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Research Article

Border Figure Detection Using a Phase Oscillator Network with Dynamical Coupling

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Oscillator networks have been developed in order to perform specific tasks related to image processing. Here we analytically investigate the existence of synchronism in a pair of phase oscillators that are short-range dynamically coupled. Then, we use these analytical results to design a network able of detecting border of black-and-white figures. Each unit composing this network is a pair of such phase oscillators and is assigned to a pixel in the image. The couplings among the units forming the network are also dynamical. Border detection emerges from the network activity.

1. Introduction

The synchronous firing of neurons seems to be important for accomplishing cognitive tasks such as attention (e.g., [1]), comprehension (e.g., [2]), coordination (e.g., [3]), perception (e.g., [4]), and sensory segmentation (e.g., [5]). Such experimental findings have inspired works about image processing (e.g., recognition, segmentation, symmetry detection) based on the synchronism of coupled oscillators (e.g., [6–11]). In addition, several neural systems, such as spinal cord (e.g., [12, 13]), hippocampus (e.g., [14, 15]), and visual cortex (e.g., [16, 17]), have been modelled by phase oscillators—one of the simplest oscillator models. In spite of its simplicity, this approach is suitable because more complex models for neurons like pulse-coupled and Hodgkin-Huxley-type models can be transformed into a phase oscillator model through coordinate changes [18]. Networks of phase oscillators can be electronically built using phase-locked loops (PLLs) (e.g., [19–22]).

Here we analytically study a phase oscillator model with dynamical coupling, and then use it in order to form a network capable of successfully detecting border of black-and-white figures. Each oscillator of our network corresponds to a first-order PLL (e.g., [8, 11, 17, 23]), which is equivalent to an overdamped pendulum.

The aim of any border detection process is to capture the main structural properties of an image, which can be useful, for instance, for image compression or diagnosis in echocardiography (e.g., [24]). Usually, this kind of image processing involves the use of partial derivatives (gradient, Laplacian; e.g., [25]), which are discretized in the space domain and in the time domain. Here we propose a scheme based on a network where each unit is formed by a pair of dynamically coupled PLLs corresponding to a pixel of the image. Border detection is obtained from the network activity. Notice that such a model can be transformed into a dedicated hardware for executing this image processing task. Also it naturally presents space discretization, because it is composed of a finite number of PLLs. This feature plus the fact that the time domain does need to be discretized can reduce the stability problems related to the algorithms employed for numerically calculating partial derivatives.

2. Model of a single unit

Consider that the temporal activities of two first-order PLLs with dynamical coupling are described by the following equations:

$$\begin{aligned}
\frac{d\theta_1}{dt} &= \omega_1 + k_{12}\sin\left(\theta_2 - \theta_1\right),\\
\frac{d\theta_2}{dt} &= \omega_2 + k_{21}\sin\left(\theta_1 - \theta_2\right),\\
\frac{dk_{12}}{dt} &= \mu\alpha_1\cos(\theta_1 - \theta_2) - \mu k_{12},\\
\frac{dk_{21}}{dt} &= \mu\alpha_2\cos(\theta_2 - \theta_1) - \mu k_{21},
\end{aligned} \tag{2.1}$$

where θ_j and $\omega_j (j = 1, 2)$ are the phase and the natural frequency of the oscillator j, respectively; k_{ij} $(i, j = 1, 2$ with $i \neq j)$ is the connection strength from the oscillator j to the oscillator i; α_j $(j = 1, 2)$ is the coefficient related to the Hebbian connection modification; and μ represents a natural (exponential) decay. The parameters ω_j, α_j, and μ are positive numbers. Thus, the connection strengths are enhanced when the oscillators are in phase, and they are weakened when they are out of phase. This kind of (synaptic) modification between two oscillators (neurons) was suggested by Hebb [26].

By defining $q \equiv \theta_2 - \theta_1$; $\omega \equiv \omega_2 - \omega_1$; $\alpha_0 \equiv (\alpha_1 + \alpha_2)/2$; $k \equiv (k_{12} + k_{21})/2$, system (2.1) can be rewritten as

$$\begin{aligned}
\frac{dq}{dt} &= f_1(q, k) = \omega - 2k\sin q,\\
\frac{dk}{dt} &= f_2(q, k) = \mu\alpha_0\cos q - \mu k.
\end{aligned} \tag{2.2}$$

The variables q and k represent the phase difference and the average connection strength, respectively. Notice that $f_j(q, k) = f_j(q + 2\pi, k)$ $(j = 1, 2)$. The formation of synchronized clusters in networks of phase oscillators described by system (2.2) was investigated by Seliger et al. [27]. In this section we study the asymptotic behaviors of this model.

