Research Article

Suboptimal Regulation of a Class of Bilinear Interconnected Systems with Finite-Time Sliding Planning Horizons

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This paper focuses on the suboptimization of a class of multivariable discrete-time bilinear systems consisting of interconnected bilinear subsystems with respect to a linear quadratic optimal regulation criterion which involves the use of state weighting terms only. Conditions which ensure the controllability of the overall system are given as a previous requirement for optimization. Three transformations of variables are made on the system equations in order to implement the scheme on an equivalent linear system. This leads to an equivalent representation of the used quadratic performance index that involves the appearance of quadratic weighting terms related to both transformed input and state variables. In this way, a Riccati-matrix sequence, allowing the synthesis of a standard feedback control law, is obtained. Finally, the proposed control scheme is tested on realistic examples.

1. Introduction

Dynamic bilinear systems have received great attention by researchers in the last decades, from the classical works of Anderson and Moore [1], Feldbaum [2], Tarn [3], or Tarn et al. [4], to the recent works of Al-Baiyat [5], Kotta et al. [6] or Garrido et al. [7]. They form a transitional class between the linear and the general nonlinear systems. The importance of such systems is well-known in diverse areas in which bilinear systems appear naturally for a variety of dynamic

processes such as those taking place in electrical systems [8], industrial processes [9], power plants [10], chemical processes [11], nuclear fusion [12], biomedical applications [13], information management [14], mechanical systems [15], and aerospace and avionics [16]. which can not be satisfactorily modeled under the assumption of linearity. On the other hand, the design of controllers for bilinear systems has been an area of major research during the recent years. This growing interest in practical applications requires the development of suitable algorithms for control problems associated with bilinear systems. Many of the results obtained rely on the optimization theory with an appropriate performance index, as it is the case of the proposed control scheme.

Well-known works as Goka et al. [17] and Tarn et al. [18] stated the controllability conditions for standard classes of discrete bilinear systems. In the first paper, an equivalent system description was derived with the equivalent input being dependent on the products of the state and the original input. Since the equivalent feed-forward loop is of a linear nature, the analysis becomes greatly simplified. In the second paper, the uniform controllability of such systems using bounded input was studied. On the other hand, it is well known that the systems are often interconnected and, in many cases, several dynamic subsystems can be distinguished for analysis purposes. This sometimes takes place in computer communication, transportation networks, control and power stations, and so on. There exists important literature dealing with both the associate multivariable and the large-scale problems [19–25]. For instance, standard adaptive control techniques were applied in [26] to compensate the undesirable deviations of the process parameters from their nominal values, being the overall process modeled as a bilinear system prior to the application of the adaptive scheme.

This paper reports suboptimal optimization techniques which are applied to bilinear models. Such models can be considered as direct extensions to the linear continuous interconnected systems stated by the work of Ramakrishna and Viswanadham [27]. A class of invariant discrete-time multivariable bilinear systems with interconnection subsystems is studied. First, the system is shown to be equivalently described by the linear feed-forward multivariable structure with multiplicative inputs including a deterministic disturbance vector. Also, other interpretations are stated. As a previous requirement for optimization, controllability results for the overall systems performance are given by extending those ones given in [17, 18, 27–31]. This is achieved through the above equivalent linear system with the use of centralized control methods. A central coordinator is supposed to be available for the local controllers to supply information to each control station from the remaining ones. That control technique is applied to the optimization of the aforementioned system class. The used performance index involves state-weighting terms only. An explicit solution of Riccati type (which is unusual for such an optimization criterion) is found out as the suboptimal solution by using manipulations on the input/state variables of the problem statement [1, 32]. The importance of this strategy arises from problems where constrains on the input rather than input weights are introduced in the optimization criterion, what leads to a feedback-type control law.

The paper is organized as follows. In Section 2, the problem statement for the class of the interconnected invariant discrete-time bilinear systems is given. Also, controllability necessary conditions obtained by the decomposition techniques are studied. Section 3 is devoted to the development of suboptimal strategies for the regulation criterion, which involves quadratic terms of the state variables. Three transformations of variables are used to turn the loss function into another one, which involves state and input quadratic terms leading to a Riccati-type suboptimal solution. The control strategies may be considered as extensions of those ones used in

[1, 29] or [25], for the linear continuous problem. The formulation is accomplished from a centralized viewpoint. In Section 4, two simulated examples are presented. The first one consists of a simulated numerical example; and the second one consists of a real regulation problem of two amplidynes acting on a DC-motor. In the second example, it is not necessary to consider constant current or voltage as it is usual in DC-motor control, with their product—namely, the electric par—a bilinear term describing the resulting dynamical system. Finally, conclusions end the paper. Appendix A is related to the meaning explanation of certain equivalent interpretations of the overall system structure. Appendix B concerns with the development of the transformations of variables involved in the redefinition of the performance index. By convenience, equations and results from the appendices are sometimes invoked in the main body of the paper order not to repeat mathematical material.

2. On the system structure and its controllability properties

2.1. Controllability of a class of bilinear systems

In [17], the following typical class of bilinear systems was considered:

$$x(k + 1) = [A + u(k)B] x(k) + Cu(k); \quad k = 0, 1, 2, \dots, \ x(\cdot) \in R^n; \ u(\cdot) \in R. \qquad (2.1)$$

Equation (2.1) can equivalently be described, if rank $(B, C) = 1$ and then rank $(B) = 1$ if C and h are both nonzero, with $B = Ch^T$ (C, h being unique n-vectors within a scalar factor), by the linear feed-forward system with multiplicative input:

$$x(k + 1) = Ax(k) + Cv(k); \quad v(k) \equiv \begin{cases} u(k)h^T x(k); & \text{if } C = 0, \\ u(k)\left[h^T x(k) + 1\right]; & \text{if } C \neq 0 \end{cases} \qquad (2.2)$$

with $k = 0, 1, 2, \dots$; and where $x(k)$ and $u(k)$ are the state and the control at time k, respectively; and A and B are real constant coefficient matrices of compatible dimensions.

Note that for arbitrary ranks of B or (B, C) in (2.1), the decomposition $B = \sum_{i=1}^{p} e_i b_i^T$ can be used, with e_i and b_i^T being, respectively, the ith unity vector (i.e., the ith component is unity and the remaining ones are zero) and the ith row vector of B. The controllability of the bilinear system was related in that paper to that of the linear system (2.2) as follows.

Theorem 2.1 (summary of results in [17]). *For the equivalent systems* (2.1)-(2.2), *the following controllability results hold.*

   (i) *If the bilinear system* (2.1) *is controllable, then the pair (A, C) in* (2.2) *is controllable.*

   (ii) *If (A^T, h) is not controllable, then the homogeneous bilinear system is not controllable.*

   (iii) *If (A, C) and (A^T, h) are both controllable pairs and if $h^T C$, $h^T A^{-1} C \neq 0$, then the homogenous bilinear system is controllable in R^n outside the origin.*

   (iv) *The inhomogeneous bilinear system is not controllable in R^n if*

      (a) *rank [controllability matrix of (A^T, h)] = rank [controllability matrix of (A^T, h);1]; $(1 \equiv n$-vector with every element being unity),*

(b) $\sum_{i=1}^{n} a_i = 1$, with $a_{(\cdot)}$ being the elements of the last row of the A-matrix in the canonical state-space representation of system (2.2).

(v) The inhomogeneous bilinear system is controllable in R^n, if (A, C) is a controllable pair, $h^T C$, $h^T A^{-1} C \neq 0$, and at least one of the conditions (iv-a) or (iv-b) is not fulfilled.

Although these results are restricted to a special class of discrete bilinear systems, many natural discrete bilinear systems [10, 15, 33–35] do satisfy these assumptions. More general controllability conditions for bilinear systems can be found in [18, 36].

Remark 2.2. More general results were derived in [30] for the case in which A^{-1} does not exist. Such a paper studies the controllability conditions for three cases of violation of the conditions given in [37] (namely, (A, C) and (A^T, h) are controllable pairs and $k_0 = $ g.c.d$\{i / h^T A^{i-1} C \neq 0, \ 0 < i \leq n^2\} = 1$) (g.c.d denotes the greatest common divisor). Such cases include the following:

(a) rank $[(A, C) = n$, rank $\Omega(h^T, A) < n$;

(b) rank $[(A, C) < n$, rank $\Omega(h^T, A) = n$; and

(c) rank $[(A; C) < n$, rank $\Omega(h^T, A) < n$, with $[(A, C) \equiv [C, AC, \ldots, A^{n-1}C]$ and $\Omega(h^T, A) \equiv [h, A^T h, \ldots, A^{(n-1)T} h]^T$.

2.2. Models for a class of bilinear systems with interconnected subsystems

In this paper, the following multivariable invariant discrete-time structure of bilinear systems (S_i) of homogeneous type is considered

$$S_i : x_i(k+1) = [A_i + u_i(k)B_i] x_i(k) + D_i z_i(k), \tag{2.3}$$

$$y_i(k) = C_i x_i(k); \quad i \in I; \ I = \{1, 2, \ldots, p\}; \ k = 0, 1, 2, \ldots \tag{2.4}$$

with the interconnection bilinear subsystems (H_i), which include linear and bilinear coupling terms, given by

$$H_i : z_i(k+1) = [M_i + u_i(k)P_i] z_i(k) + \sum_{j=1}^{p} [L_{ij} + u_i(k)M_{ij}] y_j(k); \quad i \in I; \ k = 0, 1, 2, \ldots, \tag{2.5}$$

where $x_i(k) \in R^{n_i}$, $z_i(k) \in R^{a_i}$, $y_i(k) \in R^{r_i}$, $u_i(k) \in R$ are, respectively, the state vector of the S_i-subsystems, the state vectors of the interaction H_i-subsystems, the corresponding output and input vectors at time k, and the different coefficient matrices of appropriate dimensions (see Figure 1).

