

# MATRIX VARIATE KUMMER-DIRICHLET DISTRIBUTIONS

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The multivariate Kummer-Beta and multivariate Kummer-Gamma families of distributions have been proposed and studied recently by Ng and Kotz. These distributions are extensions of Kummer-Beta and Kummer-Gamma distributions. In this article we propose and study matrix variate generalizations of multivariate Kummer-Beta and multivariate Kummer-Gamma families of distributions.

## 1. Introduction

The Kummer-Beta and Kummer-Gamma families of distributions are defined by the density functions

$$\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \left\{ {}_1F_1(\alpha; \alpha + \beta; -\lambda) \right\}^{-1} \exp(-\lambda u) u^{\alpha-1} (1-u)^{\beta-1}, \quad 0 < u < 1, \quad (1.1)$$

$$\left\{ \Gamma(\alpha) \Psi(\alpha, \alpha - \gamma + 1; \xi) \right\}^{-1} \exp(-\xi v) v^{\alpha-1} (1+v)^{-\gamma}, \quad v > 0, \quad (1.2)$$

respectively, where  $\alpha > 0$ ,  $\beta > 0$ ,  $\xi > 0$ ,  $-\infty < \gamma, \lambda < \infty$ ,  ${}_1F_1$ , and  $\Psi$  are confluent hypergeometric functions. These distributions are extensions of Gamma and Beta distributions, and for  $\alpha < 1$  (and certain values of  $\lambda$  and  $\gamma$ ) yield bimodal distributions on finite and infinite ranges, respectively. These distributions are used (i) in the Bayesian analysis of queueing system where posterior distribution of certain basic parameters in  $M/M/\infty$  queueing system is Kummer-Gamma and (ii) in common value auctions where the posterior distribution of "value of a single good" is Kummer-Beta. For properties and applications of these distributions the reader is referred to Ng and Kotz [7], Armero and Bayarri [1], and Gordy [2].

As the corresponding multivariate generalization of these distributions, we have the following  $n$ -dimensional densities:

$$\frac{\Gamma(\sum_{i=1}^n \alpha_i + \beta)}{\prod_{i=1}^n \Gamma(\alpha_i) \Gamma(\beta)} \left\{ {}_1F_1 \left( \sum_{i=1}^n \alpha_i; \sum_{i=1}^n \alpha_i + \beta; -\lambda \right) \right\}^{-1} \exp \left( -\lambda \sum_{i=1}^n u_i \right) \\ \times \prod_{i=1}^n u_i^{\alpha_i - 1} \left( 1 - \sum_{i=1}^n u_i \right)^{\beta - 1}, \quad 0 < u_i < 1, \sum_{i=1}^n u_i < 1, \tag{1.3}$$

where  $\alpha_i > 0, i = 1, \dots, n, \beta > 0, -\infty < \lambda < \infty$ , and

$$\left\{ \Gamma \left( \sum_{i=1}^n \alpha_i \right) \Psi \left( \sum_{i=1}^n \alpha_i, \sum_{i=1}^n \alpha_i - \gamma + 1; \xi \right) \right\}^{-1} \exp \left( -\xi \sum_{i=1}^n v_i \right) \\ \times \prod_{i=1}^n v_i^{\alpha_i - 1} \left( 1 + \sum_{i=1}^n v_i \right)^{-\gamma}, \quad v_i > 0, \tag{1.4}$$

where  $\alpha_i > 0, i = 1, \dots, n, \xi > 0, -\infty < \gamma < \infty$ , respectively. These distributions have been considered by Ng and Kotz [7] who refer to (1.3) and (1.4) as multivariate Kummer-Beta and multivariate Kummer-Gamma distributions, respectively. For  $\lambda = 0$ , (1.1) and (1.3) reduce to Beta and Dirichlet distributions with probability density functions

$$\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} u^{\alpha - 1} (1 - u)^{\beta - 1}, \quad 0 < u < 1, \\ \frac{\Gamma(\sum_{i=1}^n \alpha_i + \beta)}{\prod_{i=1}^n \Gamma(\alpha_i) \Gamma(\beta)} \prod_{i=1}^n u_i^{\alpha_i - 1} \left( 1 - \sum_{i=1}^n u_i \right)^{\beta - 1}, \quad 0 < u_i < 1, \sum_{i=1}^n u_i < 1, \tag{1.5}$$

respectively. Since (1.3) is an extension of Dirichlet distribution and a multivariate generalization of Kummer-Beta distribution, an appropriate nomenclature for this distribution would be *Kummer-Dirichlet distribution*. In the same vein, we may call (1.4) a Kummer-Dirichlet distribution. Further, in order to distinguish between these two distributions ((1.3) and (1.4)), we call them Kummer-Dirichlet type I and Kummer-Dirichlet type II distributions.

In this article we propose and study matrix variate generalizations of (1.3) and (1.4), respectively.

## 2. Matrix variate Kummer-Dirichlet distributions

We begin with a brief review of some definitions and notations. We adhere to standard notations (cf. Gupta and Nagar [3]). Let  $A = (a_{ij})$  be a  $p \times p$  matrix.

Then,  $A'$  denotes the transpose of  $A$ ;  $\text{tr}(A) = a_{11} + \dots + a_{pp}$ ;  $\text{etr}(A) = \exp(\text{tr}(A))$ ;  $\det(A)$  = determinant of  $A$ ;  $A > 0$  means that  $A$  is symmetric positive definite and  $A^{1/2}$  denotes the unique symmetric positive definite square root of  $A > 0$ . The multivariate gamma function  $\Gamma_p(m)$  is defined as

$$\Gamma_p(m) = \pi^{p(p-1)/4} \prod_{j=1}^p \Gamma\left(m - \frac{j-1}{2}\right), \quad \text{Re}(m) > \frac{p-1}{2}, \quad (2.1)$$

where  $\text{Re}(\cdot)$  denotes the real part of  $(\cdot)$ . It is straightforward to show that

$$\Gamma_p(m) = \int_{R>0} \det(R)^{m-(p+1)/2} \text{etr}(-R) dR, \quad \text{Re}(m) > \frac{p-1}{2}, \quad (2.2)$$

where the integral has been evaluated over the space of the  $p \times p$  symmetric positive definite matrices. The integral representation of the confluent hypergeometric function  ${}_1F_1$  is given by

$$\begin{aligned} {}_1F_1(a; b; X) &= \frac{\Gamma_p(b)}{\Gamma_p(a)\Gamma_p(b-a)} \\ &\times \int_{0 < R < I_p} \det(R)^{a-(p+1)/2} \det(I_p - R)^{b-a-(p+1)/2} \text{etr}(XR) dR, \end{aligned} \quad (2.3)$$

where  $\text{Re}(a) > (p-1)/2$  and  $\text{Re}(b-a) > (p-1)/2$ . The confluent hypergeometric function  $\Psi$  of a  $p \times p$  symmetric matrix  $X$  is defined by

$$\begin{aligned} \Psi(a, c; X) &= \frac{1}{\Gamma_p(a)} \\ &\times \int_{R>0} \text{etr}(-XR) \det(R)^{a-(p+1)/2} \det(I_p + R)^{c-a-(p+1)/2} dR, \end{aligned} \quad (2.4)$$

where  $\text{Re}(X) > 0$  and  $\text{Re}(a) > (p-1)/2$ .

Now we define the corresponding matrix variate generalizations of (1.3) and (1.4) as follows.

*Definition 2.1.* The  $p \times p$  symmetric positive definite random matrices  $U_1, \dots, U_n$  are said to have the matrix variate Kummer-Dirichlet type I distribution with parameters  $\alpha_1, \dots, \alpha_n, \beta$  and  $\Lambda$ , denoted by  $(U_1, \dots, U_n) \sim \text{KD}_p^I(\alpha_1, \dots, \alpha_n, \beta, \Lambda)$ , if their joint probability density function (pdf) is given by

$$\begin{aligned}
 & K_1(\alpha_1, \dots, \alpha_n, \beta, \Lambda) \operatorname{etr} \left( -\Lambda \sum_{i=1}^n U_i \right) \\
 & \times \prod_{i=1}^n \det(U_i)^{\alpha_i - (p+1)/2} \det \left( I_p - \sum_{i=1}^n U_i \right)^{\beta - (p+1)/2}, \quad (2.5) \\
 & 0 < U_i < I_p, \quad 0 < \sum_{i=1}^n U_i < I_p,
 \end{aligned}$$

where  $\alpha_i > (p-1)/2$ ,  $i = 1, \dots, n$ ,  $\beta > (p-1)/2$ ,  $\Lambda(p \times p)$  is symmetric and  $K_1(\alpha_1, \dots, \alpha_n, \beta, \Lambda)$  is the normalizing constant.

