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Research Article

Distributed Control of the Generalized Korteweg-de Vries-Burgers Equation

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The paper deals with the distributed control of the generalized Kortweg-de Vries-Burgers equation (GKdVB) subject to periodic boundary conditions via the Karhunen-Loève (K-L) Galerkin method. The decomposition procedure of the K-L method is presented to illustrate the use of this method in analyzing the numerical simulations data which represent the solutions to the GKdVB equation. The K-L Galerkin projection is used as a model reduction technique for nonlinear systems to derive a system of ordinary differential equations (ODEs) that mimics the dynamics of the GKdVB equation. The data coefficients derived from the ODE system are then used to approximate the solutions of the GKdVB equation. Finally, three state feedback linearization control schemes with the objective of enhancing the stability of the GKdVB equation are proposed. Simulations of the controlled system are given to illustrate the developed theory.

1. Introduction

The generalized Korteweg-de Vries-Burgers (GKdVB) equation

$$\begin{gathered} u_t - \nu u_{xx} + \mu u_{xxx} + u^{\alpha} u_x = 0, \quad x \in [0, 2\pi],\ t \geq 0, \\ u(x,0) = u_0(x), \end{gathered} \tag{1.1}$$

where $\nu, \mu \geq 0$, and α is a positive integer, is one of the simplest partial differential equations that displays nonlinearity, with fixed level of dissipation and dispersion. It has depicted many phenomena, for example strain wave and longitudinal deformation in a nonlinear elastic rod [1]. If $\alpha = 1$, and $\nu = 0$ in (1.1), the GKdVB equation becomes the classical KdV equation which was derived in 1872 by Boussinesq and Korteweg and de Vries to model the unidirectional propagation of waves in many physical systems [2, 3]. If $\alpha = 1$, the GKdVB equation becomes the KdVB equation which was used as a model for long waves in shallow water [4] and as model of unidirectional propagation of planar waves [5]. If $\alpha = 1$ and $\mu = 0$, the GKdVB equation becomes the well-known Burgers equation [6]. If $\alpha = 2$, the GKdVB equation becomes a model of some physical phenomena [7]. Moreover, the importance of the GKdVB equation for larger values of α was discussed by Benjamin et al. [7] and Bona et al. [8].

Recently, the control problem of the KdVB equation, KdV equation, and Burgers equation has been treated by many investigators; see [9–19] to name a few. Since the infinite-dimensional nature of the PDE models for fluid flow processes can be considered as an obstacle for the synthesis of practically implementable output feedback controllers, researchers have been motivated to develop model reduction techniques for the derivation of low-dimensional ordinary differential equation (ODE) models that mimic the dynamics of the PDE models [20–23].

In this paper, we present a distributed control scheme for GKdVB equation with periodic boundary conditions and the following initial condition:

$$u_0(x) = f(x) = e^{-10(0.4x-1)^2}, \tag{1.2}$$

using a reduction technique known by the Karhunen-Loéve (K-L) Gelerkin procedure. Our approach which is based on the K-L procedure is different from the one carried by Rosier and Zhang [15]. We derive a system of ODEs that mimics the dynamics of the GKdVB equation, and show that the system of ODEs has the same qualitative structure to that of the GKdVB equation. Then, we apply state feedback controllers on the ODE system to force the dynamics of the GKdVB equation to follow a certain behavior.

The paper is organized as follows. In Section 2, numerical simulations of the GKdVB equation are obtained using pseudospectral Fourier Galerkin method. Then, Karhunen-Loève decomposition is used on the numerical simulation data to extract the coherent structures for $\alpha = 1$ and $\alpha = 2$. Section 3 presents the K-L Galerkin projection method used on the GKdVB equation to extract a system of ODEs that mimics the dynamics of the GKdVB equation. Section 4 introduces three feedback linearization control schemes used on the obtained system of ODEs to enhance the convergence rate to the steady-states. Numerical results are shown in each section to illustrate the presented theory, and some concluding remarks are given in Section 5.

2. The Karhunen-Loève decomposition

The Karhunen-Loève (K-L) decomposition is a very useful and powerful statistical technique that is used in many applications. In the literature, the K-L decomposition is known by different names such as the principal component analysis (PCA) [24] , the empirical orthogonal functions [25], the quasiharmonic modes [26], the singular value decomposition (SVD) or the proper orthogonal decomposition (POD) [27], and the Hoteling transform [28]. The method was mainly used for data compression and feature identification. Among the many applications that utilized the Karhunen-Loève (K-L) decomposition, the K-L decomposition was used in fluid dynamics [27, 29], in the analysis of two-dimensional Navier-Stokes (N-S) equation [22, 30–32], and in the study of flames [33].

In this section, we use the K-L decomposition on the numerical simulation data of the GKdVB equation in order to extract the most energetic eigenfunctions or coherent structures that span the data set in an optimal way. Since K-L decomposition is heavily used in this section, we briefly describe the steps involved in this decomposition.

First, we collect the data which represents a numerical solution of the GKdVB equation at different time steps. That is, we have the following data $\{X_i\}_{i=1}^{M}$, with $X_i = [x_i^1\ x_i^2\ \cdots x_i^N]^T$ at the ith time step, where M is the number of vectors and N is the number of entries in a vector.

Next, the mean $\overline{X}$ of the data is computed such that

$$\overline{X} = \frac{1}{M}\sum_{i=1}^{M} X_i. \tag{2.1}$$

The mean is then subtracted from each X_i, $i = 1, \dots, M$. The resulting vectors

$$\widehat{X}_i = X_i - \overline{X}, \quad i = 1, \dots, M, \tag{2.2}$$

are called the caricature vectors which have zero mean.

Based on the snapshot method [29], the covariance matrix, which is a way to measure how the data is spread out, is computed. The (i, j) element of the covariance matrix C is given by

$$c_{ij} = \frac{1}{M}\langle \widehat{X}_i, \widehat{X}_j \rangle, \quad i, j = 1, \dots, M, \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product.

The eigenfunctions are computed as follows:

$$\psi_k = \sum_{i=1}^{M} \phi_i^{[k]} \widehat{X}_i, \quad k = 1, \dots, M, \tag{2.4}$$

where $\phi_i^{[k]}$ is the ith component of the kth eigenvector. These eigenfunctions form an optimal basis for the decomposition of the data set

$$\widehat{X}(x,t) = \sum_{i=1}^{M} a_i(t)\psi_i(x), \tag{2.5}$$

where $a_i(t)$ are the coefficients calculated from the projections of the sample vector onto an eigenfunction and are calculated as

$$a_i = \frac{\langle \widehat{X} \cdot \psi_i \rangle}{\langle \psi_i \cdot \psi_i \rangle}, \quad i = 1, \dots, M. \tag{2.6}$$

The energy of the data is defined as follows:

$$E = \sum_{i=1}^{M} \lambda_i, \tag{2.7}$$

where $\{\lambda_i,\ i = 1,\ldots,M\}$ is the set of eigenvalues that corresponds to the set of eigenfunctions $\{\psi_i,\ i = 1,\ldots,M\}$. To each eigenfunction, an energy percentage E_k is assigned based on the eigenfunction's associated eigenvalue:

$$E_k = \frac{\lambda_k}{E}. \tag{2.8}$$

Finally, the data can be regenerated from the optimal basis by the following representation:

$$X(x,t) \approx \overline{X} + \sum_{i=1}^{M} a_i(t)\psi_i(x). \tag{2.9}$$

In this section, the K-L decomposition is used to analyze the solution of the GKdVB equation (1.1) subject to periodic boundary conditions and the initial condition given by (1.2). The numerical solution $u(x,t)$ of the GKdVB equation is obtained by a pseudospectral Fourier Galerkin procedure. Then, $u(x,t)$ is expanded in terms of the K-L eigenfunctions ψ_n as follows:

$$u(x,t) = \sum_{n=1}^{M} a_n(t)\psi_n(x), \tag{2.10}$$

where

$$\psi_n(x) = \sum_{k=-H}^{H} c_{k,n} e^{ikx} \tag{2.11}$$

are the K-L eigenfunctions and where H depends on the spatial discretization of ψ.