The linear continuous models of [27] for Figure 1 are more general than those ones above since additional measurements from the interconnection subsystems are used to generate the inputs to the plant. Structures of this type arise from practical systems such as countercurrent heat exchangers having weak nonlinearities modeled as bilinear.

Now, the extended state vector of the overall system is defined as

$$\overline{x}(k) \equiv [\overline{x}_1^T(k), \overline{x}_2^T(k), \ldots, \overline{x}_p^T(k)]^T \in R^{n+a}, \quad \text{where } \overline{x}_i(k) \equiv [x_i^T(k), z^{iT}(k)]^T \in R^{n_i + a_i}, \tag{2.6}$$
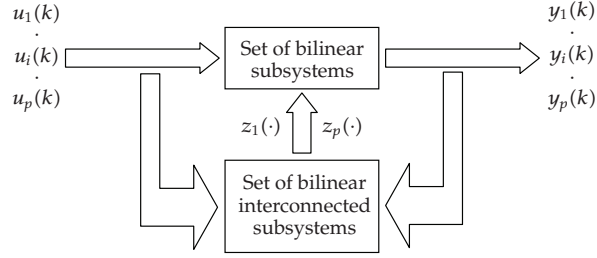
Figure 1: The class of multivariable discrete invariant-time bilinear subsystems with the interaction bilinear subsystems.

all $i \in I$ with $n \equiv \sum_{j=1}^{p} n_j$, $a \equiv \sum_{j=1}^{p} a_j$, and the extended output vector is defined as $\overline{y}(k) \equiv [y_1^T(k), y_2^T(k), \ldots, y_p^T(k)]^T \in R^r$, with $r \equiv \sum_{j=1}^{p} r_j$. Thus, the following multivariable structure, equivalent to (2.3)–(2.5), is deduced:

$$\overline{x}(k+1) = \overline{A}\,\overline{x}(k) + \sum_{i=1}^{p} u_i(k)\overline{B}_i\,\overline{x}(k); \quad k = 0, 1, 2, 3, \ldots, \tag{2.7}$$

$$\overline{y}(k) = \overline{C}\,\overline{x}(k); \quad k = 0, 1, 2, 3, \ldots, \tag{2.8}$$

where the matrices of parameters have the following structures:

$$\overline{A} = \left[\overline{A}_1^T, \overline{A}_2^T, \ldots, \overline{A}_p^T\right]^T; \qquad \overline{A}_i = \left[\overline{A}_{i1}, \overline{A}_{i2}, \ldots, \overline{A}_{ii}, \ldots, \overline{A}_{ip}\right]$$

$$\overline{A}_{ii} = \left[\begin{array}{c|c} A_i & D_i \\ \hline L_{ii}C_i & M_i \end{array}\right] \qquad \overline{A}_{ij} = \left[\begin{array}{c|c} 0 & 0 \\ \hline L_{ij}C_j & 0 \end{array}\right] \quad (\text{for } j \neq i)$$

$$\overline{B}_i = \left[0\overset{i-1}{\smile}, \ldots, 0, \widetilde{B}_i^T, 0\overset{p-1}{\smile}, \ldots, 0\right]^T; \qquad \widetilde{B}_i = \left[\widetilde{B}_{i1}, \widetilde{B}_{i2}, \ldots, \widetilde{B}_{ip}\right] \tag{2.9}$$

$$\widetilde{B}_{ii} = \left[\begin{array}{c|c} B_i & 0 \\ M_{ii}C_i & P_i \end{array}\right] \qquad \widetilde{B}_{ij} = \left[\begin{array}{c|c} 0 & 0 \\ \hline M_{ij}C_j & 0 \end{array}\right] \quad (\text{for } j \neq i)$$

$$\overline{C} = \left[\overline{C}_1^T, \overline{C}_2^T, \ldots, \overline{C}_p^T\right]; \qquad \overline{C}_i = \left[0\overset{i-1}{\smile}, \ldots, 0, C_i, 0, 0\overset{p-1}{\smile}, \ldots, 0\right]$$

for all $j, i \in I$.

Remark 2.3. In general, the number of subsystems $S_{(\cdot)}$ and the number of interaction subsystems $H_{(\cdot)}$ are not equal. Thus, assume that p_1 and p_2 ($p_1 \neq p_2$) are, respectively, the number of subsystems S_t, $t \in I_1 = \{1, 2, 3, \ldots, p_1\}$, and the number of subsystems H_s, $s \in I_2 = \{1, 2, \ldots, p_2\}$. Then, an extended state vector $\xi(\cdot)$ can be obtained similarly, if $p = \max(p_1 p_2)$ and $I = I_1 U I_2$.

Taking into account the dimensions and the structural zeros of the $\overline{B}_i$-matrices, they can be decomposed into a sum of dyads as

$$\overline{B}_i = \sum_{j=1}^{n_i+a_i} e_{j+l_i-1}\overline{b}_{j+l_i-1}^T(i); \quad \text{for all } i \in I, \tag{2.10}$$

where e_j is the unity jth vector, $\overline{b}_j^T(i)$ is the jth vector of the $\overline{B}_i$-matrix, $i \in I$, and the lower sub-scripts 1_j, all $j \in I$, are related to the first nonstructural zero rows of the $\overline{B}_i$-matrix. If they have additional (nonstructural) rows being zero, the corresponding $e_{(\cdot)}$-vectors can be neglected in (2.10). From (2.7), it is clear that the rows of the B_i-matrices have the following structures:

$$\overline{b}_j^T(i) \equiv \begin{cases} \left[\overbrace{0,\ldots,0}^{(i-1)\,\text{block}}, e_{j-l_i+1}^T B_i, \overbrace{0,\ldots,0}^{(p-i)\,\text{blocks}} \right]; & \text{if } 1_i \le j < 1_i + n_i, \\[2ex] \left[\overbrace{e_{j-l_i-n_i+1}^T M_{i1}C_1, 0,\ldots, e_{j-l_i.n_i+1}^T M_{i(i-1)}C_{i-1}, 0, e_{j-l_i-n_i+1}^T P_i}^{(i-1)\,\text{blocks}}, \right. \\[2ex] \left. \overbrace{e_{j-l_i-n_i+1}^T M_{i(i+1)}C_{i+1}, 0,\ldots, e_{j-l_i-n_i+1}^T M_{ip}C_i, 0}^{(p-1)\,\text{blocks}} \right]; & \text{if } l_i + n_{ii} \le j < 1_i + n_i + a_i, \end{cases} \tag{2.11}$$

for all $i \in I$. Then one deduces from (2.11) that

$$\overline{b}_{j+1_i-1}^T(i)\overline{x} \equiv \begin{cases} b_j^T(i)x_i(k); & \text{if } 1 \le j \le n_i, \\[1ex] p_{j-n_i}^T(i)z_i(k) + \sum_{l=1}^{p} m_{j-n_i+1}^T y_l(k); & \text{if } n_i + 1 \le j \le n_i + a_i, \\[1ex] 0; & \text{if } n_i + a_i + 1 \le j \le n + a, \end{cases} \tag{2.12}$$

with $k = 0,1,2,3,\ldots$, when $b_j^T(i)$, $p_j^T(i)$, $m_j^T(i,1)$ denote, respectively, the jth vector row of the matrices B_i, P_i, and M_i, for all I, $1 \in I$. From (2.7) through (2.12) one obtains

$$\overline{x}(k+1) = \overline{A}\,\overline{x}(k) + \sum_{i=1}^{p} \sum_{j=1}^{n_i+a_e} e_{j+l_i-1} v_{ji}(k); \quad k = 0,1,2,\ldots, \tag{2.13}$$

where the sequences of scalars $v_{ji}(k)$ are defined by

$$v_{ji}(k) \equiv \begin{cases} u_i(k)b_j^T(i)x_i(k); & \text{if } 1 \le j \le n_i, \\[1ex] u_i(k)\left[p_{j-n_i}^T(i)z_i(k) + \sum_{l=1}^{p} m_{j-n_i}^T(i,l)y_1(k) \right]; & \text{if } n_i + 1 \le j \le n_i + a_i, \end{cases} \tag{2.14}$$

for all $i \in I$, $k = 0,1,2,\ldots$, are considered as equivalent (primary) inputs for the equivalent linear systems (2.13) to be determined prior to the generation of the $u_{(\cdot)}(\cdot)$-sequence in the design implementation.

2.3. Obtaining the $u_i(k)$-sequences from the $\gamma_{ji}(k)$-sequences

Equations (2.13) and (2.14) constitute a feed-forward linear equivalent system with multiplicative inputs. But the $p\,u_i(k)$-inputs cannot, in general, be determined from the $(n+a) > p$ equivalent $v_{ji}(k)$-inputs. Then, it is necessary to take a set of p variables between the last ones, which will be optimized independently. Let us call $v_{ji}(k)$, all $i \in I$. Thus, let us define the sets

$$N_i = N_i^- U N_i^+, \quad N_i^- = \{1,2,\ldots,n_i\}, \quad N_i^+ = \{n_i+1, n_i+2,\ldots, n_i+a_i\}, \quad \text{all } i \in I, \tag{2.15}$$

such that for the kth current sampling instant,

$$b_{j_i}^T(i)x_i(k) \neq 0, \quad \text{if } x_i(k) \neq 0, \text{ if } j_i \in N_i^-,$$

$$p_{j_i-n_i}^T(i)z_i(k) + \sum_{l=1}^{p} m_{j_i-n_i}^T(i,l)y_i(k) \neq 0, \quad \text{if } j_i \in N_i^+, \tag{2.16}$$

and $k = 0, 1, 2, \ldots$.

