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The Hypersonic Navier-Stokes Approximation for the Normal Shock Structure of a Perfect Gas with the Sutherland Viscosity Law: Review and Extension

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By means of a four-region asymptotic analysis, a uniformly valid description of the Sutherland-viscosity-law Navier-Stokes structure of the normal shock wave in the hypersonic approximation is obtained.

Keywords: Asymptotic analysis; Hypersonic flow; Navier-Stokes; Normal shock; Sutherland viscosity; Uniformly valid solution

Classification Categories: 76D30; 76E30; 76K05

1 INTRODUCTION

The purpose of this paper is to review and extend the previous investigation (Bush [1]) of the structure of a one-dimensional steady shock wave in the hypersonic limit (i.e., for the case of very large values of upstream Mach number) by means of an analysis based upon

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the Navier-Stokes equations[1] for a perfect gas with constant specific heats, Sutherland-law longitudinal viscosity, and unity longitudinal Prandtl number.

The problem of determining the shock structure by means of the Navier-Stokes equations has been studied extensively by various authors (e.g., see the review articles of Illingworth [3] and Hayes [4]). There exist analytical (or closed-form) solutions for the (physically realistic) case of constant total enthalpy (which is relevant when the longitudinal Prandtl number is unity) for the longitudinal viscosity, μ'', proportional to powers of the (absolute) temperature, T^n, for $n=0, \frac{1}{2}$, and 1. (Kinetic theory predicts that $n=\frac{1}{2}$ for hard-sphere molecules and $n=1$ for Maxwellian molecules.) Since Navier-Stokes solutions for large Mach numbers depend strongly on the viscosity-temperature relation, the Sutherland viscosity law, which, over a wider range of temperatures, provides a better fit to the data than do the power laws, is used in the present analysis, as it was in [1].

Work has been done to obtain approximate Navier-Stokes solutions for the structure of shock waves when the (constant) upstream Mach number, $M_1=u_1/(\gamma R T_1)^{1/2}$, approaches infinity, and the Newtonian parameter $\varepsilon=(\gamma-1)/(\gamma+1)$, is an order-unity constant.

Sychev [5] has investigated the shock structure of a perfect gas with $\mu''\propto T$ for the case of $M_1^{-1}=0$, $\varepsilon\sim O(1)$. In the solution obtained by Sychev, the velocity prescribed at upstream infinity is actually reached at a finite value of the spatial coordinate. Upstream of this point, a constant velocity, equal to the prescribed upstream value, is assumed; this leads to a discontinuity in the derivative of the velocity. Thus, Sychev's solution is not uniformly valid over the entire range of the shock.

In the previous study [1] of the "inverse" shock-structure problem for a perfect gas obeying the Sutherland viscosity law, the difficulty of the upstream nonuniformity is overcome through the introduction of two distinct asymptotic expansions that characterize the flow as $M_1\to\infty$, $\varepsilon\sim O(1)$ and/or $\delta=(1-\varepsilon)/\varepsilon M_1^2\to 0$, $\bar{u}_2=\varepsilon(1+\delta)\sim O(1)$. The first of these, the principal expansion, is valid throughout the portion of the shock where μ''/μ_1'', through its dependence on T/T_1,

[1] The validity of an analysis of the shock-wave structure based on the Navier-Stokes equations, especially for the case of very high Mach numbers, is still open to question (e.g., see Liepmann *et al.* [2]).

tends to infinity as M_1 tends to infinity. The second one, the boundary-layer expansion, is valid in an upstream portion of the shock where T/T_1 and μ''/μ''_1 are of order unity even as M_1 tends to infinity. The first two terms of each expansion are found, and the requirements for the matching of these expansions in an intermediate region of common validity are determined. The solution of Sychev, except for the difference in the viscosity law, corresponds to the leading term of the principal expansion of Bush.

Here, the hypersonic shock-structure analysis of Bush [1] is reviewed, employing the classical definition of the origin of the shock, namely, $x=0$ when $u=u_0=\frac{1}{2}(u_1+u_2)$, rather than the definition employed previously. Thus, the leading-order principal-expansion solutions, obtained in Section 3.1 for (u_1-u) and $(u-u_2)$ of order unity as $\delta \propto M_1^{-2}$ goes to zero, are those of the original analysis, with the important exception that the constants of integration, under the classical definition of the origin, are determined to be nonzero quantities.