2.1. *The K-L decomposition for the case* $\alpha = 1$

Figure 1(a) shows the solution $u(x,t)$ of the GKdVB equation (1.1) with $\alpha = 1$ as it evolves to the steady-state solution for the initial condition given by (1.2); the time t was chosen to be 35 seconds; $dt = 0.001$ second; $\nu = 0.5$; and $\mu = 0.01$

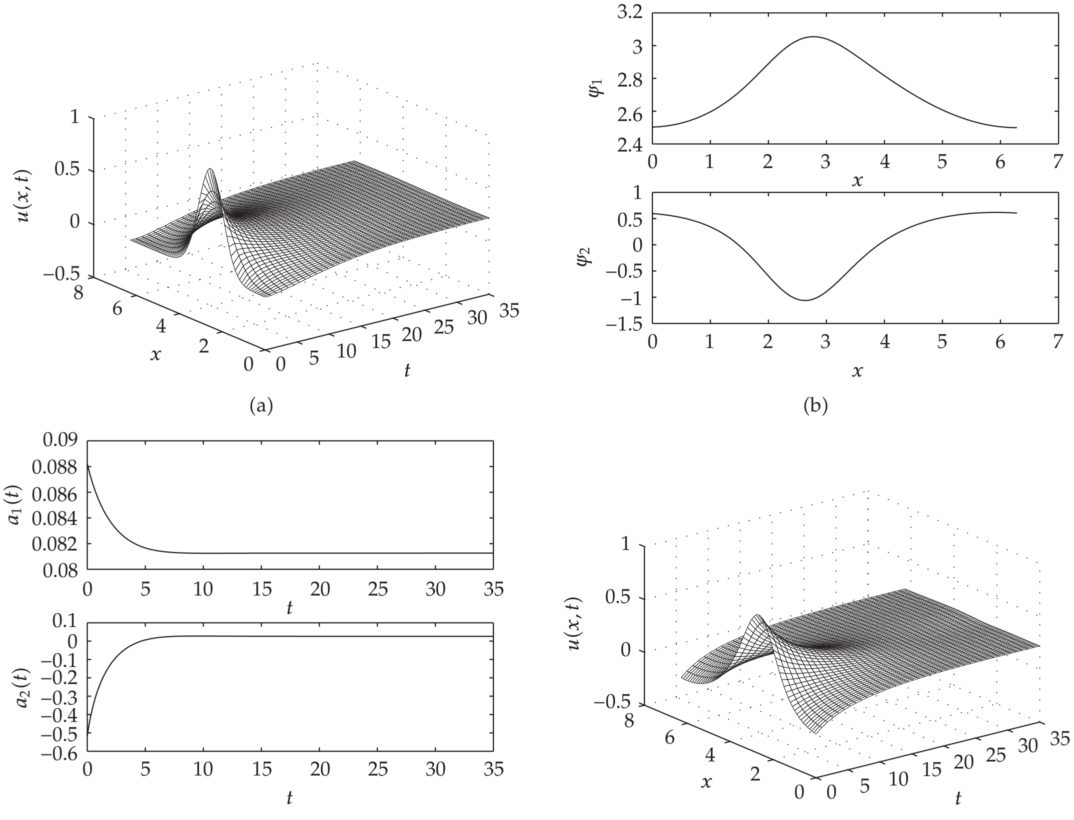
The K-L decomposition was applied on the numerical solution mentioned above. Two eigenfunctions capturing 99.6% of the energy were obtained (see Figure 1(b)). The first eigenfunction captures 95.4% of the energy and the second one captures 4.2% of the energy. Figure 1(c) presents the data coefficients that are obtained using (2.6). Figure 1(d) depicts the approximated solution of the GKdVB equation using two eigenfunctions.

Comparing Figures 1(a) and 1(d), one can conclude that two eigenfunctions are sufficient to capture the dynamics of the GKdVB equation when $\alpha = 1$.

2.2. *The K-L decomposition for the case* $\alpha = 2$

Numerical experiments show that when the degree of nonlinearity of the GKdVB equation increases by increasing the parameter α, the solution of the GKdVB equation evolves faster to the steady-state solution, and this is due to the small size of the initial condition used.

Figure 2(a) shows the solution $u(x,t)$ of the GKdVB equation (1.1) with $\alpha = 2$ as it evolves to the steady-state solution for the initial condition given by (1.2), and when $\nu = 0.5$, and $\mu = 0.01$.



(a) (b) (c) (d)

Figure 1: (a) A 3D landscape plot of the simulated solution of the GKdVB equation when $\alpha = 1$, $\nu = 0.5$, $\mu = 0.01$, and $f(x) = e^{-10(0.4x-1)^2}$. (b) The most energetic eigenfunctions of the solution of the GKdVB equation in (a). (c) The data coefficients associated with the eigenfunctions in (b). (d) A 3D landscape plot of the approximated solution using (3.2).

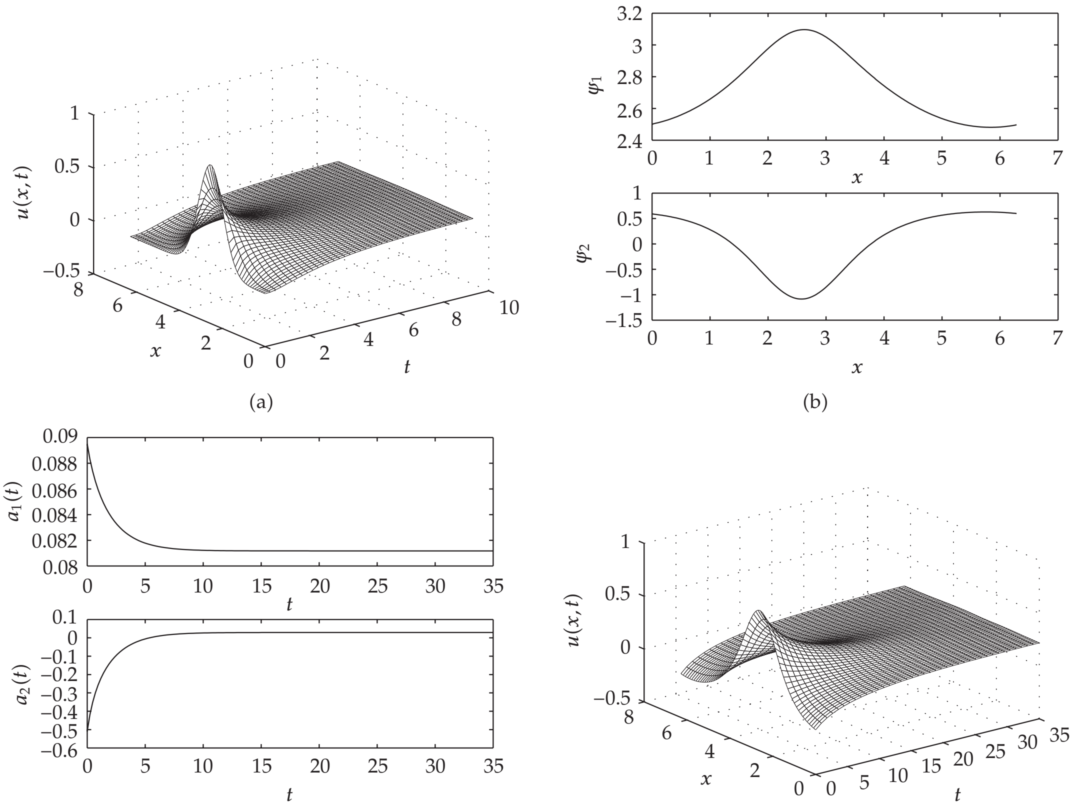
Applying the K-L decomposition on the numerical solution mentioned above, two eigenfunctions capturing 99.7% of the energy were obtained (see Figure 2(b)). The first eigenfunction captures 95.4% of the energy and the second one captures 4.3% of the energy. Figure 2(c) presents the corresponding data coefficients obtained, and Figure 2(d) depicts the approximated solution of the GKdVB equation using the above eigenfunctions.

