

# CHAINS OF KP, SEMI-INFINITE 1-TODA LATTICE HIERARCHY AND KONTSEVICH INTEGRAL

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There are well-known constructions of integrable systems that are chains of infinitely many copies of the equations of the KP hierarchy “glued” together with some additional variables, for example, the modified KP hierarchy. Another interpretation of the latter, in terms of infinite matrices, is called the 1-Toda lattice hierarchy. One way infinite reduction of this hierarchy has all the solutions in the form of sequences of expanding Wronskians. We define another chain of the KP equations, also with solutions of the Wronskian type, that is characterized by the property to stabilize with respect to a gradation. Under some constraints imposed, the tau functions of the chain are the tau functions associated with the Kontsevich integrals.

## 1. Introduction

This paper was motivated by the following arguments. There are well-known chains of infinitely many copies of the equations of the KP hierarchy “glued” together with some variables, like, for example, modified KP (see (2.2a), (2.2b), and (2.2c) below). The latter is a sequence of dressing operators of the KP hierarchy  $\{\widehat{w}_N\}$  along with “gluing” variables  $\{u_N\}$ . All these variables make a large integrable system. The chain (2.2a), (2.2b), and (2.2c) has another interpretation, in terms of infinite matrices, which is called the 1-Toda lattice hierarchy (see [2, 4, 9]). There exist different reductions of this chain, for example, modified KdV, or another reduction, a semi-infinite chain for which all  $\widehat{w}_N$  with negative  $N$  are trivial,  $\widehat{w}_N = 1$ , and  $\widehat{w}_N$  with positive  $N$  are  $P_N \partial^{-N}$  where  $P_N$  is an  $N$ th order differential operator (the corresponding matrices of the 1-Toda lattice hierarchy also are semi-infinite).

It can be shown (see Proposition 2.3) that all the solutions are sequences of well-known Wronskian solutions to KP, each  $\widehat{w}_N$  being represented by a determinant of  $N$ th order. Every next determinant is obtained from the preceding one by an extension of the Wronskian when a new function is added to the existing ones.

There is another situation where one deals with a sequence of Wronskian solutions of increasing order. This time the Wronskians are not obtained by a successive extension. The rule is more complicated. We talk about the so-called Kontsevich integral [1, 7, 8] which has its origin in quantum physics. This is an integral over the group  $U(N)$  which is a function of a matrix, invariant with respect to the matrix conjugation, that is, a function of eigenvalues  $\lambda_i$  of the matrix. The main fact about the Kontsevich integral is that it is a tau function of the KP hierarchy of the Wronskian type in variables  $t_i = \sum_k \lambda_k^{-i}$ . The dimension of the Wronskian is  $N$ . The sequence of Wronskians has, in a sense, a limit when  $N \rightarrow \infty$ . More precisely, this is a stable limit. There is some grading and the terms of a fixed weight stabilize when  $N \rightarrow \infty$ : they become independent of  $N$  when  $N$  is large enough. The stable limit belongs to the  $n$ th reduction of KP ( $n$ th GD) and, besides, satisfies the string equation.

The question we try to answer here is whether the sequence of Kontsevich tau functions is interesting by itself, not only by its limit. Is it possible to complete it with “gluing” variables to obtain a chain of related KP equations similar to (2.2a), (2.2b), and (2.2c)? The answer is positive (see Section 3.1, (3.2), (3.3), (3.4), and (3.5)). Unfortunately, we do not know a matrix version of this chain like that of the 1-Toda lattice hierarchy.

Thus, in this paper we define the “stabilizing” chain of KP, study its solutions, and demonstrate that they are exactly those which are represented by the Kontsevich integral. In the appendix we briefly show, skipping all the calculations, the way from the Wronskian solutions to the Kontsevich integral. This is actually the conversion of Itzykson and Zuber’s [7] reasoning, and the reader can find the skipped details there.

## 2. Semi-infinite 1-Toda lattice hierarchy

### 2.1.

Recall some basic facts about the modified KP and 1-Toda lattice hierarchy (see [4]). The modified KP hierarchy is a collection of the following objects ( $\partial_k = \partial/\partial t_k$ ,  $\partial = \partial_1$ ):

$$\widehat{w}_N(\partial) = 1 + w_{N1}\partial^{-1} + w_{N2}\partial^{-2} + \dots, \quad N \in \mathbb{Z} \quad (2.1)$$

and  $\{u_N\}$ ,  $N \in \mathbb{Z}$ , and the following relations:

$$(\partial + u_N)\widehat{w}_N = \widehat{w}_{N+1}\partial, \tag{2.2a}$$

$$\partial_k \widehat{w}_N = -(L_N^k)_- \widehat{w}_N, \quad \text{where } L_N = \widehat{w}_N \partial \widehat{w}_N^{-1}, \tag{2.2b}$$

$$\partial_k u_N = (L_{N+1}^k)_+ (\partial + u_N) - (\partial + u_N)(L_N^k)_+. \tag{2.2c}$$

Notice that multiplying (2.2c) by  $(\partial + u_N)^{-1}$  on the right and taking the residue we get an equivalent form of this equation

$$\partial_k u_N = -\text{res}(\partial + u_N)(L_N^k)_+ (\partial + u_N)^{-1}. \tag{2.3}$$

We construct a both way infinite matrix  $\mathbf{W}$  with elements

$$W_{ij} = \begin{cases} w_{i,i-j}, & j \leq i, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

Then the following proposition holds.

**Proposition 2.1** (see [4]). *The operators  $\{\widehat{w}_N\}$  satisfy the modified KP, along with some  $\{u_N\}$ , if and only if the matrix  $\mathbf{W}$  satisfies the discrete KP (1-Toda lattice hierarchy)*

$$\partial_k \mathbf{W} = -(\mathbf{L}^k)_- \mathbf{W}, \tag{2.5}$$

where  $\mathbf{L} = \mathbf{W}\Lambda\mathbf{W}^{-1}$ ,  $\Lambda$  is the matrix of the shift  $(\Lambda)_{ij} = \delta_{i,j-1}$ , and the subscript “-” symbolizes the strictly lower triangular part of a matrix.

Suppose that we have a special solution such that  $\widehat{w}_N = 1$  and  $u_{N-1} = 0$  when  $N \leq 0$ . Let  $P_N = \widehat{w}_N \partial^N$ . Then  $P_0 = 1$ , (2.2a) implies  $(\partial + u_N)P_N = P_{N+1}$  and therefore  $P_N$  is an  $N$ th-order differential monic operator when  $N > 0$ . Then  $\widehat{w}_N = 1 + w_{N1}\partial^{-1} + \dots + w_{NN}\partial^{-N}$ . The matrix  $\mathbf{W}$  is a direct sum of two semi-infinite blocks, one is the unity and the other is  $(W_{ij})$  with  $i$  and  $j \geq 0$ . We show that all the solutions are simply the well-known Wronskian solutions of KP.