Then from (2.14), one has the auxiliary inputs

$$v_{ji}(k) \equiv \begin{cases} [b_{j_i}^T(i)x_i(k)]^{-i}b_j^T(i)x_i(k)v_{j_i}(k); & j \in N_i^-; \ i \in I, \\ \left[p_{j_i-n_i}^T(i)z_i(k) + \sum_{l=1}^{p} m_{j_i-n_i}^T(i,l)y_1(k)\right]^{-1} \\ \quad \times \left[p_{j-n_i}^T(i)z_i(k) + \sum_{l=1}^{p} m_{j-n_i}^T(i,l)y_1(k)v_{j_i}(k)\right], & j \in N_i^+; \ i \in I; \ k = 0,1,2,3,\ldots, \end{cases} \tag{2.17}$$

with the real inputs for the nonsaturated case being

$$u_i(k) \equiv \begin{cases} [b_{j_i}^T(i)x_i(k)]^{-1}v_{j_i}(k); & \text{if } j_i \in N_i, \text{ if there exists some } j_i \in N_i^- \\ & \text{such that } b_{j_i}^T(i)x_i(k) \neq 0, \\ \\ \left[p_{j_i-n_i}^T(i)z_i(k) \right. \\ \left. \quad + \sum_{l=1}^{p} m_{j_i-n_i}^T(i,l)y_i(k)\right]^{-1}v_{j_i}(k); & \text{if } j_i \in N_i, \text{ if there exists some } j_i \in N_i^+ \\ & \text{such that } p_{j_i-n_i}^T(i)z_i(k) + \sum_{l=1}^{p} m_{j_i-n_i}^T(i,l)y_i(k) \neq 0, \\ \\ 0; & \text{if } b_{j_i}^T(i)x_i(k) = 0 \text{ for all } j_i \in N_i^- \\ & \text{and also } p_{j_i-n_i}^T(i)z_i(k) + \sum_{l=1}^{p} m_{j_i-n_i}^T(i,l)y_1(k) = 0 \\ & \text{for all } j_i \in N_i^+, \text{ all } i \in I; \ k = 0,1,2,3,\ldots. \end{cases} \tag{2.18}$$

Remark 2.4. Note that the minimum energy control (i.e., $u_i(\cdot) = 0$, some $i \in I$) has been chosen when the system becomes insensitive to the corresponding control component. In fact, any arbitrary control could be applied.

Remark 2.5. Since there are $(n + a)$ equivalent inputs, deals with an over determined problem when the $p < n + a$ inputs are solved from them. Their solution cannot generally be satisfied exactly, and there exist many possible ways of defining the "best" approximate solution if there are $p < s \leq n+a$ equivalent nonzero inputs. Also, note that the decomposition of the $\overline{B}_i$-matrices (here, into a sum of dyads, as shown in (2.10)) is not unique. Then, the least-squares technique

(which is known to give the best approximation) can be considered by taking into account the overall control term in (2.13) instead of the control itself. Namely,

$$
u_i(k) \equiv
\begin{cases}
\dfrac{\overline{v}_i^T(k)\overline{B}_i\overline{x}(k)}{\overline{x}^T(k)\overline{B}_i^T\overline{B}_i\overline{x}(k)}; & \text{if } \overline{B}_i\overline{x}(k) \neq 0, \\[4pt]
0, & \text{otherwise,}
\end{cases}
\qquad \text{all } i \in I; \ k = 0,1,2,3,\dots, \tag{2.19}
$$

where

$$
\overline{v}_i(k) \equiv
\left[
\overbrace{0,\dots,0}^{\sum_{j=1}^{i-1}(n_j+a_j)\ \text{zeros}},\quad
v_{li}(k),\dots,v_{n_i,i}(k),v_{n_i+1,i}(k),\dots,v_{n_i+a_i,i}(k),\quad
\overbrace{0,\dots,0}^{\sum_{j=i+1}^{p}(n_j+a_j)\ \text{zeros}}
\right]^T
\in R^{n+a},
$$

$$
\text{all } i \in I; \quad k = 0,1,2,3,\dots. \tag{2.20}
$$

Note that in general, the (k) vector has only nonzero components. Equation (2.19) can be rewritten as

$$
u_i(k) \equiv
\left[
\sum_{j=1}^{n_i+a_i}\left\{ b_j^T(i)x_i(k)\delta_j(i) + \left[p_{j-n_i}^T 0(i)z_i(k) + \sum_{l=1}^{p} m_{j-n_i}^T(i,l)y_1(k) \right]\overline{\delta}_j(i) \right\}^2
\right]^{-1}
$$
$$
\times
\left[
\sum_{j=1}^{n_i+a_i} v_{ji}(k)\left\{ b_j^T(i)x_i(k)\delta_j(i) + \left[p_{j-n_i}^T(i)z_i(k) + \sum_{l=1}^{p} m_{j-n_i}^T(i,l)y_1(k) \right]\overline{\delta}_j(i) \right\}
\right], \tag{2.21}
$$

where $\overline{\delta}(i) \equiv 1 - \delta_j(i)$, with $\delta_j(i) = 1$ if $1 \leq j \leq n$ and 0, if $n_i + 1 \leq j \leq n_i + a_i$ all $i \in I$; $k = 0,1,2,\dots$.

2.4. Two alternative interpretation schemes of the equivalent linear system

Equation (2.18) leads to the following equivalences to the linear system (2.13):

$$
\overline{x}(k+1) = \overline{A}\overline{x}(k) + \sum_{i=1}^{p} c_{j+l_i-1}(k)v_{j_i}(k) \tag{2.22}
$$

$$
= \overline{A}\overline{x}(k) + \sum_{i=1}^{p} e_{j+l_i-1}v_{j_i}(k) + \xi(k), \tag{2.23}
$$

where the $(n + a)$-vectors $c_{(\cdot)}(\cdot)$ and $\xi(\cdot)$ are defined in Appendix A.

Equations (2.22)-(2.23) provide two alternative interpretations of (2.13). Representation (2.22) involves an extended state-dependent control vector, while (2.23) includes a known deterministic input-dependent vector. These equivalent representations will lead to two different suboptimal control designs in the next section.

2.5. Controllability of the bilinear system with interconnections

The controllability outside the origin of the bilinear system (2.3)–(2.5) can be studied by splitting the problem as follows:

(i) controllability of the feed-forward linear system,

(ii) capability of the $u_{(\cdot)}(\cdot)$-sequences of actually acting upon the equivalent $v_{(\cdot)}(\cdot)$-sequences.

The following result on controllability conditions is useful as a previous requirement for optimization.

Theorem 2.6 (controllability of the bilinear system involving interconnections). *Controllability of the bilinear system (2.3)–(2.5) is ensured on an interval $[k', k''] \cap R^+$, $k'' = k' + \max_{1 \le i \le p}(n_i + a_i)$, if the following set of conditions holds.*

(i) *The controllability grammian of the overall system is positive definite in $[k', k'']$.*

(ii) $\max[\mathrm{rank}\,(\sum_{l=1}^{p} M_{s1}C_1)] \ge \max[\mathrm{rank}\,(L_{ij}C_j)]$ *with $1 \le s \le p$, $1 \le i \le p$ and $1 \le j \le p$.*

(iii) *All the pairs of block diagram, $[(A_i^T, M_i^T), \widetilde{B}_i]$, $i \in I$, are nonsingular.*

(iv) $(b_{j_i}^{*T}(i)\,C_i^*(k) \ne 0,\ b_{j_i}^{*T}(i)\ diagram\ (A_i^{-1}, M_i^{-1})C_i^*(k) \ne 0,\ all\ i \in I, all\ k' \le k \le k''$, *where $C_{(\cdot)}^*$ and $b_{(\cdot)}^*$ are, respectively, the subvectors of the matrices $C_{(\cdot)}$ and $B_{(\cdot)}$ in (2.3)-(2.4) associated with each subsystem $S_{(\cdot)}$.*

Sketch of the proof

Since the pair $[A + B_{(\cdot)}G_{(\cdot)}, B_{(\cdot)}]$ is controllable for all arbitrary matrix $G_{(\cdot)}$ if the pair $[A, B_{(\cdot)}]$ is controllable [29, 38], then the controllability of the pair $[A^d, \overline{B}_{(\cdot)}]$ implies the controllability of the pair $[A^d + A^\circ, \overline{B}_{(\cdot)}]$, if there exists a $G_{(\cdot)}$-matrix such that $A^\circ = \overline{B}_{(\cdot)}G_{(\cdot)}$ with $A^\circ = \overline{A} - A^d$, $A^d =$ block diagram $((A_1, M_1), (A_2, M_2), \ldots, (A_p, M_p))$. For the equality above relating $A^\circ, \overline{B}_{(\cdot)}$, and $G_{(\cdot)}$ to hold, the Rouché-Fröbenius theorem implies that condition (ii) must be fulfilled according to (2.9) and (2.11). Also, since controllability grammian of $[A^d, \overline{B}_{(\cdot)}]$ on $[k', k''] \equiv$ block diagram controllability grammian [[block diagram $(A_1, M_1), \widetilde{B}_1$], [block diagram $(A_2, M_2), \widetilde{B}_2$], ..., [block diagram $(A_p, M_p), \widetilde{B}_p$]], which is positive if proposition (i) holds. This proves sufficiency of (i)-(ii) versus controllability of the equivalent feed-forward system. Propositions (iii)-(iv) ensure from Theorem 2.1 that controllability of the equivalent linear system implies controllability of the system (2.3)-(2.4) by allowing the determination of the scalars $u_i(\cdot)$ from $v_{ji}(\cdot)$, $i \in I$, in order to drive each subsystem S_i from an arbitrary initial point to a predefined final state.

Remark 2.7. (2.5.1) If condition (iv) is such that the matrices A_i^{-1} and M_i^{-1}, some $i \in I$, the same extensions referenced in Theorem 2.1(iii) are applicable (namely, see Remark 2.2).

