ABSTRACT MECHANICAL CONNECTION AND ABELIAN RECONSTRUCTION FOR ALMOST KÄHLER MANIFOLDS

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When the phase space P of a Hamiltonian G-system (P, ω, G, J, H) has an almost Kähler structure, a preferred connection, called *abstract mechanical connection*, can be defined by declaring horizontal spaces at each point to be metric orthogonal to the tangent to the group orbit. Explicit formulas for the corresponding connection one-form $\mathcal{A}$ are derived in terms of the momentum map, symplectic and complex structures. Such connection can play the role of the *reconstruction connection* (due to the work of A. Blaom), thus significantly simplifying computations of the corresponding dynamic and geometric phases for an Abelian group G. These ideas are illustrated using the example of the resonant three-wave interaction. Explicit formulas for the connection one-form and the phases are given together with some new results on the symmetry reduction of the Poisson structure.

1. Introduction

1.1. Definitions and preliminaries

Consider a finite-dimensional symplectic manifold (P, ω). Let a Lie group G act on it canonically, that is, by preserving the symplectic form ω, and assume that this action admits an (equivariant) momentum map $J : P \to U \subset \mathfrak{g}^*$, $U \equiv J(P)$. Let a dynamical system be defined on P by some Hamiltonian H. We call (P, ω, G, J, H) a Hamiltonian G-system. Assume also that G acts on P freely and properly so that the Poisson reduction can be performed (in fact, these conditions can be slightly relaxed, cf. [9]). For background on momentum maps, Poisson reduction, etc., the reader is referred to Marsden and Ratiu [9].

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Recall that an almost Kähler manifold $(M, \omega, \mathcal{J}, \mathfrak{s})$ can be defined as a manifold M with an almost complex structure $\mathcal{J}$ and a $\mathcal{J}$-invariant (i.e., Hermitian) metric $\mathfrak{s}$, whose fundamental 2-form ω, defined by

$$\omega(x)(\boldsymbol{v}, \boldsymbol{w}) = \mathfrak{s}(x)\big(\mathcal{J}(x)\boldsymbol{v}, \boldsymbol{w}\big), \quad \forall v, w \in T_x M, \tag{1.1}$$

is closed and hence a symplectic form on M. If in addition the Nijenhuis torsion of $\mathcal{J}$ vanishes, then $\mathcal{J}$ is complex and M becomes a Kähler manifold [6]. The automorphisms of an almost Kähler structure are diffeomorphisms of M which at the same time are symplectomorphisms, almost complex maps and isometries with respect to $\omega, \mathcal{J}$, and $\mathfrak{s}$, respectively. It follows from the definition that any two of these conditions imply the third one. For the background and more information, see, for example, Kobayashi and Nomizu [6].

1.2. Reconstruction of the dynamics

The space of group orbits P/G, which is obtained by taking the quotient map $\pi : P \to P/G$ and is a smooth manifold under appropriate assumptions, inherits a Poisson structure from that of P. The Hamiltonian H drops to a reduced Hamiltonian h on P/G, and the corresponding Hamiltonian vector fields X_H and X_h, as well as their solutions x_t and y_t, respectively, are related by the projection $\pi : P \to P/G$.

Assume that y_t is periodic with period T, then for any initial condition $x_0 \in \pi^{-1}(y_0)$, the associated reconstruction phase is the unique $g \in G$ such that $x_T = g \cdot x_0$. The methods presently used to compute reconstruction phases are generally based on those established in [8]. The procedure can be sketched as follows.

If $J : P \to \mathfrak{g}^*$ is a momentum map, which we will suppose is Ad^*-equivariant, then $J(x_t) = \mu_0 \equiv J(x_0)$. Under appropriate connectedness hypotheses, the Marsden-Weinstein reduced space $J^{-1}(\mu_0)/G_{\mu_0}$ (G_{μ_0} denoting the isotropy of the coadjoint action at $\mu_0 \in \mathfrak{g}^*$) can be identified with a symplectic leaf $P_{\mu_0} \subset P/G$ containing the reduced solution curve y_t, and the projection $J^{-1}(\mu_0) \to P_{\mu_0}$ is a principal G_{μ_0}-bundle.

The first step in calculating the reconstruction phase g is to equip the bundle $J^{-1}(\mu_0) \to P_{\mu_0}$ with a principal connection α_{μ_0}, whose holonomy along the reduced curve y_t is called the associated *geometric phase* and denoted g_{geom}. The phase g is then the product $g_{\mathrm{dyn}} g_{\mathrm{geom}}$, where g_{dyn}, called the *dynamic phase*, is obtained by integrating a linear, nonautonomous, and ordinary differential equation, called the *reconstruction equation*. The coefficients in this equation are defined in terms of α_{μ_0}, the unreduced Hamiltonian vector field X_H, and an α_{μ_0}-horizontal lift of y_t to $J^{-1}(\mu_0)$. Calculating the geometric phase usually requires one to compute the curvature of α_{μ_0}.

While any connection α_{μ_0} can be used to compute g as described in the previous paragraph, a poor choice will lead to unwieldy computations (see, for example, [11]). For the so-called simple mechanical G-systems a natural choice exists (see Section 1.3); for other systems the choice is often made on a case-by-case basis.

1.3. Overview of the results

As we already mentioned, the methods presently used to compute reconstruction phases are based on those established in [8]. Though the general ideas in [8] apply for arbitrary Hamiltonian G-systems, most of the advances in the computation techniques have been done for mechanical systems on cotangent bundles T^*Q of some Riemannian manifolds Q with the metric, which determines the kinetic energy, playing the crucial role for the definition of the *mechanical connection*. Unfortunately, these settings exclude such interesting and important systems as N point vortices on a plane or on a sphere, N-wave interaction, etc., where the configuration space is *not* a cotangent bundle.

Luckily, some of these systems have a natural almost Kähler structure which we exploit in the construction of the *abstract mechanical connection*. It is defined by specifying the horizontal space to be *metric orthogonal* to the group orbits (see Section 4 for the details). The corresponding connection one-form is then obtained in terms of the momentum map, symplectic and complex structures. These expressions enable us to further simplify the computation of the reconstruction phases as described in [3] for the case of Abelian groups. The requirement of the group being Abelian is essential for the geometric phase part (see Section 4.2), but it is not used in the construction of the map L involving the abstract inertia tensor and in the expression for the dynamic phase. The Abelian property of the group makes the relation between the principal connections on Poisson and symplectic bundles trivial (see Section 3.1). It also significantly simplifies the picture of dual pairs, which underlies our constructions, and enables us to construct a very "useful" bundle $j : P/G \to \mathfrak{g}^*$. This is briefly described in Section 3.2.

In the work in progress a generalization to non-Abelian group action is being considered, as well as further simplification and links which arise in the case of the phase space being a cotangent bundle with the almost Kähler structure coming from a Riemannian metric on the configuration space. The relation between the abstract mechanical connection and the well-known mechanical connection is considered in [3], where the reconstruction phases for the cotangent bundles were analyzed, though not from the point of view of almost Kähler manifolds and corresponding abstract mechanical connections.

2. Reconstruction connection and the associated phases

In this section, we briefly overview the results on reconstruction phases obtained in [3]. We refer the reader to the original paper for detailed and comprehensive treatment of the subject. Here we are mainly interested in adopting these results to the case of almost Kähler systems, and thus we avoid giving much details to keep the presentation clear and avoid repetition.

In [3], a general formula is derived which expresses a reconstruction phase in terms of the associated reduced solution, viewed as a curve in the Poisson-reduced phase space P/G, and certain derivatives *transverse* to the symplectic leaf in P/G containing the curve. Specifically, the dynamic part of the phase depends on transverse derivative in the Poisson-reduced Hamiltonian, while the geometric part is determined by transverse derivatives in the leaf symplectic structures.

2.1. Highlights and basic assumptions

It is shown in [3] that the principal connection on the bundle $J^{-1}(\mu_0) \to P_{\mu_0}$, which plays a crucial role in the computation of the phases, is most naturally viewed as the restriction to $J^{-1}(\mu_0)$ of a certain kind of distribution $\mathcal{A}$ on P, which is called a *reconstruction connection*. To define the transverse derivatives, one then specifies a connection D on the symplectic stratification of P/G (a distribution on P/G furnishing a complement for the characteristic distribution). This connection D can be obtained by "Poisson-reducing" the connection $\mathcal{A}$.

Explicitly, assuming that as a cycle, the reduced curve y_t (see Section 1.2) is a boundary $\partial\Sigma$ ($\Sigma \subset P_{\mu_0}$ compact and oriented), the corresponding reconstruction phase is $g = g_{\mathrm{dyn}} g_{\mathrm{geom}}$, where

$$g_{\mathrm{dyn}} = \exp \int_0^T D_{\mu_0} h(y_t)\,dt, \qquad g_{\mathrm{geom}} = \exp \int_\Sigma D_{\mu_0} \omega_D. \tag{2.1}$$

In these formulas D_{μ_0} denotes a certain "exterior covariant derivative" depending on D and μ_0 that maps $\mathbb{R}$-valued p-forms on P/G to $\mathfrak{g}_{\mu_0}$-valued p-forms on P_{μ_0} ($p = 0, 1, 2, \ldots$). For example, $D_{\mu_0} h(y_t)$ is an element of $\mathfrak{g}_{\mu_0}$ that happens to measure the derivative of h in directions lying in $D(y_t)$ (i.e., in certain directions transverse to the symplectic leaves). ω_D denotes the two-form on P/G whose restriction to a given leaf gives that leaf's symplectic structure, and whose contraction with vectors in D vanishes.