**Lemma 2.2.** *Let  $\widehat{w}_N$  be a solution of the semi-infinite chain of (2.2a), (2.2b), and (2.2c) with  $N = 0, 1, 2, \dots$  and  $P_N = \widehat{w}_N \partial^N$ . There exists a sequence of linearly independent functions  $y_0, y_1, \dots$  such that  $P_N y_i = 0$  when  $i = 0, \dots, N-1$  and  $(\partial^k - \partial_k)y_i = 0$  for all  $k$  and  $i$ .*

*Proof.* Suppose the  $y_i$ 's are already constructed for  $i < N-1$  (if  $N = 1$ , nothing is supposed). Then  $P_{N-1}y_i = 0$  and  $P_N y_i = 0$  for  $i = 0, \dots, N-2$  since  $(\partial + u_{N-1})P_{N-1} = P_N$ . The kernel of the operator  $P_N$  is  $N$ -dimensional, therefore there is one more function  $y_{N-1}$  independent of  $y_0, \dots, y_{N-2}$  such that  $P_N y_{N-1} = 0$ .

First of all, we prove that if a function, in this case  $y_{N-1}$  but this is a general fact, belongs to the kernel of  $P_N$  then so does  $(\partial^k - \partial_k)y_{N-1}$ . We have

$$\begin{aligned}
 0 &= \partial_k(P_N y_{N-1}) \\
 &= (\partial_k P_N) y_{N-1} + P_N (\partial_k y_{N-1}) \\
 &= -(\widehat{w}_N \partial^k \widehat{w}_N^{-1})_- \widehat{w}_N \partial^N y_{N-1} + P_N (\partial_k y_{N-1}) \\
 &= -\widehat{w}_N \partial^N \partial^k y_{N-1} + (L_N^k)_+ P_N y_{N-1} + P_N (\partial_k y_{N-1}).
 \end{aligned} \tag{2.6}$$

The middle term of the last expression vanishes since  $P_N y_{N-1} = 0$  and  $(L_N^k)_+$  is a differential operator. Thus,  $P_N (\partial^k - \partial_k) y_{N-1} = 0$ , and  $(\partial^k - \partial_k) y_{N-1}$  is in  $\ker P_N$ . Now,

$$(\partial^k - \partial_k) y_{N-1} = \sum_{i=0}^{N-1} A_{ki} y_i, \quad A_{li} = 0. \tag{2.7}$$

The coefficients  $A_{ki}$  do not depend on  $t_1$ . We have

$$\begin{aligned}
 (\partial^l - \partial_l) (\partial^k - \partial_k) y_{N-1} &= A_{k,N-1} \sum_{i=0}^{N-1} A_{li} y_i - \sum_{i=0}^{N-1} (\partial_l A_{ki}) y_i, \\
 (\partial^k - \partial_k) (\partial^l - \partial_l) y_{N-1} &= A_{l,N-1} \sum_{i=0}^{N-1} A_{ki} y_i - \sum_{i=0}^{N-1} (\partial_k A_{li}) y_i.
 \end{aligned} \tag{2.8}$$

The operators  $(\partial^k - \partial_k)$  and  $(\partial^l - \partial_l)$  commute, hence

$$\sum_{i=0}^{N-1} (\partial_l + A_{l,N-1}) A_{ki} \cdot y_i = \sum_{i=0}^{N-1} (\partial_k + A_{k,N-1}) A_{li} \cdot y_i \tag{2.9}$$

and, by virtue of the linear independence of  $y_i$  as functions of  $t_1$ ,

$$(\partial_l + A_{l,N-1}) A_{ki} = (\partial_k + A_{k,N-1}) A_{li}. \tag{2.10}$$

This is the compatibility condition of the equations

$$(\partial_k + A_{k,N-1}) m_i = A_{ki}. \tag{2.11}$$

We can find  $m_i$ , with  $m_{N-1} = 1$ , and then

$$(\partial^k - \partial_k) y_{N-1} = \sum_{i=0}^{N-1} (\partial_k + A_{k,N-1}) m_i \cdot y_i. \tag{2.12}$$

Let  $\tilde{y}_{N-1} = \sum_0^{N-1} m_i y_i$ . It is easy to see that

$$(\partial^k - \partial_k) \tilde{y}_{N-1} = A_{k,N-1} \tilde{y}_{N-1}. \tag{2.13}$$

Now, if we solve the system

$$\partial_k \phi_{N-1} = \phi_{N-1} A_{k,N-1}, \tag{2.14}$$

consistent by virtue of (2.10), where  $i = N - 1$ , then  $y_{N-1}^* = \tilde{y}_{N-1} \phi_{N-1}$  will satisfy the equation

$$(\partial^k - \partial_k) y_{N-1}^* = 0. \tag{2.15}$$

□

*Proposition 2.3. The general construction of the solutions to the semi-infinite 1-Toda hierarchy, or, equivalently, the chain (2.2a), (2.2b), and (2.2c) is the following: let  $\{y_i\}$  be a sequence of functions of the variables  $t_1 = x, t_2, \dots$  having the property  $(\partial^k - \partial_k) y_i = 0$ . Then*

$$\widehat{w}_N = \frac{1}{W_N} \begin{vmatrix} y_0 & \cdots & y_{N-1} & 1 \\ y'_0 & \cdots & y'_{N-1} & \partial \\ \vdots & \vdots & \vdots & \vdots \\ y_0^{(N)} & \cdots & y_{N-1}^{(N)} & \partial^N \end{vmatrix} \cdot \partial^{-N}, \tag{2.16}$$

where  $W_N = W(y_0, \dots, y_{N-1})$  is the Wronskian and  $u_N = -\partial \ln(W_{N+1}/W_N)$ .

*Proof.* In one way, the proposition is almost proved by the preceding analysis. It only remains to notice that  $P_N = \widehat{w}_N \partial^N$ , where  $\widehat{w}_N$  is given by (2.16), is the monic differential operator with the kernel spanned by  $y_0, \dots, y_{N-1}$ , that  $(\partial + u_N) P_N y_N = 0$  and  $P_N y_N = W_{N+1}/W_N$ .