*Definition 2.2.* The  $p \times p$  symmetric positive definite random matrices  $V_1, \dots, V_n$  are said to have the matrix variate Kummer-Dirichlet type II distribution with parameters  $\alpha_1, \dots, \alpha_n$ ,  $\gamma$  and  $\Xi$ , denoted by  $(V_1, \dots, V_n) \sim \text{KD}_p^{\text{II}}(\alpha_1, \dots, \alpha_n, \gamma, \Xi)$ , if their joint pdf is given by

$$\begin{aligned}
 & K_2(\alpha_1, \dots, \alpha_n, \gamma, \Xi) \operatorname{etr} \left( -\Xi \sum_{i=1}^n V_i \right) \\
 & \times \prod_{i=1}^n \det(V_i)^{\alpha_i - (p+1)/2} \det \left( I_p + \sum_{i=1}^n V_i \right)^{-\gamma}, \quad V_i > 0, \quad (2.6)
 \end{aligned}$$

where  $\alpha_i > (p-1)/2$ ,  $i = 1, \dots, n$ ,  $-\infty < \gamma < \infty$ ,  $\Xi(p \times p) > 0$ , and  $K_2(\alpha_1, \dots, \alpha_n, \gamma, \Xi)$  is the normalizing constant.

The normalizing constants in (2.5) and (2.6) are given as

$$\begin{aligned}
 & \{K_1(\alpha_1, \dots, \alpha_n, \beta, \Lambda)\}^{-1} \\
 & = \int \dots \int_{\substack{0 < \sum_{i=1}^n U_i < I_p \\ U_i > 0}} \operatorname{etr} \left( -\Lambda \sum_{i=1}^n U_i \right) \\
 & \quad \times \prod_{i=1}^n \det(U_i)^{\alpha_i - (p+1)/2} \det \left( I_p - \sum_{i=1}^n U_i \right)^{\beta - (p+1)/2} \prod_{i=1}^n dU_i \\
 & = \frac{\prod_{i=1}^n \Gamma_p(\alpha_i)}{\Gamma_p(\sum_{i=1}^n \alpha_i)} \int_{0 < U < I_p} \operatorname{etr}(-\Lambda U) \det(U)^{\sum_{i=1}^n \alpha_i - (p+1)/2} \\
 & \quad \times \det(I_p - U)^{\beta - (p+1)/2} dU
 \end{aligned}$$

$$= \frac{\prod_{i=1}^n \Gamma_p(\alpha_i) \Gamma_p(\beta)}{\Gamma_p(\sum_{i=1}^n \alpha_i + \beta)} {}_1F_1\left(\sum_{i=1}^n \alpha_i; \sum_{i=1}^n \alpha_i + \beta; -\Lambda\right), \tag{2.7}$$

$$\begin{aligned} & \{\mathcal{K}_2(\alpha_1, \dots, \alpha_n, \gamma, \Xi)\}^{-1} \\ &= \int_{V_1 > 0} \cdots \int_{V_n > 0} \text{etr}\left(-\Xi \sum_{i=1}^n V_i\right) \\ & \quad \times \prod_{i=1}^n \det(V_i)^{\alpha_i - (p+1)/2} \det\left(I_p + \sum_{i=1}^n V_i\right)^{-\gamma} \prod_{i=1}^n dV_i \\ &= \frac{\prod_{i=1}^n \Gamma_p(\alpha_i)}{\Gamma_p(\sum_{i=1}^n \alpha_i)} \int_{V > 0} \text{etr}(-\Xi V) \det(V)^{\sum_{i=1}^n \alpha_i - (p+1)/2} \det(I_p + V)^{-\gamma} dV \\ &= \prod_{i=1}^n \Gamma_p(\alpha_i) \Psi\left(\sum_{i=1}^n \alpha_i, \sum_{i=1}^n \alpha_i - \gamma + \frac{p+1}{2}; \Xi\right), \end{aligned} \tag{2.8}$$

respectively, where  ${}_1F_1$  and  $\Psi$  are confluent hypergeometric functions of matrix argument.

For  $\Lambda = 0$ , the matrix variate Kummer-Dirichlet type I distribution collapses to an ordinary matrix variate Dirichlet type I distribution with pdf

$$\begin{aligned} & \frac{\Gamma_p(\sum_{i=1}^n \alpha_i + \beta)}{\prod_{i=1}^n \Gamma_p(\alpha_i) \Gamma_p(\beta)} \prod_{i=1}^n \det(U_i)^{\alpha_i - (p+1)/2} \det\left(I_p - \sum_{i=1}^n U_i\right)^{\beta - (p+1)/2}, \\ & \quad 0 < U_i < I_p, \quad 0 < \sum_{i=1}^n U_i < I_p, \end{aligned} \tag{2.9}$$

where  $\alpha_i > (p - 1)/2$ ,  $i = 1, \dots, n$ , and  $\beta > (p - 1)/2$ . A common notation to designate that  $(U_1, \dots, U_n)$  has this density is  $(U_1, \dots, U_n) \sim D_p^I(\alpha_1, \dots, \alpha_n; \beta)$ . For  $\gamma = 0$ , the matrix variate Kummer-Dirichlet type II density simplifies to the product of  $n$  matrix variate Gamma densities.

For  $p = 1$ , the densities in (2.5) and (2.6) simplify to Kummer-Dirichlet type I (multivariate Kummer-Beta) and Kummer-Dirichlet type II (multivariate Kummer-Gamma) densities, respectively. For  $n = 1$ , the matrix variate Kummer-Dirichlet type I and matrix variate Kummer-Dirichlet type II distributions reduce to the matrix variate Kummer-Beta and matrix variate Kummer-Gamma distributions, respectively. These two distributions have been studied by Nagar and Gupta [6] and Nagar and Cardeño [5]. Substituting

$n = 1$  in (2.5) and (2.6), the matrix variate Kummer-Beta and matrix variate Kummer-Gamma densities are obtained as

$$K_1(\alpha, \beta, \Lambda) \text{etr}(-\Lambda U) \det(U)^{\alpha-(p+1)/2} \times \det(I_p - U)^{\beta-(p+1)/2}, \quad 0 < U < I_p, \tag{2.10}$$

$$K_2(\alpha, \gamma, \Xi) \text{etr}(-\Xi V) \det(V)^{\alpha-(p+1)/2} \det(I_p + V)^{-\gamma}, \quad V > 0,$$

respectively, where  $\alpha > (p-1)/2$ ,  $\beta > (p-1)/2$ ,  $-\infty < \gamma < \infty$ ,  $\Lambda = \Lambda'$ , and  $\Xi(p \times p) > 0$ . These two distributions are designated by  $U \sim KB_p(\alpha, \beta, \Lambda)$  and  $V \sim KG_p(\alpha, \gamma, \Xi)$ . It may be noted that the matrix variate Kummer-Dirichlet distributions are special cases of the matrix variate Liouville distribution.

Using certain transformations, generalized matrix variate Kummer-Dirichlet distributions are generated as given in the next two theorems.

**Theorem 2.3.** *Let  $(U_1, \dots, U_n) \sim KD_p^I(\alpha_1, \dots, \alpha_n, \beta, \Lambda)$  and  $\Psi_1, \dots, \Psi_n, \Omega$  be symmetric matrices such that  $\Omega > 0$  and  $\Omega - \sum_{i=1}^n \Psi_i > 0$ . Define*

$$Z_i = \left( \Omega - \sum_{i=1}^n \Psi_i \right)^{1/2} U_i \left( \Omega - \sum_{i=1}^n \Psi_i \right)^{1/2} + \Psi_i, \quad i = 1, \dots, n. \tag{2.11}$$

*Then  $(Z_1, \dots, Z_n)$  have the generalized matrix variate Kummer-Dirichlet type I distribution with pdf*

$$\frac{K_1(\alpha_1, \dots, \alpha_n, \beta, \Lambda)}{\det(\Omega - \sum_{i=1}^n \Psi_i)^{\sum_{i=1}^n \alpha_i + \beta - (p+1)/2}} \times \frac{\prod_{i=1}^n \det(Z_i - \Psi_i)^{\alpha_i - (p+1)/2} \det(\Omega - \sum_{i=1}^n Z_i)^{\beta - (p+1)/2}}{\text{etr}\{(\Omega - \sum_{i=1}^n \Psi_i)^{-1/2} \Lambda (\Omega - \sum_{i=1}^n \Psi_i)^{-1/2} \sum_{i=1}^n (Z_i - \Psi_i)\}},$$

$$\Psi_i < Z_i < \Omega, \quad i = 1, \dots, n, \quad \sum_{i=1}^n Z_i < \Omega. \tag{2.12}$$

*Proof.* Making the transformation  $U_i = (\Omega - \sum_{i=1}^n \Psi_i)^{-1/2} (Z_i - \Psi_i) (\Omega - \sum_{i=1}^n \Psi_i)^{-1/2}$ ,  $i = 1, \dots, n$ , with Jacobian  $J(U_1, \dots, U_n \rightarrow Z_1, \dots, Z_n) = \det(\Omega - \sum_{i=1}^n \Psi_i)^{-n(p+1)/2}$  in (2.5), we get (2.12). □

If  $(Z_1, \dots, Z_n)$  has the pdf (2.12), then we write  $(Z_1, \dots, Z_n) \sim GKD_p^I(\alpha_1, \dots, \alpha_n, \beta, \Lambda; \Omega; \Psi_1, \dots, \Psi_n)$ . Note that  $GKD_p^I(\alpha_1, \dots, \alpha_n, \beta, \Lambda; I_p; 0, \dots, 0) \equiv KD_p^I(\alpha_1, \dots, \alpha_n, \beta, \Lambda)$ .