Equation (2.1) holds assuming that G_{μ_0} is Abelian, that P_{μ_0} is a non-degenerate symplectic leaf, and that D is a smooth distribution in some neighborhood of P/G. These conditions are in addition to the following assumptions which are understood to be in place throughout the paper:

- all manifolds are smooth, that is, C^∞

- the group G acts freely and properly, so that the natural projection $\pi : P \to P/G$ is a submersion
- the group action is *Hamiltonian*, that is, it admits a momentum map $J : P \to \mathfrak{g}^*$ which is Ad^*-equivariant, that is, $J(g\cdot x) = \mathrm{Ad}^*_g J(x)$.

Expressions (2.1) make sense for *any* connection D on the symplectic stratification of P/G; whence, the total reconstruction phase $g = g_{\mathrm{dyn}} g_{\mathrm{geom}}$ (which is independent of the choice of α_{μ_0}, and hence A) can be computed using *any* connection D on the symplectic stratification of P/G.

The following two subsections give a short review of the results in [3] that are relevant for our applications.

2.2. Main constructions

Definition 2.1. Call a distribution A on P a *reconstruction connection* if

(a) A is G-invariant,
(b) $\mathrm{Ker}\, T_x J = T_x(G_\mu \cdot x) \oplus A(x)$ $(x \in P,\ m \equiv J(x))$.

Here G_μ denotes the point stabilizer of the coadjoint action at $\mu \in \mathfrak{g}^*$, $T_x J$ is the tangent map, and $\oplus$ denotes the direct sum.

2.2.1. *Connections on the symplectic stratification of* P/G

Let E denote the characteristic distribution on P/G (i.e., the distribution tangent to the symplectic leaves). We call a distribution D on P/G a *connection on the symplectic stratification of* P/G if it furnishes a complement for E:

$$T(P/G) = E \oplus D. \tag{2.2}$$

Now let A be a G-invariant distribution on P. Since G acts by symplectic diffeomorphisms, the distribution A^ω, the symplectic orthogonal distribution to A, is also G-invariant. It consequently drops to a distribution $\widehat{A} \equiv \pi_*(A^\omega)$ on P/G; here π_* denotes push-forward. Conversely, if D is an arbitrary distribution on P/G, then $\widehat{D} \equiv (\pi^* D)^\omega$ is a G-invariant distribution on P; here π^* denotes pull-back. Evidently, one has

$$\widehat{\widehat{D}} = D. \tag{2.3}$$

We quote the following theorem form [3] without proof.

Theorem 2.2 (Blaom 1999). *If* A *is a general reconstruction connection, then* $\widehat{A}$ *is a connection on the symplectic stratification of* P/G. *Moreover, the map* $A \mapsto \widehat{A}$ *is a bijection from the set of reconstruction connections to the set of connections on the symplectic stratification of* P/G. *This bijection has an inverse* $D \mapsto \widehat{D}$.

If A is a reconstruction connection, one thinks of $\widehat{A}$ as its Poisson-reduced counterpart. A reconstruction connection A can be reconstructed from its reduced counterpart $D \equiv \widehat{A}$ according to $A = \widehat{D}$.

Two other lemmas from [3] are relevant to our presentation and will be used in Section 3.

Lemma 2.3. *Let* $\pi : P \to Q$ *be a Poisson submersion and let* E *denote the characteristic distribution on* Q. *If* P *is symplectic*, *and* ω *denotes the symplectic form on* P, *then*

$$\pi^* E = \operatorname{Ker} T\pi + (\operatorname{Ker} T\pi)^{\omega}. \tag{2.4}$$

Lemma 2.4. *Let* $x \in P$ *be arbitrary and define* $\mu \equiv J(x)$. *Then*

$$T_x\big(G_\mu \cdot x\big) = \big((\pi^* E)(x)\big)^{\omega}. \tag{2.5}$$

2.2.2. Transverse derivatives in P/G

Under appropriate connectedness hypotheses each reduced space $J^{-1}(\mu)/G_\mu$ may be identified with a symplectic leaf $P_\mu \subset P/G$. A connection D on the symplectic stratification of P/G allows one to define derivatives of functions on P/G transverse to P_μ. At a point in P_μ such a derivative can be identified in a natural way with an element of the isotropy algebra $\mathfrak{g}_\mu$, provided that the isotropy group G_μ is Abelian. More generally, for such μ the connection D defines an "exterior covariant derivatives" mapping $\mathbb{R}$-valued p-forms on P/G to $\mathfrak{g}_\mu$-valued p-forms on the leaf P_μ.

Let D be a fixed connection on the symplectic stratification of P/G. Fix $\mu \in U \equiv J(P)$ and assume G_μ is Abelian. Then we have the following proposition.

Proposition 2.5. *For each* $y \in P_\mu$ *there is a natural isomorphism* $D(y) \leftrightarrow \mathfrak{g}_\mu^*$ *well defined by*

$$v \longmapsto p_\mu\big(\operatorname{forg}(TJ \cdot w)\big), \tag{2.6}$$

where w *denotes any element of* $T_x P$ *with* $T\pi \cdot w = v$, *and* $x \in J^{-1}(\mu) \cap \pi^{-1}(y) = \pi_\mu^{-1}(y)$ *is arbitrary. The inverse of this map (which depends on* D, μ, *and* y*) is denoted by* $L(D, \mu, y) : \mathfrak{g}_\mu^* \to D(y)$.

The map $\operatorname{forg} : TU \to \mathfrak{g}^*$ denotes the map that "forgets base point" and $p_\mu : \mathfrak{g}^* \to \mathfrak{g}_\mu^*$ denotes the natural projection.

Definition 2.6. Suppose that f is a function on P/G defined in some neighborhood of y. Then the (D, μ)-*exterior covariant derivative* of f at y, denoted $D_\mu f(y) \in \mathfrak{g}_\mu$, is defined through

$$\langle \nu, D_\mu f(y) \rangle = \langle df, L(D,\mu,y)(\nu) \rangle \quad \forall \nu \in \mathfrak{g}^*_\mu. \tag{2.7}$$

Definition 2.7. Let σ be a differential p-form on P/G defined in a neighborhood of P_μ, and assume that G_μ is Abelian. Then the (D,μ)-*exterior covariant derivative* $D_\mu \sigma$ of σ is the $\mathfrak{g}_\mu$-valued p-form on P_μ defined through

$$\langle \nu, D_\mu \sigma(\nu_1, \dots, y_p) \rangle = d\sigma\big(L(D,\mu,y)(\nu), \nu_1, \dots, \nu_p\big), \tag{2.8}$$

where $\nu \in \mathfrak{g}^*_\mu, \nu_1, \dots, \nu_p \in T_y P_\mu$ and $y \in P_\mu$.

2.2.3. Smoothness conditions

Let A be a reconstruction connection and let D be a connection on the symplectic stratification of P/G. Then we say that, A is μ-*smooth* ($\mu \in U$) if the set

$$\{A(x) \mid x \in J^{-1}(\mu)\} \tag{2.9}$$

is a smooth sub-bundle of the tangent bundle $T(J^{-1}(\mu))$. We call D μ-*smooth* if the set

$$\{D(y) \mid y \in P_\mu\} \tag{2.10}$$

is a smooth sub-bundle of $T_{P_\mu}(P/G) \equiv \{T_y(P/G) \mid y \in P_\mu\}$.

Then, the following smoothness results hold [3]:

- D is μ-smooth if and only if A is μ-smooth.
- If D is μ-smooth, then $L(D,\mu,y)$ in (2.6) depends smoothly on $y \in P_\mu$.
- If D is μ-smooth, then $D_\mu f : P_\mu \to \mathfrak{g}_\mu$ is smooth.
- Similarly, for a p-form σ, μ-smoothness of D ensures smoothness of $D_\mu \sigma$.

2.3. Reconstruction phases

Let H be a G-invariant Hamiltonian on P, and let $h : P/G \to \mathbb{R}$ be its Poisson-reduced counterpart. With the assumptions stated in Section 2.1 satisfied, consider an integral curve $x_t \in P$ of X_H. The curve remains in the submanifold $J^{-1}(\mu_0)$ $(\mu_0 \equiv J(x_0))$ for all time t for which it is defined. The Marsden-Weinstein reduction bundle

$$\pi_{\mu_0} : J^{-1}(\mu_0) \longrightarrow P_{\mu_0} \tag{2.11}$$

is a principal G_{μ_0}-bundle. Let

$$\alpha_{\mu_0} : T\big(J^{-1}(\mu_0)\big) \longrightarrow \mathfrak{g}_{\mu_0} \tag{2.12}$$

denote the connection one-form on this bundle whose associated horizontal space at each $x \in J^{-1}(\mu_0)$ is $\mathrm{hor}_{\mu_0} \equiv A(x)$. To ensure that α_{μ_0} and hor_{μ_0} are smooth, we require that A be μ_0-smooth.