Synchronism occurs when both nodes oscillate in a common frequency, which means that $d\theta_2/dt = d\theta_1/dt$ or $\theta_2 - \theta_1 = q =$ constant. Thus, synchronism can happen if there are constants (q^*, k^*) so that if $(q(t), k(t)) = (q^*, k^*)$ then $dq/dt = 0$ and $dk/dt = 0$ for all time t. Notice that a synchronous solution is an equilibrium point (q^*, k^*) of the dynamical system (2.2). Such a solution corresponds to an intersection point of the nullclines $f_1 = 0$ $(k = \omega/(2\sin q))$ and $f_2 = 0$ $(k = \alpha_0 \cos q)$.

The stability of an equilibrium point (q^*, k^*) can usually be determined by calculating the eigenvalues λ of the Jacobian matrix corresponding to system (2.2) linearized around this point. It is asymptotically stable when all eigenvalues have negative real parts (e.g., [28]). For system (2.2), the eigenvalues λ are the roots of the polynomial

$$\lambda^2 + (\mu + 2k^* \cos q^*)\lambda + 2\mu k^* \cos q^* - 2\mu\alpha_0 (\sin q^*)^2 = 0. \tag{2.3}$$

According to the Routh-Hurwitz criterion, both roots of the polynomial $\lambda^2 + a_1\lambda + a_2 = 0$ have negative real parts if $a_1 > 0$ and $a_2 > 0$ (e.g., [29]).

When $\omega = 0$, the equilibrium points are $P_1 = (0, \alpha_0)$, $P_2 = (\pi/2, 0)$, $P_3 = (\pi, -\alpha_0)$, $P_4 = (3\pi/2, 0)$, where P_1 and P_3 are asymptotically stable, and P_2 and P_4 are unstable.

When $\omega \neq 0$, there are also four equilibrium solutions if $\omega < \alpha_0$. The points with

$$k^* = \pm \frac{\alpha_0}{\sqrt{2}} \sqrt{1 + \sqrt{1 - \left(\frac{\omega}{\alpha_0}\right)^2}} \tag{2.4}$$

are asymptotically stable; the ones with

$$k^* = \pm \frac{\alpha_0}{\sqrt{2}} \sqrt{1 - \sqrt{1 - \left(\frac{\omega}{\alpha_0}\right)^2}} \tag{2.5}$$

are unstable. When $\omega = \alpha_0$, there are only two equilibrium points because two saddle-node bifurcations (e.g., [28]) occur. When $\omega > \alpha_0$, there is not synchronism; however, there is a limit cycle, which corresponds to a closed and isolated trajectory in the state space $q \times k$.

For an autonomous two-dimensional system, Poincaré-Bendixson theorem (e.g., [28]) ensures that there is an asymptotically stable limit cycle in a region of the state space if the vector field (f_1, f_2) points inward everywhere on the boundary of this region, which must not contain any equilibrium point. For system (2.2), it is easy to verify that if $\omega > \alpha_0$, then there is such an attracting trajectory inside the region R, where R is the rectangle given by $R = \{(q, k) : 0 \leq q < 2\pi,\ \alpha_0 + (\epsilon/\mu) < k < -\alpha_0 - (\epsilon/\mu)\}$ for $\epsilon \to 0_+$. In this case, k oscillates and q grows as the time goes by, as shown in Figure 2. Figure 1 illustrates the case with synchronous (equilibrium) asymptotic solution.

In order to analytically characterize the limit cycle, suppose that the asymptotic behavior of $q(t)$ in this case is given by $q_{\text{asymp}}(t) \simeq Bt$. By inserting this approximated (linear) solution in the equation for dk/dt and integrating it, the asymptotic solution of $k(t)$ is

$$k_{\text{asymp}}(t) \simeq A \cos(Bt - \varphi), \tag{2.6}$$

where $A = \mu\alpha_0/\sqrt{\mu^2 + B^2}$ and $\sin\varphi = B/\sqrt{\mu^2 + B^2}$. By substituting this expression in the equation for dq/dt in (2.2), an equation for calculating B is found. In fact, B is the root of the polynomial