**Theorem 2.4.** Let  $(V_1, \dots, V_n) \sim \text{KD}_p^{\text{II}}(\alpha_1, \dots, \alpha_n, \gamma, \Xi)$  and  $\Psi_1, \dots, \Psi_n, \Omega$  be symmetric matrices such that  $\Omega > 0$  and  $\Omega + \sum_{i=1}^n \Psi_i > 0$ . Define

$$Y_i = \left( \Omega + \sum_{i=1}^n \Psi_i \right)^{1/2} V_i \left( \Omega + \sum_{i=1}^n \Psi_i \right)^{1/2} + \Psi_i, \quad i = 1, \dots, n. \quad (2.13)$$

Then,  $(Y_1, \dots, Y_n)$  have the generalized matrix variate Kummer-Dirichlet type II distribution with pdf

$$\frac{K_2(\alpha_1, \dots, \alpha_n, \gamma, \Xi)}{\det(\Omega + \sum_{i=1}^n \Psi_i)^{\sum_{i=1}^n \alpha_i - \gamma}} \times \frac{\prod_{i=1}^n \det(Y_i - \Psi_i)^{\alpha_i - (p+1)/2} \det(\Omega + \sum_{i=1}^n Y_i)^{-\gamma}}{\text{etr}\{(\Omega + \sum_{i=1}^n \Psi_i)^{-1/2} \Xi (\Omega + \sum_{i=1}^n \Psi_i)^{-1/2} \sum_{i=1}^n (Y_i - \Psi_i)\}},$$

$$Y_i > \Psi_i, \quad i = 1, \dots, n. \quad (2.14)$$

*Proof.* Making the transformation  $V_i = (\Omega + \sum_{i=1}^n \Psi_i)^{-1/2} (Y_i - \Psi_i) (\Omega + \sum_{i=1}^n \Psi_i)^{-1/2}$ ,  $i = 1, \dots, n$ , with the Jacobian  $J(V_1, \dots, V_n \rightarrow Y_1, \dots, Y_n) = \det(\Omega + \sum_{i=1}^n \Psi_i)^{-n(p+1)/2}$  in (2.6), we get (2.14).  $\square$

If  $(Y_1, \dots, Y_n)$  has pdf (2.14), then we write  $(Y_1, \dots, Y_n) \sim \text{GKD}_p^{\text{II}}(\alpha_1, \dots, \alpha_n, \gamma, \Xi; \Omega, \Psi_1, \dots, \Psi_n)$ . In this case  $\text{GKD}_p^{\text{II}}(\alpha_1, \dots, \alpha_n, \gamma; I_p; 0, \dots, 0) \equiv \text{KD}_p^{\text{II}}(\alpha_1, \dots, \alpha_n; \gamma, \Xi)$ .

### 3. Properties

In this section, we study certain properties of matrix variate Kummer-Dirichlet type I and II distributions. It may be noted that for  $\Lambda = \lambda I_p$ ,  $\Xi = \xi I_p$  densities (2.5) and (2.6) are orthogonally invariant. That is, for any fixed orthogonal matrix  $\Gamma(p \times p)$ , the distribution of  $(\Gamma U_1 \Gamma', \dots, \Gamma U_n \Gamma')$  is the same as the distribution of  $(U_1, \dots, U_n)$ , and similarly the distribution of  $(\Gamma V_1 \Gamma', \dots, \Gamma V_n \Gamma')$  is the same as that of  $(V_1, \dots, V_n)$ . Our next two results give marginal and conditional distributions.

**Theorem 3.1.** If  $(U_1, \dots, U_n) \sim \text{KD}_p^{\text{I}}(\alpha_1, \dots, \alpha_n, \beta, \Lambda)$ , then the joint marginal pdf of  $U_1, \dots, U_m$ ,  $m \leq n$ , is given by

$$\begin{aligned}
 & K_1 \left( \alpha_1, \dots, \alpha_m, \sum_{i=m+1}^n \alpha_i + \beta, \Lambda \right) \text{etr} \left( -\Lambda \sum_{i=1}^m U_i \right) \\
 & \times \prod_{i=1}^m \det(U_i)^{\alpha_i - (p+1)/2} \det \left( I_p - \sum_{i=1}^m U_i \right)^{\sum_{i=m+1}^n \alpha_i + \beta - (p+1)/2} \\
 & \times {}_1F_1 \left( \sum_{i=m+1}^n \alpha_i; \sum_{i=m+1}^n \alpha_i + \beta; -\Lambda \left( I_p - \sum_{i=1}^m U_i \right) \right), \\
 & 0 < U_i < I_p, 0 < \sum_{i=1}^m U_i < I_p,
 \end{aligned} \tag{3.1}$$

and the conditional density of  $(U_{m+1}, \dots, U_n) \mid (U_1, \dots, U_m)$  is given by

$$\begin{aligned}
 & \frac{K_1(\alpha_1, \dots, \alpha_n, \beta, \Lambda)}{K_1(\alpha_1, \dots, \alpha_m, \sum_{i=m+1}^n \alpha_i + \beta, \Lambda)} \\
 & \times \frac{\text{etr} \left( -\Lambda \sum_{i=m+1}^n U_i \right)}{\det \left( I_p - \sum_{i=1}^m U_i \right)^{\sum_{i=m+1}^n \alpha_i + \beta - (p+1)/2}} \\
 & \times \frac{\prod_{i=m+1}^n \det(U_i)^{\alpha_i - (p+1)/2} \det \left( I_p - \sum_{i=1}^m U_i - \sum_{i=m+1}^n U_i \right)^{\beta - (p+1)/2}}{{}_1F_1 \left( \sum_{i=m+1}^n \alpha_i; \sum_{i=m+1}^n \alpha_i + \beta; -\Lambda \left( I_p - \sum_{i=1}^m U_i \right) \right)}, \\
 & 0 < U_i < I_p - \sum_{i=1}^m U_i, i = m+1, \dots, n, \sum_{i=m+1}^n U_i < I_p - \sum_{i=1}^m U_i.
 \end{aligned} \tag{3.2}$$

*Proof.* First we find the marginal density of  $U_1, \dots, U_{n-1}$  by integrating out  $U_n$  from the joint density of  $U_1, \dots, U_n$  as

$$\begin{aligned}
 & K_1(\alpha_1, \dots, \alpha_n, \beta, \Lambda) \int_{0 < U_n < I_p - \sum_{i=1}^{n-1} U_i} \text{etr} \left( -\Lambda \sum_{i=1}^n U_i \right) \\
 & \times \prod_{i=1}^n \det(U_i)^{\alpha_i - (p+1)/2} \det \left( I_p - \sum_{i=1}^n U_i \right)^{\beta - (p+1)/2} dU_n.
 \end{aligned} \tag{3.3}$$



Now, substituting  $Z_n = (I_p - \sum_{i=1}^{n-1} U_i)^{-1/2} U_n (I_p - \sum_{i=1}^{n-1} U_i)^{-1/2}$  with Jacobian  $J(U_n \rightarrow Z_n) = \det(I_p - \sum_{i=1}^{n-1} U_i)^{(p+1)/2}$  in (3.2), we get

$$\begin{aligned} & K_1(\alpha_1, \dots, \alpha_n, \beta, \Lambda) \operatorname{etr} \left( -\Lambda \sum_{i=1}^{n-1} U_i \right) \\ & \times \prod_{i=1}^{n-1} \det(U_i)^{\alpha_i - (p+1)/2} \det \left( I_p - \sum_{i=1}^{n-1} U_i \right)^{\alpha_n + \beta - (p+1)/2} \\ & \times \int_{0 < Z_n < I_p} \operatorname{etr} \left[ - \left( I_p - \sum_{i=1}^{n-1} U_i \right)^{1/2} \Lambda \left( I_p - \sum_{i=1}^{n-1} U_i \right)^{1/2} Z_n \right] \\ & \times \det(Z_n)^{\alpha_n - (p+1)/2} \det(I_p - Z_n)^{\beta - (p+1)/2} dZ_n. \end{aligned} \tag{3.4}$$