When comparing Figures 2(a) and 2(d), one can conclude that the K-L decomposition was able to capture the large-scale dynamics of the GKdVB equation with only two eigenfunctions.

3. The K-L Galerkin projection

In order to extract a system of ODEs that mimics the dynamics of the original GKdVB equation (1.1), we first write the original PDE as follows:

$$\frac{\partial u}{\partial t} = D(u), \tag{3.1}$$



(a) (b) (c) (d)

Figure 2: (a) A 3D landscape plot of the simulated solution of the GKdVB equation when $\alpha = 2$, $\nu = 0.5$, $\mu = 0.01$, and $f(x) = e^{-10(0.4x-1)^2}$. (b) The most energetic eigenfunctions of the solution of the GKdVB equation in (a). (c) The data coefficients associated with the eigenfunctions in (b). (d) A 3D landscape plot of the approximated solution using (3.2).

with given initial and boundary conditions, where "D" is a differential operator, and $u(x,t)$ is an approximation solution that can be written in the following form:

$$u(x,t) = \sum_{i=1}^{K} a_i(t)\varphi_i(x). \tag{3.2}$$

In (3.2), $a_i(t)$ is the ith solution of the system of ODEs and can be computed in a way that minimize the residual error produced by the approximate solution above, $\varphi_i(x)$ is the ith eigenfunction from the K-L decomposition, and K is the number of the most energetic eigenfunctions.

The system of ODEs can be derived by projecting the normalized eigenfunctions onto the PDE as follows:

$$\dot{a}_i(t) = \left\langle D\left(\sum_{i=1}^{K} a_i(t)\varphi_i(x)\right), \varphi_i(x)\right\rangle, \quad i = 1,\ldots,K \tag{3.3}$$

with initial condition

$$a_i(0) = \langle u(x,0), \psi_i(x) \rangle, \quad i = 1, \ldots, K, \tag{3.4}$$

where $u(x,0)$ is known from the original PDE.

3.1. The K-L Galerkin projection for the case $\alpha = 1$

Using (3.2) on the GKdVB equation (1.1) with $\alpha = 1$,

$$u_t = \nu u_{xx} - \mu u_{xxx} - u u_x, \tag{3.5}$$

and choosing the numbers of eigenfunctions $K = 2$, we obtain the following:

$$\sum_{i=1}^{2} \dot{a}_i(t)\psi_i(x) = \nu \sum_{i=1}^{2} a_i(t)\psi_i''(x) - \mu \sum_{i=1}^{2} a_i(t)\psi_i'''(x) - \left(\sum_{i=1}^{2} a_i(t)\psi_i(x)\right)\left(\sum_{j=1}^{2} a_j(t)\psi_j'(x)\right), \tag{3.6}$$

where $\dot{a}_i(t)$ is the derivative with respect to time and $\psi_i'(x)$ is the first derivative with respect to x. Now, taking the Euclidean inner product of the above equation with ψ_k, $k = 1,2$ and using the orthogonality property of ψ_i's:

$$\langle \psi_i, \psi_j \rangle = \int_0^{2\pi} \psi_i(x)\psi_j(x)dx = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j, \end{cases} \tag{3.7}$$

we obtain the following system of ODEs:

$$\dot{a}_k(t) = \nu \sum_{i=1}^{2} a_i(t)\langle \psi_k, \psi_i'' \rangle - \mu \sum_{i=1}^{2} a_i(t)\langle \psi_k, \psi_i''' \rangle - \sum_{i=1}^{2}\sum_{j=1}^{2} a_i(t)a_j(t)\langle \psi_k, \psi_i\psi_j' \rangle, \quad k = 1,2. \tag{3.8}$$

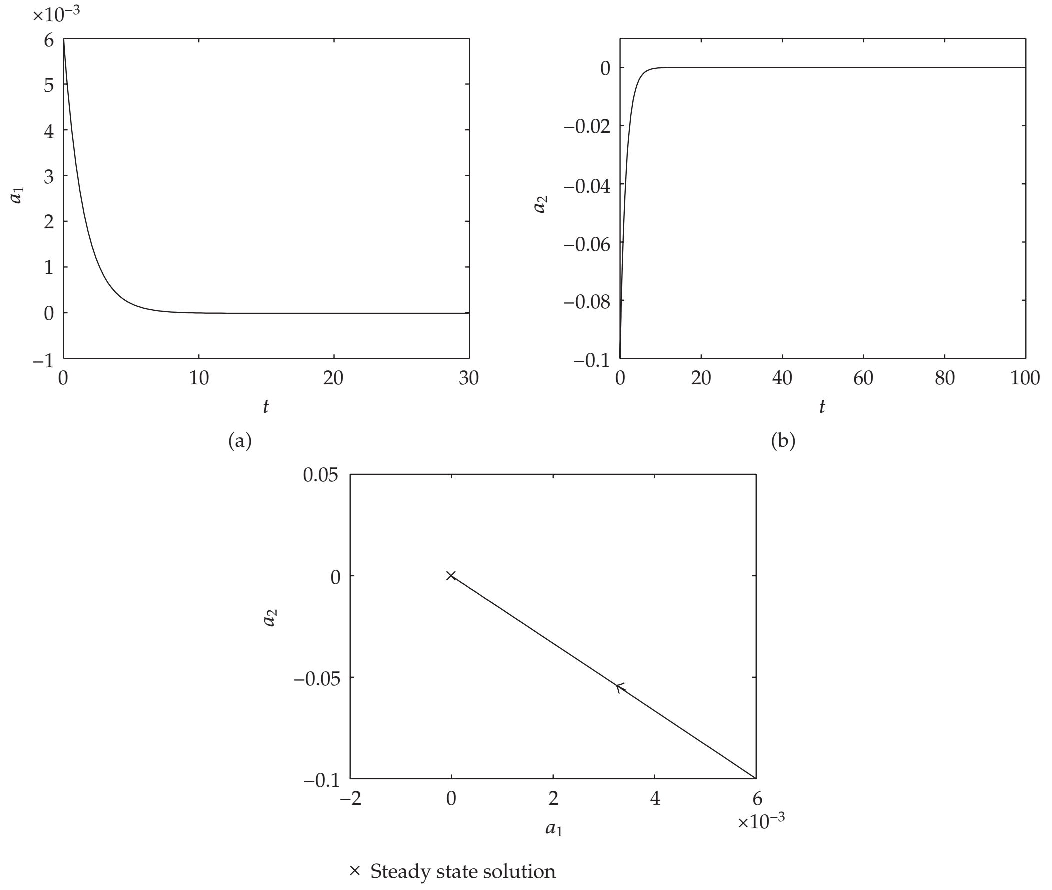
Substituting the eigenfunctions obtained for the case $\alpha = 1$ in (3.8), we get

$$\begin{aligned} \dot{a}_1 &= -0.0054188768\nu a_1 - 0.0175592926 a_1 a_2 + 0.0806774017\nu a_2 \\ &\quad + 0.0208940165\mu a_2 + 0.0047853181 a_2^2, \\ \dot{a}_2 &= 0.0806774017\nu a_1 - 0.0208940165\mu a_1 + 0.0175592854 a_1^2 \\ &\quad - 0.0047853181 a_1 a_2 - 1.3396124415\nu a_2. \end{aligned} \tag{3.9}$$

The solution of the above system can be obtained numerically using any ODE solver. Using MATLAB ODE solver, a solution of the system was found for the initial conditions $a_1(0) = 0.006$ and $a_2(0) = -0.1$ (see Figure 3).

Using (3.2) with the data coefficients computed above and the normalized eigenfunctions obtained from the K-L decomposition, we obtain an approximate solution of the GKdVB equation. Figure 4 shows the approximated solution of the GKdVB equation when $\alpha = 1$.

Comparing Figure 1(d) with Figure 4, one can conclude that the Galerkin projection method gives a reasonable approximation of the solution of the GKdVB equation when $\alpha = 1$.