These principal-expansion solutions (for the coordinate as a function of the velocity) are found to be not uniformly valid as the upstream and downstream terminal states are approached, i.e., as (u_1-u) and $(u-u_2)$ go to zero, as δ goes to zero. The upstream nonuniformity of the principal expansion was recognized and addressed through the introduction of the boundary-layer expansion in the original analysis of Bush; the downstream nonuniformity was recognized but was not addressed.

In Section 3.2, an upstream expansion, which corresponds closely to the original boundary-layer expansion, is constructed to remove the upstream nonuniformity of the principal expansion. However, it is determined that the leading-order upstream-expansion solutions are, themselves, not uniformly valid as the upstream terminal state is approached. The upstream nonuniformity of the boundary-layer-expansion solutions was not addressed in [1]. A far-upstream expansion is presented in Section 3.3. The leading-order solutions for this far-upstream expansion match to those of the upstream expansion and yield a uniformly valid approach to the upstream terminal state.

The solutions of the downstream expansion, constructed in Section 3.4, both match to those of the principal expansion, removing the downstream nonuniformity of this principal expansion, and provide a uniform approach to the downstream terminal state.

2 STATEMENT OF PROBLEM

In this determination of the structure of a hypersonic normal shock wave, it is taken that (1) the flow is steady and one-dimensional, with the spatial coordinate denoted by x and the corresponding velocity component by u; (2) the flow quantities go from uniform state ① at upstream infinity ($x \to -\infty$) through the shock wave to uniform state ② at downstream infinity ($x \to +\infty$), with the terminal uniform states related by the Rankine-Hugoniot conditions; (3) the Navier-Stokes equations are valid, with the viscous stress and heat flux, respectively, given by $\tau = \mu''(\mathrm{d}u/\mathrm{d}x)$ and $q = -k(\mathrm{d}T/\mathrm{d}x)$; (4) the fluid under consideration is a thermally and calorically perfect gas, so that $p = \rho RT$, and the internal energy and enthalpy, respectively, can be expressed as $e = c_v T$ and $h = c_p T$, where c_v and c_p are the constant specific heats, with $\gamma = (c_p/c_v)$, the ratio of the specific heats; (5) the (realistic) constant-total-enthalpy case is relevant, wherein the constant longitudinal Prandtl number, $\mathrm{Pr}'' = (\mu'' c_p/k)$, is equal to unity; (6) the longitudinal-viscosity and thermal-conductivity coefficients obey the Sutherland law, i.e., $(\mu''/\mu_1''), (k/k_1) = (T/T_1)^{1/2}\{(1+\theta)/[1+\theta(T/T_1)^{-1}]\}$, θ being the Sutherland parameter; and (7) the upstream Mach number, $M_1 = u_1/(\gamma R T_1)^{1/2}$, goes to infinity, and the Newtonian parameter, $\varepsilon = (\gamma-1)/(\gamma+1)$, is of order unity, such that $\delta = (1-\varepsilon)/\varepsilon M_1^2$ goes to zero.

Under the preceding restrictions, if the following nondimensional quantities are introduced:

$$\xi = \frac{\rho_1 u_1 x}{\mu_1''}, \tag{2.1}$$

$$\bar{u} = \frac{u}{u_1}, \quad \text{with } \bar{u}_1 = 1,\ \bar{u}_2 = \frac{(\gamma-1)}{(\gamma+1)}\left[1 + \frac{2}{(\gamma-1)M_1^2}\right] = \varepsilon(1+\delta),$$

$$\bar{T} = \bar{T}(\bar{u}) = \frac{T}{T_1} = \left[1 + \frac{(\gamma-1)M_1^2}{2}(1-\bar{u}^2)\right] = [1 + \delta^{-1}(1-\bar{u}^2)],$$

$$\bar{\mu}'' = \bar{\mu}''(\bar{T}(\bar{u})) = \frac{\mu''}{\mu_1''} = \frac{(1+\theta)\bar{T}^{3/2}}{(\bar{T}+\theta)} = \frac{(1+\theta)[1+\delta^{-1}(1-\bar{u}^2)]^{3/2}}{[(1+\theta)+\delta^{-1}(1-\bar{u}^2)]}, \tag{2.2}$$

then the "inverse" boundary-value problem of Bush [1] for the hypersonic shock structure is