Let $y_t \in P_{\mu_0}$ denote the integral curve of the reduced Hamiltonian vector field X_h on P/G that has $y_0 = \pi(x_0) \in P_{\mu_0}$ as its initial point. Then as X_H and X_h are π-related, we have $y_t = \pi(x_t)$ for all t.

Let $d_t \in J^{-1}(\mu_0)$ denote the hor_{μ_0}-horizontal lift of y_t having x_0 as its initial point d_0. Supposing that y_t is periodic with period T, we have

$$d_T = g_{\mathrm{geom}} \cdot x_0, \qquad x_T = g_{\mathrm{dyn}} \cdot d_T, \tag{2.13}$$

for some uniquely defined $g_{\mathrm{geom}}, g_{\mathrm{dyn}} \in G_{\mu_0}$ called *geometric* and *dynamic* phases associated with the reduced solution Y_t. The product $g_{\mathrm{total}} = g_{\mathrm{geom}} g_{\mathrm{dyn}}$ is called the *total* phase. It does not depend on $A = \widehat{D}$, but depends only on y_0, the flow of X_H, and the period T.

2.3.1. Dynamic phases

It is well known (see [8]) that the dynamic phase is given by the solution of the following initial value problem, known as the *reconstruction equation*:

$$\dot{g}_t = g_t \xi_t, \quad \text{where } \xi_t \equiv \alpha_{\mu_0}\big(X_H(d_t)\big), \quad g_0 = \mathrm{Id}. \tag{2.14}$$

Here $g_t \xi_t$ denotes the tangent action of g_t.

Corollary 3.6 of [3] states that, assuming G_{μ_0} is Abelian, the dynamic phase is given by

$$g_{\mathrm{dyn}} = \exp \int_0^T D_{\mu_0} h(y_t)\, dt. \tag{2.15}$$

2.3.2. Geometric phases

Recall that the geometric phase g_{geom} associated with a solution x_t is the holonomy of a principal connection α_{μ_0} on $J^{-1}(\mu_0) \to P_{\mu_0}$ along the corresponding reduced solution curve $y_t = \pi(x_t) \in P_{\mu_0}$. Assuming G_{μ_0} is Abelian, the holonomy of appropriate curves is determined by the curvature of α_{μ_0}. It is well known (see [8]) that if the cycle y_t is in fact a boundary $\partial\Sigma$ ($\Sigma \subset P_{\mu_0}$ compact and oriented), then

$$g_{\mathrm{geom}} = \exp\left(- \int_\Sigma \Omega_{\mu_0} \right), \tag{2.16}$$

where Ω_{μ_0} is the curvature of α_{μ_0}, viewed as a $\mathfrak{g}_{\mu_0}$-valued two-form on the reduced space P_{μ_0}.

Theorem C of [3] shows that *all* curvature information on α_{μ_0} is encoded in (i) the connection D on the symplectic stratification of P/G corresponding to the reconstruction connection A, together with (ii) the Poisson structure on P/G.

The connection D allows to "assemble" the reduced symplectic structures ω_Λ ($\Lambda \subset P/G$ a symplectic leaf) into a single two-form ω_D on P/G by decreeing that

$$\omega_D(u,v) \equiv \omega_\Lambda\big(p_D u, p_D v\big), \quad u,v \in T(P/G), \tag{2.17}$$

where Λ denotes the leaf to which the common base point of u and v belongs, and where $p_D : T(P/G) \to E$ denotes the projection along D onto the characteristic distribution E.

We remark that in general ω_D need not be smooth, but if P_{μ_0} is a nondegenerate symplectic leaf, then ω_D is smooth wherever D is of constant rank and smooth. Then, Corollary 4.5 of [3] states that assuming G_{μ_0} is Abelian and ω_D is smooth in a neighborhood of P_{μ_0}, the geometric phase is given by

$$g_{\text{geom}} = \exp\int_\Sigma D_{\mu_0}\omega_D. \tag{2.18}$$

3. Connections on various bundles for Abelian groups

In this section, the relation between connections on Poisson and symplectic bundles is analyzed. This establishes the validity of the application of results in [3] to our settings in the case of Abelian groups G, so that the metric orthogonal spaces to the group orbit in the whole tangent T_xP as well as within the kernel $\operatorname{Ker} TJ(x) \subset T_xP$ both constitute valid horizontal spaces for Poisson and symplectic bundles, respectively.

In Section 3.2, the formalism of dual pairs is introduced into the picture. The symplectic leaf correspondence theorem brings insight into the structure of various bundles and relates the corresponding connections. For the Abelian case, it gives a new interpretation of the connection on symplectic stratification D as a connection on the bundle $j : P/G \to U \subset \mathfrak{g}^*$ of symplectic leaves over the dual of the Lie algebra (see Section 3.2).

3.1. Connections on Poisson and symplectic bundles

Consider the relation between a connection on the Poisson reduction bundle $P \to P/G$ and connections on each of the symplectic Marsden-Weinstein reduction bundles $J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu$ for different $\mu \in \mathfrak{g}^*$. This relation can be easily established in the case of an Abelian group G when $G_\mu \equiv G$ and both bundles have similar fibers.

Recall that a connection on the bundle $P \to P/G$ is a Lie algebra valued one-form $\mathcal{A}$ on P that is G-equivariant $g \cdot \mathcal{A} = \mathrm{Ad}_g \cdot \mathcal{A}$ and satisfies $\mathcal{A}(\xi_P) = \xi,\ \forall \xi \in \mathfrak{g}$. The corresponding horizontal space is defined by $\mathrm{hor} = \operatorname{Ker}\mathcal{A}$. The following theorem then holds.

Theorem 3.1. *For the case of an Abelian group* G, *a connection* $\mathcal{A}$ *on the Poisson bundle induces connections* α_μ *on symplectic Marsden-Weinstein* *bundles for regular momentum values* μ. *In particular*, *it defines a reconstruction connection* A *on* P. *Moreover*, *the connections on the symplectic stratification of* P/G *corresponding to* A *and to* $\mathcal{A}$ *coincide*, *that is*,

$$\widehat{A} = D = \widehat{\mathcal{A}}. \tag{3.1}$$

Proof. Choose a regular value $\mu \in \mathfrak{g}^*$ such that the symplectic reduction at μ is defined. Define induced horizontal and vertical spaces at $x \in J^{-1}(\mu)$ by the intersections with $\operatorname{Ker} TJ$:

$$\mathrm{hor}_\mu = \mathrm{hor} \cap \operatorname{Ker} TJ, \qquad \mathrm{ver}_\mu = \mathrm{ver} \cap \operatorname{Ker} TJ. \tag{3.2}$$

By definition, $\mathrm{hor}_\mu \cap \mathrm{ver}_\mu = 0$. As G is Abelian, $G_\mu = G$, and $\operatorname{Ker} T\pi \subset (\operatorname{Ker} T\pi)^\omega = \operatorname{Ker} TJ$, so that $\mathrm{ver}_\mu = \operatorname{Ker} T\pi$. Using the following set-theoretical identity $(A+B) \cap C = A + B \cap C$ if $A \subset C$, we obtain

$$\begin{aligned} \operatorname{Ker} TJ &= (\operatorname{Ker} T\pi + \mathrm{hor}) \cap \operatorname{Ker} TJ \\ &= \operatorname{Ker} T\pi + \mathrm{hor} \cap \operatorname{Ker} TJ \\ &= \mathrm{ver}_\mu + \mathrm{hor}_\mu. \end{aligned} \tag{3.3}$$

Hence, $\operatorname{Ker} TJ = \mathrm{ver}_\mu \oplus \mathrm{hor}_\mu$. The corresponding connection one-form α_μ is defined by the horizontal space via $\operatorname{Ker} \alpha_\mu = \mathrm{hor}_\mu$. The collection of these α_μ then define a reconstruction connection A as defined in Section 2. It is G-invariant because $\mathcal{A}$ is G-invariant for Abelian groups.

Finally, for the connections on the symplectic stratification of P/G determined by connections on Poisson and symplectic bundles, that is, by $\mathcal{A}$ and A, respectively, we have at $y = \pi(x)$

$$\begin{aligned} D'(y) &\equiv \widehat{\mathcal{A}}(x) = T\pi\big(\mathrm{hor}^\omega(x)\big), \\ D(y) &\equiv \widehat{A}(x) = T\pi\big((\mathrm{hor}_\mu)^\omega\big) = T\pi\big((\mathrm{hor} \cap \operatorname{Ker} TJ)^\omega\big) \\ &= T\pi\big((\mathrm{hor})^\omega + (\operatorname{Ker} TJ)^\omega\big) = T\pi\big((\mathrm{hor})^\omega + \operatorname{Ker} T\pi\big) = T\pi\big(\mathrm{hor}^\omega(x)\big), \end{aligned} \tag{3.4}$$

where $x \in J^{-1}(\mu)$ with $y = \pi(x)$ and we have used that $(\operatorname{Ker} TJ)^\omega = \operatorname{Ker} T\pi$. Comparing the last two expressions we conclude that $D = D'$. □

This result enables us to go back and forth between connections on Poisson and symplectic bundles for Abelian groups; in particular, it will let us apply results of [3] for the reconstruction phases and use the abstract mechanical connection (defined in Section 4) as a reconstruction connection.