(a)

(b)

(c)

Figure 3: Generated solutions of a_1 and a_2 from the K-L Galerkin ODE system when $\alpha = 1$ with the initial conditions $a_1(0) = 0.006$ and $a_2(0) = -0.1$.

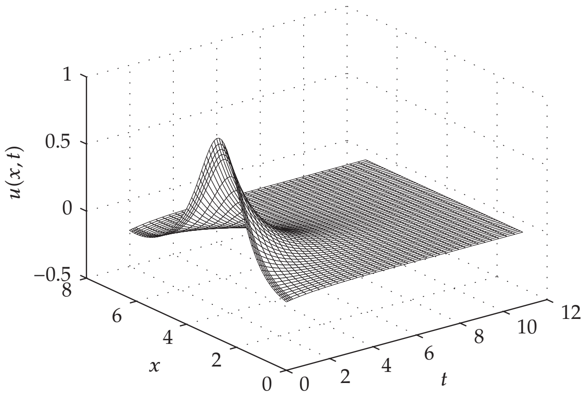


Figure 4: Approximated solution of the GKDVB equation generated by the K-L Galerkin ODE system when $\alpha = 1$ with the initial conditions $a_1(0) = 0.006$ and $a_2(0) = -0.1$.

3.2. The K-L Galerkin projection for the case $\alpha = 2$

Using the procedure illustrated in Section 3.1 above, a system of ODEs is obtained for the case $\alpha = 2$:

$$\begin{aligned}\dot{a}_k(t) &= \nu\sum_{i=1}^{2} a_i(t)\langle \psi_k, \psi_i'' \rangle - \mu\sum_{i=1}^{2} a_i(t)\langle \psi_k, \psi_i''' \rangle \\ &\quad - \sum_{i=1}^{2}\sum_{j=1}^{2} a_i^2(t)a_j(t)\langle \psi_k, \psi_i^2\psi_j' \rangle - 2\sum_{i\neq j=1}^{2} a_i(t)a_j^2(t)\langle \psi_k, \psi_i\psi_j\psi_j' \rangle.\end{aligned} \tag{3.10}$$

Substituting the eigenfunctions obtained for the case $\alpha = 2$ in (3.10), we get

$$\begin{aligned}\dot{a}_1 &= -0.00613709a_1^2a_2 + 0.00428373a_1a_2^2 - 0.00374423a_2^3 \\ &\quad + 0.00830501\mu a_2 - 0.0066366916\nu a_1 + 0.0918063477\nu a_2, \\ \dot{a}_2 &= 0.00613709a_1^3 - 0.00428373a_1^2a_2 + 0.00374423a_1a_2^2 \\ &\quad - 0.00830501\mu a_1 + 0.0918063477\nu a_1 - 1.3533719227\nu a_2.\end{aligned} \tag{3.11}$$

Figure 5 shows the general behavior of the solution of system (3.11) computed by the MATLAB ODE solver with the initial conditions $a_1(0) = 35$ and $a_2(0) = -30$, and Figure 6 presents an approximation of the solution of the GKdVB equation when $\alpha = 2$. Comparing Figure 2(d) with Figure 6, one can deduce that the system of ODEs computed by the Galerkin projection method mimics the dynamics of the GKdVB equation when $\alpha = 2$.

4. Feedback linearization control scheme for the GKdVB equation

In this section, we analyze the GKdVB equation using distributed control. The idea of using distributed control on PDEs was investigated in [20, 34–38]. Three schemes are introduced in this section.

4.1. A feedback linearization control scheme for the GKdVB equation

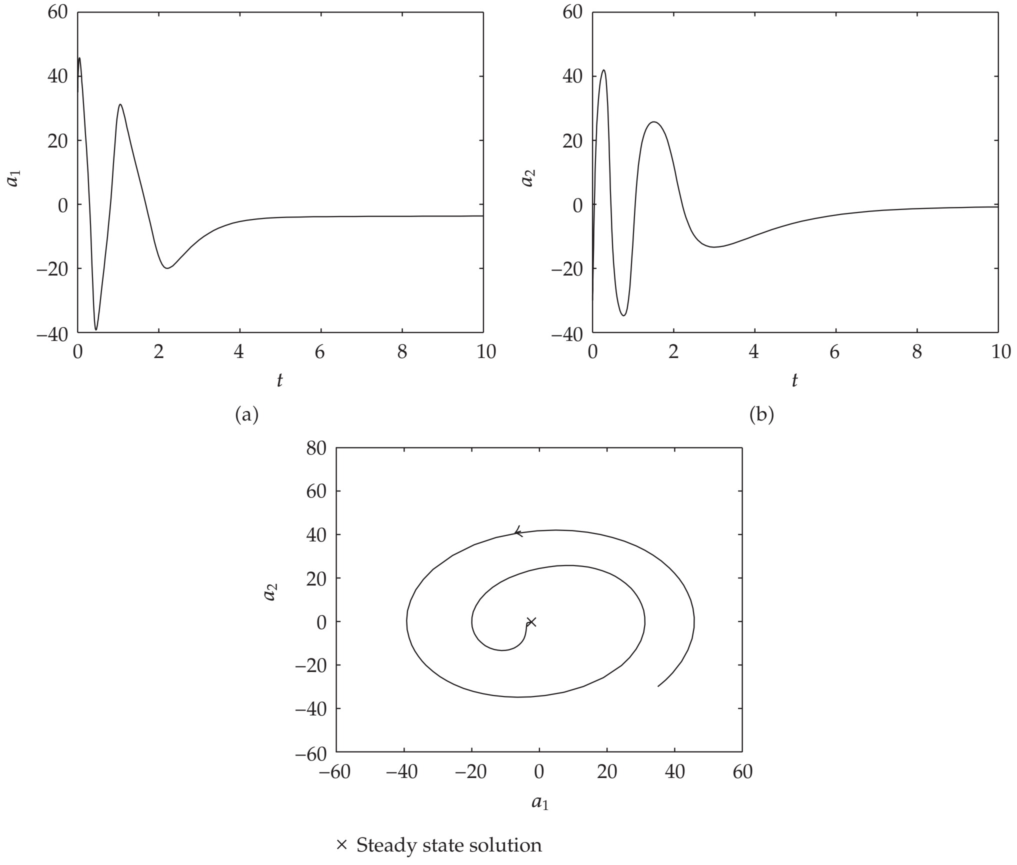
The GKdVB equation with distributed control can be written as

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial x^2} - \mu\frac{\partial^3 u}{\partial x^3} - u^\alpha\frac{\partial u}{\partial x} + \sum_{i=1}^{n_a} b_i v_i(t), \tag{4.1}$$

where $b_i(x)$ is the actuator distribution function, $v_i(t)$ is the ith input, and n_a is the number of actuators which will be chosen to be 2.

As mentioned before in (3.2), the solution of the GKdVB equation can be expressed in terms of the K-L eigenfunctions ψ_i, $i = 1, 2$. Hence, (4.1) becomes

$$\sum_{i=1}^{2} \dot{a}_i(t)\psi_i(x) = \nu\sum_{i=1}^{2} a_i(t)\psi_i''(x) - \mu\sum_{i=1}^{2} a_i(t)\psi_i'''(x) - \left(\sum_{i=1}^{2} a_i(t)\psi_i(x)\right)^\alpha \left(\sum_{i=1}^{2} a_i(t)\psi_i'(x)\right) + \sum_{i=1}^{n_a} b_i v_i(t). \tag{4.2}$$



(a)

(b)

(c)

Figure 5: Generated solutions of a_1 and a_2 from the K-L Galerkin ODE system when $\alpha = 2$ with the initial conditions $a_1(0) = 35$ and $a_2(0) = -30$.