The converse follows from the fact that  $\widehat{w}_N$  given by (2.16), as it is well known, is a dressing operator of the KP hierarchy, the operators  $(\partial + u_N) P_N$ , where  $u_N = -\ln(W_{N+1}/W_N)$  and  $P_{N+1}$ , have the same kernel spanned by  $y_0, \dots, y_N$  and, therefore, coincide. □

**2.2.**

The property  $(\partial^k - \partial_k) y_i = 0$  implies

$$y_i \left( t_1 - \frac{1}{z}, t_2 - \frac{1}{2z^2}, t_3 - \frac{1}{3z^3}, \dots \right) = \left( 1 - \frac{1}{z} \partial \right) y_i(t_1, t_2, t_3, \dots) \tag{2.17}$$

or  $y_i(t - [z^{-1}]) = (1 - \partial/z) y_i(t)$ , using the common notation. Indeed,

$$\begin{aligned} \exp \left( \sum_1^\infty -(kz^k)^{-1} \partial_k \right) y_i &= \left( \exp \sum_1^\infty -(kz^k)^{-1} \partial^k \right) y_i \\ &= \exp \ln \left( 1 - \frac{1}{z} \partial \right) y_i = \left( 1 - \frac{1}{z} \partial \right) y_i. \end{aligned} \tag{2.18}$$

The Baker, or wave function is  $\widehat{w}(\partial) \exp \sum_1^\infty t_k \xi^k = \widehat{w}(z) \exp \sum_1^\infty t_k z^k$  where

$$\widehat{w}_N(z) = \frac{1}{W_N} \begin{vmatrix} y_0 & \cdots & y_{N-1} & z^{-N} \\ y'_0 & \cdots & y'_{N-1} & z^{-N+1} \\ \vdots & \vdots & \vdots & \vdots \\ y_0^{(N)} & \cdots & y_{(N-1)}^{(N)} & 1 \end{vmatrix}. \quad (2.19)$$

Subtracting the second row divided by  $z$  from the first one, then the third divided by  $z$  from the second one, and so forth, we obtain zeros in the last column except the last element which is 1. In the  $i$ th row there will be elements  $y_j^{(i)} - z^{-1} y_j^{(i+1)} = y_j^{(i)}(t - [z^{-1}])$ , according to (2.17). Thus,

$$\widehat{w}_N(z) = \frac{W_N(t - [z^{-1}])}{W_N(t)} \quad (2.20)$$

which means that  $W_N(t)$  is the tau function  $\tau_N(t)$  corresponding to  $\widehat{w}_N(z)$ , and  $u_N = -\ln(\tau_{N+1}/\tau_N)$ .

What are the functions  $y_i(t)$  with the property  $(\partial^k - \partial_k)y_i = 0$ ? For example,  $\exp \sum_1^\infty t_k \alpha^k$ , linear combinations of several such functions, and integrals  $\int f(\alpha) \exp(\sum_1^\infty t_k \alpha^k) d \sum(\alpha)$ . Using the Schur polynomials  $p_k(t_1, t_2, \dots)$  defined by  $\exp \sum_1^\infty t_k \alpha^k = \sum_0^\infty p_k(t) \alpha^k$ , we see that all the above examples have a form  $y_i(t) = \sum c_{ik} p_k(t)$ . Conversely, any series of this form has the property  $(\partial^k - \partial_k)y_i = 0$  since the Schur polynomials have it, as it is easy to see. Apparently, this is, in a sense, the most general form of such functions.

### 3. Stabilizing chain

*Definition 3.1.* The stabilizing chain is a collection of the following objects:

$$\begin{aligned} \widehat{w}_N(\partial) &= 1 + w_{N1} \partial^{-1} + \cdots + w_{NN} \partial^{-N} \\ &= P_N(\partial) \cdot \partial^{-N}, \quad N = 0, 1, 2, 3, \dots, \end{aligned} \quad (3.1)$$

and  $u_N, v_{N+1}, N = 0, 1, 2, 3, \dots$ , and the following relations:

$$(\partial + u_N) \cdot \widehat{w}_N = (\partial + v_{N+1}) \cdot \widehat{w}_{N+1}, \quad (3.2)$$

$$\partial_k \widehat{w}_N = - (L_N^k)_- \widehat{w}_N, \quad \text{where } L_N = \widehat{w}_N \partial \widehat{w}_N^{-1}, \quad (3.3)$$

$$\partial_k u_N = - \text{res}((\partial + u_N)(L_N^k)_+ (\partial + u_N)^{-1}), \quad (3.4)$$

$$\partial_k v_{N+1} = - \text{res}((\partial + v_{N+1})(L_{N+1}^k)_+ (\partial + v_{N+1})^{-1}). \quad (3.5)$$

Equations (3.4) and (3.5) can also be written as

$$\begin{aligned} \partial_k u_N &= \text{res } L_N^k - \text{res } ((\partial + u_N)L_N^k(\partial + u_N)^{-1}), \\ \partial_k v_{N+1} &= \text{res } L_{N+1}^k - \text{res } ((\partial + v_{N+1})L_{N+1}^k(\partial + v_{N+1})^{-1}), \end{aligned} \tag{3.6}$$

where

$$\text{res } ((\partial + u_N)L_N^k(\partial + u_N)^{-1}) = \text{res } ((\partial + v_{N+1})L_{N+1}^k(\partial + v_{N+1})^{-1}) \tag{3.7}$$

by virtue of (3.2).

It can happen that starting from some term  $u_N = v_{N+1}$  and all  $\widehat{w}_N$  are equal, the chain stabilizes. Then  $w_{NN} = 0$ . Also, it can happen that the chain contains constant segments and after that again begins to change. These segments can be just skipped. We assume that all  $w_{NN} \neq 0$ . Nevertheless, some tendency to stabilization remains, as we will see below. A grading will be introduced so that all quantities will be sums of terms of different weights. We will see that terms of a given weight stabilize, the greater is the weight the later the stabilization occurs. The stabilization is actually the most important feature of this chain allowing one to consider the stable limits when  $N$  tends to  $\infty$ . This is used, for example, in the Kontsevich integral.

The chain is well defined if one proves that

- (1) the right-hand side of (3.3) is a pseudodifferential operator ( $\Psi$ DO) of the form  $\sum_1^N a_j \partial^{-i}$ ,
- (2) vector fields  $\partial_k$  defined by (3.3), (3.4), and (3.5) respect the relation (3.2),
- (3) vector fields  $\partial_k$  commute.

Equation (3.3) defines a copy of the KP hierarchy for each  $n$ . It is clear that the operator in the right-hand side is negative. Rewriting it as  $\partial_k \widehat{w}_N = -\widehat{w}_N \partial^k + (L_N^k)_+ \widehat{w}_N$ , one can see that it contains only powers  $\partial^{-i}$  with  $i \leq N$ . Thus, (1) is proven.