$$\frac{d\xi}{d\bar{u}} = -\frac{(1+\varepsilon)\bar{u}}{(1-\bar{u})(\bar{u}-[\varepsilon(1+\delta)])} \cdot \frac{(1+\theta)[1+\delta^{-1}(1-\bar{u}^2)]^{3/2}}{[(1+\theta)+\delta^{-1}(1-\bar{u}^2)]}, \tag{2.3}$$

$$\xi \to -\infty \quad \text{as } \bar{u} \to \bar{u}_1 = 1, \tag{2.4a}$$

$$\xi \to +\infty \quad \text{as } \bar{u} \to \bar{u}_2 = \varepsilon(1+\delta), \tag{2.4b}$$

$$\begin{aligned} M_1 &= \frac{u_1}{(\gamma R T_1)^{1/2}} \to \infty, \quad \varepsilon = \frac{(\gamma-1)}{(\gamma+1)} \sim O(1): \\ \delta &= \frac{2}{(\gamma-1)M_1^2} = \frac{(1-\varepsilon)}{\varepsilon M_1^2} \to 0, \quad \bar{u}_2 = \varepsilon(1+\delta) \sim O(1). \end{aligned} \tag{2.5}$$

3 HYPERSONIC ASYMPTOTIC EXPANSIONS

To find the shock structure for (non-Newtonian) hypersonic flow, it is necessary to obtain the solution of (2.3), subject to the (farfield) boundary conditions of (2.4), for the limits of (2.5).[2] Expansions in terms of δ are constructed below.

3.1 The Principal Expansion[3]

For $\delta \to 0$, $\bar{u}_2 \sim O(1)$ and $(1-\bar{u})$, $(\bar{u}-\bar{u}_2) \sim O(1)$, such that

$$\bar{T} = \left[\frac{(1-\bar{u}^2)}{\delta}\right]\left\{1+\left[\frac{\delta}{(1-\bar{u}^2)}\right]\right\} \to \infty,$$

$$\bar{\mu}'' = (1+\theta)\left[\frac{(1-\bar{u}^2)}{\delta}\right]^{1/2}\left\{1-\left(\theta-\frac{1}{2}\right)\left[\frac{\delta}{(1-\bar{u}^2)}\right]+O\left(\left[\frac{\delta}{(1-\bar{u}^2)}\right]^2\right)\right\} \to \infty,$$

with the introduction of the coordinate

$$\eta = \frac{\delta^{1/2}}{(1+\varepsilon)(1+\theta)}\xi, \tag{3.1}$$

[2] Here, $\bar{u}_2 = \varepsilon(1+\delta)$ appears as an order-unity parameter; its expansion in δ is irrelevant for the method of solution employed.

[3] This principal expansion analysis follows that of Bush [1].

(2.3) may be expanded as

$$\frac{d\eta}{d\bar{u}} = -\frac{\bar{u}(1+\bar{u})^{1/2}}{(1-\bar{u})^{1/2}(\bar{u}-\bar{u}_2)}\left\{1-\left(\theta-\frac{1}{2}\right)\left[\frac{\delta}{(1-\bar{u}^2)}\right]+O\left(\left[\frac{\delta}{(1-\bar{u}^2)}\right]^2\right)\right\}. \tag{3.2}$$

Inspection of (3.2)[4] shows that η has an asymptotic expansion, denoted here as the principal expansion, of the form

$$\eta(\bar{u};\bar{u}_2,\theta,\delta) \cong \left[\eta_0(\bar{u};\bar{u}_2)+\delta\frac{(\theta-1/2)}{(1-\bar{u}_2)}\eta_1(\bar{u};\bar{u}_2)+\cdots\right]. \tag{3.3}$$

