\begin{aligned} \|S(x) - T(x)\| &= 4^{-n}\|S(2^n x) - T(2^n x)\| \\ &\leq 4^{-n}(\|S(2^n x) - f(2^n x)\| + \|f(2^n x) - T(2^n x)\|) \\ &\leq \frac{1}{4}\sum_{i=0}^{\infty}\frac{\phi(2^i 2^n x, 0)}{4^{n+i}} \end{aligned}$$

for all $x \in X$. By letting $n \to \infty$ in the preceding inequality, we immediately find the uniqueness of $T$. This completes the proof of the theorem. □

Throughout this paper, let $B$ be a unital Banach algebra with norm $|\cdot|$, and let ${}_B\mathbb{B}_1$ and ${}_B\mathbb{B}_2$ be left Banach $B$-modules with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. A quadratic mapping $Q : {}_B\mathbb{B}_1 \to {}_{\mathbb{B}}\mathbb{B}_2$ is called $B$-quadratic if

$$Q(ax) = a^2 Q(x), \quad \forall a \in B, \forall x \in {}_B\mathbb{B}_1.$$

**Corollary 3.2.** *Let $\phi : {}_B\mathbb{B}_1 \times {}_B\mathbb{B}_1 \to \mathbb{R}^+$ be a function satisfying (3.1) and (3.2) for all $x,\ y \in {}_B\mathbb{B}_1$. Suppose that a mapping $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfies*

$$\left\| f(2\alpha x + \alpha y) + f(2\alpha x - \alpha y) - \alpha^2 f(x+y) - \alpha^2 f(x-y) - 6\alpha^2 f(x)\right\| \leq \phi(x, y)$$

*for all* $\alpha \in B$ $(|\alpha| = 1)$ *and for all* $x, y \in {}_B\mathbb{B}_1$*, and* $f$ *is measurable or* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in {}_B\mathbb{B}_1$*. Then there exists a unique* $B$*-quadratic mapping* $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$*, defined by (3.5), which satisfies the equation (1.3) and the inequality (3.4) for all* $x \in {}_B\mathbb{B}_1$*.*

*Proof.* By Theorem 3.1, it follows from the inequality of the statement for $\alpha = 1$ that there exists a unique quadratic mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfying the inequality (3.4) for all $x \in {}_B\mathbb{B}_1$. Under the assumption that $f$ is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, by the same reasoning as the proof of [5], the quadratic mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfies

$$T(tx) = t^2 T(x), \quad \forall x \in {}_B\mathbb{B}_1, \forall t \in \mathbb{R}.$$

That is, $T$ is $\mathbb{R}$-quadratic. For each fixed $\alpha \in B$ $(|\alpha| = 1)$, replacing $f$ by $T$ and setting $y = 0$ in (1.3), we have $T(\alpha x) = \alpha^2 T(x)$ for all $x \in {}_B B_1$. The last relation is also true for $\alpha = 0$. For each element $a \in B$ $(a \neq 0)$, $a = |a| \cdot \frac{a}{|a|}$. Since $T$ is $\mathbb{R}$-quadratic and $T(\alpha x) = \alpha^2 T(x)$ for each element $\alpha \in B(|\alpha| = 1)$,

$$\begin{aligned} T(ax) &= T\left(|a| \cdot \frac{a}{|a|}x\right) \\ &= |a|^2 \cdot T\left(\frac{a}{|a|}x\right) \\ &= |a|^2 \cdot \frac{a^2}{|a|^2} \cdot T(x) \\ &= a^2 T(x), \quad \forall a \in B(a \neq 0), \ \forall x \in {}_B\mathbb{B}_1. \end{aligned}$$

So the unique $\mathbb{R}$-quadratic mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ is also $B$-quadratic, as desired. This completes the proof of the corollary. □

Since $\mathbb{C}$ is a Banach algebra, the Banach spaces $E_1$ and $E_2$ are considered as Banach modules over $\mathbb{C}$. Thus we have the following corollary.