Now, one has to prove that

$$\partial_k ((\partial + u_N)\widehat{w}_N - (\partial + v_{N+1})\widehat{w}_{N+1}) = 0 \tag{3.8}$$

if (3.2) holds. We have

$$\begin{aligned} &\partial_k ((\partial + u_N)\widehat{w}_N - (\partial + v_{N+1})\widehat{w}_{N+1}) \\ &= (\text{res } L_N^k - \text{res } ((\partial + u_N)L_N^k(\partial + u_N)^{-1}))\widehat{w}_N \\ &\quad - (\text{res } L_{N+1}^k - \text{res } ((\partial + v_{N+1})L_{N+1}^k(\partial + v_{N+1})^{-1}))\widehat{w}_{N+1} \\ &\quad - (\partial + u_N)(L_N^k)_- \widehat{w}_N + (\partial + v_{N+1})(L_{N+1}^k)_- \widehat{w}_{N+1}. \end{aligned} \tag{3.9}$$

The last two terms are transformed as

$$\begin{aligned}
& -(\partial + u_N)(L_N^k)_- \widehat{w}_N \\
& = -((\partial + u_N)L_N^k)_- \widehat{w}_N - \text{res}(L_N^k) \widehat{w}_N \\
& = -((\partial + u_N)L_N^k(\partial + u_N)^{-1}(\partial + u_N))_- \widehat{w}_N - \text{res}(L_N^k) \widehat{w}_N \quad (3.10) \\
& = -((\partial + u_N)L_N^k(\partial + u_N)^{-1})_- (\partial + u_N) \widehat{w}_N - \text{res}(L_N^k) \widehat{w}_N \\
& \quad + \text{res}((\partial + u_N)L_N^k(\partial + u_N)^{-1}) \widehat{w}_N.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& (\partial + v_{N+1})(L_{N+1}^k)_- \widehat{w}_{N+1} \\
& = ((\partial + v_{N+1})L_{N+1}^k(\partial + v_{N+1})^{-1})_- (\partial + v_{N+1}) \widehat{w}_{N+1} \\
& \quad + \text{res}(L_{N+1}^k) \widehat{w}_{N+1} - \text{res}((\partial + v_{N+1})L_{N+1}^k(\partial + v_{N+1})^{-1}) \widehat{w}_{N+1}. \quad (3.11)
\end{aligned}$$

Taking into account (3.2) and

$$(\partial + v_{N+1})L_{N+1}^k(\partial + v_{N+1})^{-1} = (\partial + u_N)L_N^k(\partial + u_N)^{-1}, \quad (3.12)$$

the sum of all the terms is zero.

Before we prove (3), the relations (3.4) and (3.5) will be presented in a different form.

**Lemma 3.2.** *The following remarkable formula holds*

$$\left(\partial - \frac{w'}{w}\right)^{-1} = \sum_0^{\infty} \partial^{-k-1} \frac{w^{(k)}}{w}, \quad (3.13)$$

where  $w$  is a function.

*Proof.* We multiply the right-hand side of (3.13) by  $(\partial - w'/w)$ :

$$\begin{aligned}
& \sum_0^{\infty} \partial^{-k-1} \frac{w^{(k)}}{w} \left(\partial - \frac{w'}{w}\right) \\
& = \sum_0^{\infty} \partial^{-k} \frac{w^{(k)}}{w} - \sum_0^{\infty} \partial^{-k-1} \left(\frac{w^{(k)}}{w}\right)' - \sum_0^{\infty} \partial^{-k-1} \frac{w^{(k)} w'}{w^2} \quad (3.14) \\
& = \sum_0^{\infty} \partial^{-k} \frac{w^{(k)}}{w} - \sum_0^{\infty} \partial^{-k-1} \frac{w^{(k+1)}}{w} = 1.
\end{aligned}$$

□



Corollary 3.3. For any polynomial  $P(\partial)$ , the following holds:

$$\begin{aligned} \operatorname{res} \partial^k \left( \partial - \frac{w'}{w} \right)^{-1} &= \frac{w^{(k)}}{w}, \\ \operatorname{res} P(\partial) \cdot \left( \partial - \frac{w'}{w} \right)^{-1} &= \frac{P(\partial)w}{w}. \end{aligned} \tag{3.15}$$

Notice that (3.2) implies that  $(\partial + v_{N+1})w_{N+1,N+1} = 0$  where  $(\partial + v_{N+1})w_{N+1,N+1}$  is understood as a result of the action of the operator  $(\partial + v_{N+1})$  on the function  $w_{N+1,N+1}$  (not a product!). Indeed, this is the coefficient in  $\partial^{-N-1}$  of the expression in the right-hand side while the left-hand side does not contain this term. Thus,

$$v_{N+1} = -\frac{w'_{N+1,N+1}}{w_{N+1,N+1}} = -\partial \ln w_{N+1,N+1}. \tag{3.16}$$

Now,

$$\begin{aligned} \partial_k v_{N+1} &= -\operatorname{res} \left( \left( \partial - \frac{w'_{N+1,N+1}}{w_{N+1,N+1}} \right) (L_{N+1}^k)_+ \left( \partial - \frac{w'_{N+1,N+1}}{w_{N+1,N+1}} \right)^{-1} \right) \\ &= -\frac{(\partial - w'_{N+1,N+1}/w_{N+1,N+1})(L_{N+1}^k)_+ w_{N+1,N+1}}{w_{N+1,N+1}} \\ &= -\partial \frac{(L_{N+1}^k)_+ w_{N+1,N+1}}{w_{N+1,N+1}}. \end{aligned} \tag{3.17}$$

Subtracting (3.6), we have

$$\partial_k (u_N - v_{N+1}) = \operatorname{res} L_N^k - \operatorname{res} L_{N+1}^k. \tag{3.18}$$

These two equations are equivalent to (3.4) and (3.5).

An alternative way to get (3.17) is the following: equation (3.3) implies  $\partial_k \widehat{w}_{N+1} = -\widehat{w}_{N+1} \partial^k + (L_{N+1}^k)_+ \widehat{w}_{N+1}$  and

$$\partial_k w_{N+1,N+1} = -(L_{N+1}^k)_+ w_{N+1,N+1}. \tag{3.19}$$

Then (3.17) easily follows from (3.16).