(2.5.2) Theorem 2.6(iv) ensures the existence of an admissible control sequence which leads to the system outside the hyperplane of $b_{l_i+j_i-1}^{*T}(i)x_i(k) \ne 0$ of insensitivity of the subsystem S_i to the $u_i(\cdot)$-control [17, 28]. A weak additive signal can be added to the right-hand side of (2.3) at the kth sample in case of the failure of condition (iv) in Theorem 2.6 at the next step.

3. Optimization techniques

In this section, a regulation criterion which involves quadratic weighting terms in the state variables is given [1, 39]. The loss function used is

$$J_N = \frac{1}{2}\sum_{k=0}^{N} x^T(k)\overline{Q}(k)\overline{x}(k); \quad \overline{Q}(\cdot) = \overline{Q}^T(\cdot) \geq 0\,;\ N < \infty, \tag{3.1}$$

where "≥ 0" denotes positive semidefiniteness. The reason of choosing a finite-time horizon N is then seen from a practical context viewpoint.

　　In the sequel, the various inverse matrices which take place are assumed to exist. Because of their structures, this hypothesis is not strong.

3.1. Centralized control

The overall system comprises subsystems, which are interconnected, and the implementation design must deal with the interactions that exist. Centralized control techniques are used for optimization. In the centralized approach, contrarily to the decentralized case [19, 27, 40], a central coordinator exists to take into account the interactions; namely, the coordinator is supposed to be available to the local controller to supply information to each control station from the remaining ones.

　　The philosophy involved is to define transformed state and input variables, which allows the redefinition of the loss function including quadratic terms of the redefined state and input variables. The idea of using transformations of variables was first pointed out in [1] for the linear continuous case. Its major interest appears in the case of constrained input sequences because, despite this fact, a recursive Riccati expression can be found leading to the optimal feedback solution. Thus, three transformations of variables are made on the equivalent feed-forward linear system of (2.22) (subsequently (2.23) will be invoked) to redefine the state vector equation of standard type (see (B.2) in Appendix B); that is,

$$f(k+1) = \overline{A}^{(3)}(k)\,f(k) + C^{(3)}(k)v^{(3)}(k); \quad k = 0,1,2,\dots. \tag{3.2}$$

On the other hand, such transformations are applied to the loss function (3.1) to obtain a standard performance index on a given planning horizon

$$J_N = \frac{1}{2}\sum_{k=0}^{N}\left[f^T(k)\overline{Q}^{(3)}(k)f(k) + v^{(3)^T}(k)\overline{R}^{(3)}(k)v^{(3)}(k)\right]; \quad N < \infty. \tag{3.3}$$

　　The necessary manipulation on the (2.22)–(3.1) to derive the (3.2)-(3.3) is detailed in Appendix B.

　　Now, since (3.3) is a quadratic criterion of standard type (i.e., it involves quadratic terms in both the input and the state variables), a Riccati-matrix sequence may be found, which leads to the optimal solution. The main optimization results are now summarized.

Theorem 3.1 (optimization of the system equivalent inputs). *Assume that Theorem 2.6 holds. Then, the optimal equivalent input sequence for the equivalent feed-forward linear system with respect to the loss function* (3.1) *is given by the following expression:*

$$
\begin{aligned}
v_{j_i}^*(k) = v_{j_i}^{*(3)}(k+1) &- \left[C_i^T(k)\overline{Q}(k+1)C_i(k)\right]^{-1}C_i^T(k)\overline{Q}(k+1)\overline{A}\,\overline{x}(k) \\
&- e_{j_i}^T\overline{R}^{(3)^{-1}}(k+1)\overline{W}^{(2)^T}(k+1)\overline{A}\overline{x}(k)\,; \quad i \in I; \; k = 0,1,2,\ldots,N,
\end{aligned}
\tag{3.4}
$$

where the optimal equivalent redefined input sequence is

$$
\begin{aligned}
v_{j_i}^{(3)*}(k+1) = -e_{j_i}^T\Big[\overline{R}^{(3)}(k+1) &+ C^{(3)^T}(k+1)P(k+2)C^{(3)}(k+1)\Big]^{-1} \\
&\times C^{(3)^T}(k+1)P(k+2)\overline{A}^{(3)}(k+1)\overline{A}\,\overline{x}(k); \quad k = 0,1,2,\ldots,N
\end{aligned}
\tag{3.5}
$$

with $P(\cdot)$ *being the recursive Riccati matrices; that is,*

$$
\begin{aligned}
P(k) = \overline{Q}^{(3)}(k) + \overline{A}^{(3)}(k)&P(k+1) \\
&\times \left\{I - C^{(3)}(k)\left[\overline{R}^{(3)}(k) + C^{(3)}(k)P(k+1)C^{(3)}(k)\right]^{-1}C^{(3)}(k)P(k+1)\right\}\overline{A}^{(3)}(k);
\end{aligned}
\tag{3.6}
$$

$$
P(N) = \overline{Q}^{(3)}(N); \quad k = 0,1,\ldots,N.
$$

Outline of the proof

It is obvious by direct substitution while dealing with the aforementioned optimization strategies [1, 29, 38] applied to the feed-forward linear equivalent system of (2.22).

Corollary 3.2 (generation of the optimal input sequences). *Under the same hypotheses as in Theorem 3.1, consider the sequences of scalars*

$$
\overline{u}_i(k) \equiv
\begin{cases}
\left[b_{j_i}^T(i)x_i(k)\right]^{-1}v_{j_i}^*(k); \; \text{all } j \in N_i, & \text{if } j_i \in N_i^-, \\[2ex]
\Bigg[p_{j_i-n_i}^T(i)z_i(k) \\
\quad + \displaystyle\sum_{l=1}^{p}m_{j_i-n_i}^T(i,l)y_1(k)\Bigg]^{-1}v_{j_i}^*(k); \; \text{all } j \in N_i, & \text{if } j_i \in N_i^+, \\[2ex]
0; & \text{if } b_{j_i}^T(i)x_i(k) = 0, \\[2ex]
& \quad p_{j_i-n_i}^T(i)z_i(k) + \displaystyle\sum_{l=1}^{p}m_{j_i-n_i}^T(i,l)y_1(k) = 0 \\[1ex]
& \quad \forall j \in N_i; \; k = 0,1,2,\ldots,N
\end{cases}
\tag{3.7}
$$

with $v_{j_i}^*(k)$, $k = 0,1,2,\ldots$ *being the* j_ith *component of the optimal equivalent inputs obtained from Theorem 3.1.*

Then, the following propositions hold.

(a) *The optimal input sequence with respect to the loss function* (3.1) *is* $u_i^*(k) = u_i(k)$, $k = 0, 1, 2, \ldots$ *if the interval* $(-\infty, \infty) \subset \mathbb{R}$ *is allowed as input definition domain, all* $i \in I$.

(b) *If* $|u_i(k)| \leq M < \infty$, *all* $i \in I$, *then the optimal input sequence with respect to the loss function* (3.1) *is* $u_i^* = \text{sat}[\,\overline{u}_i(k)\,]$, *all* $i \in I$, $k = 0, 1, 2, \ldots$; *where the saturation function is defined by*

$$\text{sat}[g(x)] \equiv \begin{cases} M \, \text{sgn}\,[g(x)]; & \text{if } |g(x)| > M, \\ g(x); & \text{otherwise.} \end{cases} \tag{3.8}$$

Proof. It follows from **Theorem 3.1** and the fact that, from (3.7), propositions (a)-(b) in **Corollary 3.2** imply that $u_{(\cdot)}(\cdot)$ belongs to the boundary of its definition domain (and vice versa), and the fact that the optimal Hamiltonian function associated with the loss function (3.1) is a strictly convex function of the equivalent input sequence $v_{(\cdot)}(\cdot)$. $\qquad\square$

Remark 3.3. (3.1.1) Note that the equivalent and the true input sequences are obtained in a centralized way because of the coupling between the different S_i through the states of the remaining S_j, $i, j \in I$.

(3.1.2) Also, note that (2.7) can be rewritten as

$$\overline{x}(k+1) = \overline{A}\,\overline{x}(k) + \sum_{j=1}^{p} H_j(k) u_j(k); \quad k = 0, 1, 2, \ldots, \tag{3.9}$$

where

$$H_i(k) \equiv \overline{B}_i \overline{x}(k); \quad \text{all } i \in I; \ k = 0, 1, 2, \ldots. \tag{3.10}$$

By applying optimization techniques to (3.9), one could obtain, at first, an optimal $u_{(\cdot)}(\cdot)$-sequence. But this would be nonsense because the Riccati-matrix sequence would be dependent on the future measurements of the state vector.

(3.1.3) Due to the structure of the $c_{(\cdot)}$-vectors which are state-dependent (see (A.1)) and so unknown in advance, both the optimal equivalent and the true inputs cannot be determined through Riccati-matrix sequence from the results in **Theorem 3.1** and **Corollary 3.2**.

Suboptimal schemes are now presented which are alternative to the optimal one of **Theorem 3.1**.

3.2. Centralized suboptimal schemes

3.2.1. Centralized suboptimal modified scheme 1

It consists of the following variations on the optimal scheme from **Theorem 3.1** and **Corollary 3.2**:

(i) to use finite (sufficient small for the problem coherence) time-sliding optimization horizons. Namely, the time horizon for optimization is one-step advanced when the input is determined at each step. Only the first input associated with each optimization horizon is in fact applied;

(ii) to approximate the $C_{(\cdot)}(\cdot)$-vectors by their values by using each current value at the first sample of each optimization horizon; that is, $C(t) = C(k)$, all $k \leq t \leq k + N$; $k = 0, 1, 2, \ldots$.