Term-by-term integration of (3.2) shows that the leading terms of this expansion are

$$\eta_0 = \left[(1+\bar{u}_2)\cos^{-1}(\bar{u})+(1-\bar{u}_2^2)^{1/2}-\bar{u}_2\left(\frac{1+\bar{u}_2}{1-\bar{u}_2}\right)^{1/2}\log\left(\frac{1-V}{1+V}\right)\right]-A_0,$$

$$\eta_1 = \left[\left(\frac{1+\bar{u}}{1-\bar{u}}\right)^{1/2}+\frac{\bar{u}_2}{(1-\bar{u}_2)^{1/2}}\log\left(\frac{1-V}{1+V}\right)\right]-A_1, \quad \ldots,$$

$$\text{with}\quad V=\left[\left(\frac{1+\bar{u}_2}{1-\bar{u}_2}\right)\left(\frac{1-\bar{u}}{1+\bar{u}}\right)\right]^{1/2}, \text{ and}$$

$$A_0, A_1, \ldots = \text{consts. (to be specified).} \tag{3.4}$$

In order that $\eta \to 0$ as $\bar{u} \to \frac{1}{2}(1+\bar{u}_2)$, the classical fixing of the origin of the spatial coordinate, the constants of integration[5] are determined to be

$$A_0 = \left[(1+\bar{u}_2)\cos^{-1}\left\{\frac{1}{2}(1+\bar{u}_2)\right\}+\frac{1}{2}\{(1-\bar{u}_2)(3+\bar{u}_2)\}^{1/2}\right.$$
$$\left.-\bar{u}_2\left(\frac{1+\bar{u}_2}{1-\bar{u}_2}\right)^{1/2}\log\left(\frac{1-V_a}{1+V_a}\right)\right] > 0,$$

$$A_1 = \left[\left(\frac{3+\bar{u}_2}{1-\bar{u}_2}\right)^{1/2}+\frac{\bar{u}_2}{(1-\bar{u}_2^2)^{1/2}}\log\left(\frac{1-V_a}{1+V_a}\right)\right], \quad \ldots,$$

$$\text{with } V_a = \left(\frac{1+\bar{u}_2}{3+\bar{u}_2}\right)^{1/2}. \tag{3.5}$$

[4] For air, $\theta \approx 0.505$; nevertheless, no advantage is taken of the smallness of the parameter $(\theta-\frac{1}{2})$.

[5] In Bush [1], the constants corresponding to $A_0, A_1, \ldots$ are set equal to zero.

As the upstream state is approached, $\bar{u} \to \bar{u}_1 = 1$, and the expansion of (3.3)–(3.5) can be written as

$$\begin{aligned}\eta \cong & -\left\{A_0 + \Delta\left(\theta - \frac{1}{2}\right)A_1 + \cdots\right\} \\ & + \frac{2\sqrt{2}}{(1-\bar{u}_2)^{1/2}}\left\{\left[v^{1/2} - \frac{1}{12}(1-5\bar{u}_2)v^{3/2} + \cdots\right]\right. \\ & \left. + \tfrac{1}{2}\,\Delta(\theta - \tfrac{1}{2})\left[v^{-1/2} - \tfrac{1}{4}(1+3\bar{u}_2)v^{1/2} + \cdots\right] + \cdots\right\}, \\ & \text{with } v = \frac{(1-\bar{u})}{(1-\bar{u}_2)} \to 0,\ \Delta = \frac{\delta}{(1-\bar{u}_2)} \to 0. \end{aligned} \tag{3.6}$$

From (3.6), it is seen that $\eta_0 = -A_0 < 0$ for $v = 0$. Thus, to leading order of approximation, the upstream boundary condition is not satisfied in the hypersonic limit. Further, with $\theta > \frac{1}{2}$, $\eta_1 \to +\infty$ (algebraically) as $v \to 0$. Clearly, the solution of (3.3)–(3.5) cannot be expected to be valid in any upstream region where v is very close to zero; the principal expansion holds only for $\Delta/v \ll 1$. It is necessary to identify expansions that are valid in the upstream part of the shock. The upstream and far-upstream expansions that complement the principal expansion are presented in Sections 3.2 and 3.3, respectively.

As the downstream state is approached, $\bar{u} \to \bar{u}_2 = \varepsilon(1+\delta) \sim O(1)$, and the principal expansion yields

$$\begin{aligned}\eta \cong & -\bar{u}_2\left(\frac{1+\bar{u}_2}{1-\bar{u}_2}\right)^{1/2} \\ & \times \left\{[\log w + P_0 + \cdots] - \Delta\frac{(\theta - 1/2)}{(1+\bar{u}_2)}[\log w + P_1 + \cdots] + \cdots\right\}, \\ & \text{with } w = \frac{(u-\bar{u}_2)}{(1-\bar{u}_2)} \to 0,\ \ \Delta = \frac{\delta}{(1-\bar{u}_2)} \to 0, \text{ and} \\ & P_0, P_1, \ldots = \text{consts.} \sim O(1). \end{aligned} \tag{3.7}$$

Although $\eta_0 \to +\infty$ (logarithmically) as $w \to 0$, the behavior of η_1 is such that the principal expansion is not valid for $z = \Delta\log(w^{-1}) = -\Delta\log w \sim O(1)$ as $w \to 0$ and $\Delta \to 0$. The uniformly valid downstream expansion, necessary to complete the uniformly valid picture of the shock structure, is presented in Section 3.4.