*Proof of the commutativity of  $\{\partial_k\}$ .* They commute in their action on all  $\widehat{w}_N$  as KP vector fields. Therefore (3.16) yields  $\partial_k \partial_l v_{N+1} = \partial_l \partial_k v_{N+1}$ . Finally,

$$\begin{aligned}
 \partial_l \partial_k (u_N - v_{N+1}) &= \text{res} \left( [(L_N^l)_+, L_N^k] - [(L_{N+1}^l)_+, L_{N+1}^k] \right) \\
 &= -\text{res} \left( [(L_N^l)_-, L_N^k] - [(L_{N+1}^l)_-, L_{N+1}^k] \right) \\
 &= -\text{res} \left( [(L_N^l)_-, (L_N^k)_+] - [(L_{N+1}^l)_-, (L_{N+1}^k)_+] \right) \\
 &= -\text{res} \left( [L_N^l, (L_N^k)_+] - [L_{N+1}^l, (L_{N+1}^k)_+] \right) \\
 &= -\text{res} \left( [(L_N^k)_+, L_N^l] - [(L_{N+1}^k)_+, L_{N+1}^l] \right) \\
 &= \partial_k \partial_l (u_N - v_{N+1})
 \end{aligned} \tag{3.20}$$

whence  $\partial_l \partial_k u_N = \partial_k \partial_l u_N$ . □

#### 4. Solutions to the chain

##### 4.1.

Let  $y_{0N}, \dots, y_{N-1N}$  be a basis of the kernel of the differential operator  $P_N = \widehat{w}_N \partial^N$  such that  $P_N y_{iN} = 0$ .

*Lemma 4.1. Passing, if needed, to linear combinations of  $y_{iN}$  with coefficients depending only on  $t_2, t_3, \dots$ , one can always achieve*

$$\partial_k y_{iN} = \partial^k y_{iN}, \tag{4.1}$$

$$y_{iN} = y'_{i,N+1}, \quad i = 0, \dots, N-1. \tag{4.2}$$

(The relation (4.4) is what makes this chain different from the one considered in Section 3 (modified KP) where it was  $y_{iN} = y_{i,N+1}$  instead.)

*Proof.* Suppose  $y_{iN}$ , where  $i = 0, \dots, N-1$ , are already constructed. Let  $y_{0,N+1}, \dots, y_{N,N+1}$  be a basis of the kernel of the operator  $P_{N+1}$  such that  $P_{N+1} y_{i,N+1} = 0$ . The functions  $y'_{0,N+1}, \dots, y'_{N,N+1}$  which belong to the kernel of  $(\partial + u_N)P_N$  are linearly independent, otherwise there would be a linear combination of  $y_{i,N+1}$  belonging to the kernel of  $P_{N+1}$  which is constant (with respect to  $t_1 = x$ ) while we know that  $P_{N+1} 1 = w_{N+1,N+1} \neq 0$  by assumption. Hence, at least one of these functions does not belong to  $\ker P_N$ , let it be  $y'_{N,N+1}$  such that  $P_N y'_{N,N+1} \neq 0$ . Since all  $P_N y'_{i,N+1}$  belong to the 1-dimensional kernel of  $\partial + u_N$ , there must be constants  $\alpha_i$  such that  $P_N (y'_{i,N+1} - \alpha_i y'_{N,N+1}) = 0$ . (When we speak about constants, we mean constants with respect to  $t_1 = x$  depending, maybe, on higher times.) Thus  $(y_{i,N+1} - \alpha_i y_{N,N+1})'$  form a basis of the kernel of  $P_N$ . There exist their linear combinations  $(y_{i,N+1}^{(1)})'$  coinciding with  $y_{iN}$ :  $(y_{i,N+1}^{(1)})' = y_{iN}$  where  $i = 0, \dots, N-1$ . This yields  $\partial(\partial^k - \partial_k) y_{i,N+1}^{(1)} = 0$  and  $(\partial^k - \partial_k) y_{i,N+1}^{(1)} = c_{ki} = \text{constant}$ . As in Lemma 2.2, we can prove that  $(\partial^k - \partial_k) y_{i,N+1}^{(1)} \in \ker P_{N+1}$ ,

therefore  $c_{ki} = 0$  since constants do not belong to the kernel. It remains to consider  $y_{N,N+1}$ . Since  $(\partial^k - \partial_k)y_{N,N+1} = \sum A_i y_{i,N+1}$ , the same reasoning as in Lemma 2.2 will do the rest.  $\square$

**Proposition 4.2.** *All solutions to the chain (3.2), (3.3), (3.4), and (3.5) have the following structure. Let  $y_{iN}$ , where  $N = 1, 2, \dots$  and  $i = 0, \dots, N-1$ , be arbitrary functions of variables  $t_1 = x, t_2, \dots$  satisfying the relations*

$$\partial_k y_{iN} = \partial^k y_{iN}, \tag{4.3}$$

$$y_{iN} = y'_{i,N+1}, \quad i = 0, \dots, N-1. \tag{4.4}$$

Then

$$\widehat{w}_N = \frac{1}{W_N} \begin{vmatrix} y_{0N} & \cdots & y_{N-1,N} & 1 \\ y'_{0N} & \cdots & y'_{N-1,N} & \partial \\ \vdots & \vdots & \vdots & \vdots \\ y_{0N}^{(N)} & \cdots & y_{N-1,N}^{(N)} & \partial^N \end{vmatrix} \cdot \partial^{-N}, \tag{4.5}$$

where  $W_N = W_N(y_{0N}, \dots, y_{N-1,N})$  is the Wronskian. Besides,

$$u_N = -\partial \ln \frac{1}{W_N} \begin{vmatrix} y_{0N} & \cdots & y_{N-1,N} & y_{N,N} \\ y'_{0N} & \cdots & y'_{N-1,N} & y'_{N,N} \\ \vdots & \vdots & \vdots & \vdots \\ y_{0N}^{(N)} & \cdots & y_{N-1,N}^{(N)} & y_{N,N}^{(N)} \end{vmatrix}, \tag{4.6}$$

where  $y_{N,N} = y'_{N,N+1}$  by definition and

$$v_{N+1} = -\partial \ln \frac{1}{W_{N+1}} \begin{vmatrix} y_{0N} & \cdots & y_{N-1,N} & y_{N,N} \\ y'_{0N} & \cdots & y'_{N-1,N} & y'_{N,N} \\ \vdots & \vdots & \vdots & \vdots \\ y_{0N}^{(N)} & \cdots & y_{N-1,N}^{(N)} & y_{N,N}^{(N)} \end{vmatrix}. \tag{4.7}$$

*Proof.* In one way, the proposition follows from Lemma 4.1. Indeed,  $P_N = \widehat{w}_N \partial^N$  given by (4.5) is the unique differential monic operator having the kernel spanned by  $y_{0N}, \dots, y_{N-1,N}$ , the latter always can be assumed to be satisfying (4.3) and (4.4). Further,  $u_N = -\partial \ln P_N \partial y_{N,N+1}$  and  $v_{N+1,N+1} = -\partial \ln P_{N+1} 1$  which easily yields (4.6) and (4.7).

Conversely, one must prove that if  $\{y_{iN}\}$  have the properties (4.3) and (4.4), then (4.5), (4.6), and (4.7) present a solution to the chain (3.2), (3.3), (3.4), and (3.5).