But

$$\begin{aligned} & K_1(\alpha_1, \dots, \alpha_n, \beta, \Lambda) \\ & \times \int_{0 < Z_n < I_p} \operatorname{etr} \left[ - \left( I_p - \sum_{i=1}^{n-1} U_i \right)^{1/2} \Lambda \left( I_p - \sum_{i=1}^{n-1} U_i \right)^{1/2} Z_n \right] \\ & \times \det(Z_n)^{\alpha_n - (p+1)/2} \det(I_p - Z_n)^{\beta - (p+1)/2} dZ_n \\ & = K_1(\alpha_1, \dots, \alpha_n, \beta, \Lambda) \frac{\Gamma_p(\alpha_n) \Gamma_p(\beta)}{\Gamma_p(\alpha_n + \beta)} {}_1F_1 \left( \alpha_n; \alpha_n + \beta; -\Lambda \left( I_p - \sum_{i=1}^{n-1} U_i \right) \right) \\ & = K_1(\alpha_1, \dots, \alpha_{n-1}, \alpha_n + \beta, \Lambda) {}_1F_1 \left( \alpha_n; \alpha_n + \beta; -\Lambda \left( I_p - \sum_{i=1}^{n-1} U_i \right) \right). \end{aligned} \tag{3.5}$$

Hence, we get the joint density of  $(U_1, \dots, U_{n-1})$  as

$$\begin{aligned} & K_1(\alpha_1, \dots, \alpha_{n-1}, \alpha_n + \beta, \Lambda) \operatorname{etr} \left( -\Lambda \sum_{i=1}^{n-1} U_i \right) \prod_{i=1}^{n-1} \det(U_i)^{\alpha_i - (p+1)/2} \\ & \times \det \left( I_p - \sum_{i=1}^{n-1} U_i \right)^{\alpha_n + \beta - (p+1)/2} {}_1F_1 \left( \alpha_n; \alpha_n + \beta; -\Lambda \left( I_p - \sum_{i=1}^{n-1} U_i \right) \right). \end{aligned} \tag{3.6}$$

Repeating this procedure  $n - m$  times gives the marginal density of  $(U_1, \dots, U_m)$  as

$$\begin{aligned}
 & K_1 \left( \alpha_1, \dots, \alpha_m, \sum_{i=m+1}^n \alpha_i + \beta, \Lambda \right) \text{etr} \left( -\Lambda \sum_{i=1}^m U_i \right) \\
 & \times \prod_{i=1}^m \det(U_i)^{\alpha_i - (p+1)/2} \det \left( I_p - \sum_{i=1}^m U_i \right)^{\sum_{i=m+1}^n \alpha_i + \beta - (p+1)/2} \\
 & \times {}_1F_1 \left( \sum_{i=m+1}^n \alpha_i; \sum_{i=m+1}^n \alpha_i + \beta; -\Lambda \left( I_p - \sum_{i=1}^m U_i \right) \right).
 \end{aligned} \tag{3.7}$$

Now, the second part of the theorem follows immediately. □

**Corollary 3.2.** *If  $(U_1, \dots, U_n) \sim \text{KD}_p^I(\alpha_1, \dots, \alpha_n, \beta, \Lambda)$ , then the marginal pdf of  $U_i$ ,  $i = 1, \dots, n$  is given by*

$$\begin{aligned}
 & K_1 \left( \alpha_i, \sum_{j=1(\neq i)}^n \alpha_j + \beta, \Lambda \right) \text{etr} \left( -\Lambda U_i \right) \det(U_i)^{\alpha_i - (p+1)/2} \\
 & \times \det(I_p - U_i)^{\sum_{j=1(\neq i)}^n \alpha_j + \beta - (p+1)/2} \\
 & \times {}_1F_1 \left( \sum_{j=1(\neq i)}^n \alpha_j; \sum_{j=1(\neq i)}^n \alpha_j + \beta; -\Lambda(I_p - U_i) \right), \quad 0 < U_i < I_p.
 \end{aligned} \tag{3.8}$$

It is interesting to note that the marginal density of  $U_i$  does not belong to the Kummer-Beta family and differs by an additional factor containing confluent hypergeometric function  ${}_1F_1$ .

In Theorem 3.3 we give results on marginal and conditional distributions for Kummer-Dirichlet type II distribution. Before doing so, we need to give an integral that will be used in the derivation of marginal distribution. From (2.6) and (2.8), we have

$$\begin{aligned}
 & \int_{X>0} \int_{Y>0} \text{etr}[-\Xi(X+Y)] \det(Y)^{a_1 - (p+1)/2} \\
 & \quad \times \det(X)^{a_2 - (p+1)/2} \det(I_p + X + Y)^{-b} dX dY \\
 & = \Gamma_p(a_1) \Gamma_p(a_2) \Psi \left( a_1 + a_2, a_1 + a_2 - b + \frac{p+1}{2}; \Xi \right),
 \end{aligned} \tag{3.9}$$

where  $\text{Re}(a_1) > (p - 1)/2$ ,  $\text{Re}(a_2) > (p - 1)/2$  and  $\text{Re}(\Xi) > 0$ . Substituting

$W = (I_p + X)^{-1/2} Y(I_p + X)^{-1/2}$  with the Jacobian  $J(Y \rightarrow W) = \det(I_p + X)^{(p+1)/2}$  in (3.9) and integrating  $W$ , we obtain

$$\begin{aligned} & \int_{X>0} \text{etr}(-\Xi X) \det(X)^{a_2-(p+1)/2} \det(I_p + X)^{a_1-b} \\ & \quad \times \Psi\left(a_1, a_1 - b + \frac{p+1}{2}; \Xi(I_p + X)\right) dX \quad (3.10) \\ & = \Gamma_p(a_2) \Psi\left(a_1 + a_2, a_1 + a_2 - b + \frac{p+1}{2}; \Xi\right). \end{aligned}$$

Now we turn to our problem of finding the marginal and conditional distributions.

**Theorem 3.3.** *If  $(V_1, \dots, V_n) \sim \text{KD}_p^{\text{II}}(\alpha_1, \dots, \alpha_n, \gamma, \Xi)$ , then the joint marginal pdf of  $V_1, \dots, V_m$ ,  $m \leq n$ , is given by*

$$\begin{aligned} & \Gamma_p\left(\sum_{i=m+1}^n \alpha_i\right) K_2\left(\alpha_1, \dots, \alpha_m, \sum_{i=m+1}^n \alpha_i, \gamma, \Xi\right) \text{etr}\left(-\Xi \sum_{i=1}^m V_i\right) \\ & \quad \times \prod_{i=1}^m \det(V_i)^{\alpha_i-(p+1)/2} \det\left(I_p + \sum_{i=1}^m V_i\right)^{-\gamma + \sum_{i=m+1}^n \alpha_i} \quad (3.11) \\ & \quad \times \Psi\left(\sum_{i=m+1}^n \alpha_i, \sum_{i=m+1}^n \alpha_i - \gamma + \frac{p+1}{2}; \Xi\left(I_p + \sum_{j=1}^m V_j\right)\right), \\ & \quad V_j > 0, j = 1, \dots, m, \end{aligned}$$

and the conditional density of  $(V_{m+1}, \dots, V_n) | (V_1, \dots, V_m)$  is given by

$$\begin{aligned} & \frac{K_2(\alpha_1, \dots, \alpha_n, \gamma, \Xi)}{\Gamma_p\left(\sum_{i=m+1}^n \alpha_i\right) K_2\left(\alpha_1, \dots, \alpha_m, \sum_{i=m+1}^n \alpha_i, \gamma, \Xi\right)} \\ & \quad \times \frac{\text{etr}\left(-\Xi \sum_{i=m+1}^n V_i\right)}{\det\left(I_p + \sum_{i=1}^m V_i\right)^{-\gamma + \sum_{i=m+1}^n \alpha_i}} \\ & \quad \times \frac{\prod_{i=m+1}^n \det(V_i)^{\alpha_i-(p+1)/2} \det\left(I_p + \sum_{i=1}^m V_i + \sum_{i=m+1}^n V_i\right)^{-\gamma}}{\Psi\left(\sum_{i=m+1}^n \alpha_i, \sum_{i=m+1}^n \alpha_i - \gamma + (p+1)/2; \Xi\left(I_p + \sum_{j=1}^m V_j\right)\right)}, \\ & \quad V_i > 0, i = m+1, \dots, n. \quad (3.12) \end{aligned}$$

*Proof.* In this case, to obtain the marginal density of  $V_1, \dots, V_{n-1}$ , we substitute  $W_n = (I_p + \sum_{i=1}^{n-1} V_i)^{-1/2} V_n (I_p + \sum_{i=1}^{n-1} V_i)^{-1/2}$  with the