3.2. Connections on dual pairs

Recall the notion of dual pairs introduced by Weinstein [13]. Consider a symplectic manifold (P, ω), Poisson manifolds Q_1, Q_2, and Poisson maps ρ_i :

$P \to Q_i$, $i = 1,2$. If for almost all $x \in P$, $(\operatorname{Ker} T\rho_1(x))^\omega = \operatorname{Ker} T\rho_2(x)$, the diagram $Q_1 \xleftarrow{\rho_1} P \xrightarrow{\rho_2} Q_2$ is called a *dual pair*. The dual pair is called *full*, if ρ_1, ρ_2 are surjective submersion. If $Q_1 \xleftarrow{\rho_1} P \xrightarrow{\rho_2} Q_2$ is a full dual pair, then the spaces of Casimir functions on Q_1 and Q_2 are in bijective correspondence, that is, $\operatorname{Cas}(Q_1) \circ \rho_1 = \operatorname{Cas}(Q_2) \circ \rho_2$ (Weinstein [13]).

It was shown in Adam and Ratiu [1] that for a symplectic manifold (P, ω) with a Hamiltonian action of a Lie group G having an equivariant momentum map $J : P \to U \subset \mathfrak{g}^*$, $U \equiv J(P)$, such that $\pi : P \to P/G$ and J are surjective submersion, $P/G \xleftarrow{\pi} P \xrightarrow{J} U$ is a full dual pair. The Poisson reduced space P/G, being a base of a principle G-bundle, is itself foliated by symplectic leaves Σ_y through points $y \in P/G$. We denote the space of symplectic leaves by $\mathcal{S}$. With the proper connectedness assumptions, these leaves are precisely the symplectic reduced spaces $P_\mu = J^{-1}(\mu)/G_\mu$ (note that G_μ can be different for different values of μ).

On the other hand, P is foliated by the level sets of the momentum map $J^{-1}(\mu)$, for different $\mu \in \mathfrak{g}^*$, with the dual of the Lie algebra itself being a foliation by coadjoint orbits $\mathcal{O}_\mu$ through μ. It follows from the symplectic leaf correspondence theorem [13] that, under the assumptions in the previous paragraph, the base space of this foliation is in one-to-one correspondence with $\mathcal{S}$, the space of symplectic leaves of the Poisson reduced space P/G. A natural one-to-one correspondence between the symplectic leaves in each leg of a dual pair has been described in Weinstein [13], together with a sketch of the proof. Here, we state the symplectic leaf correspondence theorem and refer for a detailed and comprehensive proof to Blaom [4].

Theorem 3.2. *Let* P *be a symplectic manifold and* $Q_1 \xleftarrow{\rho_1} P \xrightarrow{\rho_2} Q_2$ *a full dual pair. Assume that each leg* $\rho_j : P \to Q_j$, $j = 1,2$ *satisfies the property that pre-images of connected sets are connected. Let* $\mathcal{F}_j$ *denote the set of symplectic leaves in* Q_j. *Then*, *under the assumptions outlined in Section 3.2 above*, *there exists a bijection* $\mathcal{F}_1 \to \mathcal{F}_2$ *given by*

$$\Sigma_1 \longmapsto \rho_2\big(\rho_1^{-1}(\Sigma_1)\big) \tag{3.5}$$

having inverse

$$\Sigma_2 \longmapsto \rho_1\big(\rho_2^{-1}(\Sigma_2)\big). \tag{3.6}$$

This theorem enables us to define a leaf-to-leaf bijection that maps symplectic leaves Σ_y (which are diffeomorphic to symplectic reduced spaces P_μ) to coadjoint orbits $\mathcal{O}_\mu$ in the dual of the Lie algebra, $\mu = J(x)$. Yet another realization of the symplectic leaves Σ_y is given by the orbit reduction theorem [10] which establishes one-to-one correspondence between orbit reduced spaces $P_{\mathcal{O}_\mu} = J^{-1}(\mathcal{O}_\mu)/G$ and symplectic reduced spaces $P_\mu = J^{-1}(\mu)/G_\mu$.

In the case of an Abelian group G, the coadjoint orbits are trivial, that is, $\mathcal{O}_\mu = \{\mu\}$ and $G_\mu = G$, so that $P_{\mathcal{O}_\mu} \equiv P_\mu$ and $\mathcal{S} \cong \mathfrak{g}^*$. It follows then from the reduction lemma (cf. [10]) that G-orbits of any point $x \in P$ are *isotropic*, that is, $T_x(G\cdot x) \subset (T_x(G\cdot x))^{\omega}$ or, equivalently, $\operatorname{Ker} T_x\pi \subset (\operatorname{Ker} T_x\pi)^{\omega}$. Moreover, the bijection $\mathcal{F}_1 \to \mathcal{F}_2$ becomes a well-defined map of the manifolds, $j : P/G \to U$ which can be obtained through factoring the momentum map. Indeed, the equivariance of the momentum map $J : P \to U \subset \mathfrak{g}^*$ amounts in the Abelian case to invariance. It therefore factors through $\pi : P \to P/G$, delivering a map $j : P/G \to U$ making the diagram in Figure 3.1 commute.

$$\begin{tikzcd}
 & P \arrow[dl, "J"'] \arrow[dr, "\pi"] & \\
\mathfrak{g}^* & & P/G \arrow[ll, "j"']
\end{tikzcd}$$

Figure 3.1. The momentum map J factors through delivering a map $j : P/G \to U \subset \mathfrak{g}^*$.

The map j is a submersion since J is a submersion (under our hypothesis of a free action). Since the coadjoint orbits are points, the symplectic leaves in P/G are simply the fibers of j, that is, $P_\mu = j^{-1}(\mu)$, $\mu \in U$.

Thus, with this interpretation, the connection on the symplectic stratification D can be thought of as an (Ehresmann) connection on the bundle $j : P/G \to U \subset \mathfrak{g}^*$. Theorem 2.1 of [3] as well as the results of Section 3 establish a relation between the connections A and D on the bundles $\pi : P \to P/G$ and $j : P/G \to U \subset \mathfrak{g}^*$, respectively.

Finally, the tangent map Tj delivers the isomorphism of Proposition 2.5, where now L does not depend explicitly on μ as G is Abelian; that is, $\mathfrak{g}^*_\mu = \mathfrak{g}^*$, $p_\mu \equiv \mathrm{Id}$, and the dependence on μ enters only through $\mu = j(y)$.

Lemma 3.3. *Let* $L(D,y) : \mathfrak{g}^* \to D(y)$ *be defined by (2.6), then its inverse is given by the tangent map* Tj *restricted to the distribution* D

$$L^{-1} = \operatorname{forg}\left(Tj|_D\right) : D \longrightarrow \mathfrak{g}^*, \tag{3.7}$$

where the map $\operatorname{forg} : TU \to \mathfrak{g}^*$ *denotes the map that "forgets base point."*

Proof. The proof readily follows from the fact that the momentum map factors through the quotient map, so that $TJ = Tj \circ T\pi$, and the definition of the map L for any $y \in P_\mu$ given by (2.6), where w is any vector in T_xP that satisfies $T\pi \cdot w = v$, with $v \in D(y)$ and $x \in J^{-1}(\mu) \cap \pi^{-1}(x)$:

$$\begin{aligned} v \longmapsto L^{-1}(D,y)\cdot v &\equiv p_\mu\big(\operatorname{forg}(TJ\cdot w)\big) \\ &= \operatorname{forg}\big(Tj \circ T\pi \cdot w\big) = \operatorname{forg}\big(Tj \cdot v\big). \end{aligned} \tag{3.8}$$

□

4. Abstract mechanical connection

Let P be an almost Kähler manifold with a complex structure $\mathcal{J} : T_xP \to T_xP$, such that $\mathcal{J}^2 = -1$, a symplectic form ω and a $\mathcal{J}$-invariant Riemannian metric $\mathfrak{s}$ with the standard relation between these structures, (1.1),

$$\omega(v,w) = \mathfrak{s}(\mathcal{J}v,w) \quad \forall v,w \in T_xP. \tag{4.1}$$

Let a Lie group G act on P freely and properly (see [12] for some interesting results on how to relax the regularity conditions) by isometries of the almost Kähler structure, that is, it preserves Riemannian, symplectic, and almost complex forms. The quotient manifold then has a unique Poisson structure such that the canonical projection $\pi : P \to P/G$ is a Poisson map. Assume that the G action admits an equivariant momentum map J and that $P/G \xleftarrow{\pi} P \xrightarrow{J} U \subset \mathfrak{g}^*$ is a full dual pair, that is, π and J are surjective submersions. Though we are not interested here in the results for Kähler reduction (The reader is referred to [5, 12], for example, for Marsden-Weinstein reduction on Kähler manifolds.) we notice that the almost complex structure can be dropped to the quotient space P/G. We keep the same notation for the reduced object but write $\mathcal{J}(y)$, where $y = \pi(x)$, to indicate that it can be computed at any $x \in \pi^{-1}(y)$.