Indeed, KP equations (3.3) can be obtained in a standard way: differentiating  $\widehat{w}_N \partial^N y_{iN} = 0$  with respect to  $t_k$ , one gets

$$(\partial_k \widehat{w}_N) \partial^N y_{iN} + \widehat{w}_N \partial^N \partial^k y_{iN} = 0 \tag{4.8}$$

or

$$(\partial_k \widehat{w}_N) \partial^N y_{iN} + (L_N^k)_- \widehat{w}_N \partial^N y_{iN} = - (L_N^k)_+ \widehat{w}_N \partial^N y_{iN} \tag{4.9}$$

The operator  $(L_N^k)_- \widehat{w}_N \partial^N$  has an order less than  $N$ . On the other hand, this is a differential operator since it is equal to  $\widehat{w}_N \partial^N - (L_N^k)_+ \widehat{w}_N \partial^N = P_N - (L_N^k)_+ P_N$ . Thus, the differential operator  $(\partial_k \widehat{w}_N) \partial^N + (L_N^k)_- \widehat{w}_N \partial^N$  of order less than  $N$  has an  $N$ -dimensional kernel and must vanish.

The operators  $(\partial + u_N) P_N \partial$  and  $(\partial + v_{N+1}) P_{N+1}$  have the same kernels spanned by  $y_{0,N+1}, \dots, y_{N,N+1}, 1$ , therefore they coincide. We have (3.2).

From  $\partial_k \widehat{w}_N = -\widehat{w}_N \partial^k + (L_N^k)_+ \widehat{w}_N$ , we have  $\partial_k w_{NN} = (L_N^k)_+ w_{NN}$  and

$$\partial_k v_{N+1} = -\partial \left[ \frac{(L_N^k)_+ w_{NN}}{w_{NN}} \right]. \tag{4.10}$$

Finally, applying the operator  $\partial_k$  to (3.2) and equating terms of zero degree in  $\partial$ , we obtain  $\partial_k u_N - \text{res} L_N^k = \partial_k v_{N+1} - \text{res} L_{N+1}^k$ . The last two equations are equivalent to (3.4) and (3.5) which completes the proof.  $\square$

Tau functions are  $\tau_N = W_N$ .

### 4.2. Solutions in the form of series in Schur polynomials: stabilization

Recall that the Schur polynomials are defined by

$$\exp \left( \sum_1^\infty t_k z^k \right) = \sum_0^\infty p_k(t) z^k. \tag{4.11}$$

A grading can be introduced being prescribed that the variable  $t_i$  has the weight  $i$ , the weight of  $z$  is  $-1$ . Then the polynomial  $p_k(t)$  is of weight  $k$ . It is easy to verify that the Schur polynomials have the properties

$$\partial_i p_k = p_{k-i} = \partial^i p_k, \tag{4.12}$$

$$p_k(t) - \zeta^{-1} \partial p_k(t) = p_k \left( t_1 - \frac{\zeta^{-1}}{1}, \dots, t_r - \frac{\zeta^{-r}}{r}, \dots \right),$$

which can be obtained from definition (4.11).

Let

$$y_{iN} = \sum_{m=0}^\infty c_m^{(i)} p_{m+N-i}, \quad i = 0, \dots, N-1 \tag{4.13}$$

with some coefficients  $c_m^{(i)}$  being  $c_0^{(i)} = 1$ . Then (4.3) and (4.4) of Proposition 4.2 hold and (4.5), (4.6), and (4.7) define solutions for any sets of coefficients  $c_m^{(i)}$ . In particular, up to a nonimportant sign,

$$\tau_N = \begin{vmatrix} y_{0N}^{(N-1)} & \cdots & y_{0N} \\ y_{1N}^{(N-1)} & \cdots & y_{1N} \\ \vdots & \vdots & \vdots \\ y_{N-1,N}^{(N-1)} & \cdots & y_{N-1,N} \end{vmatrix}. \tag{4.14}$$

Equation (4.13) implies

$$\tau_N = \sum_{m_0, \dots, m_{N-1}} c_{m_0}^{(0)} \cdots c_{m_{N-1}}^{(N-1)} \begin{vmatrix} p_{m_0} & \cdots & p_{m_0+N-1} \\ p_{m_1-1} & \cdots & p_{m_1+N-2} \\ \vdots & \vdots & \vdots \\ p_{m_{N-1}-(N-1)} & \cdots & p_{m_{N-1}} \end{vmatrix} \tag{4.15}$$

(it is assumed that  $p$  with negative subscripts vanish).

**Proposition 4.3 (Itzykson and Zuber).** *The tau functions (4.15) have the stabilization property: terms of a weight  $l$  do not depend on  $N$  when  $N > l$ .*

*Proof.* The diagonal terms of the determinant are  $p_{m_0}, p_{m_1}, \dots, p_{m_{N-1}}$ . All the terms of the determinant are of equal weights, namely,  $m_0 + m_1 + \dots + m_{N-1} = l$ . We consider a determinant of weight  $l$  and prove that all  $m_i$  with  $i \geq l$  vanish unless the determinant vanishes. Suppose that there is some  $i \geq l$  such that  $m_i \neq 0$ . The elements of the determinant which are located in the  $i$ th column above  $p_{m_i}$  have the following subscripts:  $m_0 + i, m_1 + i - 1, \dots, m_{i-1} + 1$ . Together with  $m_i$ , there are  $i + 1$  nonzero integers with a sum

$$m_0 + m_1 + \dots + m_{i-1} + m_i + \sum_{j=1}^i j \leq l + \sum_{j=1}^i j < i + 1 + \sum_{j=1}^i j = \sum_{j=1}^{i+1} j, \tag{4.16}$$

that is, less than the sum of the first  $i + 1$  integers. This implies that at least two of them coincide. Then the corresponding rows coincide, and the determinant vanishes.

Thus, if a determinant does not vanish, then, starting from the  $l$ th row, all the diagonal elements are equal to  $p_0 = 1$ , and all the elements to the left of the diagonal vanish. The determinant of weight  $l$  reduces to a minor of  $l$ th order in the upper left corner, and the terms of weight  $l$  are

$$\sum_{m_0 + \dots + m_{l-1} = l} c_{m_0}^{(0)} \cdots c_{m_{l-1}}^{(l-1)} \begin{vmatrix} p_{m_0} & \cdots & p_{m_0+l-1} \\ p_{m_1-1} & \cdots & p_{m_1+l-2} \\ \vdots & \vdots & \vdots \\ p_{m_{l-1}-(l-1)} & \cdots & p_{m_{l-1}} \end{vmatrix} \tag{4.17}$$

that does not depend on  $N$ . □

## Appendix

### From the stabilizing chain to the Kontsevich integral

#### A.1.

It is well known that the so-called Kontsevich integral which originates in quantum field theory is a tau function of the type (4.15) with some special coefficients  $c_m^{(i)}$ . We briefly sketch here the way from the general solution (4.15) to the Kontsevich integral if two additional requirements are imposed: the stable limit of  $\tau_N$  must belong to an  $n$ th restriction of KP hierarchy (the  $n$ th GD) and satisfy the string equation. All the skipped details of calculations can be found in Itzykson and Zuber [7] on which we base our presentation. The only difference is that we do this in reverse order: not from matrix integrals to tau functions (4.15) but vice versa. It is interesting to see what kind of reasoning and motivation could lead one from integrable systems to matrix integrals of the type studied by Kontsevich and to make sure that nothing essential is lost.