**Corollary 3.3.** *Let* $E_1$ *and* $E_2$ *be Banach spaces over the complex field* $\mathbb{C}$*, and let* $\varepsilon \geq 0$ *be a real number. Suppose that a mapping* $f : E_1 \to E_2$ *satisfies*

$$\|f(2\alpha x + \alpha y) + f(2\alpha x - \alpha y) - \alpha^2 f(x + y) - \alpha^2 f(x - y) - 6\alpha^2 f(x)\| \leq \varepsilon$$

*for all* $\alpha \in \mathbb{C}$ $(|\alpha| = 1)$ *and for all* $x, y \in E_1$*, and* $f$ *is measurable or* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in E_1$*. Then there exists a unique* $\mathbb{C}$*-quadratic mapping* $T : E_1 \to E_2$ *which satisfies the equation (1.3) and the inequality*

$$\|f(x) - T(x)\| \leq \frac{\varepsilon}{6}$$

*for all* $x \in E_1$*.*

The S. Czerwik [4] theorem for the functional equation (1.1) states that if a function $f : G \to Y$, where $G$ is an abelian group and $Y$ a Banach space, satisfies the inequality $\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$ for $p \neq 2$ and for all $x, y \in G$, then there exists a unique quadratic function $q$ such that $\|f(x) - q(x)\| \leq \frac{\varepsilon\|x\|^p}{|4-2^p|} + \frac{\|f(0)\|}{3}$ for all $x \in G$, and for all $x \in G - \{0\}$ and $\|f(0)\| = 0$ if $p < 0$. From the main theorem 3.1, we obtain the following corollary concerning the stability of the equation (1.3). We note that $p$ need not be equal to $q$.

**Corollary 3.4.** *Let* $X$ *and* $Y$ *be a real normed space and a Banach space, respectively, and let* $\varepsilon, p, q$ *be real numbers such that* $\varepsilon \geq 0$*,* $q > 0$ *and either* $p, q < 2$ *or* $p, q > 2$*. Suppose that a function* $f : X \to Y$ *satisfies*

$$\|f(2x + y) + f(2x - y) - f(x + y) - f(x - y) - 6f(x)\| \leq \varepsilon(\|x\|^p + \|y\|^q)$$

*for all $x, y \in X$. Then there exists a unique quadratic function $T : X \to Y$ which satisfies the equation (1.3) and the inequality*

$$\|f(x) - T(x)\| \le \frac{\varepsilon}{2|4 - 2^p|}\|x\|^p$$

*for all $x \in X$ and for all $x \in X - \{0\}$ if $p < 0$. The function $T$ is given by*

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n} \text{ if } p, q < 2 \qquad \left(T(x) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) \text{ if } p, q > 2\right)$$

*for all $x \in X$. Further, if for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to $Y$ is continuous, then $T(rx) = r^2T(x)$ for all $r \in \mathbb{R}$.*

The proof of the last assertion in the above corollary goes through in the same way as that of [4].

The Skof-Cholewa [3] theorem for the functional equation (1.1) states that if a function $f : G \to Y$, where $G$ is an abelian group and $Y$ a Banach space, satisfies the inequality $\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \le \varepsilon$ for all $x, y \in G$, then there exists a unique quadratic function $q$ such that $\|f(x) - q(x)\| \le \frac{\varepsilon}{2}$ for all $x \in G$. But we have a much better possible upper bound concerning the stability theorem for the functional equation (1.3) as follows. The following corollary is an immediate consequence of Theorem 3.1.

**Corollary 3.5.** *Let $X$ and $Y$ be a real normed space and a Banach space, respectively, and let $\varepsilon \ge 0$ be a real number. Suppose that a function $f : X \to Y$ satisfies*

$$\|f(2x + y) + f(2x - y) - f(x + y) - f(x - y) - 6f(x)\| \le \varepsilon \tag{3.10}$$

*for all $x, y \in X$. Then there exists a unique quadratic function $T : X \to Y$ defined by $T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$ which satisfies the equation (1.3) and the inequality*

$$\|f(x) - T(x)\| \le \frac{\varepsilon}{6} \tag{3.11}$$

*for all $x \in X$. Further, if for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to $Y$ is continuous, then $T(rx) = r^2T(x)$ for all $r \in \mathbb{R}$.*

**Remark 3.6.** If we write $y = x$ in the inequality of (3.3), we get

$$\|f(3x) - 5f(x) - f(2x)\| \le \phi(x, x) + \|f(0)\|. \tag{3.12}$$

Combining (3.12) with (3.6), we have

$$\|f(3x) - 9f(x)\| \le \phi(x, x) + \frac{\phi(x, 0)}{2} + \|f(0)\|. \tag{3.13}$$

We can easily show the following relation by induction on $n$ together with (3.13)

$$\left\|\frac{f(3^n x)}{9^n} - f(x)\right\| \le \frac{1}{9}\sum_{i=0}^{n-1} \frac{1}{9^i}\left[\phi(3^i x, 3^i x) + \frac{\phi(3^i x, 0)}{2} + \|f(0)\|\right]$$

for all $x \in X$.