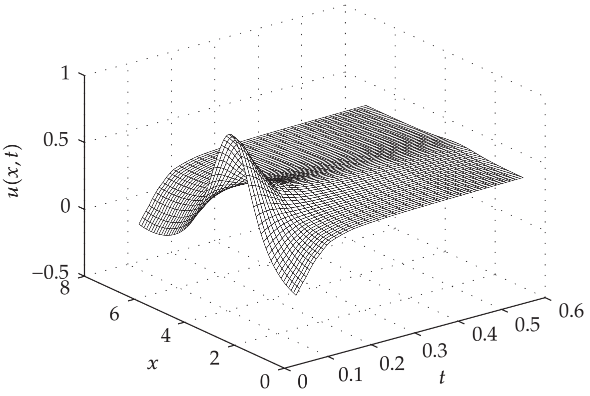


Figure 6: Approximated solution of the GKDVB equation generated by the K-L Galerkin ODE system when $\alpha = 2$ with the initial conditions $a_1(0) = 35$ and $a_2(0) = -30$.

Using the Galerkin projection method, the GKdVB equation is transformed into the following system of ODEs:

$$\dot{a}_k(t) = \nu \sum_{i=1}^{2} a_i(t) \langle \psi_k, \psi_i'' \rangle - \mu \sum_{i=1}^{2} a_i(t) \langle \psi_k, \psi_i''' \rangle - g(t) + \sum_{i=1}^{2} \beta_i^k v_i(t), \tag{4.3}$$

where

$$\begin{gathered} g(t) = \left\langle \left(\sum_{i=1}^{2} a_i(t) \psi_i(x) \right)^{\alpha} \left(\sum_{i=1}^{2} a_i(t) \psi_i'(x) \right), \psi_k \right\rangle, \quad k = 1, 2, \\ \beta_i^k = \int_0^{2\pi} \psi_k b_i dx. \end{gathered} \tag{4.4}$$

Define

$$\eta_k(t) = \nu \sum_{i=1}^{2} a_i(t) \langle \psi_k, \psi_i'' \rangle - \mu \sum_{i=1}^{2} a_i(t) \langle \psi_k, \psi_i''' \rangle - g(t), \tag{4.5}$$

$$w_k(t) = \sum_{i=1}^{2} \beta_i^k v_i(t), \quad k = 1, 2, \tag{4.6}$$

then the system of ODEs (4.3) becomes

$$\dot{a}_k = \eta_k(t) + w_k(t), \quad k = 1, 2. \tag{4.7}$$

Proposition 4.1. *Let ξ_1 and ξ_2 be two positive real numbers, then the following feedback linearization control scheme:*

$$w_k(t) = -\eta_k(t) - \xi_k a_k(t), \quad k = 1, 2, \tag{4.8}$$

renders the system of ODEs in (4.7) *exponentially stable.*

Proof. Substituting (4.8) in (4.7), we get

$$\dot{a}_k(t) = -\xi_k a_k(t), \quad k = 1, 2, \tag{4.9}$$

or

$$a_k(t) = a_k(0) e^{-\xi_k t}, \quad k = 1, 2, \tag{4.10}$$

since $\xi_1, \xi_2 > 0$, then $a_k(t)$ will converge exponentially to zero as $t \to \infty$. Therefore, the system of ODEs (4.7) with the controller given in (4.8) is exponentially stable. □

Remark 4.2. We can force the solution of the system to converge to any desired fixed point. This can be achieved by making the following change of variables:

$$\tilde{a}_k(t) = a_k(t) + c_k, \quad k = 1, 2, \tag{4.11}$$

where (c_1, c_2) correspond to the coordinates of the desired fixed point.

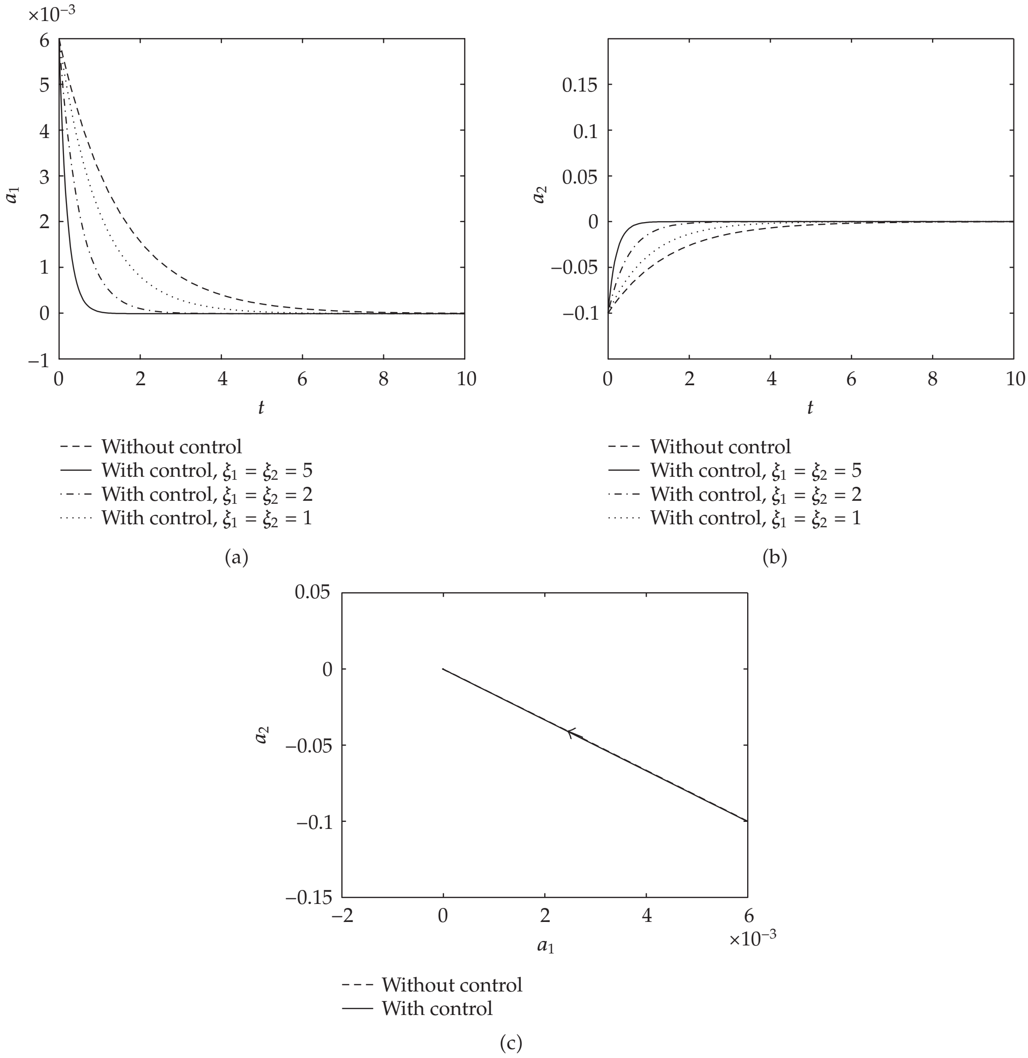


Figure 7: The solutions of the controlled system of ODEs compared to the solutions of the uncontrolled ODE system when $\alpha = 1$, $a_1(0) = 0.006$, $a_2(0) = -0.1$ with the initial condition given by (1.2).

Now, from (4.6), we have the following:

$$\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} \beta_1^1 & \beta_2^1 \\ \beta_1^2 & \beta_2^2 \end{pmatrix} \cdot \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}. \tag{4.12}$$

Using the result of the above proposition, system (4.12) can be written by

$$\begin{pmatrix} -\eta_1(t) - \xi_1 a_1(t) \\ -\eta_2(t) - \xi_2 a_2(t) \end{pmatrix} = \begin{pmatrix} \beta_1^1 & \beta_2^1 \\ \beta_1^2 & \beta_2^2 \end{pmatrix} \cdot \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}. \tag{4.13}$$

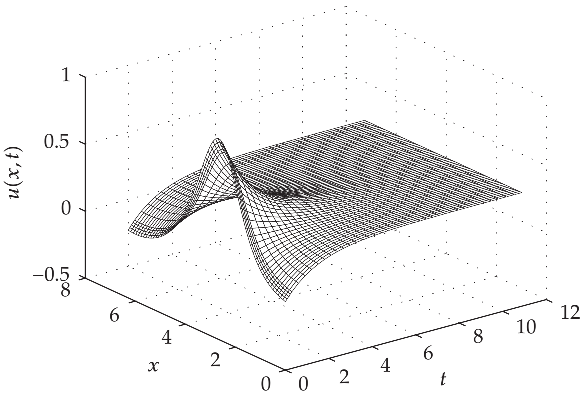


Figure 8: A 3D landscape of the approximated controlled solution of the GKdVB equation when $\alpha = 1$, $\xi_1 = \xi_2 = 5$, with the initial condition given by (1.2).