Jacobian  $J(V_n \rightarrow W_n) = \det(I_p + \sum_{i=1}^{n-1} V_i)^{(p+1)/2}$ . Thus, the joint density of  $V_1, \dots, V_{n-1}$  is obtained as

$$\begin{aligned}
& K_2(\alpha_1, \dots, \alpha_n, \gamma, \Xi) \operatorname{etr} \left( -\Xi \sum_{i=1}^{n-1} V_i \right) \\
& \quad \times \prod_{i=1}^{n-1} \det(V_i)^{\alpha_i - (p+1)/2} \det \left( I_p + \sum_{i=1}^{n-1} V_i \right)^{-\gamma + \alpha_n} \\
& \quad \times \int_{W_n > 0} \operatorname{etr} \left[ - \left( I_p + \sum_{i=1}^{n-1} V_i \right)^{1/2} \Xi \left( I_p + \sum_{i=1}^{n-1} V_i \right)^{1/2} W_n \right] \\
& \quad \quad \times \det(W_n)^{\alpha_n - (p+1)/2} \det(I_p + W_n)^{-\gamma} dW_n \quad (3.13) \\
& = \Gamma_p(\alpha_n) K_2(\alpha_1, \dots, \alpha_n, \gamma, \Xi) \operatorname{etr} \left( -\Xi \sum_{i=1}^{n-1} V_i \right) \\
& \quad \times \prod_{i=1}^{n-1} \det(V_i)^{\alpha_i - (p+1)/2} \det \left( I_p + \sum_{i=1}^{n-1} V_i \right)^{-\gamma + \alpha_n} \\
& \quad \times \Psi \left( \alpha_n, \alpha_n - \gamma + \frac{p+1}{2}; \Xi \left( I_p + \sum_{i=1}^{n-1} V_i \right) \right).
\end{aligned}$$

Further, substituting  $W_{n-1} = (I_p + \sum_{i=1}^{n-2} V_i)^{-1/2} V_{n-1} (I_p + \sum_{i=1}^{n-2} V_i)^{-1/2}$  with the Jacobian  $J(V_{n-1} \rightarrow W_{n-1}) = \det(I_p + \sum_{i=1}^{n-2} V_i)^{(p+1)/2}$  in (3.13) and integrating  $W_{n-1}$  using (3.10), we get the joint marginal density of  $V_1, \dots, V_{n-2}$  as

$$\begin{aligned}
& \Gamma_p(\alpha_n) K_2(\alpha_1, \dots, \alpha_n, \gamma, \Xi) \operatorname{etr} \left( -\Xi \sum_{i=1}^{n-2} V_i \right) \\
& \quad \times \prod_{i=1}^{n-2} \det(V_i)^{\alpha_i - (p+1)/2} \det \left( I_p + \sum_{i=1}^{n-2} V_i \right)^{-\gamma + \alpha_n + \alpha_{n-1}} \\
& \quad \times \int_{W_{n-1} > 0} \operatorname{etr} \left[ - \left( I_p + \sum_{i=1}^{n-2} V_i \right)^{1/2} \Xi \left( I_p + \sum_{i=1}^{n-2} V_i \right)^{1/2} W_{n-1} \right] \\
& \quad \quad \times \det(W_{n-1})^{\alpha_{n-1} - (p+1)/2} \det(I_p + W_{n-1})^{-\gamma + \alpha_n}
\end{aligned}$$

$$\begin{aligned}
 & \times \Psi \left( \alpha_n, \alpha_n - \gamma + \frac{p+1}{2}; \left( I_p + \sum_{i=1}^{n-2} V_i \right)^{1/2} \right. \\
 & \quad \left. \times \Xi \left( I_p + \sum_{i=1}^{n-2} V_i \right)^{1/2} W_{n-1} \right) dW_{n-1} \\
 & = \Gamma_p(\alpha_n) \Gamma_p(\alpha_{n-1}) K_2(\alpha_1, \dots, \alpha_n, \gamma, \Xi) \operatorname{etr} \left( -\Xi \sum_{i=1}^{n-2} V_i \right) \\
 & \quad \times \prod_{i=1}^{n-2} \det(V_i)^{\alpha_i - (p+1)/2} \det \left( I_p + \sum_{i=1}^{n-2} V_i \right)^{-\gamma + \alpha_n + \alpha_{n-1}} \\
 & \quad \times \Psi \left( \alpha_n + \alpha_{n-1}, \alpha_n + \alpha_{n-1} - \gamma + \frac{p+1}{2}; \Xi \left( I_p + \sum_{i=1}^{n-2} V_i \right) \right).
 \end{aligned} \tag{3.14}$$

Integrating out  $V_{n-2}, \dots, V_{m+1}$  similarly, we get the marginal density of  $V_1, \dots, V_m$  as

$$\begin{aligned}
 & \prod_{i=m+1}^n \Gamma_p(\alpha_i) K_2(\alpha_1, \dots, \alpha_n, \gamma, \Xi) \operatorname{etr} \left( -\Xi \sum_{i=1}^m V_i \right) \\
 & \quad \times \prod_{i=1}^m \det(V_i)^{\alpha_i - (p+1)/2} \det \left( I_p + \sum_{i=1}^m V_i \right)^{-\gamma + \sum_{i=m+1}^n \alpha_i} \\
 & \quad \times \Psi \left( \sum_{i=m+1}^n \alpha_i, \sum_{i=m+1}^n \alpha_i - \gamma + \frac{p+1}{2}; \Xi \left( I_p + \sum_{i=1}^m V_i \right) \right).
 \end{aligned} \tag{3.15}$$

The final expression of the marginal density of  $V_1, \dots, V_m$  is obtained by noting that

$$\begin{aligned}
 & \prod_{i=m+1}^n \Gamma_p(\alpha_i) K_2(\alpha_1, \dots, \alpha_n, \gamma, \Xi) \\
 & = \Gamma_p \left( \sum_{i=m+1}^n \alpha_i \right) K_2 \left( \alpha_1, \dots, \alpha_m, \sum_{i=m+1}^n \alpha_i, \gamma, \Xi \right).
 \end{aligned} \tag{3.16}$$

The derivation of the conditional density is now straightforward. □

Corollary 3.4. If  $(V_1, \dots, V_n) \sim \text{KD}_p^{\text{II}}(\alpha_1, \dots, \alpha_n, \gamma, \Xi)$ , then the density of  $V_i$ ,  $i = 1, \dots, n$  is given by

$$\begin{aligned} & \Gamma_p \left( \sum_{j=1(\neq i)}^n \alpha_j \right) K_2 \left( \alpha_i, \sum_{j=1(\neq i)}^n \alpha_j, \gamma, \Xi \right) \text{etr}(-\Xi V_i) \\ & \quad \times \det(V_i)^{\alpha_i - (p+1)/2} \det(I_p + V_i)^{-\gamma + \sum_{j=1(\neq i)}^n \alpha_j} \\ & \quad \times \Psi \left( \sum_{j=1(\neq i)}^n \alpha_j, \sum_{j=1(\neq i)}^n \alpha_j - \gamma + \frac{p+1}{2}; \Xi(I_p + V_i) \right), \quad V_i > 0. \end{aligned} \tag{3.17}$$

Note that the marginal density of  $V_i$  differs from the Kummer-Gamma density. It is a pdf with an additional factor containing confluent hypergeometric function  $\Psi$ .

Theorem 3.5. Let  $(U_1, \dots, U_n) \sim \text{KD}_p^{\text{I}}(\alpha_1, \dots, \alpha_n, \beta, I_p)$  and define

$$W_i = \left( I_p - \sum_{i=1}^m U_i \right)^{-1/2} U_i \left( I_p - \sum_{i=1}^m U_i \right)^{-1/2}, \quad i = m+1, \dots, n. \tag{3.18}$$

Then the joint density of  $(W_{m+1}, \dots, W_n)$  is given by

$$\begin{aligned} & \frac{\Gamma_p(\sum_{j=m+1}^n \alpha_j + \beta)}{\prod_{i=m+1}^n \Gamma_p(\alpha_i) \Gamma_p(\beta)} \left\{ {}_1F_1 \left( \sum_{i=1}^n \alpha_i; \sum_{i=1}^n \alpha_i + \beta; -I_p \right) \right\}^{-1} \\ & \quad \times \text{etr} \left( - \sum_{i=m+1}^n W_i \right) \\ & \quad \times \prod_{i=m+1}^n \det(W_i)^{\alpha_i - (p+1)/2} \det \left( I_p - \sum_{i=m+1}^n W_i \right)^{\beta - (p+1)/2} \\ & \quad \times {}_1F_1 \left( \sum_{i=1}^m \alpha_i; \sum_{j=1}^n \alpha_j + \beta; - \left( I_p - \sum_{i=m+1}^n W_i \right) \right), \\ & \quad 0 < W_i < I_p, \sum_{i=1}^m W_i < I_p. \end{aligned} \tag{3.19}$$

*Proof.* Transforming  $W_i = (I_p - \sum_{i=1}^m U_i)^{-1/2} U_i (I_p - \sum_{i=1}^m U_i)^{-1/2}$ ,  $i = m+1, \dots, n$  with Jacobian  $J(U_{m+1}, \dots, U_n \rightarrow W_{m+1}, \dots, W_n) = \det(I_p -$