In Theorem 3.1, let $\phi : X^2 \to \mathbb{R}^+$ be a function such that

$$\sum_{i=0}^{\infty} \frac{\phi(3^i x, 3^i x) + \phi(3^i x, 0)}{9^i} \qquad \left(\sum_{i=1}^{\infty} 9^i \left[\phi\left(\frac{x}{3^i}, \frac{x}{3^i}\right) + \phi\left(\frac{x}{3^i}, 0\right)\right], \text{ respectively}\right)$$

converges and

$$\lim_{n\to\infty} \frac{\phi(3^n x, 3^n y)}{9^n} = 0 \qquad \left(\lim_{n\to\infty} 9^n \phi\left(\frac{x}{3^n}, \frac{y}{3^n}\right) = 0\right)$$

for all $x,\ y \in X$. Note that in the second case $f(0) = 0$ since $\phi(0,0) = 0$. Then, using the last inequality and the same argument of Theorem 3.1, we can find the unique quadratic function $T$ defined by $T(x) = \lim_{n\to\infty} 3^{-2n} f(3^n x)$ which satisfies (1.3) and the inequality

$$\|f(x) - T(x)\| \le \frac{1}{9}\sum_{i=0}^{\infty} \frac{1}{9^i}\left[\phi(3^i x, 3^i x) + \frac{\phi(3^i x, 0)}{2}\right] + \frac{\|f(0)\|}{8} \tag{3.14}$$

$$\left(\|f(x) - T(x)\| \le \frac{1}{9}\sum_{i=1}^{\infty} 9^i\left[\phi\left(\frac{x}{3^i}, \frac{x}{3^i}\right) + \frac{\phi\left(\frac{x}{3^i}, 0\right)}{2}\right]\right)$$

for all $x \in X$. Thus we obtain an alternative result of Theorem 3.1. In Theorem 3.1, we have a simpler possible upper bound (3.4) than that of (3.14). The advantage of the inequality (3.4) compared to (3.14) is that the right hand side of (3.4) has no term for $\|f(0)\|$.

As a consequence of the above Remark 3.6, we have the following corollary. Because of the restricted condition $0 < p$, we have $f(0) = 0$.

**Corollary 3.7.** *Let $X$ and $Y$ be a real normed space and a Banach space, respectively, and let $\varepsilon \ge 0$, $0 < p \ne 2$ be real numbers. Suppose that a function $f : X \to Y$ satisfies*

$$\|f(2x+y) + f(2x-y) - f(x+y) - f(x-y) - 6f(x)\| \le \varepsilon(\|x\|^p + \|y\|^p)$$

*for all $x, y \in X$. Then there exists a unique quadratic function $T : X \to Y$ which satisfies the equation (1.3) and the inequality*

$$\|f(x) - T(x)\| \le \frac{5\varepsilon}{2|9 - 3^p|}\|x\|^p$$

*for all $x \in X$. The function $T$ is given by*

$$T(x) = \lim_{n\to\infty} \frac{f(3^n x)}{9^n} \quad \text{if } 0 < p < 2 \qquad \left(T(x) = \lim_{n\to\infty} 9^n f\left(\frac{x}{3^n}\right) \quad \text{if } p > 2\right)$$

*for all $x \in X$. Further, if for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to $Y$ is continuous, then $T(rx) = r^2 T(x)$ for all $r \in \mathbb{R}$.*

**Remark 3.8.** If we put $y = x = 0$ in the inequality of (3.10), we get $6\|f(0)\| \le \varepsilon$. Applying Remark 3.6 to (3.10), we know that there exists a unique quadratic function $T : X \to Y$ defined by $T(x) = \lim_{n\to\infty} \frac{f(3^n x)}{9^n}$ which satisfies the equation (1.3) and the inequality

$$\|f(x) - T(x)\| \le \frac{3\varepsilon}{16} + \frac{\|f(0)\|}{8} \le \frac{5\varepsilon}{24}$$

for all $x \in X$. But we have a better possible upper bound (3.11) than that of the last inequality.