Provided that the coefficients β_i^j, for $(i, j = 1, 2)$, are well chosen (i.e., $\beta_1^1\beta_2^2 - \beta_1^2\beta_2^1 \neq 0$), then the controllers $v_1(t)$ and $v_2(t)$ in (4.1) are determined by

$$\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} \beta_1^1 & \beta_2^1 \\ \beta_1^2 & \beta_2^2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -\eta_1(t) - \xi_1 a_1(t) \\ -\eta_2(t) - \xi_2 a_2(t) \end{pmatrix}. \tag{4.14}$$

The solution of the controlled system of ODEs given by (4.7) can be calculated easily by any ODE solver. The above controller was tested numerically using the MATLAB ODE solver on the GKdVB equation for $\alpha = 1$ and $\alpha = 2$ with the initial condition given by (1.2). Figure 7 shows the plots of the solutions' ($a_1(t)$ and $a_2(t)$) profiles of the above controller for different values of ξ's; these profiles are also compared to the solution of the ODE system produced by the Galerkin projection when $\alpha = 1$. It should be noted that, when the value of ξ increases, the solution evolves faster to the fixed point, and a better control is obtained. We present in Figure 8 a 3D landscape of the approximated solution of the GKdVB equation when $\alpha = 1$ with ξ_1 and ξ_2 being chosen to be 5.

The solutions of the controller given by (4.7) for different values of ξ's were compared numerically to the solution of the ODE system produced by the Galerkin projection when $\alpha = 2$ (see Figure 9). Figure 10 depicts a 3D landscape approximation solution of the GKdVB equation when $\alpha = 2$ with ξ_1 and ξ_2 being chosen to be 0.2.

4.2. Another control scheme for the GKdVB equation

In the control scheme given by (4.8), all the terms of η_k, $k = 1, 2$, were canceled, whereas some of the elements of η_k, $k = 1, 2$, have a stabilizing effect on the dynamics and hence there is no need to cancel them. In this section, we design another version of the feedback controller given by (4.8).

As mentioned before, the solution of the GKdVB equation can be expressed in terms of the K-L eigenfunctions φ_i, $i = 1, 2$, as follows:

$$u(x,t) = \sum_{n=1}^{2} \sum_{l=-h}^{h} a_n(t) c_{l,n} e^{ilx}, \tag{4.15}$$

and h is an integer which depends on the spatial discretization of ψ_n's. Using the above representation of the K-L eigenfunctions in (4.3), we get the following system of ODEs:

$$\dot{a}_k(t) = \nu \sum_{i=1}^{2} \sum_{l=-h}^{h} - l^2 c_{l,i} c_{l,k} a_i(t) + \mu \sum_{i=1}^{2} \sum_{l=-h}^{h} i l^3 c_{l,i} c_{l,k} a_i(t) - g(t) + \sum_{i=1}^{2} \beta_i^k v_i(t). \tag{4.16}$$

Let

$$A_0 = \begin{pmatrix} \sum_{l=-h}^{h} l^2 c_{l,1}^2 & \sum_{l=-h}^{h} l^2 c_{l,1} c_{l,2} \\ \sum_{l=-h}^{h} l^2 c_{l,2} c_{l,1} & \sum_{l=-h}^{h} l^2 c_{l,2}^2 \end{pmatrix},$$

$$a(t) = \begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix}, \qquad \tilde{f}(t) = \begin{pmatrix} \tilde{f}_1(t) \\ \tilde{f}_2(t) \end{pmatrix}, \qquad w(t) = \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}, \tag{4.17}$$

where $\tilde{f}_k(t)$ is

$$\tilde{f}_k(t) = \mu \sum_{i=1}^{2} \sum_{l=-h}^{h} i l^3 c_{l,i} c_{l,k} a_i(t) - g(t), \tag{4.18}$$

where $k = 1, 2$. Then, the system of ODEs can be written as

$$\dot{a}(t) = -\nu A_0 a(t) + \tilde{f}(t) + w(t). \tag{4.19}$$

The matrix A_0 can be easily computed, and it is easy to check that the matrix A_0 is positive definite.

Proposition 4.3. *The controller*

$$w(t) = -\tilde{f}(t) \tag{4.20}$$

renders the system of ODEs exponentially stable.

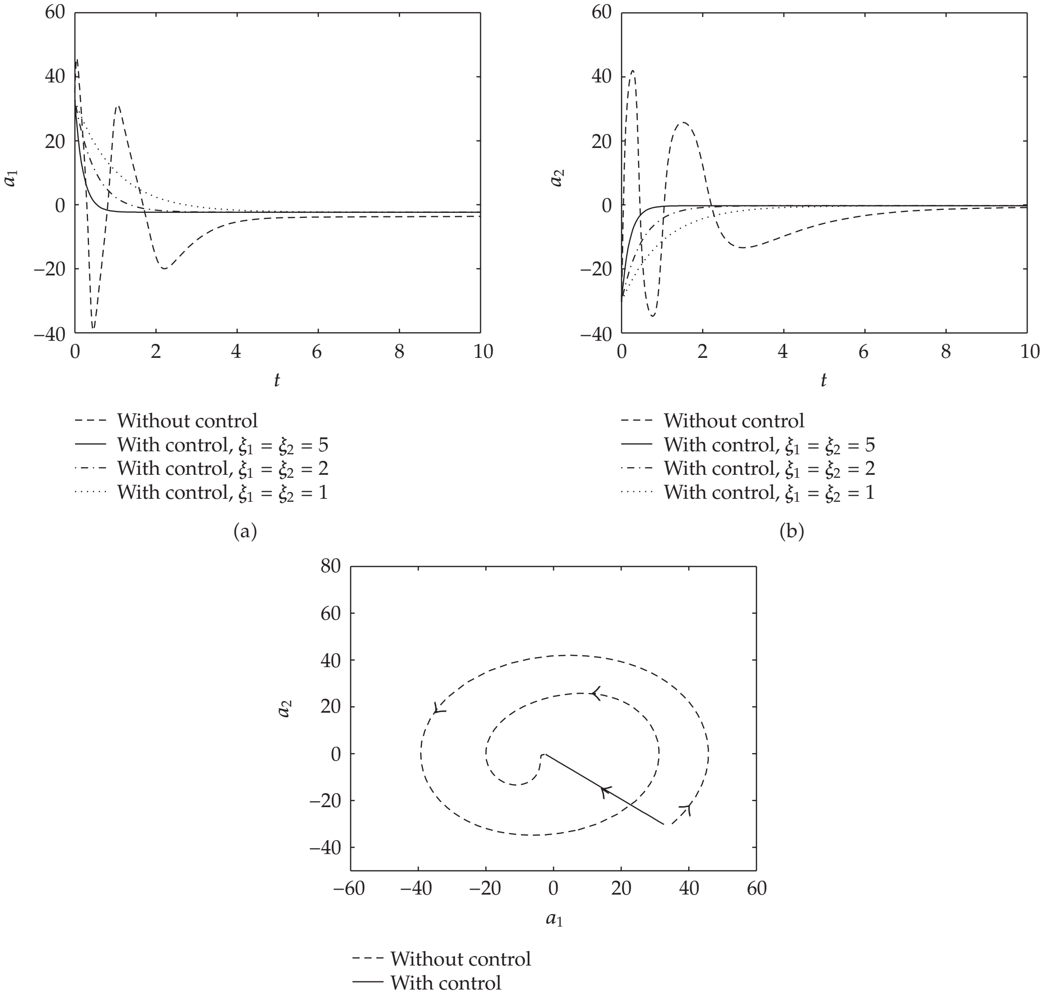
Proof. Substituting (4.20) in (4.19), we obtain the following:

$$\dot{a}(t) = -\nu A_0 a_k(t) \tag{4.21}$$

or

$$a(t) = a(0) e^{-\nu A_0 t}, \tag{4.22}$$

since the matrix "$-A_0$" is negative definite, then $a(t)$ converges exponentially to zero as $t \to \infty$. Therefore, the system of ODEs (4.19) with the controller given by (4.20) is exponentially stable. □



(a)

(b)

(c)

Figure 9: The solutions of the controlled system of ODEs compared to the solutions of the uncontrolled ODE system when $\alpha = 2$, $a_1(0) = 35$, $a_2(0) = -30$, with the initial condition given by (1.2).