$$B^3 - \omega B^2 + (\mu^2 + \mu\alpha_0)B - \omega\mu^2 = 0. \tag{2.7}$$

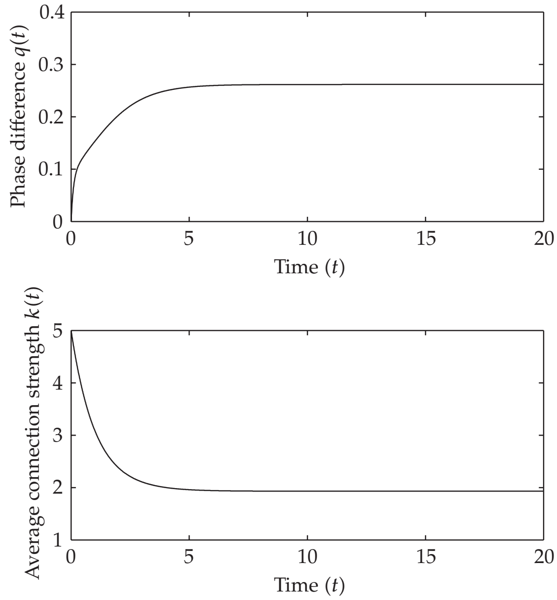

Figure 1: Temporal evolution of $q(t)$ and $k(t)$ obtained by numerically integrating system (2.2). Parameter values: $\omega = 1$, $\alpha = 2$, $\mu = 1$. Initial conditions: $q(0) = 0$, $k(0) = 5$. In this case, the asymptotic solution corresponds to the equilibrium point $(q^*, k^*) = (0.26, 1.93)$.

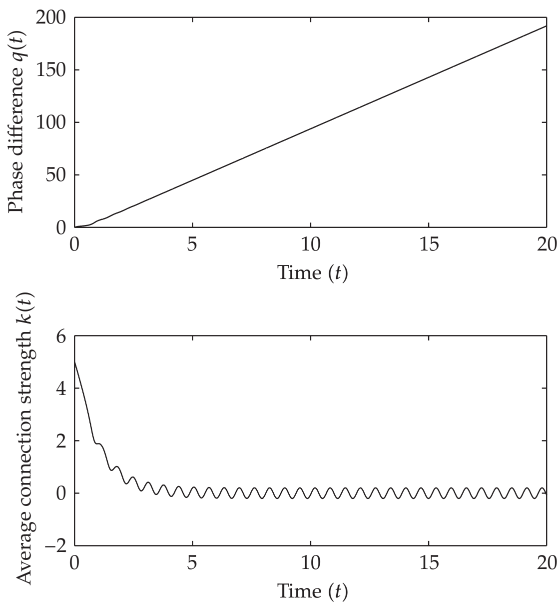

Figure 2: Temporal evolution of $q(t)$ and $k(t)$ obtained by numerically integrating system (2.2). Parameter values: $\omega = 10$, $\alpha = 2$, $\mu = 1$. Initial conditions: $q(0) = 0$, $k(0) = 5$. In this case, the asymptotic solution corresponds to the limit cycle, where $k(t)$ oscillates with frequency $B \simeq 9.8$ and amplitude $A \simeq 0.20$, and $q(t)$ linearly grows with slope A.

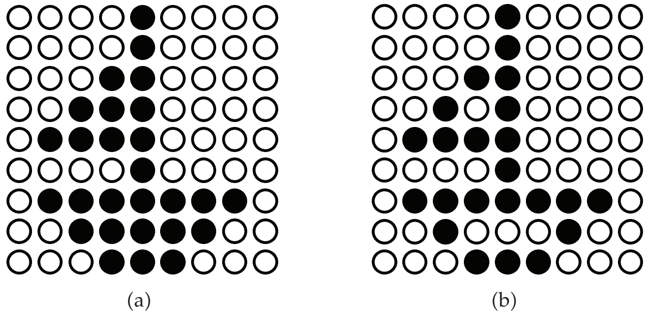

Figure 3: (a) At $t = 0$, this is the image presented to the network with 9×9 units. (b) At $t = 40$, the border of the image is dynamically determined.

Observe that when $\omega \gg 1$, then the angular velocity B can be written as $B \simeq \omega + \Delta$, with $|\Delta| \ll 1$. The value of Δ is estimated by

$$\Delta \simeq -\frac{\mu\alpha_0\omega}{\omega^2 + \mu^2 + \mu\alpha_0}. \tag{2.8}$$

Consequently, the oscillation amplitude A of $k_{\text{asymp}}(t)$ when $\omega \gg 1$ is

$$A \simeq \frac{\mu\alpha_0}{\omega}. \tag{2.9}$$

For instance, for $\omega = 10$, $\alpha_0 = 2$, and $\mu = 1$, these approximated expressions give $B \simeq 9.8$ and $A \simeq 0.20$, which are in good agreement with the numerical solutions of system (2.2) shown in Figure 2.