$\sum_{i=1}^m U_i)^{(n-m)(p+1)/2}$ , in the joint density of  $(U_1, \dots, U_n)$ , we get

$$\begin{aligned}
 & K_1(\alpha_1, \dots, \alpha_n, \beta, I_p) \text{etr} \left[ - \sum_{i=m+1}^n W_i - \left( \sum_{i=1}^m U_i \right) \left( I_p - \sum_{i=m+1}^n W_i \right) \right] \\
 & \times \prod_{i=1}^m \det(U_i)^{\alpha_i - (p+1)/2} \det \left( I_p - \sum_{i=1}^m U_i \right)^{\sum_{j=m+1}^n \alpha_j + \beta - (p+1)/2} \\
 & \times \prod_{i=m+1}^n \det(W_i)^{\alpha_i - (p+1)/2} \det \left( I_p - \sum_{i=m+1}^n W_i \right)^{\beta - (p+1)/2}, \\
 & 0 < U_i < I_p, \quad i = m+1, \dots, n, \quad \sum_{i=m+1}^n U_i < I_p, \\
 & 0 < W_i < I_p, \quad i = 1, \dots, m, \quad \sum_{i=1}^m W_i < I_p.
 \end{aligned} \tag{3.20}$$

Now, integrating  $U_1, \dots, U_m$ ,

$$\begin{aligned}
 & \int_{\substack{0 < \sum_{i=1}^m U_i < I_p \\ 0 < U_i < I_p}} \dots \int \text{etr} \left[ - \left( \sum_{i=1}^m U_i \right) \left( I_p - \sum_{i=m+1}^n W_i \right) \right] \\
 & \times \prod_{i=1}^m \det(U_i)^{\alpha_i - (p+1)/2} \\
 & \times \det \left( I_p - \sum_{i=1}^m U_i \right)^{\sum_{i=m+1}^n \alpha_i + \beta - (p+1)/2} \prod_{i=1}^m dU_i \\
 & = \frac{\prod_{i=1}^m \Gamma_p(\alpha_i)}{\Gamma_p(\sum_{i=1}^m \alpha_i)} \int_{0 < U < I_p} \text{etr} \left[ - \left( I_p - \sum_{i=m+1}^n W_i \right) U \right] \\
 & \quad \times \det(U)^{\sum_{i=1}^m \alpha_i - (p+1)/2} \\
 & \quad \times \det(I_p - U)^{\sum_{i=m+1}^n \alpha_i + \beta - (p+1)/2} dU \\
 & = \frac{\prod_{i=1}^m \Gamma_p(\alpha_i) \Gamma_p(\sum_{i=m+1}^n \alpha_i + \beta)}{\Gamma_p(\sum_{i=1}^n \alpha_i + \beta)} \\
 & \quad \times {}_1F_1 \left( \sum_{i=1}^m \alpha_i; \sum_{i=1}^n \alpha_i + \beta; - \left( I_p - \sum_{i=m+1}^n W_i \right) \right),
 \end{aligned} \tag{3.21}$$

and using

$$\begin{aligned}
 K_1(\alpha_1, \dots, \alpha_n, \beta, I_p) & \frac{\prod_{i=1}^m \Gamma_p(\alpha_i) \Gamma_p(\sum_{i=m+1}^n \alpha_i + \beta)}{\Gamma_p(\sum_{i=1}^n \alpha_i + \beta)} \\
 & = \frac{\Gamma_p(\sum_{i=m+1}^n \alpha_i + \beta)}{\prod_{i=m+1}^n \Gamma_p(\alpha_i) \Gamma_p(\beta)} \left\{ {}_1F_1 \left( \sum_{i=1}^n \alpha_i; \sum_{i=1}^n \alpha_i + \beta; -I_p \right) \right\}^{-1}, \tag{3.22}
 \end{aligned}$$

we get the desired result. □

**Theorem 3.6.** Let  $(V_1, \dots, V_n) \sim \text{KD}_p^{\text{II}}(\alpha_1, \dots, \alpha_n, \gamma, I_p)$  and define

$$Z_i = \left( I_p + \sum_{i=1}^m V_i \right)^{-1/2} V_i \left( I_p + \sum_{i=1}^m V_i \right)^{-1/2}, \quad i = m+1, \dots, n. \tag{3.23}$$

Then the pdf of  $(Z_{m+1}, \dots, Z_n)$  is given by

$$\begin{aligned}
 & \left\{ \prod_{i=m+1}^n \Gamma_p(\alpha_i) \Psi \left( \sum_{i=1}^n \alpha_i, \sum_{i=1}^n \alpha_i - \gamma + \frac{p+1}{2}; I_p \right) \right\}^{-1} \\
 & \times \text{etr} \left( - \sum_{i=m+1}^n Z_i \right) \prod_{i=m+1}^n \det(Z_i)^{\alpha_i - (p+1)/2} \det \left( I_p + \sum_{i=m+1}^n Z_i \right)^{-\gamma} \\
 & \times \Psi \left( \sum_{i=1}^m \alpha_i, \sum_{i=1}^n \alpha_i - \gamma + \frac{p+1}{2}; \left( I_p + \sum_{i=m+1}^n Z_i \right) \right), \quad Z_i > 0. \tag{3.24}
 \end{aligned}$$

*Proof.* The proof is similar to the proof of Theorem 3.5. □

**Theorem 3.7.** Let  $(U_1, \dots, U_n) \sim \text{KD}_p^{\text{I}}(\alpha_1, \dots, \alpha_n, \beta, \Lambda)$  and define  $U = \sum_{i=1}^n U_i$  and  $X_i = U^{-1/2} U_i U^{-1/2}$ ,  $i = 1, \dots, n-1$ . Then

- (i)  $(X_1, \dots, X_{n-1})$  and  $U$  are independent,
- (ii)  $(X_1, \dots, X_{n-1}) \sim D_p^{\text{I}}(\alpha_1, \dots, \alpha_{n-1}; \alpha_n)$ , and
- (iii)  $U \sim \text{KB}_p(\sum_{i=1}^n \alpha_i, \beta, \Lambda)$ .

*Proof.* Substituting  $U_i = U^{1/2} X_i U^{1/2}$ ,  $i = 1, \dots, n-1$  and  $U_n = U^{1/2} (I_p - \sum_{i=1}^{n-1} X_i) U^{1/2}$  with the Jacobian  $J(U_1, \dots, U_{n-1}, U_n \rightarrow X_1, \dots, X_{n-1}, U) =$



$\det(\mathbf{U})^{(n-1)(p+1)/2}$  in the joint density of  $(\mathbf{U}_1, \dots, \mathbf{U}_n)$ , we get the desired result.  $\square$

**Theorem 3.8.** Let  $(V_1, \dots, V_n) \sim \text{KD}_p^{\text{II}}(\alpha_1, \dots, \alpha_n, \gamma, \Xi)$  and define  $V = \sum_{i=1}^n V_i$  and  $Y_i = V^{-1/2} V_i V^{-1/2}$ ,  $i = 1, \dots, n-1$ . Then

- (i)  $(Y_1, \dots, Y_{n-1})$  and  $V$  are independent,
- (ii)  $(Y_1, \dots, Y_{n-1}) \sim \text{D}_p^{\text{I}}(\alpha_1, \dots, \alpha_{n-1}; \alpha_n)$ , and
- (iii)  $V \sim \text{KG}_p(\sum_{i=1}^n \alpha_i, \gamma, \Xi)$ .

*Proof.* The proof is similar to the proof of Theorem 3.7.  $\square$

In Theorems 3.9 and 3.10, we derive the joint pdfs of partial sums of random matrices distributed as matrix variate Kummer-Dirichlet type I or II.

**Theorem 3.9.** Let  $(\mathbf{U}_1, \dots, \mathbf{U}_n) \sim \text{KD}_p^{\text{I}}(\alpha_1, \dots, \alpha_n, \beta, \Lambda)$  and define

$$\mathbf{U}_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} \mathbf{U}_j, \quad \alpha_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} \alpha_j, \quad n_0^* = 0, \quad n_i^* = \sum_{j=1}^i n_j, \quad i = 1, \dots, \ell. \tag{3.25}$$

Then  $(\mathbf{U}_{(1)}, \dots, \mathbf{U}_{(\ell)}) \sim \text{KD}_p^{\text{I}}(\alpha_{(1)}, \dots, \alpha_{(\ell)}, \beta, \Lambda)$ .