4.1. Main constructions

Definition 4.1. The *abstract locked inertia tensor* $\mathbb{I}(x) : \mathfrak{g} \to \mathfrak{g}^*$, $\forall x \in P$, is defined by the following expression:

$$\left\langle \mathbb{I}(x) \cdot \xi, \eta \right\rangle = \mathfrak{s}\left(\xi_P(x), \eta_P(x)\right) \tag{4.2}$$

for any Lie algebra elements $\xi, \eta \in \mathfrak{g}$, where ξ_P, η_P are the corresponding infinitesimal generators, that is, vector fields on P.

The abstract locked inertia tensor is, obviously, an isomorphism for any $x \in P$ for which the group action is free. For a general Lie group, it is G-equivariant in the sense of a map $\mathbb{I} : P \to L(\mathfrak{g}, \mathfrak{g}^*)$, namely

$$\mathbb{I}(g \cdot x) \cdot \mathrm{Ad}_g \xi = \mathrm{Ad}^*_{g^{-1}} \mathbb{I}(x) \cdot \xi. \tag{4.3}$$

For an Abelian group, the abstract locked inertia tensor is, in fact, G-invariant and, hence, can be dropped to the quotient P/G. We use the same notation for the reduced object but write $\mathbb{I}(y)$, where $y = \pi(x)$, to indicate that it can be computed at any $x \in \pi^{-1}(y)$.

Definition 4.2. For any choice of a principle connection on P/G, define the *induced metric* $\mathfrak{s}'$ on P/G in the following way. Let $a, b \in T_y(P/G)$ and let $\tilde{a}, \tilde{b}$ be their corresponding pre-images in the horizontal subspace, that is,

$\tilde{a}, \tilde{b} \in \mathrm{hor}(x)$, $\pi(\tilde{a}) = a$, and $\pi(\tilde{a}) = b$, where $x \in \pi^{-1}(y)$. As the metric is G-invariant we can define $\mathfrak{s}'(a,b) = \mathfrak{s}_x(\tilde{a}, \tilde{b})$, for any $x \in \pi^{-1}(y)$.

Definition 4.3. The *abstract mechanical connection* on the principle G-bundle $P \to P/G$ is defined by specifying a horizontal space within T_xP at each point $x \in P$ to be metric-orthogonal to the tangent to the group orbits

$$\mathrm{hor}(x) = \big\{ v \in T_xP \mid \mathfrak{s}\big(v, \xi_P(x)\big) = 0 \ \forall \xi \in \mathfrak{g} \big\}. \tag{4.4}$$

The connection one-form $\mathcal{A}$ is determined by $\mathrm{Ker}\mathcal{A}(x) = \mathrm{hor}(x)$; an explicit expression for it is given by the following theorem.

Theorem 4.4. *Abstract mechanical connection on an almost Kähler manifold is given by*

$$\mathcal{A}(x) \cdot w = \mathbb{I}^{-1}(x) \cdot \mathfrak{s}\big(\omega^{\#}\big(dJ(x)\big), w\big) \quad \forall w \in T_xP. \tag{4.5}$$

Proof. For any tangent vector $w \in T_xP$ and any Lie algebra element $\eta \in \mathfrak{g}$:

$$\mathfrak{s}\big(w, \eta_P\big) = \mathfrak{s}\big(w_v + w_h, \eta_P\big) = \mathfrak{s}\big(w_v, \eta_P\big) = \mathfrak{s}\big(\xi_P^w, \eta_P\big) = \big\langle \mathbb{I}(x)\xi^w, \eta \big\rangle, \tag{4.6}$$

where $w_v = \xi_P^w$ for some $\xi^w \in \mathfrak{g}$ is a vertical (fiber) component, w_h is a horizontal component, and $\mathfrak{s}(w_h, \eta_P) = 0$ by definition.

By definition of the momentum map $\eta_P = \omega^{\#}(d\langle J(x), \eta\rangle)$, so that

$$\mathfrak{s}\big(w, \eta_P\big) = \mathfrak{s}\big(w, \omega^{\#}\big(d\langle J(x), \eta\rangle\big)\big) = \big\langle \mathfrak{s}\big(\omega^{\#}(dJ), w\big), \eta \big\rangle, \tag{4.7}$$

where dJ is thought of as a $\mathfrak{g}^*$-valued one-form on P, and we have used the fact that the pairing between $\mathfrak{g}$ and its dual is independent of $x \in P$. Thus,

$$\big\langle \mathbb{I}(x)\xi^w, \eta \big\rangle = \big\langle \mathfrak{s}\big(\omega^{\#}(dJ), w\big), \eta \big\rangle, \tag{4.8}$$

and the result follows from the nondegeneracy of the pairing.

To verify that $\mathcal{A}$ indeed defines a connection, we check that it satisfies $\mathcal{A}(\xi_P(x)) = \xi \ \forall \xi \in \mathfrak{g}$ and is G-equivariant. Consider the pairing of $\mathbb{I}(x) \cdot \mathcal{A}(\xi_P(x))$ with an arbitrary element from the Lie algebra $\eta \in \mathfrak{g}$ and use Definitions 4.1, 4.2, and 4.3 of the connection and the abstract locked inertia tensor:

$$\begin{aligned} \big\langle \mathbb{I}(x) \cdot \mathcal{A}\big(\xi_P(x)\big), \eta \big\rangle &= \big\langle \mathfrak{s}\big(\omega^{\#}\big(dJ(x)\big), \xi_P(x)\big), \eta \big\rangle \\ &= \mathfrak{s}\big(\xi_P, \eta_P\big) = \big\langle \mathbb{I}(x)\xi, \eta \big\rangle. \end{aligned} \tag{4.9}$$

From the nondegeneracy of the pairing, it follows that $\mathcal{A}(\xi_P(x)) = \xi$. The G-equivariance means that $\Phi_g^* \mathcal{A} = \mathrm{Ad}_g \mathcal{A}$ and follows from equivariance of the momentum map and equivariance of the abstract locked inertia tensor in the sense of a map $\mathbb{I} : P \to L(\mathfrak{g}, \mathfrak{g}^*)$ (see (4.3)). ☐

Corollary 4.5. *The connection one-form can be written as follow*:

$$\mathcal{A}(x)\cdot w = \mathbb{I}^{-1}(x)\cdot \mathrm{forg}\big(TJ(x)(\mathcal{J}w)\big) \quad \forall w \in T_xP. \tag{4.10}$$

Then,

$$\mathrm{hor}(x) = \mathrm{Ker}\big(TJ(x)\circ \mathcal{J}\big). \tag{4.11}$$

Proof. Using (1.1), $\mathcal{J}^2 = -1$ and omitting x for simplicity, we have $\forall w \in T_xP$

$$\begin{aligned}\mathcal{A}\cdot w &= \mathbb{I}^{-1}\cdot \mathfrak{s}\big(\omega^{\#}(dJ), -\mathcal{J}^2 w\big)\\ &= \mathbb{I}^{-1}\cdot \omega\big(\omega^{\#}(dJ), \mathcal{J}w\big)\\ &= \mathbb{I}^{-1}\cdot \mathrm{forg}\big(TJ(\mathcal{J}w)\big),\end{aligned} \tag{4.12}$$

where for the last equality we used the definition of a symplectic form and considered the one-form d_xJ as a tangent map $\mathrm{forg}\circ TJ$ acting on vectors in T_xP. □

Lemma 4.6. *For the choice of the abstract mechanical connection* $\mathcal{A}$ *on* P *with* $\mathrm{hor} = (\mathrm{Ker}\, T\pi)^{\perp}$, *the following holds*:

$$\mathrm{hor}^{\omega} = (\mathrm{Ker}\, TJ)^{\perp} = \mathcal{J}(\mathrm{Ker}\, T\pi). \tag{4.13}$$

Proof. The proof follows readily from (4.11) of Corollary 4.5 and the ω-orthogonality of $\mathrm{Ker}\, TJ$ and $\mathrm{Ker}\, T\pi$:

$$\begin{aligned}\mathrm{hor}^{\omega} = \big(\mathrm{Ker}(TJ\circ\mathcal{J})\big)^{\omega} = \big(\mathcal{J}(\mathrm{Ker}\, TJ)\big)^{\omega} &= \Big(\big((\mathrm{Ker}\, TJ)^{\perp}\big)^{\omega}\Big)^{\omega}\\ = (\mathrm{Ker}\, TJ)^{\perp} = \big((\mathrm{Ker}\, T\pi)^{\omega}\big)^{\perp} &= \mathcal{J}\big(\mathrm{Ker}(TJ)\big),\end{aligned} \tag{4.14}$$

where we used that $((W)^{\omega})^{\perp} = \mathcal{J}(W)$ for a subspace $W \in T_xP$. □

Below we present two alternative proofs of this lemma which provide an interesting insight into the issue; these proofs can be skipped on the first reading.