3. Network for detecting figure border

Our two-dimensional network for detecting figure border is composed of units consisting of a pair of phase oscillators forced by an external input I. In this network, the coefficients α_1 and α_2 can vary with the time. Thus, system (2.2) is rewritten as

$$\begin{aligned}\frac{dq}{dt} &= I + \omega - 2k\sin q,\\ \frac{dk}{dt} &= \mu\alpha(t)\cos q - \mu k.\end{aligned} \tag{3.1}$$

Assume that a black-and-white figure will be presented to this network, as illustrated in Figure 3(a). The variable I represents the input for each unit according to the following rules: $I = 0$ corresponds to a white part of the figure; $I = 9$ corresponds to a black part. Notice that the input can be translated into a new natural frequency Ω by defining $I + \omega \equiv \Omega$. Thus, system (3.1) is reduced to system (2.2) if $\alpha(t)$ is a constant.

Each unit is coupled with its four closest neighbors (left, right, up, and down), which is usually known as four-neighborhood in image processing literature (e.g., [30]) or von

Neumann neighborhood in cellular automaton literature (e.g., [31]). As a consequence of this coupling, the value of $\alpha(t)$ for each cell is given by

$$\alpha(t) = \begin{cases} \alpha_0 & \text{if } 0 \le t \le T, \\ p\alpha_0 & \text{if } t > T. \end{cases} \tag{3.2}$$

The parameter T is a settling time; $p \equiv 1 + qx$ where q is a positive number (here $q = 14$); $x = G(y)$ where $G(y) = 0$ if $y = 0$; and $G(y) = 1$ if $y \neq 0$. The value of y is obtained by

$$y = \prod_{i=1}^{4} \left[k_i(T) - k_i(T - \delta T) \right], \tag{3.3}$$

where δT is the time step (here $\delta T = 0.01$) of the integration method (here fourth-order Runge-Kutta method) used for numerically solving the dynamical system. The variable $k_i(t)$ corresponds to the average connection strength of the closest neighbor i $(i = 1, 2, 3, 4)$. If the neighbor j does not exist, then $k_j(t) \equiv 0$.

An image is presented to the network at $t = 0$. For $0 \le t \le T$, all the units behave as if they were isolated, because the value of $\alpha(= \alpha_0)$ for each unit is independent of the neighbor activity. By taking for all units $\omega = 1$, $\alpha_0 = 2$, $\mu = 1$, $T = 20$, $q(0) = 0$, and $k(0) = 5$, the units with $I = 0$ will tend to a stationary solution and the units with $I = 9$ will tend to a limit cycle, as explained in the last section. For these parameter values, the permanent regime was already reached when $t = T$, as shown in Figures 1 and 2.

At $t = T$, the value of α for each unit can be changed. Expressions (3.2) and (3.3) imply what follows. If all of its four neighbors are in a limit cycle $(y \neq 0)$, then the value of α is increased from 2 to 30; if at least one neighbor is in a stationary state or if the unit does not have a complete von Neumann neighborhood $(y = 0)$, then the value of α is kept equal to 2. The limit cycle is characterized by $k(t)$ oscillating with frequency B and amplitude A, and $q(t)$ linearly growing with slope A, as presented in the last section.

For $\alpha = 30$ and $\Omega = 10$, $q(t)$ and $k(t)$ tend to a stationary activity, which in our model corresponds to the white color. Thus, at $t = 2T = 40$, the unique units remaining in oscillatory activity (the black color) are the ones in the figure border. Figure 3(b) shows the contour of Figure 3(a), after the units have reached the (new) permanent regime.

The higher the value of μ is, the shorter the transitory phase will be. Hence, border detection can be made faster by increasing the value of this parameter.

4. Conclusion

Networks governed by differential (e.g., [6–8, 10, 11]) or difference equations (e.g., [9, 32]) have been employed for image processing. Here we used a network with local and dynamical coupling for identifying contour of black-and-white images. The unit of such a network is the variant of the Kuramoto model [33] for two oscillators (neurons), which was proposed by Seliger et al. [27]. In our scheme, after applying the input corresponding to the figure, the value of α for each unit is switched or not, depending on the neighborhood activity. This scheme could be implemented using first-order PLLs. This investigation could be extended for higher-order PLLs, which could be designed in order to shorten the transient.

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