*Proof.* Make the transformation

$$\mathbf{U}_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} \mathbf{U}_j, \quad \mathbf{W}_j = \mathbf{U}_{(i)}^{-1/2} \mathbf{U}_j \mathbf{U}_{(i)}^{-1/2}, \tag{3.26}$$

$j = n_{i-1}^* + 1, \dots, n_i^* - 1$ ,  $i = 1, \dots, \ell$ . The Jacobian of this transformation is given by

$$\begin{aligned} & J(\mathbf{U}_1, \dots, \mathbf{U}_n \longrightarrow \mathbf{W}_1, \dots, \mathbf{W}_{n_1-1}, \mathbf{U}_{(1)}, \dots, \mathbf{W}_{n_{\ell-1}^*+1}, \dots, \mathbf{W}_{n-1}, \mathbf{U}_{(\ell)}) \\ &= \prod_{i=1}^{\ell} J(\mathbf{U}_{n_{i-1}^*+1}, \dots, \mathbf{U}_{n_i^*} \longrightarrow \mathbf{W}_{n_{i-1}^*+1}, \dots, \mathbf{W}_{n_i^*-1}, \mathbf{U}_{(i)}) \\ &= \prod_{i=1}^{\ell} \det(\mathbf{U}_{(i)})^{(n_i-1)(p+1)/2}. \end{aligned} \tag{3.27}$$

Now, substituting from (3.26) and (3.27) in the joint density of  $(\mathbf{U}_1, \dots, \mathbf{U}_n)$  given by (2.5), we get the joint density of  $\mathbf{W}_{n_{i-1}^*+1}, \dots, \mathbf{W}_{n_i^*-1}, \mathbf{U}_{(i)}$ ,

where  $i = 1, \dots, \ell$ , as

$$\begin{aligned}
 & K_1(\alpha_1, \dots, \alpha_n, \beta, \Lambda) \operatorname{etr} \left( -\Lambda \sum_{i=1}^{\ell} \mathbf{U}_{(i)} \right) \\
 & \times \prod_{i=1}^{\ell} \det(\mathbf{U}_{(i)})^{\alpha_{(i)} - (p+1)/2} \det \left( \mathbf{I}_p - \sum_{i=1}^{\ell} \mathbf{U}_{(i)} \right)^{\beta - (p+1)/2} \\
 & \times \prod_{i=1}^{\ell} \left\{ \prod_{j=n_{i-1}^*+1}^{n_i^*-1} \det(\mathbf{W}_j)^{\alpha_j - (p+1)/2} \right. \\
 & \left. \times \det \left( \mathbf{I}_p - \sum_{j=n_{i-1}^*+1}^{n_i^*-1} \mathbf{W}_j \right)^{\alpha_{n_i^*-1} - (p+1)/2} \right\},
 \end{aligned} \tag{3.28}$$

where  $0 < \mathbf{U}_{(i)} < \mathbf{I}_p$ ,  $\sum_{i=1}^{\ell} \mathbf{U}_{(i)} < \mathbf{I}_p$ ,  $0 < \mathbf{W}_j < \mathbf{I}_p$ ,  $j = n_{i-1}^* + 1, \dots, n_i^* - 1$ ,  $\sum_{j=n_{i-1}^*+1}^{n_i^*-1} \mathbf{W}_j < \mathbf{I}_p$ ,  $i = 1, \dots, \ell$ . From (3.28), it is easy to see that  $(\mathbf{U}_{(1)}, \dots, \mathbf{U}_{(\ell)})$  and  $(\mathbf{W}_{n_{i-1}^*+1}, \dots, \mathbf{W}_{n_i^*-1})$ ,  $i = 1, \dots, \ell$ , are independently distributed. Further,  $(\mathbf{U}_{(1)}, \dots, \mathbf{U}_{(\ell)}) \sim \text{KD}_p^{\text{I}}(\alpha_{(1)}, \dots, \alpha_{(\ell)}, \beta, \Lambda)$  and  $(\mathbf{W}_{n_{i-1}^*+1}, \dots, \mathbf{W}_{n_i^*-1}) \sim \text{D}_p^{\text{I}}(\alpha_{n_{i-1}^*+1}, \dots, \alpha_{n_i^*-1}; \alpha_{n_i^*})$ , where  $i = 1, \dots, \ell$ .  $\square$

When  $\ell = 1$ ,  $\sum_{i=1}^n \mathbf{U}_i \sim \text{KB}_p(\sum_{i=1}^n \alpha_i, \beta, \Lambda)$ .

**Theorem 3.10.** Let  $(\mathbf{V}_1, \dots, \mathbf{V}_n) \sim \text{KD}_p^{\text{II}}(\alpha_1, \dots, \alpha_n, \gamma, \Xi)$  and define

$$\mathbf{V}_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} \mathbf{V}_j, \quad \alpha_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} \alpha_j, \quad n_0^* = 0, \quad n_i^* = \sum_{j=1}^i n_j, \quad i = 1, \dots, \ell. \tag{3.29}$$

Then  $(\mathbf{V}_{(1)}, \dots, \mathbf{V}_{(\ell)}) \sim \text{KD}_p^{\text{II}}(\alpha_{(1)}, \dots, \alpha_{(\ell)}, \gamma, \Xi)$ .

*Proof.* Make the transformation

$$\mathbf{V}_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} \mathbf{V}_j, \quad \mathbf{Z}_j = \mathbf{V}_{(i)}^{-1/2} \mathbf{V}_j \mathbf{V}_{(i)}^{-1/2}, \tag{3.30}$$

where  $j = n_{i-1}^* + 1, \dots, n_i^* - 1$ ,  $i = 1, \dots, \ell$ . The Jacobian of this transformation

is given by

$$\begin{aligned}
 & J(V_1, \dots, V_n \longrightarrow Z_1, \dots, Z_{n_1-1}, V_{(1)}, \dots, Z_{n_{\ell}^*_{i-1}+1}, \dots, Z_{n-1}, V_{(\ell)}) \\
 &= \prod_{i=1}^{\ell} J(V_{n_{i-1}^*+1}, \dots, V_{n_i^*} \longrightarrow Z_{n_{i-1}^*+1}, \dots, Z_{n_i^*-1}, V_{(i)}) \tag{3.31} \\
 &= \prod_{i=1}^{\ell} \det(V_{(i)})^{(n_i-1)(p+1)/2}.
 \end{aligned}$$

Now, substituting from (3.30) and (3.31) in the joint density of  $(V_1, \dots, V_n)$  given by (2.6), it can easily be shown that  $(V_{(1)}, \dots, V_{(\ell)})$  and  $(Z_{n_{i-1}^*+1}, \dots, Z_{n_i^*-1})$ ,  $i = 1, \dots, \ell$ , are independently distributed. Further,  $(V_{(1)}, \dots, V_{(\ell)}) \sim \text{KD}_p^{\text{II}}(\alpha_{(1)}, \dots, \alpha_{(\ell)}, \gamma, \Xi)$  and  $(Z_{n_{i-1}^*+1}, \dots, Z_{n_i^*-1}) \sim \text{D}_p^{\text{I}}(\alpha_{n_{i-1}^*+1}, \dots, \alpha_{n_i^*-1}; \alpha_{n_i^*})$ ,  $i = 1, \dots, \ell$ .  $\square$

When  $\ell = 1$ , the distribution of  $\sum_{i=1}^n V_i$  is Kummer-Gamma with parameters  $\sum_{i=1}^n \alpha_i$ ,  $\gamma$  and  $\Xi$ . From the joint density of  $U_1, \dots, U_n$ , we have

$$\begin{aligned}
 & \mathbb{E} \left[ \prod_{i=1}^n \det(U_i)^{r_i} \right] \\
 &= K_1(\alpha_1, \dots, \alpha_n, \beta, \Lambda) \int_{\substack{0 < \sum_{i=1}^n U_i < I_p \\ 0 < U_i < I_p}} \dots \int \text{etr} \left( -\Lambda \sum_{i=1}^n U_i \right) \\
 &\quad \times \prod_{i=1}^n \det(U_i)^{\alpha_i+r_i-(p+1)/2} \det \left( I_p - \sum_{i=1}^n U_i \right)^{\beta-(p+1)/2} \prod_{i=1}^n dU_i \\
 &= \frac{K_1(\alpha_1, \dots, \alpha_n, \beta, \Lambda)}{K_1(\alpha_1+r_1, \dots, \alpha_n+r_n, \beta, \Lambda)}, \quad \text{Re}(\alpha_i+r_i) > \frac{p-1}{2} \\
 &= \frac{\prod_{i=1}^n \Gamma_p(\alpha_i+r_i) \Gamma_p(\sum_{i=1}^n \alpha_i+\beta)}{\prod_{i=1}^n \Gamma_p(\alpha_i) \Gamma_p[\sum_{i=1}^n (\alpha_i+r_i)+\beta]} \\
 &\quad \times \frac{{}_1F_1(\sum_{i=1}^n (\alpha_i+r_i); \sum_{i=1}^n (\alpha_i+r_i)+\beta; -\Lambda)}{{}_1F_1(\sum_{i=1}^n \alpha_i; \sum_{i=1}^n \alpha_i+\beta; -\Lambda)}, \\
 &\quad \text{Re}(\alpha_i+r_i) > \frac{p-1}{2}, \tag{3.32}
 \end{aligned}$$