First, we need to make the stable limit of  $\tau_N$  belonging to the  $n$ th restriction of KP which is equivalent to the independence of the  $n$ th time,  $t_n$ .

Usually, when one wishes to make a tau function (4.14) independent of  $t_n$ , one requires that  $\partial_n y_{iN} = \alpha_{iN} y_{iN}$  where  $\alpha_{iN}$  are some numbers. Then  $y_{iN} = \exp(\alpha_{iN} t_n) y_{iN}(0)$  where  $y_{iN}(0)$  does not depend on  $t_n$ , and  $\tau_N = \exp(\alpha_{0N} + \dots + \alpha_{N_1, N}) t_n \cdot \tau_N(0)$  where  $\tau_N(0)$  does not depend on  $t_n$ . The exponential factor can be dropped since a tau function is determined up to a multiplication by an exponential of any linear combination of the time variables.

Now, in the problem we are talking about, we do not necessarily wish to make all  $\tau_N$  independent of  $t_n$ , only their stable limit. Then it suffices instead of the "horizontal quasi-periodicity"  $\partial_n y_{iN} = y_{iN}^{(n)} = \alpha_{iN} y_{iN}$  to require the "vertical periodicity"  $\partial_n y_{iN} = y_{i+n, N}$ . In terms of the series (4.13), this means that

$$c_m^{(i+n)} = c_m^{(i)}. \quad (\text{A.1})$$

The proof is in [7]. The idea is clear: when a row which is not one of the  $n$  last rows is differentiated, then the resulting determinant has two equal rows. If we consider only terms of a fixed weight, then they depend only on a minor of a fixed size in the upper left corner for all  $N$  large enough, and these term vanish though the whole determinant does not.

#### A.2.

Considering the determinant in (4.15), we can recognize primitive characters of the group  $GL(n)$  or  $U(n)$  where

$$t_i = \sum_k \frac{1}{i} \epsilon_k^i \tag{A.2}$$

and  $\epsilon_k$  are the eigenvalues of a matrix for which this character is evaluated (see [11]); it is supposed that  $m_0 \geq m_1 \geq \dots \geq m_{N-1}$ . The latter can always be achieved by a permutation and relabeling of the indices.

It is not easy to understand why  $\tau$ -functions happen to be related to characters (a good explanation of this fact can lead to new profound theories). However, we can extract lessons from this relationship. First of all, the  $\tau$ -function is given as an expansion in a series in characters. Hence it can be considered as a function on the unitary or the general linear group invariant with respect to the conjugation. In the end we will have an explicit formula giving this function which is the Kontsevich integral.

Secondly, the benefit of the usage of variables  $\epsilon_k$  instead of  $t_i$  is obvious: the elaborated techniques of the theory of characters can be applied to the  $\tau$ -function as well. We will use also an inverse matrix with the eigenvalues  $\lambda_i = \epsilon_i^{-1}$ . Thus, we have

$$t_i = \sum_k \frac{1}{i} \lambda_k^{-i}. \tag{A.3}$$

This change of variables is called the *Miwa transformation*.

It is easy to show that

$$p_l(t) = \sum_{l_1 + \dots + l_N = l} \epsilon_1^{l_1} \epsilon_2^{l_2} \dots \epsilon_N^{l_N}. \tag{A.4}$$

This is the Newton formula expressing complete symmetric functions of variables  $\epsilon_k$  in terms of sums of their powers,  $t_i$ .

Introduce a notation

$$|g_0(\lambda), \dots, g_{N-1}(\lambda)| = \det(g_j(\lambda_i)). \tag{A.5}$$

Then it can be proven that

$$\tau_N = \frac{|f_0(\lambda), f_1(\lambda)\lambda, f_2(\lambda)\lambda^2, \dots, f_{N-1}(\lambda)\lambda^{N-1}|}{|1, \lambda, \lambda^2, \dots, \lambda^{N-1}|}, \tag{A.6}$$

where  $f_i(\lambda) = \sum_0^\infty c_m^{(i)} \lambda^{-m}$  being  $f_i = f_{i+n}$ .

**A.3.**

Up to this point there were no restrictions imposed on the coefficients  $c_m^{(k)}$  except the periodicity (A.1). Now we try to satisfy the string equation (cf. [3, 5, 10]). The string equation is closely related to the so-called additional symmetries of the KP hierarchy (each equation of the hierarchy provides

a symmetry for all other equations, they are the main symmetries, while an additional symmetry is not contained in the hierarchy itself). The string equation is equivalent to the fact that  $\tau$  does not depend on the additional variable  $t_{-n+1,1}^*$ . However, it is known that the derivative  $\partial_{-n+1,1}^*$  is defined not uniquely: there is still a possibility for gauge transformations and for a shift of variables  $t_i \mapsto t_i + \alpha_i$ .

First of all, the operator  $\partial_{-n+1,1}^*$  acts on  $\tau$ , as the operator

$$W_{-n}^{(2)} = \sum_{i+j=n} ij t_i t_j - 2 \sum_1^\infty t_{i+n} \partial_i + (n-1) n t_n + C. \tag{A.7}$$

This has to be expressed in terms of new variables,  $\lambda$ 's or  $\epsilon$ 's. The last two terms are not important: an arbitrary linear term in  $t_i$  and a constant can be added, this is precisely a gauge transformation. The result is

$$W_{-n}^{(2)} = \sum_{i,j>0; i+j=n} \sum_r \frac{1}{\lambda_r^i} \sum_s \frac{1}{\lambda_s^j} - 2 \sum_k \frac{1}{\lambda_k^{n-1}} \frac{\partial}{\partial \lambda_k}. \tag{A.8}$$

If a possibility of a shift  $t_i \mapsto t_i + \alpha_i$  is taking into account, then

$$\tau_N = \frac{(|f_0(\lambda), f_1(\lambda)\lambda, f_2(\lambda)\lambda^2, \dots, f_{N-1}(\lambda)\lambda^{N-1} | \prod_i \exp \sum_j \alpha_j \lambda_i^j)_{\leq N-1}}{|1, \lambda, \lambda^2, \dots, \lambda^{N-1}|}, \tag{A.9}$$

where the subscript  $\leq N-1$  means taking powers of  $\lambda_i$  not larger than  $N-1$ .