3.2.2. Centralized suboptimal modified scheme 2

It is based on the application of (2.23) and (A.2) in Appendix A, by implementing the optimization scheme while neglecting the dependence of $\xi(\cdot)$ on the $\nu^{(3)}_{(\cdot)}(\cdot)$-optimal equivalent input sequence. This leads to a suboptimal Riccati-matrix sequence, which does not depend on the state vector, and thus, implementable. The presence of the deterministic disturbance $\xi(\cdot)$ obliges to use an additional vector (denoted by $\rho(\cdot)$) in the optimization procedure in Theorem 3.1 in order to maintain a Riccati-type solution. The costate associated with the Hamiltonian of the loss function (3.1) verifies the so-called "modified Riccati transformation" [26]; namely, $\eta(t)$ (costate) $\equiv P(t)f(t) - \rho(t)$. Thus, by taking into account that now $f(k) \equiv \overline{A}\overline{x}(k-1) + \xi(k-1)$, one finds out the following result instead of Theorem 3.1. See Appendix B for particular mathematical details.

Theorem 3.4 (optimization of the equivalent inputs for the system representation including a deterministic disturbance). *Under the same assumptions as in Theorem 3.1, it follows that the optimal equivalent input sequence for the equivalent feed-forward linear representation involving a deterministic disturbance with respect to the loss function* (3.1) *is given by*

$$
\nu^*_{j_i}(k) = \nu^{(3)*}_{j_i}(k+1) - \left[e_{j_i}^T \overline{Q}(k+1)e_{j_i}\right]^{-1} e_{j_i}^T \overline{Q}(k+1)
$$
$$
- e_{j_i}^T \overline{R}^{(3)^{-1}}(k+1)\overline{W}^{(2)^T}(k+1)\left[\overline{A}\overline{x}(k) + \xi(k)\right]; \quad \text{all } i \in I; \; k = 0,1,2,\ldots, \tag{3.11}
$$

where the optimal equivalent redefined input sequence is

$$
\nu^{*(3)}_{j_i}(k) = e_{j_i}^T \left[\overline{R}^{(3)}(k) + C^{(3)^T}(k)P(k+1)C^{(3)}(k)\right]^{-1} C^{(3)^T}(k)
$$
$$
\times \left[\rho(k+1) - P(k+1)\left\{A^{(3)}(k)\left[\overline{A}\overline{x}(k-1) + \xi(k-1)\right] + \xi(k)\right\}\right]; \quad \text{all } i \in I; \; k = 1,2,3,\ldots. \tag{3.12}
$$

The recursive Riccati-matrix sequence is now given by

$$
P(k) = \overline{Q}^{(3)}(k) + \overline{A}^{(3)}(k)
$$
$$
\times \left\{P(k+1) - P(k+1)C^{(3)}(k)\left[\overline{R}^{(3)}(k) + C^{(3)T}(k)P(k+1)C^{(3)}(k)\right]^{-1}C^{(3)}(k)P(k+1)\right\}\overline{A}^{(3)}(k);
$$
$$
P(N) = \overline{Q}^{(3)}(N); \quad k = 0,1,2,\ldots,N, \tag{3.13}
$$

$$
\rho(k+1) = \left\{I + \left[\overline{Q}^{(3)}(k) - P(k)\right]\overline{A}^{(3)-1}(k)\overline{R}^{(3)-1}(k)\overline{C}^{(3)}(k)\overline{C}^{(3)T}(k)\overline{A}^{(3)-T}(k)\right\}
$$
$$
- \overline{A}^{(3)-T}(k)\left\{\rho(k) + \left[P(k) - \overline{Q}^{(3)}(k)\right]\overline{A}^{(3)-1}(k)\xi(k)\right\}; \tag{3.14}
$$
$$
\rho(N) = 0; \quad k = 0,1,2,\ldots,N.
$$

Proof (outline). It follows immediately from applying the three transformations of variables of Appendix B to (2.23) and (A.2), and from the fact that $f(k) \equiv \overline{A}\overline{x}(k-1) + \xi(k-1)$, or

equivalently

$$f(k+1) = \overline{A}^{(3)}(k)f(k) + C^{(3)}(k)v^{(3)}(k) + \xi(k); \quad k = 0,1,2,\ldots, \tag{3.15}$$

and the fact that the $C^{(3)}(\cdot)$ vectors are now

$$C^{(3)}(k) = C^{(3)} \equiv \left[\, \overline{A}\, e_{j_1+l_1-1}, \overline{A}e_{j_2+l_2-1}, \ldots, \overline{A}e_{j_p+l_p-1}\right] \quad \text{with } k = 0,1,2,3,\ldots. \tag{3.16}$$

$\square$

Remark 3.5. (3.2.1) Note that Corollary 3.2 also applies here to obtain the optimal system input sequence under constraints (see (3.7) and (3.8)).

(3.2.2) The $\rho(\cdot)$-vector (related to the deterministic disturbance in the modified Riccati transformation is dependent on the $v^{(3)}(\cdot)$-sequence. To solve such a dependence, the $\rho(k+1)$-vector of (3.14) can be decomposed into

$$\rho(k+1) = m(k)\rho(k) + \overline{M}(k)\xi(k); \quad k = 0,1,2,\ldots, \tag{3.17}$$

being $M(\cdot)$, $\overline{M}(\cdot) \in R^{(n+a)x(n+a)}$ matrices defined by

$$M(k) = \left[I + \left[\overline{Q}^{(3)}(k) - P(k)\right] \overline{A}^{(3)^{-1}}(k)\, \overline{R}^{(3)^{-1}}(k)\, C^{(3)}(k)\, C^{(3)^T}(k)\, \overline{A}^{(3)^{-T}}(k)\right]^{-1}; \quad k = 0,1,2,3,\ldots,$$

$$\overline{M}(k) = \left[I + \left[\overline{Q}^{(3)}(k) - P(k)\right] \overline{A}^{(3)^{-1}}(k)\, \overline{R}^{(3)^{-1}}(k)\, C^{(3)}(k)\, C^{(3)^T}(k)\, \overline{A}^{(3)^{-T}}(k)\right]^{-1}$$

$$\times \overline{A}^{(3)^{-T}}(k)\left[P(k) - \overline{Q}^{(3)}(k)\right]\overline{A}^{(3)^{-1}}(k); \quad k = 0,1,2,3,\ldots. \tag{3.18}$$

Thus, the following result yields.

Corollary 3.6 (useful implementability result for obtaining the optimal equivalent input). *If the same assumptions from Theorem 3.4 are fulfilled, then the optimal equivalent input for the system interpretation including a deterministic disturbance with respect to the loss function* (3.1) *can be rewritten as follows (see* (3.11)*):*

$$v_{j_i}^*(k) = \Big\{1 + e_{j_i}^T\Big\{\left[\overline{R}^{(3)}(k+1) + C^{(3)^T}(k+1)P(k+2)C^{(3)}(k+1)\right]^{-1}C^{(3)^T}(k+1)P(k+2)\overline{A}^{(3)}(k+1)$$

$$+ \overline{R}^{(3)^{-1}}(k+1)\overline{W}^{(2)}(k+1)F(k+1) + \left[e_{j_i}^T\overline{Q}(k+1)e_{j_i}\right]^{-1}\Big\}e_{j_i}\overline{Q}(k+1)F(k+1)\Big\}^{-1}$$

$$\times \Big\{e_{j_i}^T\left[\overline{R}^{(3)}(k+1) + C^{(3)^T}(k+1)P(k+2)C^{(3)}(k+1)\right]^{-1}C^{(3)^T}(k+1)$$

$$\times \Big\{\left[\overline{M}(k+1) - P(k+2)\right]\xi(k+1) + M(k+1)\rho(k+1) - P(k+2)\overline{A}^{(3)}(k+1)\overline{A}\,\overline{x}(k)\Big\}$$

$$- \left[e_{j_i}^T\overline{Q}(k+1)e_{j_i}\right]^{-1}e_{j_i}^T\overline{Q}(k+1)\overline{A}\overline{x}(k)$$

$$- e_{j_i}^T\overline{R}^{(3)^{-1}}(k+1)\overline{W}^{(2)^T}(k)\overline{A}\,\overline{x}(k)\Big\}; \quad \text{all } i \in I; \ k = 0,1,2,3,\ldots, \tag{3.19}$$

where the matrix $F(\cdot) \in R^{(n+a)\times p}$ is partitioned as

$$F(k) \equiv \left[F_1(k), F_2(k), \ldots, F_p(k)\right]; \quad k = 0, 1, 2, 3, \ldots. \tag{3.20}$$

So that each vector $F_i(k) \in R^{n+a}$

$$F_i(k) \equiv \sum_{i=1}^{p} \sum_{j=1, j\neq i}^{n_i+a_i} \left\{ b_{j_i-n_i}^T(i)x_i(k)\delta_{j_i} + \left[p_{j_i-n_i}^T(i)z_i(k) + \sum_{l=1}^{p} m_{j_i-n_i}^T(i,l)y_1(k) \right]\bar{\delta}_{j_i} \right\}^{-1}$$

$$\times \left\{ b_{j-n_i}^T(i)x_i(k)\delta_{j_i} + \left[p_{j-n_i}^T(i)z_i(k) + \sum_{l=1}^{p} m_{j-n_i}^T(i,l)y_1(k) \right]\bar{\delta}_{j_i} \right\}; \quad all \ i \in I, \ k = 0, 1, 2, \ldots.$$

$$\tag{3.21}$$

Proof (outline). Equation (3.19) can be obtained after direct calculations if (3.17) and (3.18) are substituted into (3.11) while taking Remark 3.5 into account. □

3.3. Considerations about implementation

The following remarks must be pointed out.

(1) The system implementation scheme, which has been reported in the paper, has appeared to be suboptimal twice. First of all, the optimization scheme is dependent on the state/input vectors (some estimates must be made) according to the two alternative interpretations of the feed-forward linear system (2.13). This fact is due to the nature of the decomposition methods which have been applied and also due to the over determination problem one must deal with when the system inputs are generated from the associate equivalent ones. The performance degradation (i.e., the optimality losses being inherent to the applied suboptimization procedures) could be studied through direct calculations. However, the hypotheses that have been taken into account allow the system implementation.