## 4. STABILITY OF (1.4)

We will investigate the Hyers-Ulam-Rassias stability problem for the functional equation (1.4). Thus we find the condition that there exists a true quadratic function near an approximately quadratic function.

**Theorem 4.1.** *Let $\phi : X^2 \to \mathbb{R}^+$ be a function such that*

$$\sum_{i=0}^{\infty} \frac{1}{9^i}\left[\frac{\phi(3^i x, 3^i x)}{2} + 2\phi(3^i x, 0)\right] \tag{4.1}$$

$$\left(\sum_{i=1}^{\infty} 9^i\left[\frac{1}{2}\phi\left(\frac{x}{3^i}, \frac{x}{3^i}\right) + 2\phi\left(\frac{x}{3^i}, 0\right)\right], \ \textit{respectively}\right)$$

*converges and*

$$\lim_{n\to\infty}\frac{\phi(3^n x, 3^n y)}{9^n}=0 \qquad \left(\lim_{n\to\infty} 9^n\phi\left(\frac{x}{3^n},\frac{y}{3^n}\right)=0\right) \tag{4.2}$$

*for all* $x,\ y\in X$. *Suppose that a function* $f: X\to Y$ *satisfies*

$$\|f(2x+y)+f(x+2y)-4f(x+y)-f(x)-f(y)\|\le\phi(x,y) \tag{4.3}$$

*for all* $x,y\in X$. *Then there exists a unique quadratic function* $T: X\to Y$ *which satisfies the equation (1.4) and the inequality*

$$\|f(x)-T(x)\|\le\frac{1}{9}\sum_{i=0}^{\infty}\frac{1}{9^i}\left[\frac{\phi(3^i x,3^i x)}{2}+2\phi(3^i x,0)\right]+\frac{\|f(0)\|}{4} \tag{4.4}$$

$$\left(\|f(x)-T(x)\|\le\frac{1}{9}\sum_{i=1}^{\infty}9^i\left[\frac{1}{2}\phi\left(\frac{x}{3^i},\frac{x}{3^i}\right)+2\phi\left(\frac{x}{3^i},0\right)\right],\right)$$

*for all* $x\in X$. *The function* $T$ *is given by*

$$T(x)=\lim_{n\to\infty}\frac{f(3^n x)}{9^n} \qquad \left(T(x)=\lim_{n\to\infty}9^n f\left(\frac{x}{3^n}\right)\right) \tag{4.5}$$

*for all* $x\in X$.

*Proof.* If we write $y=x$ in the inequality of (4.3), we get

$$\|f(3x)-2f(2x)-f(x)\|\le\frac{1}{2}\phi(x,x). \tag{4.6}$$

Putting $y=0$ in (4.3) and multiplying by $2$, we have

$$\|2f(2x)-8f(x)\|\le 2\phi(x,0)+2\|f(0)\| \tag{4.7}$$

for all $x\in X$. Adding the inequality (4.6) with (4.7) and then dividing by $9$, we get

$$\left\|\frac{f(3x)}{9}-f(x)\right\|\le\frac{1}{9}\left[\frac{\phi(x,x)}{2}+2\phi(x,0)+2\|f(0)\|\right] \tag{4.8}$$

for all $x\in X$. Using the induction on $n$, we obtain that

$$\begin{aligned}\left\|\frac{f(3^n x)}{9^n}-f(x)\right\| &\le \frac{1}{9}\sum_{i=0}^{n-1}\frac{1}{9^i}\left[\frac{\phi(3^i x,3^i x)}{2}+2\phi(3^i x,0)+2\|f(0)\|\right]\\ &\le \frac{1}{9}\sum_{i=0}^{\infty}\frac{1}{9^i}\left[\frac{\phi(3^i x,3^i x)}{2}+2\phi(3^i x,0)\right]+\frac{\|f(0)\|}{4}\end{aligned} \tag{4.9}$$

for all $x\in X$.