Alternative proof. By definition, $w \in \mathrm{hor}^{\omega}(x)$ if and only if

$$\omega(v, w) = \mathfrak{s}(v, \mathcal{J}w) = 0 \quad \forall v \in \mathrm{hor}(x). \tag{4.15}$$

Thus, $w \in \mathrm{hor}^{\omega}(x) \Leftrightarrow \mathcal{J}w \in (\mathrm{hor}(x))^{\perp}$, or

$$w \in \big(\mathrm{hor}^{\omega}(x)\big)^{\perp} \iff \mathcal{J}w \in \mathrm{hor}(x) = \big(\mathrm{Ker}\, T\pi(x)\big)^{\perp}. \tag{4.16}$$

On the other hand, $u \in (\mathrm{Ker}\, T\pi(x))^{\omega} = \mathrm{Ker}\, TJ(x)$ if and only if

$$\omega\big(u, \xi_P(x)\big) = \mathfrak{s}\big(\xi_P(x), \mathcal{J}u\big) = 0 \quad \forall \xi \in \mathfrak{g}. \tag{4.17}$$

Thus,

$$u \in \operatorname{Ker} TJ(x) \Longleftrightarrow \mathcal{J}u \in \big(T_x(G\cdot x)\big)^{\perp} \equiv \big(\operatorname{Ker} T\pi(x)\big)^{\perp}. \tag{4.18}$$

Comparing conditions for u and w, we conclude that $\mathrm{hor}^{\omega}(x) = (\operatorname{Ker} TJ(x))^{\perp}$. Then, (4.18) follows from $\operatorname{Ker} TJ = (\operatorname{Ker} T\pi)^{\omega}$ and $((\operatorname{Ker} T\pi)^{\omega})^{\perp} = \mathcal{J}(\operatorname{Ker} T\pi)$. ☐

Alternative proof. First notice that

$$\mathrm{hor}^{\omega} = \big((\operatorname{Ker} T\pi)^{\perp}\big)^{\omega} = \big((\operatorname{Ker} T\pi)^{\omega}\big)^{\perp} = (\operatorname{Ker} TJ)^{\perp}. \tag{4.19}$$

The last equality in (4.19) follows from the following argument. Let $A \subset T_xP$, then $a \in A^{\perp} \Leftrightarrow \mathfrak{s}(a,b) = 0\ \forall b \in A$. Similarly, $c \in (A^{\perp})^{\omega} \Leftrightarrow \omega(c,a) = 0\ \forall a \in A^{\perp}$. But $0 = \omega(c,a) = \mathfrak{s}(\mathcal{J}c, a)\ \forall a \in A^{\perp}$ implies that

$$c \in \big(A^{\perp}\big)^{\omega} \Longleftrightarrow \mathcal{J}c \in \big(A^{\perp}\big)^{\perp} \equiv A. \tag{4.20}$$

This is equivalent to $c \in \mathcal{J}(A) \Leftrightarrow c \in (A^{\perp})^{\omega}$, so that $(A^{\perp})^{\omega} = \mathcal{J}(A)$ and $((\operatorname{Ker} T\pi)^{\perp})^{\omega} = \mathcal{J}(\operatorname{Ker} T\pi)$. ☐

Define for any point $\nu \in \mathfrak{g}^*$ a one-form $\mathcal{A}_{\nu}(x) = \langle \nu, \mathcal{A}(x)\rangle$ on P.

Lemma 4.7. *Identifying vectors and one-forms on* P *via Riemannian metric*

$$\big(\mathcal{A}_{\nu}(x)\big)^{\#} = \big(\mathbb{I}^{-1}(x)\cdot\nu\big)_P. \tag{4.21}$$

Proof. Using (4.5) we obtain $\forall w \in T_xP$

$$\begin{aligned} \mathcal{A}_{\nu}\cdot w &= \big\langle \nu, \mathbb{I}^{-1}\cdot \mathfrak{s}\big(\omega^{\#}(dJ), w\big)\big\rangle \\ &= \mathfrak{s}\big(\omega^{\#}\big(d\langle J, \mathbb{I}^{-1}\nu\rangle\big), w\big) = \mathfrak{s}\big(\big(\mathbb{I}^{-1}\nu\big)_P, w\big). \end{aligned} \tag{4.22}$$

☐

4.2. Abelian groups and reconstruction phases

In the rest of this section we assume that the Lie group G is Abelian. A simple corollary of Theorem 3.1 implies that metric orthogonal horizontal spaces on the Poisson bundle $P \to P/G$ induce metric orthogonal horizontal spaces on symplectic bundles $J^{-1}(\mu) \to P_{\mu}$ for regular μ. Hence, by analogy, the reconstruction connection A corresponding to $\mathcal{A}$ by means of Theorem 3.1 can be called an *abstract mechanical reconstruction connection*. The same theorem gives also the corresponding connection on the symplectic stratification $D = \widehat{A}$ by specifying its horizontal spaces to be $T\pi(\mathrm{hor}^{\omega})$. The following results significantly simplify explicit computations of these spaces, that is, the distribution D.

Theorem 4.8. *For the choice of the abstract mechanical connection* $\mathcal{A}$ *on* P, *the distribution* D, *which corresponds to the connection on the symplectic stratification* $j : P/G \to \mathfrak{g}^*$, *is metric orthogonal to the characteristic distribution* E *in the metric* $\mathfrak{s}'$ *induced on the quotient* P/G. *Moreover, the distribution* D *can be explicitly constructed using the infinitesimal generator vector fields* ξ_P *according to the following expression*:

$$D(y) = T\pi\big(\mathcal{J}\big(\operatorname{Ker} T\pi(x)\big)\big), \tag{4.23}$$

where $x \in \pi^{-1}(y)$ *and* $\operatorname{Ker} T\pi(x) = \operatorname{span}\{\xi_P(x)\}$.

Proof. Consider any vectors $v \in D(y)$ and $w \in E(y) \equiv T_y\Sigma_y$. By definition of the induced metric $\mathfrak{s}'(v,w) = \mathfrak{s}(\tilde{v},\tilde{w})$, where $\tilde{v},\tilde{w} \in \operatorname{hor}(x)$ are horizontal components of the pre-images: $T\pi(\tilde{v}) = v$, $T\pi(\tilde{w}) = w$, and $x \in \pi^{-1}(y)$.

From $T\pi(\tilde{v}) = v \in T\pi(\operatorname{hor}^{\omega})$ it follows that $\tilde{v} \in \operatorname{hor}^{\omega} + \operatorname{Ker} T\pi$. But $\tilde{v} \in \operatorname{hor} \equiv (\operatorname{Ker} T\pi)^{\perp}$, so that by Lemma 4.6,

$$\tilde{v} \in \operatorname{hor}^{\omega} \cap (\operatorname{Ker} T\pi)^{\perp} \equiv (\operatorname{Ker} TJ)^{\perp} \cap (\operatorname{Ker} T\pi)^{\perp}. \tag{4.24}$$

For the vector $w \in E(y)$ it holds $Tj(w) = 0$ and, hence, by the commutativity of the diagram in Figure 3.1, $TJ(\tilde{w}) = 0$ for any of its pre-images. In particular, for the horizontal pre-image $\tilde{w} \in \operatorname{hor}$ we have

$$\tilde{w} \in \operatorname{Ker} TJ \cap (\operatorname{Ker} T\pi)^{\perp}. \tag{4.25}$$

From the expressions for $\tilde{v}$ and $\tilde{w}$, it follows that $\mathfrak{s}'(v,w) = \mathfrak{s}(\tilde{v},\tilde{w}) = 0$.

Finally, (4.23) follows from $D = T\pi((\operatorname{hor})^{\omega})$ and Lemma 4.6. □

4.2.1. Transverse derivatives

Here we give a new construction of the map L defined by Proposition 2.5 which is crucial for the definition of the transverse derivatives, and hence for the computation of the phases. Our construction is based on Lemma 3.3 and depends implicitly on the choice of the abstract mechanical connection.

Definition 4.9. For each point $y \in P/G$ define a map $N(D,y) : \mathfrak{g} \to D(y)$ by

$$\xi \longmapsto T\pi\big(\mathcal{J}\big(\xi_P(x)\big)\big), \tag{4.26}$$

where ξ is a Lie algebra element, $\xi_P(x)$ is its corresponding infinitesimal generator at $x \in \pi^{-1}(y) \subset P$, and $\mathcal{J}$ is the almost complex structure on P.

From (4.26) it follows that N is a linear map as all maps used in its definition are linear. From the symplectic leaf correspondence theorem and the fact that G is Abelian and finite dimensional it follows that the dimension of

$D(y)$ (which equals the co-dimension of the leaf Σ_y) equals the dimension of the algebra $\mathfrak{g}$. On the other hand,

$$\dim\big(\mathrm{hor}^{\omega}(x)\big)=\dim\big(\mathcal{J}\big(\operatorname{Ker}T\pi(x)\big)\big)=\dim\big(\operatorname{Ker}T\pi(x)\big)=\dim\mathfrak{g}. \tag{4.27}$$

Hence, from the fact that $D=T\pi(\mathrm{hor}^{\omega})$ the following lemma follows.

Lemma 4.10. *For each $y\in P/G$ the map N is an isomorphism between the Lie algebra $\mathfrak{g}$ and the transverse space $D(y)$ defined at y by the distribution D on the symplectic stratification $j:P/G\to\mathfrak{g}^*$.*

Lemma 4.11. *For an Abelian group G, the map $L(D,y)$ defined in Proposition 2.5 is given by the following composition*:

$$L(D,y)=N(D,y)\circ\mathbb{I}^{-1}(y):\mathfrak{g}^*\longrightarrow D(y), \tag{4.28}$$

where $\mathbb{I}$ is the abstract locked inertia tensor.