where the last step has been obtained using (2.7). Further

$$\begin{aligned}
 & \mathbb{E} \left[ \det \left( I_p - \sum_{i=1}^n U_i \right)^h \right] \\
 &= K_1(\alpha_1, \dots, \alpha_n, \beta, \Lambda) \int \cdots \int_{\substack{0 < \sum_{i=1}^n U_i < I_p \\ U_i > 0}} \text{etr} \left( -\Lambda \sum_{i=1}^n U_i \right) \\
 & \quad \times \prod_{i=1}^n \det(U_i)^{\alpha_i - (p+1)/2} \det \left( I_p - \sum_{i=1}^n U_i \right)^{\beta + h - (p+1)/2} \prod_{i=1}^n dU_i \\
 &= \frac{K_1(\alpha_1, \dots, \alpha_n, \beta, \Lambda)}{K_1(\alpha_1, \dots, \alpha_n, \beta + h, \Lambda)}, \quad \text{Re}(h) > -\beta + \frac{p-1}{2} \\
 &= \frac{\Gamma_p(\beta + h) \Gamma_p(\sum_{i=1}^n \alpha_i + \beta)}{\Gamma_p(\beta) \Gamma_p(\sum_{i=1}^n \alpha_i + \beta + h)} \frac{{}_1F_1(\sum_{i=1}^n \alpha_i; \sum_{i=1}^n \alpha_i + \beta + h; -\Lambda)}{{}_1F_1(\sum_{i=1}^n \alpha_i; \sum_{i=1}^n \alpha_i + \beta; -\Lambda)}, \\
 & \quad \text{Re}(h) > -\beta + \frac{p-1}{2}.
 \end{aligned} \tag{3.33}$$

Alternately, the above moment expression can be obtained by noting that  $\sum_{i=1}^n U_i$  has Kummer-Beta distribution. Similarly, for Kummer-Dirichlet type II matrices

$$\begin{aligned}
 & \mathbb{E} \left[ \prod_{i=1}^n \det(V_i)^{r_i} \right] \\
 &= \prod_{i=1}^n \frac{\Gamma_p(\alpha_i + r_i)}{\Gamma_p(\alpha_i)} \frac{\Psi(\sum_{i=1}^n (\alpha_i + r_i), \sum_{i=1}^n (\alpha_i + r_i) - \gamma + (p+1)/2; \Xi)}{\Psi(\sum_{i=1}^n \alpha_i, \sum_{i=1}^n \alpha_i - \gamma + (p+1)/2; \Xi)}, \\
 & \quad \text{Re}(\alpha_i + r_i) > \frac{p-1}{2}, \\
 & \mathbb{E} \left[ \det \left( I_p + \sum_{i=1}^n V_i \right)^{-h} \right] = \frac{\Psi(\sum_{i=1}^n \alpha_i, \sum_{i=1}^n \alpha_i - \gamma - h + (p+1)/2; \Xi)}{\Psi(\sum_{i=1}^n \alpha_i, \sum_{i=1}^n \alpha_i - \gamma + (p+1)/2; \Xi)}.
 \end{aligned} \tag{3.34}$$

Next two results give certain asymptotic distributions for the Kummer-Dirichlet type I and type II distributions (see Javier and Gupta [4]).

**Theorem 3.11.** *Let  $(U_1, \dots, U_n) \sim \text{KD}_p^I(\alpha_1, \dots, \alpha_n, \beta, \beta\Lambda)$  and  $W = (W_1, \dots, W_n)$  be defined by  $W_i = \beta U_i$ ,  $i = 1, \dots, n$ . Then  $W$  is asymptotically distributed as a product of independent matrix variate gamma densities;*

more specifically

$$\lim_{\beta \rightarrow \infty} f(W) = \prod_{i=1}^n \frac{\det(W_i)^{\alpha_i - (p+1)/2} \text{etr}[-(I_p + \Lambda)W_i]}{\det(I_p + \Lambda)^{\alpha_i} \Gamma_p(\alpha_i)}, \quad (3.35)$$

where  $f(W)$  denotes the density of the matrix  $W$ .

*Proof.* In the joint density of  $(U_1, \dots, U_n)$  given by (2.5) transform  $W_i = \beta U_i$ ,  $i = 1, \dots, n$  with the Jacobian  $J(U_1, \dots, U_n \rightarrow W_1, \dots, W_n) = \beta^{-n p(p+1)/2}$ . The density of  $W = (W_1, \dots, W_n)$  is given by

$$\begin{aligned} f(W) &= \frac{\Gamma_p(\sum_{i=1}^n \alpha_i + \beta)}{\Gamma_p(\beta)} \beta^{-p \sum_{i=1}^n \alpha_i} \left\{ {}_1F_1 \left( \sum_{i=1}^n \alpha_i; \sum_{i=1}^n \alpha_i + \beta; -\beta \Lambda \right) \right\}^{-1} \\ &\times \text{etr} \left( -\Lambda \sum_{i=1}^n W_i \right) \left\{ \prod_{i=1}^n \frac{\det(W_i)^{\alpha_i - (p+1)/2}}{\Gamma_p(\alpha_i)} \right\} \\ &\times \det \left( I_p - \frac{1}{\beta} \sum_{i=1}^n W_i \right)^{\beta - (p+1)/2}. \end{aligned} \quad (3.36)$$

The result follows, since

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \frac{\Gamma_p(\sum_{i=1}^n \alpha_i + \beta)}{\Gamma_p(\beta)} \beta^{-p \sum_{i=1}^n \alpha_i} &= 1, \\ \lim_{\beta \rightarrow \infty} {}_1F_1 \left( \sum_{i=1}^n \alpha_i; \sum_{i=1}^n \alpha_i + \beta; -\beta \Lambda \right) &= {}_1F_0 \left( \sum_{i=1}^n \alpha_i; -\Lambda \right) = \det(I_p + \Lambda)^{-\sum_{i=1}^n \alpha_i}, \\ \lim_{\beta \rightarrow \infty} \det \left( I_p - \frac{1}{\beta} \sum_{i=1}^n W_i \right)^{\beta - (p+1)/2} &= \text{etr} \left( -\sum_{i=1}^n W_i \right). \end{aligned} \quad (3.37)$$

□

An analogous result for Kummer-Dirichlet type II distribution is shown to be the following.

**Theorem 3.12.** Let  $(V_1, \dots, V_n) \sim \text{KD}_p^{\text{II}}(\alpha_1, \dots, \alpha_n, \gamma, |\gamma| \Xi)$ ,  $\gamma \neq 0$ , and  $W = (W_1, \dots, W_n)$  be defined by  $W_i = |\gamma| V_i$ ,  $i = 1, \dots, n$ . Then,  $W$  is asymptotically distributed as a product of independent matrix variate gamma

*densities; more specifically*

$$\lim_{|\gamma| \rightarrow \infty} g(W) = \prod_{i=1}^n \frac{\det(W_i)^{\alpha_i - (p+1)/2} \text{etr}[-(I_p + \Xi)W_i]}{\det(I_p + \Xi)^{\alpha_i} \Gamma_p(\alpha_i)}, \quad (3.38)$$

where  $g(W)$  denotes the density of the matrix  $W$ .

*Proof.* We prove the result for  $\gamma > 0$ . The proof for  $\gamma < 0$  follows similar steps. The density of  $W = (W_1, \dots, W_n)$  is given by

$$\begin{aligned} g(W) &= \gamma^{-p \sum_{i=1}^n \alpha_i} \left\{ \Psi \left( \sum_{i=1}^n \alpha_i, \sum_{i=1}^n \alpha_i - \gamma + \frac{p+1}{2}; \gamma \Xi \right) \right\}^{-1} \\ &\times \text{etr} \left( -\Xi \sum_{i=1}^n W_i \right) \left\{ \prod_{i=1}^n \frac{\det(W_i)^{\alpha_i - (p+1)/2}}{\Gamma_p(\alpha_i)} \right\} \\ &\times \det \left( I_p + \frac{1}{\gamma} \sum_{i=1}^n W_i \right)^{-\gamma}. \end{aligned} \quad (3.39)$$

The result follows, since

$$\begin{aligned} \lim_{\gamma \rightarrow \infty} \gamma^{p \sum_{i=1}^n \alpha_i} \Psi \left( \sum_{i=1}^n \alpha_i, \sum_{i=1}^n \alpha_i - \gamma + \frac{p+1}{2}; \gamma \Xi \right) &= \det(I_p + \Xi)^{-\sum_{i=1}^n \alpha_i}, \\ \lim_{\gamma \rightarrow \infty} \det \left( I_p + \frac{1}{\gamma} \sum_{i=1}^n W_i \right)^{-\gamma} &= \text{etr} \left( -\sum_{i=1}^n W_i \right). \end{aligned} \quad (3.40)$$

□

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## Special Issue on Decision Support for Intermodal Transport

### Call for Papers

Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

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The mathematical models may be of the optimization type or of the evaluation type to gain an insight in intermodal operations. The mathematical models aim to support decisions on the strategic, tactical, and operational levels. The decision-makers belong to the various players in the intermodal transport world, namely, drayage operators, terminal operators, network operators, or intermodal operators.

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