3.2 The Upstream Expansion[6]

The foregoing principal expansion indicates that it is valid for $(1-\bar{u}) \gg \delta$, but it is not valid in an upstream region where $(1-\bar{u}) \sim O(\delta)$. To study the solution in such an upstream region, a new independent variable, s, defined by

$$s = \frac{v}{\Delta} = \frac{(1-\bar{u})}{\delta} \sim O(1) \quad \text{for } (1-\bar{u}) \to 0 \text{ and } \delta \to 0, \tag{3.8}$$

is introduced. In turn, for $s \sim O(1)$ and $\delta \to 0$, such that $\bar{T} = [(1+2s) - \delta s^2] \sim O(1), \ldots$, and for η defined by (3.1), (2.3) may be expanded as

$$\frac{d\eta}{ds} = \frac{\delta^{1/2}}{(1-\bar{u}_2)} \frac{(1+2s)^{3/2}}{s((1+\theta)+2s)} \times \left\{ 1 + \delta \left[\frac{\bar{u}_2 s}{(1-\bar{u}_2)} - \frac{3}{2} \frac{s^2}{(1+2s)} + \frac{s^2}{((1+\theta)+2s)} \right] + O(\delta^2) \right\}. \tag{3.9}$$

Term-by-term integration of (3.9) yields the following upstream representation for the solution:

$$\eta = -\left\{ A_0 + \delta \frac{(\theta - 1/2)}{(1-\bar{u}_2)} A_1 + O(\delta^2) \right\} + \frac{\delta^{1/2}}{(1-\bar{u}_2)} \{ \zeta_0 + \delta \zeta_1 + O(\delta^2) \}, \tag{3.10}$$

where

$$\zeta_0 = \left[2(1+2s)^{1/2} + \frac{1}{(1+\theta)} \log \left\{ \frac{(1+2s)^{1/2} - 1}{(1+2s)^{1/2} + 1} \right\} + \frac{2\theta^{3/2}}{(1+\theta)} \cot^{-1} \left\{ \left(\frac{1+2s}{\theta} \right)^{1/2} \right\} \right] + (1-\bar{u}_2)^{1/2} C_0,$$

$$\zeta_1 = -\frac{s(1+2s)^{3/2}}{2((1+\theta)+2s)} + \frac{1}{2} \left(\frac{1+\bar{u}_2}{1-\bar{u}_2} \right) \left[\frac{1}{3}(1+2s)^{3/2} - \theta(1+2s)^{1/2} - \theta^{3/2} \cot^{-1} \left\{ \left(\frac{1+2s}{\theta} \right)^{1/2} \right\} \right] + \frac{C_1}{(1-\bar{u}_2)^{1/2}}, \quad \ldots,$$

with $C_0, C_1, \ldots =$ consts. (to be specified). (3.11)

[6] This upstream expansion corresponds to the boundary-layer expansion of Bush [1].

Downstream, as $s \to \infty$, (3.10) and (3.11) produce

$$\eta \cong -\{A_0 + \Delta(\theta - \tfrac{1}{2})A_1 + \cdots\} + \Delta^{1/2}\{C_0 + \Delta C_1 + \cdots\}$$
$$+ \Delta^{1/2} \frac{2\sqrt{2}}{(1-\bar{u}_2)^{1/2}} \left\{ \left[s^{1/2} + \frac{1}{2}\left(\theta - \frac{1}{2}\right)s^{-1/2} + \cdots \right] \right.$$
$$\left. - \Delta\left[\tfrac{1}{12}(1 - 5\bar{u}_2)s^{3/2} + \tfrac{1}{8}(\theta - \tfrac{1}{2})(1 + 3\bar{u}_2)s^{1/2} + \cdots\right] + \cdots \right\}. \quad (3.12)$$

A comparison of (3.6) and (3.12) indicates that the principal and upstream expansions match in an intermediate region, where $v \to 0$ and $s = v/\Delta \to \infty$ as $\Delta \to 0$, if $C_0, C_1, \ldots = 0$.