Repeating the similar argument of Theorem 3.1, we obtain the desired result. The proof of assertion indicated by parentheses in the theorem is similarly proved and we omit it. In this case, $f(0)=0$ since $\phi(0,0)=0$ by assumption. This completes the proof of the theorem. □

The proof of the following corollary is similar to that of Corollary 3.2.

**Corollary 4.2.** *Let* $\phi: {}_B\mathbb{B}_1\times{}_B\mathbb{B}_1\to\mathbb{R}^+$ *be a function satisfying (4.1) and (4.2) for all* $x,\ y\in {}_B\mathbb{B}_1$. *Suppose that a mapping* $f: {}_B\mathbb{B}_1\to{}_B\mathbb{B}_2$ *satisfies*

$$\left\|f(2\alpha x+\alpha y)+f(\alpha x+2\alpha y)-4\alpha^2 f(x+y)-\alpha^2 f(x)-\alpha^2 f(y)\right\|\le\phi(x,y) \tag{4.10}$$

*for all* $\alpha\in B$ $(|\alpha|=1)$ *and for all* $x,y\in{}_B\mathbb{B}_1$, *and* $f$ *is measurable or* $f(tx)$ *is continuous in* $t\in\mathbb{R}$ *for each fixed* $x\in{}_B\mathbb{B}_1$. *Then there exists a unique* $B$*-quadratic mapping* $T: {}_B\mathbb{B}_1\to{}_B\mathbb{B}_2$, *defined by (4.5), which satisfies the equation (1.4) and the inequality (4.4) for all* $x\in{}_B\mathbb{B}_1$.

**Corollary 4.3.** *Let $E_1$ and $E_2$ be Banach spaces over the complex field $\mathbb{C}$, and let $\varepsilon \geq 0$ be a real number. Suppose that a mapping $f : E_1 \to E_2$ satisfies*

$$\left\| f(2\alpha x + \alpha y) + f(\alpha x + 2\alpha y) - 4\alpha^2 f(x+y) - \alpha^2 f(x) - \alpha^2 f(y) \right\| \leq \varepsilon$$

*for all $\alpha \in \mathbb{C}$ ($|\alpha| = 1$) and for all $x, y \in E_1$, and $f$ is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Then there exists a unique $\mathbb{C}$-quadratic mapping $T : E_1 \to E_2$ which satisfies the equation (1.3) and the inequality*

$$\|f(x) - T(x)\| \leq \frac{5\varepsilon}{16}$$

*for all $x \in E_1$.*

In Theorem 4.1, we obtain the alternative result if the conditions of $\phi$ are replaced by the following.

**Remark 4.4.** Let $\phi : X^2 \to \mathbb{R}^+$ be a function such that

$$\sum_{i=0}^{\infty} \frac{1}{4^i}\phi(2^i x, 0) \qquad \left( \sum_{i=1}^{\infty} 4^i \phi\left(\frac{x}{2^i}, 0\right), \text{ respectively} \right)$$

converges and

$$\lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{4^n} = 0 \qquad \left( \lim_{n\to\infty} 4^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \right)$$

for all $x,\ y \in X$. Suppose that a function $f : X \to Y$ satisfies

$$\|f(2x+y) + f(x+2y) - 4f(x+y) - f(x) - f(y)\| \leq \phi(x,y)$$

for all $x, y \in X$. Then there exists a unique quadratic function $T : X \to Y$ which satisfies the equation (1.4) and the inequality

$$\|f(x) - T(x)\| \leq \frac{1}{4}\sum_{i=0}^{\infty} \frac{1}{4^i}\phi(2^i x, 0) + \frac{\|f(0)\|}{3} \tag{4.11}$$

$$\left( \|f(x) - T(x)\| \leq \frac{1}{4}\sum_{i=1}^{\infty} 4^i \phi\left(\frac{x}{2^i}, 0\right) \right)$$

for all $x \in X$. The function $T$ is given by

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n} \qquad \left( T(x) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) \right)$$

for all $x \in X$.

From Remark 4.4, we obtain the following corollary concerning the stability of the equation (1.4). We note that $p$ need not be equal to $q$ and $\|f(0)\| = 0$ if $p > 0$.