Proof. By the definition of the momentum map, $J_{\xi}\equiv\langle J,\xi\rangle$ is a Hamiltonian for the vector field ξ_P of the infinitesimal transformations, that is, for any vector $u\in T_xP$

$$\omega(x)\big(\xi_P,u\big)=d_xJ_{\xi}(u). \tag{4.29}$$

The one-form d_xJ_{ξ} can be thought of as the tangent map TJ acting on vectors in T_xP and paired with $\xi\in\mathfrak{g}$. Take u to be $\mathcal{J}(\eta_P)$ for some infinitesimal generator η_P corresponding to $\eta\in\mathfrak{g}$. Then,

$$\begin{aligned}\omega(x)\big(\xi_P,\mathcal{J}(\eta_P)\big)&=d_xJ_{\xi}\big(\mathcal{J}(\eta_P)\big)=\big\langle d_xJ\big(\mathcal{J}(\eta_P)\big),\xi\big\rangle\\&=\big\langle\,\mathrm{forg}\,\big(TJ\big(\mathcal{J}(\eta_P)\big)\big),\xi\big\rangle\\&=\big\langle\,\mathrm{forg}\,\big(Tj\circ T\pi\big(\mathcal{J}(\eta_P)\big)\big),\xi\big\rangle\\&=\big\langle\,\mathrm{forg}\,\big(Tj\circ N(\eta)\big),\xi\big\rangle,\end{aligned} \tag{4.30}$$

where we used the definition of the map N given by (4.26).

On the other hand,

$$\begin{aligned}\omega(x)\big(\xi_P,\mathcal{J}(\eta_P)\big)&=-\omega(x)\big(\mathcal{J}(\eta_P),\xi_P\big)=\mathfrak{s}(x)\big(\eta_P,\xi_P\big)\\&=\big\langle\mathbb{I}(x)\eta,\xi\big\rangle=\big\langle\mathbb{I}\big([x]\big)\eta,\xi\big\rangle=\big\langle\mathbb{I}(y)\eta,\xi\big\rangle.\end{aligned} \tag{4.31}$$

Alternatively, this expression can be obtained from Corollary 4.5 using an explicit form of the connection one-form given by (4.10).

From (4.30) and (4.31) and the nondegeneracy of the pairing we conclude that $\mathrm{forg}(Tj\circ N)=\mathbb{I}$, then from Lemma 3.3 it follows that

$$L(D,y)=\mathrm{forg}\,\big(Tj|_D\big)^{-1}=N(D,y)\circ\mathbb{I}^{-1}(y). \tag{4.32}$$

Notice that L is an isomorphism as both N and $\mathbb{I}$ are. $\square$

4.2.2. Dynamic phase

Recall that according to (2.15), the infinitesimal dynamic phase is given by the transverse derivative of the reduced Hamiltonian, which we simplify using formula (4.32) for the map L.

Theorem 4.12. *The ν-component of the infinitesimal dynamic phase ξ_{dyn}, for any $\nu \in \mathfrak{g}^*$, can be expressed via the abstract locked inertia tensor and the almost complex structure according to*

$$\big\langle \nu, \xi_{\mathrm{dyn}}(y)\big\rangle = \big\langle \nu, D_\mu h(y)\big\rangle = dh\Big(T\pi\big(\mathcal{J}([x])\big(\big(\mathbb{I}^{-1}([x])\cdot\nu\big)_P\big)\big)\Big), \tag{4.33}$$

where $x \in [x] = \pi^{-1}(y)$ *and* $\mu = j(y)$.

Proof. The proof is quite straightforward and relies on the constructions discussed in this section. Using the definition of the transverse derivative, Lemma 4.11 and G-invariance of the abstract locked inertia tensor, and the almost complex structure, we obtain

$$\begin{aligned}\big\langle \nu, \xi_{\mathrm{dyn}}(y)\big\rangle &= \big\langle \nu, D_\mu h(y)\big\rangle = dh\big(L(D,\mu,y)\cdot\nu\big)\\ &= dh\big(N(D,y)\circ \mathbb{I}^{-1}(x)\cdot\nu\big)\\ &= dh\Big(T\pi\big(\mathcal{J}(x)\big(\big(\mathbb{I}^{-1}(x)\cdot\nu\big)_P\big)\big)\Big),\end{aligned} \tag{4.34}$$

where the last equality follows from (4.26).

As it was pointed out earlier, both $\mathcal{J}$ and $\mathbb{I}$ are G-invariant and, hence, can be dropped to the quotient P/G, so that (4.33) can be computed at any $x \in [x] \equiv y$. □

Remark 4.13. Equation (4.33) is equivalent to

$$\big\langle \nu, \xi_{\mathrm{dyn}}(y)\big\rangle = dH\big(\mathcal{J}(x)\big(\big(\mathbb{I}^{-1}(x)\cdot\nu\big)_P\big)\big), \tag{4.35}$$

where $x \in \pi^{-1}(y)$ and $\mu = j(y)$.

Notice that (4.33) does not depend on the choice of $x \in \pi^{-1}(y)$. This agrees with the general philosophy of [3] that all information about the phases is contained in the reduced quantities. Yet, for the explicit computations it might be convenient to work with the objects in the unreduced space. Alternatively, when one has a good model of the reduced space P/G, one can compute a basis $\boldsymbol{v}_k$ of the distribution D at any $y \in P/G$ using isomorphism L and Lemma 4.11 corresponding to a basis e_i of $\mathfrak{g}^*$. Then, for any $\nu = \sum \nu^i e_i \in \mathfrak{g}^*$, the corresponding ν-component of the dynamic phase is given by the derivative of the reduced Hamiltonian in the direction $v = \sum \nu^k \boldsymbol{v}_k$, that is, $\langle \nu, \xi_{\mathrm{dyn}}(y)\rangle = dh(\sum \nu^k \boldsymbol{v}_k)$.

4.2.3. Geometric phase

Assuming the μ-regularity of the distribution D (see Section 2), the geometric phase is given by the transverse derivative of the assembled reduced symplectic form ω_D according to (2.17). In this section, we give an explicit construction of this form ω_D using the horizontal lifts with respect to the abstract mechanical connection A and the *unreduced* symplectic form ω. This allows us to circumvent explicit computations of the curvature of the connection one-form that is used in (2.16) and, in some cases, also the computations of the reduced symplectic form that is used in (2.17).

Definition 4.14. For an Abelian group G, define a closed "horizontal" two-form ω' on P/G according to

$$\omega'(y)(u,v) := \omega(x)(\tilde{u},\tilde{v}) \quad \forall u,v \in T_y(P/G),\ y \in P_\mu, \tag{4.36}$$

where $\tilde{u},\tilde{v} \in \mathcal{A}(x) \equiv \text{hor}(x) \equiv (\text{Ker}\, T\pi)^\perp$ with $T\pi(\tilde{u}) = u$, $T\pi(\tilde{v}) = v$, $x \in \pi^{-1}(y)$.

From the G-invariance of the symplectic form ω as well as of the horizontal distribution $\mathcal{A}$ we conclude that ω' is well defined.

Theorem 4.15. *The two-form ω' coincides with the assembled two-form ω_D on P/G:*

$$\omega' = \omega_D. \tag{4.37}$$

Proof. We start with the definition of the two-form ω' Definition 4.14 of the two-form and shall demonstrate that the following three special cases hold for any $y \in P/G$:

(1) $\omega'|_E = \omega_\mu$, here E is the characteristic distribution and $\mu = j(y)$,
(2) $\omega'|_D = 0$,
(3) $\omega'(u,v) = 0$ for any $u \in E(y) \equiv T_y P_\mu$ and $v \in D(y)$,

which all together prove the statement of the theorem, according to the definition of the assembled form (2.17).

(1) From the definition of the reduced symplectic form in the Marsden-Weinstein reduction it follows that

$$\omega_\mu(y)(u,v) = \omega(x)(\breve{u},\breve{v}), \tag{4.38}$$

where $x \in \pi^{-1}(y) \cap J^{-1}(\mu)$ and $\breve{u},\breve{v} \in A(x)$, that is, the pre-images lie in the horizontal space of the reconstruction connection A. Recall that in our case, A denotes the metric orthogonal to the group orbit within the kernel of TJ:

$$A(x) = \big(T_x(G\cdot x)\big)^\perp \cap \text{Ker}\, TJ(x) = \big(\text{Ker}\, T\pi(x)\big)^\perp \cap \text{Ker}\, TJ(x). \tag{4.39}$$

From Lemma 2.3 and the fact that G-orbits of any point $x \in P$ are isotropic for an Abelian group, it follows that

$$\pi^* E = \operatorname{Ker} T\pi + (\operatorname{Ker} T\pi)^{\omega} = (\operatorname{Ker} T\pi)^{\omega}. \tag{4.40}$$

Hence, using the definition on the two-form ω', Definition 4.14 of the two-form, for any vectors $u, v \in E = T_y P_\mu$ lying in the characteristic distribution at $y \in P_\mu$, their pre-images $\tilde{u}, \tilde{v} \in \mathcal{A}(x)$ satisfy

$$\tilde{u}, \tilde{v} \in \mathcal{A}(x) \cap (\operatorname{Ker} T\pi)^{\omega} = (\operatorname{Ker} T\pi)^{\perp} \cap \operatorname{Ker} TJ = A(x), \tag{4.41}$$

so that

$$\omega'(y)(u, v) := \omega(x)(\tilde{u}, \tilde{v}) = \omega(x)(\breve{u}, \breve{v}) = \omega_\mu(y)(u, v). \tag{4.42}$$