Upstream, as $s \to 0$, from (3.10) and (3.11), it is found that, with $C_0, C_1, \ldots = 0$,

$$\eta \cong -\{A_0 + \Delta(\theta - \tfrac{1}{2})A_1 + \cdots\} + \Delta^{1/2}\{B_0 + \Delta B_1 + \cdots\}$$
$$+ \frac{\Delta^{1/2}}{(1+\theta)(1-\bar{u}_2)^{1/2}} \left\{ \left[\log s + \frac{(1+3\theta)}{(1+\theta)} s + \cdots \right] + \Delta[\bar{u}_2 s + \cdots] + \cdots \right\},$$
$$\text{with } B_0 = \frac{1}{(1-\bar{u}_2)^{1/2}} \left[2 + \frac{2\theta^{3/2}\cot^{-1}(\theta^{-1/2}) - \log 2}{(1+\theta)} \right],$$
$$B_1 = -\frac{(1+\bar{u}_2)}{2(1-\bar{u}_2)^{1/2}} \left[(\theta - \tfrac{1}{3}) + \theta^{3/2}\cot^{-1}(\theta^{-1/2}) \right], \quad \ldots. \quad (3.13)$$

This result indicates that the upstream boundary condition is not satisfied uniformly in this upstream region, even though $\zeta_0 \to -\infty$ (logarithmically) as $s \to 0$.

3.3 The Far-Upstream Expansion

An examination of (3.13) suggests that the upstream boundary condition can be satisfied uniformly in a far-upstream region, characterized by the velocity variable

$$t = \Delta^{1/2}\log(s^{-1}) = -\Delta^{1/2}\log s \sim O(1) \quad \text{for } s \to 0 \text{ and } \Delta \to 0. \quad (3.14)$$

For $t \sim O(1)$ and $\Delta \to 0$, such that $\bar{T} = [1 + O(\exp(-\Delta^{-1/2}))] \approx 1, \ldots,$ the shock-structure equation, (2.3), subject to (3.1), reduces to

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = -\frac{1}{(1+\theta)(1-\bar{u}_2)^{1/2}}\left\{1 + O(\exp(-\Delta^{-1/2}))\right\}. \tag{3.15}$$

Integration of (3.15), taking into account the results of (3.13), yields

$$\eta \cong \left\{-\left[\frac{t}{(1+\theta)(1-\bar{u}_2)^{1/2}} + A_0\right] + \Delta^{1/2}B_0 - \Delta(\theta - \tfrac{1}{2})A_1 \right.$$
$$\left. + \Delta^{3/2}B_1 + \cdots\right\}. \tag{3.16}$$

This representation of the solution uniformly satisfies the upstream boundary condition, i.e., $\eta \to -\infty$ as $t \to +\infty$ and $\Delta \to 0$ (and/or $\bar{u} \to \bar{u}_1 = 1$ and $\delta \to 0$). The upstream behavior of the "direct" solution of the shock structure is

$$\frac{(\bar{u}-\bar{u}_2)}{(1-\bar{u}_2)} \cong 1 - \Delta \exp\left\{-\Delta^{-1/2}K(\eta_u - \bar{A})\right\}(1+\cdots) \to 1$$
$$\text{as } \eta_u = -\eta \to \infty \text{ and } \Delta \to 0,$$
$$\text{with } \bar{A} \cong [A_0 - \Delta^{1/2}B_0 + \Delta(\theta - \tfrac{1}{2})A_1 - \Delta^{3/2}B_1 + \cdots] \sim O(1),$$
$$K = [(1+\theta)(1-\bar{u}_2)^{1/2}] \sim O(1). \tag{3.17}$$

3.4 The Downstream Expansion

Directing attention to the downstream behavior of the shock, based on the results of (3.7), here, it is taken that there is a downstream region, characterized by the velocity variable

$$z = \Delta \log(w^{-1}) = -\Delta \log w \sim O(1)$$
$$\text{as} \quad w = \frac{(\bar{u}-\bar{u}_2)}{(1-\bar{u}_2)} \to 0 \quad \text{and} \quad \Delta = \frac{\delta}{(1-\bar{u}_2)} \to 0. \tag{3.18}$$