**Corollary 4.5.** *Let $X$ and $Y$ be a real normed space and a Banach space, respectively, and let $\varepsilon, p, q$ be real numbers such that $\varepsilon \geq 0$, $q > 0$ and either $p, q < 2$ or $p, q > 2$. Suppose that a function $f : X \to Y$ satisfies*

$$\|f(2x+y) + f(x+2y) - 4f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^q)$$

*for all $x, y \in X$. Then there exists a unique quadratic function $T : X \to Y$ which satisfies the equation (1.4) and the inequality*

$$\|f(x) - T(x)\| \leq \frac{\varepsilon}{|4 - 2^p|}\|x\|^p + \frac{\|f(0)\|}{3}$$

*for all* $x \in X$ *and for all* $x \in X - \{0\}$ *if* $p < 0$. *The function* $T$ *is given by*

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n} \ \text{ if } p, q < 2 \qquad \left(T(x) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) \ \text{ if } p, q > 2\right)$$

*for all* $x \in X$. *Further, if for each fixed* $x \in X$ *the mapping* $t \mapsto f(tx)$ *from* $\mathbb{R}$ *to* $Y$ *is continuous, then* $T(rx) = r^2 T(x)$ *for all* $r \in \mathbb{R}$.

As a consequence of the above Theorem 4.1, we have the following.

**Corollary 4.6.** *Let* $X$ *and* $Y$ *be a real normed space and a Banach space, respectively, and let* $\varepsilon \geq 0$, $0 < p \neq 2$ *be real numbers. Suppose that a function* $f : X \to Y$ *satisfies*

$$\|f(2x + y) + f(x + 2y) - 4f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

*for all* $x, y \in X$. *Then there exists a unique quadratic function* $T : X \to Y$ *which satisfies the equation (1.4) and the inequality*

$$\|f(x) - T(x)\| \leq \frac{3\varepsilon}{|9 - 3^p|} \|x\|^p$$

*for all* $x \in X$. *The function* $T$ *is given by*

$$T(x) = \lim_{n\to\infty} \frac{f(3^n x)}{9^n} \ \text{ if } 0 < p < 2 \qquad \left(T(x) = \lim_{n\to\infty} 9^n f\left(\frac{x}{3^n}\right) \ \text{ if } p > 2\right)$$

*for all* $x \in X$. *Further, if for each fixed* $x \in X$ *the mapping* $t \mapsto f(tx)$ *from* $\mathbb{R}$ *to* $Y$ *is continuous, then* $T(rx) = r^2 T(x)$ *for all* $r \in \mathbb{R}$.

The following corollary is an immediate consequence of Theorem 4.1.

**Corollary 4.7.** *Let* $X$ *and* $Y$ *be a real normed space and a Banach space, respectively, and let* $\varepsilon \geq 0$ *be a real number. Suppose that a function* $f : X \to Y$ *satisfies*

$$\|f(2x + y) + f(x + 2y) - 4f(x + y) - f(x) - f(y)\| \leq \varepsilon \tag{4.12}$$

*for all* $x, y \in X$. *Then there exists a unique quadratic function* $T : X \to Y$ *defined by* $T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$ *which satisfies the equation (1.4) and the inequality*

$$\|f(x) - T(x)\| \leq \frac{5\varepsilon}{16} \tag{4.13}$$

*for all* $x \in X$. *Further, if for each fixed* $x \in X$ *the mapping* $t \mapsto f(tx)$ *from* $\mathbb{R}$ *to* $Y$ *is continuous, then* $T(rx) = r^2 T(x)$ *for all* $r \in \mathbb{R}$.

**Remark 4.8.** If we put $y = x = 0$ in the inequality of (4.12), we get $4\|f(0)\| \leq \varepsilon$. Applying Remark 4.4 to (4.12), we know that there exists a unique quadratic function $T : X \to Y$ defined by $T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$ which satisfies the equation (1.4) and the inequality

$$\|f(x) - T(x)\| \leq \frac{\varepsilon}{3} + \frac{\|f(0)\|}{3} \leq \frac{5\varepsilon}{12}$$

for all $x \in X$. But we have a better possible upper bound (4.13) than that of the last inequality.

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