(2) In summary, the steps that the designer ought to follow in the implementation environment are as follows:

(a) to set up a finite-time sliding optimization horizon, $[k, k + N]$, $1 \leq N < \infty$, integers,

(b) to test the system controllability according to Theorem 2.6 and/or Corollary 3.2,

(c) to estimate the state vector (suboptimal modified scheme 1 of Section 3.2.1) and the equivalent deterministic disturbance (suboptimal modified scheme 2 of Section 3.2.2) on the optimization horizon $[k, k + N]$. To do that, the estimated system is chosen from (3.2) as

$$\widehat{f}_{k+1} = \widehat{\bar{A}}_{1,k+1}^{(3)} \widehat{\bar{f}}_k + \widehat{C}_{1,k}^{(3)} v_{k-1}^{(3)},$$

$$\widehat{f}_{k+j} = \widehat{\bar{A}}_{1,k+j}^{(3)} \widehat{f}_{k+j-1} + \widehat{C}_{k+j-1}^{(3)} v_{k+j-1}^{(3)}; \quad j \in [2, N], \tag{3.22}$$

and the equivalent input estimates become

$$\widehat{v}_k^{(3)}[k] = \widehat{v}_{k-1}^{(3)}[k-1] = v_{k-1}^{(3)}, \tag{3.23}$$

$$\widehat{v}_{k+j}^{(3)}[k] = \widehat{v}_{k+j}^{(3)}[k-1]; \quad \text{all integer } k \neq j, \ j \in [1, N], \tag{3.24}$$

$$\widehat{v}_{k+j}^{(3)}[k] = 0; \quad \text{if } k = j, \tag{3.25}$$

(d) to compute the Riccati matrix from (3.6) or (3.13), respectively, and the system matrices from (B.6) and (B.8) of Appendix B, or (B.8) and (3.16), respectively.

(e) to generate the suboptimal input sequence $v_{j_i}^*(k)$, all i $\in I, k = 0, 1, 2, \ldots$ from (3.4) or (3.11), respectively.

(f) to obtain the suboptimal input sequence $u_i^*(k)$, all i $\in I$, $k = 0, 1, 2, 3, \ldots, N$, by applying Corollary 3.2. One must have only input controls as inputs to the system; namely, p inputs.

Remark 3.7. (3.3.1) Note that in (3.22)–(3.25), the following system representation is assumed to be

$$f_{k+1} = \overline{A}_k^{(3)} f_k + C_k^{(3)} v_k^{(3)}; \quad k = 0, 1, 2, \ldots \tag{3.26}$$

instead of (3.2), and the subscript "1" in the matrices is related to the suboptimal modified scheme 1. Similar expressions can be derived for the alternative interpretation scheme (3.15) of Section 3.2.2.

(3.3.2) Also, note that such "fictitious" states must be estimated in order to later reupdate the input sequence. In fact, only the $v^{(3)}(k-1)$ equivalent input is applied in each optimization horizon.

(3.3.3) The usual asymptotic stability tests via *Lyapunov*'s theorem (which is applied to linear time-invariant optimal systems with respect to quadratic criteria) are not applicable here because the system under study is neither invariant nor linear. Besides, standard stability proofs for optimal regulators cannot be applied to the optimal scheme, which has been obtained because of its nonimplementability. The suboptimal modified schemes can lead to instability (or at least stability can not be proved) when $N \to \infty$. But this fact does not affect the system stability in the current context of this paper since a finite-time sliding for the optimization horizon is used.

In particular, these circumstances make the implementation of a decentralized design quite difficult because some links of the equivalent feed-forward system (2.13) are cut off. However, stability can be ensured if $\overline{A}$ is a *Hurwitz*'s matrix and $u(\cdot)$ is a bounded sequence, if a saturation type rule is applied when computing the optimal redefined equivalent inputs $v_{(\cdot)}^{(3)}(\cdot)$ in the various schemes.

Table 1: Matrix entries of the simulated system structure.

$$A_1 \equiv \begin{bmatrix} 1-0.1557T & 0 \\ 0 & 1-0.1800T \end{bmatrix}; B_1 \equiv \begin{bmatrix} 0 & 0 \\ -0.0189T & 0 \end{bmatrix}; D_1 \equiv \begin{bmatrix} 0 & 0 & 0 & (-0.0557+0.0043T)T & (-0.0557+0.0043T)T \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_1 \equiv \begin{bmatrix} 1-100.0000T & 0 \\ 13.3333T & 1-85.8333T \end{bmatrix}; P_1 \equiv \begin{bmatrix} 0 & (80.0000-8000.0000T)T \\ 0 & 1066.6640T \end{bmatrix}; L_{11} \equiv \begin{bmatrix} 0 & 0 \\ (-0.0062+0.5322T)T & 0 \end{bmatrix}$$

$$M_2 \equiv \begin{bmatrix} 1-28.6145T & 1950.3287T & 35.7332T \\ 1.9554T & 1-231.1652T & -4.1281T \\ 0.0260T & -0.0056T & 1+0.1706T \end{bmatrix}; L_{21} \equiv \begin{bmatrix} (35.7322-4536.9199T)T & 0 \\ (-4.1281+512.0844T)T & 0 \\ (-0.0056+0.0575T)T & 0 \end{bmatrix}$$

$$M_3 \equiv \begin{bmatrix} 1-2.0000T & 0 \\ 6.8570T & 1-75.2380T \end{bmatrix}; L_{21} \equiv \begin{bmatrix} 0 & 0 \\ (-6.9022+44.5212T)T & 0 \end{bmatrix}$$

4. Numerical simulation

4.1. Example 1

This first example implemented deals with the following ninth-order discrete model:

$$S_1 : x_1(k+1) \equiv \left[A_1(T) + u_1(k)B_1(T)\right]x_1(k) + D_1(T)z_1(k); \quad u_1 \in R,$$

$$y_1(k) \equiv x_1(k); \quad x_1(\cdot) \in R^{2x1},$$

$$H_1 : z_1(k+1) \equiv \left[M_1(T) + u_1(k)P_1(T)\right]z_1(k) + L_{11}(T)x_1(k); \quad z_1(\cdot) \in R^{2x1}, \quad (4.1)$$

$$H_2 : z_2(k+1) \equiv M_2(T)z_2(k) + L_{21}(T)x_1(k); \quad z_2(\cdot) \in R^{3x1},$$

$$H_3 : z_3(k+1) \equiv M_3(T)z_3(k) + L_{31}(T)x_1(k); \quad z_3(\cdot) \in R^{2x1},$$

all $k \geq 0$, $k \in z$. Table 1 displays the entries of the constant discrete matrices related to the composite homogeneous bilinear structures S_1, H_i, $i = 1, 2, 3$, in (4.1). The simulations have been performed with the centralized suboptimal modified scheme 1, including free control type (namely, without saturation) only according to Corollary 3.2.

The steady state of the signal trajectory under no control leads to the following parameters: $y_{1,st}^* = 21.8$, $y_{2,st}^* = 238.63$, $u_{st}^* = 400$, $N_{st}^* = 57$ samples. The system performance is studied as a function of the optimization horizon, N. Also, the plant settings are: Initial conditions, $x_1(0) = [20.0, 250.0]^T$, $z_1(0) = [0.0, 0.0]^T$, $z_2(0) = [0.0, 0.0, 0.0]^T$, and $z_3(0) = [0.0, 0.0]^T$; sampling period $T = 0.03$ seconds and working horizon WH = 150 samples.

The obtained results are shown in Table 2 where the deviations (percent) are related to the normal operation signals (i.e., $y_{1,st}^*$, $y_{2,st}^*$ and u_{st}^*) and the CCT is the average time the computer needs to perform a computation iteration of the optimization horizon.

Exhaustive simulations about variations of the optimization horizon have shown that the discrete control improves the system response from a better transient characteristic point of view. However, there exist deviations of the steady-state signal tracking related to the no-control operation mode signals. Also, the experiments become time-consuming and memory-storage expensive as the optimization horizon does increase. For instance, some of the
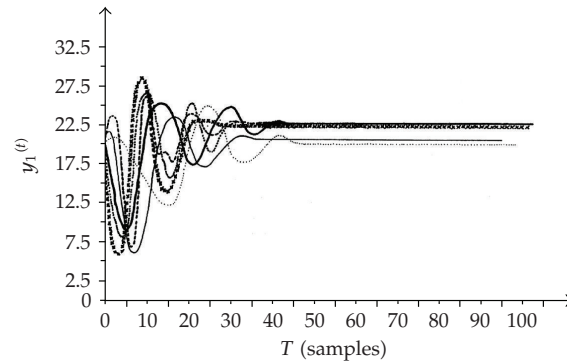
(a) Relation between the transient characteristics of the first output component and the optimization horizon
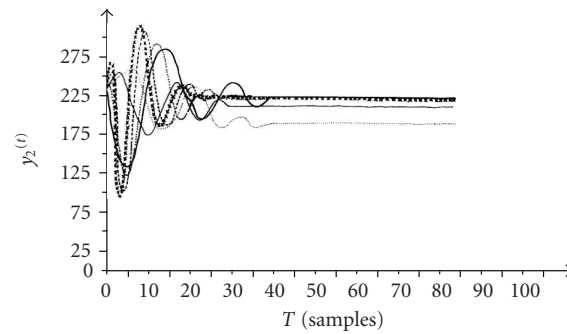
(b) Second output component transient characteristics as a function of the optimization horizon
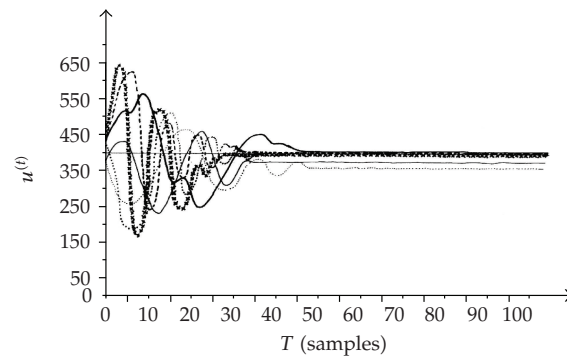
(c) Input control performance as affected by the optimization horizon

Figure 2

simulated examples have been plotted in Figure 2, consisting of the two components of the plant output and the waveform of the control input to the system (see Table 2 for the definitions of the involved signals related to the optimization horizon sizes).