Skipping all calculations, the string equation implies the following recurrent system of equations for  $f_k$ :

$$\lambda^k f_k = D^k f_0, \quad k = 0, 1, \dots, n-1, \tag{A.10}$$

where

$$D = \sum_j \alpha_j j \lambda^{j-n} - \frac{n-1}{2} \frac{1}{\lambda^n} + \frac{1}{\lambda^{n-1}} \frac{\partial}{\partial \lambda}. \tag{A.11}$$

If we recall that  $f_0 = f_n$ , then for the first term we have

$$D^n f_0 = \lambda^n f_0 \tag{A.12}$$

or

$$\left( \sum_j \alpha_j j \lambda^{j-n} - \frac{n-1}{2} \frac{1}{\lambda^n} + \frac{1}{\lambda^{n-1}} \frac{\partial}{\partial \lambda} \right)^n f_0 = \lambda^n f_0, \tag{A.13}$$

which must hold identically in  $\lambda$ , can be satisfied by some  $f_0 = \sum_0^\infty c_m^{(0)} \lambda^{-m}$  if and only if  $\sum_j \alpha_j j \lambda^{j-n} = \lambda$ , that is,  $\sum_j \alpha_j \lambda^j = \lambda^{n+1}/(n+1)$ .



It is possible to perform some scaling transformation  $\lambda \mapsto \alpha\lambda$ ,  $D \mapsto \alpha^{-1}D$ . This is not so important, but just in order that our formulas coincide with those in [7] we take  $\alpha = n^{1/(n+1)}$ . Then

$$\begin{aligned}
 D &= \lambda + \frac{n-1}{2n\lambda^n} + \frac{1}{n\lambda^{n-1}} \frac{\partial}{\partial \lambda} \\
 &= \lambda^{(n-1)/2} \exp\left(-\frac{n}{n+1}\lambda^{n+1}\right) \frac{\partial}{\partial \lambda^n} \exp\left(\frac{n}{n+1}\lambda^{n+1}\right) \lambda^{-(n-1)/2}.
 \end{aligned}
 \tag{A.14}$$

The function  $g_0 = \exp(n\lambda^{n+1}/(n+1))\lambda^{-(n-1)/2}f_0$  satisfies the equation

$$\left(\frac{\partial}{\partial \lambda^n}\right)^n g_0 = \lambda^n g_0;
 \tag{A.15}$$

which, being written in the variable  $\mu = \lambda^n$ , is a generalization of the Airy equation (the latter is a special case with  $n = 2$ ). A solution can be found by the Laplace method. The solution is

$$f_0 = \lambda^{(n-1)/2} \exp\left(-\frac{n}{n+1}\lambda^{n+1}\right) \int \exp\left(\lambda^n m - \frac{m^{n+1}}{n+1}\right) dm.
 \tag{A.16}$$

It is easy to see that

$$D^k f_0 = \lambda^{(n-1)/2} \exp\left(-\frac{n}{n+1}\lambda^{n+1}\right) \int m^k \exp\left(\lambda^n m - \frac{m^{n+1}}{n+1}\right) dm.
 \tag{A.17}$$

The  $\tau$ -function is

$$\begin{aligned}
 \tau_N &= i^{N(N+1)/2} \prod_k \lambda_k^{(n-1)/2} \exp\left(-\sum_k \frac{n}{n+1} \lambda_k^{n+1}\right) \\
 &\quad \times \int \dots \int dm_1 \dots dm_N \prod_{r>s} \frac{m_r - m_s}{\lambda_r - \lambda_s} \exp \sum_k \left( i\lambda_k^n m_k - \frac{(im_k)^{n+1}}{n+1} \right).
 \end{aligned}
 \tag{A.18}$$

We do not discuss here the problems of the choice of the contours of integration and of the convergence.

**A.4.**

In a sense, the problem is already solved, the solution is explicitly written. However, we remember the above remark that it is natural to consider a  $\tau$ -function as a function on a matrix group invariant under conjugation,  $\lambda_k$  being eigenvalues of a matrix, since originally it was written as a series in characters of the group. If we want to restrict ourselves to real values of the time variables  $t_i$ , that is, real  $\lambda_k$ , then the function is restricted to matrices

with real eigenvalues, for example, Hermitian. Our intention now is to write the formula (A.18) more directly in terms of matrices themselves, not of their eigenvalues.

The theory of representations has a tool for that, the Harish-Chandra formula [6]. Let  $\Phi(X)$  be a function of Hermitian matrices  $X$ , invariant under conjugations, that is, depending only on eigenvalues of  $X$ . We consider integrals over the space of Hermitian matrices (h.m.)  $\int_{\text{h.m.}} \Phi(X) \exp(-\text{itr } XY) dX$  where  $Y$  is a Hermitian matrix with eigenvalues  $\mu_k$ , and  $dX = \prod_{i,j} dX_{ij}$ . If  $X = U M U^{-1}$  where  $U$  is unitary and  $M = \text{diag } m_1, \dots, m_N$  then the function can be partly integrated, with respect to the "angle" variables  $U$  and only integral over diagonal matrices  $M$  remains

$$\begin{aligned} & \int_{\text{h.m.}} \Phi(X) \exp(\text{itr } XY) dX \\ &= (2\pi i)^{N(N-1)} \int \dots \int dm_1 \dots dm_N \prod_{r>s} \frac{m_r - m_s}{\mu_r - \mu_s} \Phi(M) \exp\left(i \sum m_k \mu_k\right). \end{aligned} \quad (\text{A.19})$$

Using these techniques, one can show that

$$\tau_N = \text{const} \frac{\int \exp(\text{tr}(-n.l.(Z + \Lambda)^{n+1}/(n+1))) dZ}{\int \exp(\text{tr}(-\text{quad.}(Z + \Lambda)^{n+1}/(n+1))) dZ}, \quad (\text{A.20})$$

where  $n.l.(Z + \Lambda)^{n+1}$  symbolizes all terms of degree higher than 1 in the expansion of  $(Z + \Lambda)^{n+1}$  in powers of  $Z$  while  $\text{quad.}(Z + \Lambda)^{n+1}$  stands for the quadratic term;  $\Lambda$  is a matrix with eigenvalues  $\lambda_k$ .

The expression (A.20) is called the *Kontsevich integral* (more precisely, its generalization from  $n = 2$  to any  $n$ ).

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