Using the result of the above proposition, then the system

$$\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} \beta_1^1 & \beta_2^1 \\ \beta_1^2 & \beta_2^2 \end{pmatrix} \cdot \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} \tag{4.23}$$

can be presented as

$$\begin{pmatrix} -\widetilde{f}_1(t) \\ -\widetilde{f}_2(t) \end{pmatrix} = \begin{pmatrix} \beta_1^1 & \beta_2^1 \\ \beta_1^2 & \beta_2^2 \end{pmatrix} \cdot \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}. \tag{4.24}$$

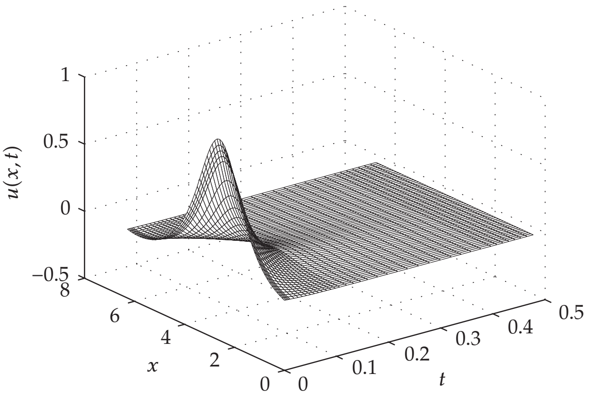


Figure 10: A 3D landscape of the approximated controlled solution of the GKdVB equation when $\alpha = 2$, $\xi_1 = \xi_2 = 0.2$, with the initial condition given by (1.2).

Provided that the coefficients β_i^j, for $(i, j = 1, 2)$, are well chosen (i.e., $\beta_1^1\beta_2^2 - \beta_1^2\beta_2^1 \neq 0$), then the controllers $v_1(t)$ and $v_2(t)$ in (4.1) are determined as follows:

$$\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} \beta_1^1 & \beta_2^1 \\ \beta_1^2 & \beta_2^2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -\widetilde{f}_1(t) \\ -\widetilde{f}_2(t) \end{pmatrix}. \tag{4.25}$$

4.3. A control scheme for the GKdVB equation using a single actuator

Sections 4.2 and 4.3 addressed the control problem of the GKdVD equation when the system uses two actuators (i.e., two control inputs). This section discusses the control of the GKdVB equation when only one actuator is used.

The GKdVB equation with one actuator can be written as

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} - \mu \frac{\partial^3 u}{\partial x^3} - u^\alpha \frac{\partial u}{\partial x} + b(x)v(t), \quad 0 \le x \le 2\pi, \tag{4.26}$$

where $v(t)$ is the control input and $b(x)$ acts to distribute the control throughout the spatial domain $[0, 2\pi]$.

The derivation of the system of ordinary differential equations (ODEs) based on K-L Galerkin projection results in

$$\dot{a}_k(t) = \nu \sum_{i=1}^{2} a_i(t) \langle \varphi_k, \varphi_i'' \rangle - \mu \sum_{i=1}^{2} a_i(t) \langle \varphi_k, \varphi_i''' \rangle - \left\langle \left(\sum_{i=1}^{2} a_i(t) \varphi_i(x) \right)^{\alpha} \left(\sum_{i=1}^{2} a_i(t) \varphi_i'(x) \right), \varphi_k \right\rangle + \beta_k v(t), \tag{4.27}$$

where

$$\beta_k = \int_0^{2\pi} \varphi_k(x) b(x) dx, \quad k = 1, 2. \tag{4.28}$$

Hence, the behavior of the GKdVB equation for the case $\alpha = 2$ in (4.26) can be approximated by the following system of ODEs:

$$\begin{aligned}\dot{a}_1(t) &= -0.00613709a_1^2a_2 + 0.00428373a_1a_2^2 - 0.00374423a_2^3 + 0.00830501\mu a_2\\&\quad - 0.0066366916\nu a_1 + 0.0918063477\nu a_2 + \beta_1 v(t),\\\dot{a}_2(t) &= 0.00613709a_1^3 - 0.00428373a_1^2a_2 + 0.00374423a_1a_2^2 - 0.00830501\mu a_1\\&\quad + 0.0918063477\nu a_1 - 1.3533719227\nu a_2 + \beta_2 v(t).\end{aligned} \tag{4.29}$$

Proposition 4.4. *The state feedback controller*

$$v(t) = -k_1a_1(t) - k_2a_2(t) \tag{4.30}$$

with the design parameters k_1 & k_2 such that $k_1\beta_1 > 0$, $k_2\beta_2 > 0$, and $k_1\beta_2 = k_2\beta_1$ renders the ODE system in (4.29) *asymptotically stable.*

Proof . Consider the following Lyapunov function candidate:

$$V(t) = \frac{1}{2}\left(a_1^2(t) + a_2^2(t)\right). \tag{4.31}$$

Note that $V(t) > 0$ if $(a_1(t), a_2(t)) \neq (0,0)$ and $V(t) = 0$ if $(a_1(t), a_2(t)) = (0,0)$. Taking the derivative of $V(t)$ with respect to time and using (4.29) and (4.30), it follows that

$$\begin{aligned}\dot{V} &= \dot{a}_1a_1 + \dot{a}_2a_2\\&= \big(-0.00613709a_1^2a_2 + 0.00428373a_1a_2^2 - 0.00374423a_2^3 + 0.00830501\mu a_2\\&\qquad - 0.0066366916\nu a_1 + 0.0918063477\nu a_2 + \beta_1 v(t)\big) a_1\\&\quad + \big(0.00613709a_1^3 - 0.00428373a_1^2a_2 + 0.00374423a_1a_2^2 - 0.00830501\mu a_1\\&\qquad + 0.0918063477\nu a_1 - 1.3533719227\nu a_2 + \beta_2 v(t)\big) a_2\\&\leq -\left(\beta_1a_1 + \beta_2a_2\right)\left(k_1a_1 + k_2a_2\right)\\&\leq -\beta_2k_2\left(a_2 + \frac{\beta_2k_1 + \beta_1k_2}{2\beta_2k_2}a_1\right)^2\\&\leq 0.\end{aligned} \tag{4.32}$$

Note that $\dot{V}$ is negative definite. Hence, by Lyapunov theory, the controller scheme given by (4.30) guarantees the asymptotic stability of the GKdVB equation. □

5. Concluding remarks

In this paper, we have analyzed the control problem of the GKdVB equation subject to periodic boundary conditions by applying a distributed control strategy. The Karhunen-Loève Galerkin method was used to produce systems of ODEs which mimic the dynamics of the GKdVB equation for $\alpha = 1$ and $\alpha = 2$. Then, we used three state feedback linearization control schemes on the system of the ODEs that render it exponentially stable. Simulation results are presented to show the effectiveness of the developed control schemes.

For future work, we will look into the development of adaptive and optimal control schemes for the GKdVB equation, and the design of boundary controllers for different values of α.

References

[1] G. J. Liu and G. S. Duan, “Solitary wave solutions for equations of longitudinal wave in a nonlinear elastic rod,” *Journal of Henan Normal University. Natural Science*, vol. 29, no. 3, pp. 101–103, 2001.

[2] J. L. Bona, S. M. Sun, and B.-Y. Zhang, “A non-homogeneous boundary-value problem for the Korteweg-de Vries equation in a quarter plane,” *Transactions of the American Mathematical Society*, vol. 354, no. 2, pp. 427–490, 2002.