Table 2: Typical results of the discretely controlled plant without saturation as a function of the optimization horizon.

Wave-form plot	Optimizaton horizon N (samples)	Steady-state response N_{st} (samples)	First component $y_{1,st}$ Value	Percent deviation	Second component $y_{2,st}$ Value	Percent deviation	Steady-state control response U_{st}	Percent deviation	Const function J_N	CCT (seconds)
——	1	47	22.51	+ 3.26	237.51	− 0.47	405.38	+ 1.35	15392.76	1.132
.....	2	35	22.33	+ 2.43	238.15	− 0.2	400.43	+ 0.12	12468.19	1.121
- - -	3	32	22.14	+ 1.56	237.32	− 0.55	403.61	+ 0.92	10319.78	0.991
xxx	4	28	21.87	+ 0.32	236.49	− 0.9	398.27	− 0.43	14215.84	1.001
none	5	32	22.05	+ 1.15	238.01	− 0.26	399.15	− 0.21	13854.21	0.997
none	6	33	21.14	− 3.03	238.18	− 0.19	382.54	− 4.37	14129.83	0.946
—	7	35	20.25	− 7.11	225.06	− 5.09	375.82	− 6.05	15639.08	0.94
none	8	41	20.03	− 8.12	210.87	− 11.63	366.39	− 8.4	10378.51	0.984
none	9	45	19.98	− 8.35	205.17	− 14.02	351.41	− 9.65	13227.83	1
....	10	48	19.81	− 9.13	201.39	− 15.61	360.23	− 9.94	16863.71	0.945
none	15	59	15.74	− 23.21	195.43	− 18.1	328.51	− 17.87	17459.29	0.954
none	20	76	15.92	− 26.97	187.31	− 21.51	310.32	− 22.42	29243.62	0.935
none	10	115	13.61	−37.57	132.75	− 44.37	200.39	− 49.9	58346.43	0.957

4.2. Example 2

A discrete bilinear system which corresponds to the discretization of a bilinear continuous one for small sampling period and zero-order hold (ZOH) is given as defined in Section 2.2, by

$$x(k + 1) = \left[A(T) + u(k)N(T)\right] x(k) + E(T) u(k) + w(k),$$
$$y(k) = C x(k),$$

(4.2)

where $x(k) \equiv [x_1^T(k), x_2^T(k), x_3^T(k)]^T$, $y(k) \equiv [y_1^T(k), 0, 0]^T$ and there exists a deterministic disturbance $w(k) \equiv [w_1^T(k), 0, 0]^T$ so that $w_1(k) \equiv S_1(T) u^2(k) + [S_2(T) + u(k) S_3(T)] q(k)$, being T the sampling period, $(\cdot)^T$ denotes transpose and the sampling instant is defined as $t_{k+1} = t_k + T$.

In (4.2), the coefficients of the block matrices are composed of the following structures:

$$A(T) = \begin{bmatrix} A_1(T) & 0 & M_1(T) \\ P_2(T) & A_2(T) & R_2(T) \\ P_3(T) & R_3(T) & A_3(T) \end{bmatrix};$$

$$N(T) = \begin{bmatrix} N_1(T) & 0 & S_2(T) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

(4.3)

$$E(T) = \begin{bmatrix} E_1(T) \\ E_2(T) \\ E_3(T) \end{bmatrix};$$

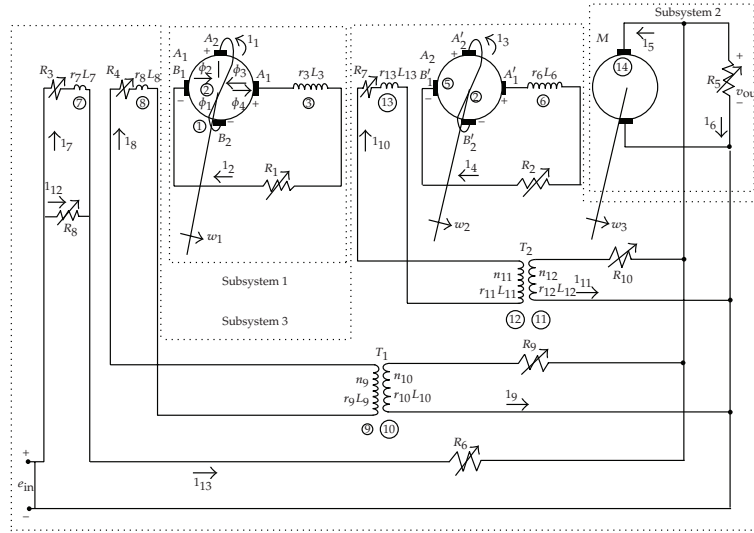$$C = \begin{bmatrix} I & 0 & 0 \end{bmatrix}.$$

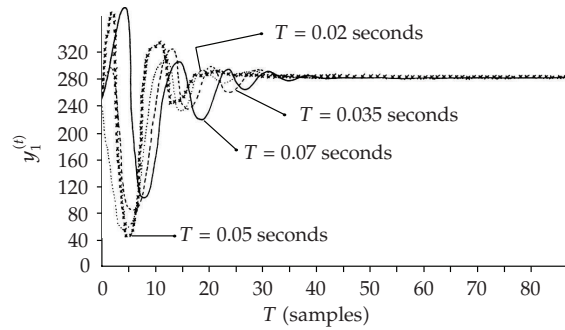**Figure 3:** Electromechanical system with interconnected subsystems.

Figure 4: Motor angular speed output as function of the sampling period with respect to a step reference signal of 277 rad/seg.

In particular, (4.2) and (4.3) correspond to the linearized discretization of an electromechanical system consisting of two amplidynes acting on the DC-motor shown in Figure 3, with a regulation mechanism incorporated as introduced in Section 3. This regulation scheme is not restricted to considering either the relevant control current or voltage to be constant, which emphasizes the interest of the bilinear modelling.

Figure 4 plots a typical tracking response output of the system as a function of the sampling period, for a fixed optimization horizon $N = 50$ samples and $\overline{Q}$ being unity in the quadratic regulation performance criterion (see (3.1)).

The control action implemented can be used as an alternative to the traditional strategies of linear control on electrical DC-machines. In general, it has been noticed that the discrete control implies better transients related to the linear approach because those transient responses are faster, that is, present lower settling times with greater shooting parameters. Besides, it has also been observed that the discrete control performance improves, in general,

as the optimization horizon is increased for different experiments, with respect to that of the nominal plant when no control action is reached.

5. Conclusions

A multivariable invariant discrete-time bilinear system being composed of interconnected subsystems has been studied. An equivalent feed-forward linear system with equivalent inputs, which are derived from products state-input, has been given. Then, the system has been suboptimized with respect to a quadratic finite-time optimization horizon in order to drive each subsystem from any arbitrary initial point to a predefined final state.

The suboptimization has been made by neglecting either the time-dependence of the control vectors on the state vector (modified suboptimal scheme 1) or the dependence of a deterministic disturbance vector on the equivalent input sequence (modified suboptimal scheme 2). These approaches have effect only on the system implementability rather than on its stability. Besides, stability proofs lead to drawbacks when the optimization horizon is infinite because of the suboptimal real implementation. This is also translated into drawbacks when implementing decentralized expected results. The proposed suboptimal schemes have been proven by means of realistic examples.

Appendices

A. Vector variation for the two alternative interpretation schemes of the feed-forward linear systems

Section 2.4 deals with two alternative representations of the feed-forward linear system (2.13). This appendix is devoted to the derivation of the vectors $C_{(\cdot)}(\cdot)$ and $\xi(\cdot)$ involved in this section.

Substituting (2.18) into (2.13) and grouping terms, one obtains

$$
c_{j+l_i-1}(k) \equiv e_{j_i+l_i-1} + \sum_{j=1/j \neq j_i}^{n_i+a_i} \left\{ \left\{ b_{j_i}^T(i)x_i(k)\delta_{j_i} + \left[p_{j_i-n_i}^T(i)z_i(k) + \sum_{l=1}^{p} m_{j_i-n_i}^T(i,l)y_1(k) \right]\overline{\delta}_{j_i} \right\}^{-1} \right.
$$
$$
\left. \times \left\{ b_j^T(i)x_i(k)\delta_{j_i} + \left[p_{j-n_i}^T(i)z_i(k) + \sum_{l=1}^{p} m_{j-n_i}^T(i,l)y_1(k) \right]\overline{\delta}_{j_i} \right\} e_{j+l_i-1} \right\},
$$
(A.1)

or alternatively

$$
\xi(k) \equiv
\begin{cases}
\sum_{i=1}^{p} \sum_{j=1/j \neq j_i}^{n_i+a_i} \left\{ b_{j_i}^T(i)x_i(k)\delta_{j_i} + \left[p_{j_i-n_i}^T(i)z_i(k) + \sum_{l=1}^{p} m_{j_i-n_i}^T(i,l)y_1(k) \right]\overline{\delta}_{j_i} \right\}^{-1} \\
\qquad \times \left\{ b_j^T(i)x_i(k)\delta_{j_i} + \left[p_{j-n_i}^T(i)z_i(k) + \sum_{l=1}^{p} m_{j-n_i}^T(i,l)y_1(k) \right]\overline{\delta}_{j_i} \right\} v_{j_i}(k)e_{j+l_i-1} \right\}; \\
\qquad \text{if at least an admissible } j_i \in N_i \text{ exists at the current sampling instant,} \\
0; \qquad \text{otherwise,}
\end{cases}
$$
(A.2)

where $\overline{\delta}_{j_i} \equiv 1 - \overline{\delta}_{j_i}$, with $\delta_{j_i} = 1$, if $j_i \in N_i$, and $\delta_{j_i} = 0$, if $j_i \in N_i^+$, all $i \in I$, $k = 0,1,2,\ldots$.