In the limit of $z \sim O(1)$ and $\Delta \to 0$, such that $\bar{T} = \Delta^{-1}[(1+\bar{u}_2) + \Delta + O(\exp(-\Delta^{-1}))] \to \infty, \ldots$, (2.3), subject to (3.1), yields

$$\frac{d\eta}{dz} \cong \Delta^{-1}\bar{u}_2\left(\frac{1+\bar{u}_2}{1-\bar{u}_2}\right)^{1/2}\left\{1 - \Delta\left[\frac{(\theta - 1/2)}{(1+\bar{u}_2)}\right] + \cdots\right\}. \quad (3.19)$$

Integration of (3.19), taking into account the results of (3.7), produces the downstream expansion

$$\eta \cong \Delta^{-1}\bar{u}_2\left(\frac{1+\bar{u}_2}{1-\bar{u}_2}\right)^{1/2}\left\{z\left[1 - \Delta\frac{(\theta - 1/2)}{(1+\bar{u}_2)} + \cdots\right] - \Delta\left[P_0 - \Delta\frac{(\theta - 1/2)}{(1+\bar{u}_2)}P_1 + \cdots\right]\right\}. \quad (3.20)$$

It is seen that this representation of the solution uniformly satisfies the downstream boundary condition, i.e., $\chi = \Delta\eta \to +\infty$ as $z \to +\infty$ and $\Delta \to 0$ (and/or $\bar{u} \to \bar{u}_2 = \varepsilon(1+\delta) \sim O(1)$ and $\delta \to 0$). The downstream behavior of the "direct" solution of the shock structure is determined to be

$$\frac{(\bar{u} - \bar{u}_2)}{(1-\bar{u}_2)} \cong \exp\{-L(\eta_d - \bar{P})\}(1 + \cdots) \to 0$$

as $\eta_d = \eta \to +\infty$ and $\Delta \to 0$,

$$\text{with } \bar{P} \cong \bar{u}_2\left(\frac{1+\bar{u}_2}{1-\bar{u}_2}\right)^{1/2}\left[P_0 - \Delta\frac{(\theta - 1/2)}{(1+\bar{u}_2)}P_1 + \cdots\right] \sim O(1),$$

$$L \cong \frac{1}{\bar{u}_2}\left(\frac{1-\bar{u}_2}{1+\bar{u}_2}\right)^{1/2}\left[1 + \Delta\frac{(\theta - 1/2)}{(1+\bar{u}_2)} + \cdots\right] \sim O(1). \quad (3.21)$$

4 RESULTS AND DISCUSSION

The foregoing asymptotic analysis for the model boundary-value problem of (2.3)–(2.5) reveals that a four-region structure is required in order to obtain uniformly valid solutions from the upstream state to the downstream state. The (near-downstream) principal and the (near-)upstream regions, of Sections 3.1 and 3.2, respectively, must be complemented by the far-upstream and the (far-)downstream

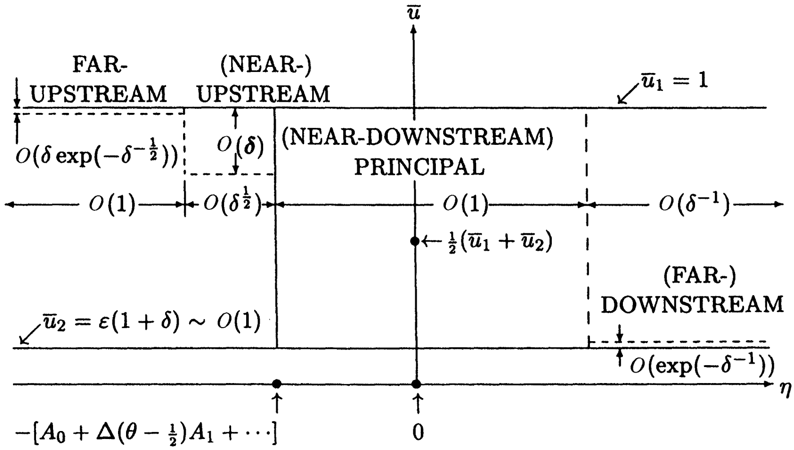

FIGURE 1 Schematic diagram of the four-region shock structure.

regions, of Sections 3.3 and 3.4, respectively. The details of this structure are shown schematically in Fig. 1.

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