[3] B. Y. Zhang, “Boundary stabilization of the Korteweg-de Vries equation,” in *Control and Estimation of Distributed Parameter Systems: Nonlinear Phenomena (Vorau, 1993)*, vol. 118 of *International Series of Numerical Mathematics*, pp. 371–389, Birkhäuser, Basel, Switzerland, 1994.

[4] V. Boyko, “On new generalizations of the Burgers and Korteweg-de Vries equations,” *Symmetry in Nonlinear Mathematics Physics*, vol. 1-2, pp. 122–129, 1997.

[5] G. Karch, “Self-similar large time behavior of solutions to Korteweg-de Vries-Burgers equation,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 35, no. 2, pp. 199–219, 1999.

[6] J. M. Burgers, “A mathematical model illustrating the theory of turbulence,” in *Advances in Applied Mechanics*, R. von Mises and T. von Kármán, Eds., pp. 171–199, Academic Press, New York, NY, USA, 1948.

[7] T. B. Benjamin, I. L. Bona, and J. J. Mahony, “Model equations for long waves in nonlinear dispersive systems,” *Philosophical Transactions of the Royal Society of London. Series A*, vol. 272, no. 1220, pp. 47–78, 1972.

[8] J. L. Bona, W. G. Pritchard, and L. R. Scott, “An evaluation of a model equation for water waves,” *Philosophical Transactions of the Royal Society of London. Series A*, vol. 302, no. 1471, pp. 457–510, 1981.

[9] A. Balogh and M. Krstić, “Global boundary stabilization and regularization of Burgers’ equation,” in *Proceedings of the American Control Conference*, vol. 3, pp. 1712–1716, San Diego, Calif, USA, June 1999.

[10] A. Balogh and M. Krstić, “Boundary control of the Korteweg-de Vries-Burgers equation: further results on stabilization and well-posedness, with numerical demonstration,” *IEEE Transaction on Automatic Control*, vol. 45, no. 9, pp. 1739–1745, 2000.

[11] G. Perla Menzala, C. F. Vasconcellos, and E. Zuazua, “Stabilization of the Korteweg-de Vries equation with localized damping,” *Quarterly of Applied Mathematics*, vol. 60, no. 1, pp. 111–129, 2002.

[12] A. F. Pazoto, “Unique continuation and decay for the Korteweg-de Vries equation with localized damping,” *ESAIM: Control, Optimization and Calculus of Variations*, vol. 11, no. 3, pp. 473–486, 2005.

[13] L. Rosier, “Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain,” *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 2, pp. 33–55, 1997.

[14] L. Rosier, “Exact boundary controllability for the linear Korteweg-de Vries equation—a numerical study,” in *Control and Partial Differential Equations (Marseille-Luminy, 1997)*, vol. 4 of *ESAIM Proceedings*, pp. 255–267, SIAM, Paris, France, 1998.

[15] L. Rosier and B.-Y. Zhang, “Global stabilization of the generalized Korteweg-de Vries equation posed on a finite domain,” *SIAM Journal on Control and Optimization*, vol. 45, no. 3, pp. 927–956, 2006.

[16] N. Smaoui, “Nonlinear boundary control of the generalized Burgers equation,” *Nonlinear Dynamics*, vol. 37, no. 1, pp. 75–86, 2004.

[17] N. Smaoui, “Boundary and distributed control of the viscous Burgers equation,” *Journal of Computational and Applied Mathematics*, vol. 182, no. 1, pp. 91–104, 2005.

[18] N. Smaoui and R. H. Al-Jamal, “A nonlinear boundary control for the dynamics of the generalized Korteweg-de Vries-Burgers equation,” *Kuwait Journal of Science & Engineering*, vol. 34, no. 2A, pp. 57–76, 2007.

[19] N. Smaoui, M. Zribi, and A. Almulla, “Sliding mode control of the forced generalized Burgers equation,” *IMA Journal of Mathematical Control and Information*, vol. 23, no. 3, pp. 301–323, 2006.

[20] P. D. Christofides, *Nonlinear and Robust Control of PDE Systems: Methods and Applications to Transport-Reaction Processes*, Systems and Control: Foundations and Applications, Birkhäuser, Boston, Mass, USA, 2001.

[21] A. E. Dean, I. G. Kevrekidis, G. E. Karniadakis, and S. A. Orszag, "Low-dimensional models for complex geometry flows: application to grooved channels and circular cylinders," *Physics of Fluids A*, vol. 3, no. 10, pp. 2337–2354, 1991.

[22] N. Smaoui and D. Armbruster, "Symmetry and the Karhunen-Loève analysis," *SIAM Journal of Scientific Computing*, vol. 18, no. 5, pp. 1526–1532, 1997.

[23] E. S. Titi, "On approximate inertial manifolds to the Navier-Stokes equations," *Journal of Mathematical Analysis and Applications*, vol. 149, no. 2, pp. 540–557, 1990.

[24] I. T. Jollife, *Principal Component Analysis*, Springer, New York, NY, USA, 1986.

[25] E. N. Lorenz, "Deterministic nonperiodic flow," *Journal of the Atmospheric Sciences*, vol. 20, no. 2, pp. 130–141, 1963.

[26] C. L. Brooks, M. Karplus, and B. M. Pettitt, *Proteins: A Theoretical Perspective of Dynamics, Structures and Thermodynamics*, John Wiley & Sons, New York, NY, USA, 1988.

[27] J. L. Lumley, "The structure of inhomogeneous turbulent flows," in *Atmospheric Turbulence and Radio Wave Propagation*, A. M. Yaglom and V. I. Tatarski, Eds., Nauka, Moscow, Russia, 1967.

[28] R. C. Gonzalez and P. Wintz, *Digital Image Processing*, Addison-Wesley, Reading, Mass, USA, 2nd edition, 1987.

[29] L. Sirovich, "Turbulence and dynamics of coherent structures—I: coherent structures," *Quarterly of Applied Mathematics*, vol. 45, no. 3, pp. 561–571, 1987.

[30] D. Armbruster, R. Heiland, E. Kostelich, and B. Nicolaenko, "Phase-space analysis of bursting behavior in Kalmogorov flow," *Physica D*, vol. 58, no. 1–4, pp. 392–401, 1992.

[31] D. Armbruster, B. Nicolaenko, N. Smaoui, and Pascal Chossat, "Symmetries and dynamics for 2-D Navier-Stokes flow," *Physica D*, vol. 95, no. 1, pp. 81–93, 1996.

[32] N. Smaoui, "A model for the unstable manifold of the bursting behavior in the 2D Navier-Stokes flow," *SIAM Journal on Scientific Computing*, vol. 23, no. 3, pp. 824–839, 2001.

[33] N. Smaoui and S. Al-Yakoob, "Analyzing the dynamics of cellular flames using Karhunen-Loève decomposition and autoassociative neural networks," *SIAM Journal of Scientific Computing*, vol. 24, no. 5, pp. 1790–1808, 2003.

[34] A. Armaou and P. D. Christofides, "Feedback control of the Kuramoto-Sivashinsky equation," *Physica D*, vol. 137, no. 1-2, pp. 49–61, 2000.

[35] A. Armaou and P. D. Christofides, "Wave suppression by nonlinear finite-dimensional control," *Chemical Engineering Science*, vol. 55, no. 14, pp. 2627–2640, 2000.

[36] J. Baker and P. D. Christofides, "Finite-dimensional approximation and control of non-linear parabolic PDE systems," *International Journal of Control*, vol. 73, no. 5, pp. 439–456, 2000.

[37] P. D. Christofides and A. Armaou, "Global stabilization of the Kuramoto-Sivashinsky equation via distributed output feedback control," *Systems & Control Letters*, vol. 39, no. 4, pp. 283–294, 2000.

[38] S. Y. Shvartsman and I. G. Kevrekidis, "Nonlinear model reduction for control of distributed systems: a computer-assisted study," *AIChE Journal*, vol. 44, no. 7, pp. 1579–1595, 1998.

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