B. Derivation of the optimization equations

In the sequel, in order not to repeat tedious notation, the j_i-index (related to the system inputs and associate equations of the feed-forward linear system (2.13)) will be denoted by the subscript i, $i \in I$. Also, three modified auxiliary inputs $v^{(1)}(k)$, $v^{(2)}(k)$, and $v^{(3)}(k)$ are calculated from $v(k)$, and introduced due to the bilinear terms.

B.1. First transformation

Let us define new variables as

$$C_j^{(1)}(k) \equiv \overline{A}C_j(k-1); \qquad v_j^{(1)}(k) \equiv v_j(k-1),$$

$$f(k) \equiv \overline{A}\overline{x}(k-1) = \overline{x}(k) - \sum_{i=1}^{p} C_i(k-1)v_i(k-1) \tag{B.1}$$

with $j \in I$ and $k \geq 1$.

Substituting (B.1) into (2.22), one obtains

$$f(k+1) = \overline{A}f(k) + \sum_{i=1}^{p} C_i^{(1)}(k)v_i^{(1)}(k); \quad k = 0,1,2,3,\ldots. \tag{B.2}$$

Also, the loss function (3.1) can be equivalently rewritten as

$$J_N = \frac{1}{2}\left\{\sum_{k=0}^{N} f^T(k)\left[\overline{Q}(k) - \sum_{i=1}^{p} r_i^{-1}(k)h_i(k)h_i^T(k)\right]f(k)\right.$$

$$+ \sum_{i=1}^{p}\left\{r_i(k)\left[v_i^{(1)}(k) + r_i^{-1}(k)h_i^T(k)f(k)\right]^2\right.$$

$$\left.\left. + 2\sum_{j=1/j>i}^{p} C_i^{(1)^T}(k)\overline{A}^{-T}h_j(k)v_i^{(1)}(k)v_j^{(1)}(k)\right\}\right\}; \quad N < \infty, \tag{B.3}$$

where

$$r_j(k) \equiv C_j^{(1)^T}(k)\overline{A}^{-T}\overline{Q}(k)\overline{A}^{-1}C_j^{(1)}(k) = C_j^T(k-1)\overline{Q}(k)C_j(k-1),$$

$$h_j(k) \equiv \overline{Q}(k)\overline{A}^{-1}C_j^{(1)}(k) = \overline{Q}(k)C_j(k-1), \tag{B.4}$$

all $j \in I$; $k = 1,2,3,\ldots.$

B.2. Second transformation

The following new variables are defined as follows:

$$C_j^{(2)}(k) \equiv C_j^{(1)}(k),$$

$$v_j^{(2)}(k) \equiv v_j^{(1)}(k) + r_j^{-1}(k)h_j^T(k)f(k)$$

$$= v_j(k-1) + \overline{x}^T(k-1)\overline{A}^T\overline{Q}(k)C_j(k-1)\left[C_j^T(k-1)\overline{Q}(k)C_j(k-1)\right]^{-1},$$

$$\overline{A}^{(2)}(k) \equiv \overline{A} - \sum_{i=1}^{p} r_i^{-1}(k) C_i^{(1)}(k) h_i^T(k)$$

$$= \overline{A} - \sum_{i=1}^{p} \left[C_i^T(k-1)\overline{Q}(k)C_i(k-1) \right]^{-1} \overline{A}\, C_i(k-1)C_i^T(k-1)\overline{Q}(k),$$

(B.5)

all $j \in I$; $k = 1, 2, 3, \ldots$.

From (B.5), one has equivalently to (B.3);

$$f(k+1) = \overline{A}^{(2)}(k)f(k) + C^{(2)}(k)v^{(2)}(k); \quad k = 0, 1, 2, 3, \ldots,$$

(B.6)

where

$$C^{(2)}(\cdot) \equiv \left[C_1^{(2)}(\cdot), C_2^{(2)}(\cdot), \ldots, C_p^{(2)}(\cdot) \right] \in R^{(n+a)x(n+a)},$$

$$v^{(2)}(\cdot) \equiv \left[v_1^{(2)}(\cdot), v_2^{(2)}(\cdot), \ldots, v_p^{(2)}(\cdot) \right] \in R^p.$$

(B.7)

Also, taking into account the mentioned (B.5), (B.3) becomes as follows:

$$J_N = \frac{1}{2} \sum_{k=0}^{N} \left\{ f^T(k)\overline{Q}^{(2)}(k)f(k) + \sum_{i=1}^{p} \left[r_i(k)v_i^{(2)^2}(k) + 2 \sum_{j=1/j>i}^{p} p_{ij}(k)v_i^2(k)v_j^2(k) \right] \right.$$
$$\left. -2\sum_{i=1}^{p}\sum_{j=1/j>i}^{p} f^T(k)p_{ij}(k)\left[t_j^T(k)v_i^2(k) + t_i^T(k)v_j^2(k) \right] \right\}; \quad N < \infty$$

(B.8)

with

$$p_{ij}(k) = p_{ji}(k) \equiv C_i^{(2)^T}\overline{A}^{-T}h_j(k) = C_i^T(k-1)\overline{Q}(k)C_j(k-1); \quad j \neq i,$$

$$t_i(k) \equiv r_i^{-1}(k)h_i^T(k) = \left[C_i^T(k-1)\overline{Q}(k)C_i(k-1) \right]^{-1} C_i^T(k-1)\overline{Q}(k),$$

(B.9)

for all $i, j \in I$; $k = 0, 1, 2, \ldots$.

Equation (B.8) can be compactly rewritten as

$$J_n = \frac{1}{2} \sum_{k=0}^{N} \left[f^T(k)\overline{Q}^{(2)}(k)f(k) + v^{(2)^T}(k)\overline{R}^{(2)}(k)v^{(2)}(k) + 2f^T(k)\overline{W}^{(2)}(k)v^{(2)}(k) \right]; \quad N < \infty,$$

(B.10)

where the matrices involved are

$$
\overline{Q}^{(2)}(k) \equiv \overline{Q}(k) - \sum_{i=1}^{p} \left\{ \left[C_i^T(k-1)\overline{Q}(k)C_i(k-1)\right]^{-1}\overline{Q}(k)C_i(k-1)C_i^T(k-1)\overline{Q}(k) \right.
$$

$$
- 2\sum_{'j=1/j>i}^{p} \left\{ \left[C_j^T(k-1)\overline{Q}(k)C_i(k-1)\right]^{-1}\left[C_j^T(k-1)\overline{Q}(k)C_j(k-1)\right]^{-1} \right.
$$

$$
\left. \left. \times C_i(k-1)\overline{Q}(k)C_j(k-1)\overline{Q}(k)C_i(k-1)C_j^T(k-1)\overline{Q}(k) \right\} \right\}
$$

(B.11)

$$
\overline{R}^{(2)}(k) = \left[\overline{R}_{ij}^{(2)}(k) \right];
$$

(B.12)

$$
\overline{R}_{ij}^{(2)}(k) = \overline{R}_{ji}^{(2)}(k) \equiv \begin{cases} r_i(k), & \text{if } j = i, \\ p_{ij}(k), & \text{if } j \neq i, \end{cases}
$$

$$
\overline{W}^{(2)}(k) \equiv \overline{W}_1(k)\overline{W}_2(k)
$$

(B.13)

with

$$
\overline{W}_1(k) \equiv \left[t_1^T(k), t_2^T(k), \ldots, t_p^T(k) \right],
$$

$$
\overline{W}_2(k) \equiv \left[\overline{W}_{ij}(k) \right];
$$

(B.14)

$$
\overline{W}_{ij}(k) = \overline{W}_{ji}(k) \equiv \begin{cases} 0 & \text{if } j = i \\ -p_{ij}(k), & \text{if } j \neq i \end{cases}
$$

$r_i(\cdot)$, $p_{ij}(\cdot)$, and $t_i(\cdot)$; $j \neq i$, in (B.12) to (B.14); $i, j \in I$, are given, respectively, in (B.4) and (B.9).

Note that the factorization of matrix $\overline{W}^{(2)}(\cdot)$ in (B.13) is possible from (B.8).

B.3. Third transformation

Although it is not necessary, this transformation of variables provides immediately a loss function (redefinition of (3.1)), which includes weighting terms associated with both the state vector and the transformed input. Let us define them as follows:

$$
C^{(3)}(k) \equiv C^{(2)}(k),
$$

$$
v^{(3)}(k) \equiv v^{(2)}(k) + \overline{R}^{(2)^{-1}}(k)\overline{W}^{(2)^T}(k)f(k),
$$

(B.15)

$$
\overline{A}^{(3)}(k) \equiv \overline{A}^{(2)}(k) - C^{(2)}(k)\overline{R}^{(2)^{-1}}(k)\overline{W}^{(2)T}(k) \quad \forall k = 0, 1, 2, 3, \ldots. \tag{B.16}
$$

Substitution of (B.15) and (B.16) into (B.6) yields directly (3.2). Also, substituting (B.15) into (B.10), one obtains (3.3) if the weighting matrices are defined by

$$
\overline{Q}^{(3)}(k) \equiv \overline{Q}^{(2)}(k) - \overline{W}^{(2)}(k)\overline{R}^{(2)^{-1}}(k)\overline{W}^{(2)T}(k),
$$

$$
\overline{R}^{(3)}(k) \equiv \overline{R}^{(2)}(k)
$$

(B.17)

with $\overline{Q}^{(3)}(\cdot) \geq 0$ and $\overline{R}^{(3)} > 0$.

Acknowledgments

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