(2) Let $u, v \in D(y)$, then $\tilde{u}, \tilde{v} \in \pi^* D = A^{\omega}$, by the definition of the reconstruction connection. But $\tilde{u}, \tilde{v} \in (\operatorname{Ker} T\pi)^{\perp}$, so that

$$\begin{aligned} \tilde{u}, \tilde{v} \in A^{\omega} \cap (\operatorname{Ker} T\pi)^{\perp} &= \big((\operatorname{Ker} T\pi)^{\perp} \cap \operatorname{Ker} TJ\big)^{\omega} \cap (\operatorname{Ker} T\pi)^{\perp} \\ &= \Big(\big((\operatorname{Ker} T\pi)^{\omega}\big)^{\perp} + \operatorname{Ker} T\pi\Big) \cap (\operatorname{Ker} T\pi)^{\perp}. \end{aligned} \tag{4.43}$$

Using the modularity property and the fact that $\operatorname{Ker} T\pi$ is isotropic, that is, $\operatorname{Ker} T\pi \subset (\operatorname{Ker} T\pi)^{\omega}$, and, hence,

$$(\operatorname{Ker} T\pi)^{\perp} \supset \big((\operatorname{Ker} T\pi)^{\omega}\big)^{\perp}, \tag{4.44}$$

we obtain that

$$\tilde{u}, \tilde{v} \in A^{\omega} \cap (\operatorname{Ker} T\pi)^{\perp} = \big((\operatorname{Ker} T\pi)^{\omega}\big)^{\perp} \cap (\operatorname{Ker} T\pi)^{\perp} = \big((\operatorname{Ker} T\pi)^{\omega}\big)^{\perp}, \tag{4.45}$$

but this space is also isotropic, that is, it is contained in its symplectic orthogonal because of (4.44)

$$\big((\operatorname{Ker} T\pi)^{\omega}\big)^{\perp} = \big((\operatorname{Ker} T\pi)^{\perp}\big)^{\omega} \subset \Big(\big((\operatorname{Ker} T\pi)^{\omega}\big)^{\perp}\Big)^{\omega}. \tag{4.46}$$

Thus, $\omega'|_D = 0$.

(3) Finally, combining the two arguments (1) and (2), for any $u \in T_y E$ and $v \in D(y)$, $\tilde{u} \in A$ and $\tilde{v} \in ((\operatorname{Ker} T\pi)^{\omega})^{\perp}$. But,

$$A^{\omega} = \big((\operatorname{Ker} T\pi)^{\perp} \cap \operatorname{Ker} TJ\big)^{\omega} = \big((\operatorname{Ker} T\pi)^{\omega}\big)^{\perp} + (\operatorname{Ker} TJ)^{\omega}, \tag{4.47}$$

so that $\tilde{v} \in A^{\omega}$ and $\omega(\tilde{u}, \tilde{v}) = 0$. □

Corollary 4.16. *The infinitesimal geometric reconstruction phase is computed according to*

$$\xi_{\text{geom}}(y) = D_\mu \omega'(y), \tag{4.48}$$

where the transverse derivative $D_\mu \omega'$ *is computed using Lemma 4.11.*

5. Application: resonant three-wave interaction

The three-wave equations describe the resonant quadratic nonlinear interaction of three waves and are obtained as amplitude equations in an asymptotic reduction of primitive equations in optics, fluid dynamics, and plasma physics. It was first analyzed by Alber, Luther, Marsden, and Robbins in [2] and later in [7]. Here we only quote the results relevant for the definition of the connection and the computation of phases and refer the reader to [2] for the detailed description. Some results for the Poisson reduction obtained here (such expressions for the Casimirs C_1 and C_2 as well as formulas (5.11) for the reduced Poisson bracket and (5.15) for the reduced symplectic structure) are original and were not presented in [2]. We use the canonical Hamiltonian structure and ignore an alternative Lie-Poisson description of this system.

5.1. The phase space and its Kähler structure

The phase space P of the system is $\mathbb{C}^3$ with appropriately weighed standard Kähler structure. In particular, a γ_i-weighed canonical Poisson bracket on $\mathbb{C}^3$ is used. This bracket has the real and imaginary parts of each complex dynamic variable q_i as conjugate variables. The corresponding symplectic structure is written as follows:

$$\omega(z,w) = -\sum_k \frac{1}{s_k\gamma_k}\,\mathrm{Im}\left(z_k\bar{w}_k\right), \tag{5.1}$$

where $z,w \in T_q\mathbb{C}^3$ and s_k are sign variables.

Similarly, define a weighted metric on P

$$\mathfrak{s}(z,w) = \sum_k \frac{1}{s_k\gamma_k}\,\mathrm{Re}\left(z_k\bar{w}_k\right), \tag{5.2}$$

and the standard complex structure $\mathcal{J}(z) = iz$. The Kähler structure then contains $\mathfrak{s}$ and ω as real and imaginary parts, respectively.

5.2. The symmetry group and momentum map

Consider the action of an Abelian group T^2 on $\mathbb{C}^3$ given by

$$\left(q_1,q_2,q_3\right) \longmapsto \left(\exp^{-i\xi^1} q_1, \exp^{-i(\xi^1+\xi^2)} q_2, \exp^{-i\xi^2} q_3\right), \tag{5.3}$$

where $\xi = (\xi_1,\xi_2)$ is an element of the Lie algebra $t^2 \equiv \mathbb{R}^2$. The vector fields of the infinitesimal transformations corresponding to ξ_1,ξ_2 are given by

$$\begin{aligned} \xi^1_P(q) = \left(-i\xi_1 q_1, -i\xi_1 q_2, 0\right),\ \xi^2_P(q) \\ = \left(0, -i\xi_2 q_2, -i\xi_2 q_3\right) \in T_q\mathbb{C}^3. \end{aligned} \tag{5.4}$$

Points of the form $(q_1, 0, 0), (0, q_2, 0), (0, 0, q_3)$ have nontrivial isotropy subgroups, and thus account for singularities in the reduced space, that is, as the action is not free, the reduced space fails to be a smooth manifold (cf. [9]). Henceforth, we ignore these points and restrict ourselves to the set of regular points in $\mathbb{C}^3$.

The momentum map for this action was computed in [2] and is given by

$$J(q_1, q_2, q_3) = (K_1, K_2) = \left(\frac{1}{2} \left(\frac{|q_1|^2}{s_1\gamma_1} + \frac{|q_2|^2}{s_2\gamma_2} \right), \frac{1}{2} \left(\frac{|q_2|^2}{s_2\gamma_2} + \frac{|q_3|^2}{s_3\gamma_3} \right) \right). \tag{5.5}$$

We keep the notations (K_1, K_2) for the values of the momentum map to be consistent with [9]; they play the role of $\mu = J(x)$ which was used throughout the paper.

It is checked directly using (5.3) that the momentum map is G-invariant. For further applications, we note that even though J is not analytic, we can consider its differential as a real-valued map of the tangent space $T\mathbb{C}^3$ to $T\mathbb{R}^2$.

5.3. The Hamiltonian

The Hamiltonian for the three-wave interaction is

$$H = -\frac{1}{2}(\bar{q}_1 q_2 \bar{q}_3 + q_1 \bar{q}_2 q_3). \tag{5.6}$$

Hamilton's equations are $\dot{q}_k = \{q_k, H\}$ and it is straightforward to check that in complex notations they are given by

$$\frac{dq_k}{dt} = -2is_k\gamma_k \frac{\partial H}{\partial \bar{q}_k}. \tag{5.7}$$

5.4. Poisson reduction

It was shown in [2] that the following quantities constitute invariants for the T^2 action

$$X + iY = q_1 \bar{q}_2 q_3, \qquad Z_1 = |q_1|^2 - |q_2|^2, \qquad Z_2 = |q_2|^2 - |q_3|^2. \tag{5.8}$$

They provide coordinates for the four-dimensional (real) orbit space. The symplectic leaves in it are two-dimensional. This follows from the leaf correspondence theorem, as T^2 is Abelian, and each point in it being a coadjoint orbit has co-dimension 2. One can define two Casimirs on $\mathbb{C}^3/T^2$, for example,

$$C_1 = (X^2 + Y^2) - \kappa_4 (2s_2\gamma_2 K_1 + Z_1)(2s_3\gamma_3 K_2 + Z_2)(2s_2\gamma_2 K_2 - Z_2), \tag{5.9}$$

where $\kappa_4 = (s_1\gamma_1 s_2\gamma_2 s_3\gamma_3)/(s_1\gamma_1 + s_2\gamma_2)(s_2\gamma_2 + s_3\gamma_3)^2$, and

$$C_2 = (Z_1 - 2s_1\gamma_1 K_1)(s_2\gamma_2 + s_3\gamma_3) + (Z_2 + 2s_3\gamma_3 K_2)(s_1\gamma_1 + s_2\gamma_2), \tag{5.10}$$

which can be obtained by a pull-back of properly defined Casimirs on $\mathfrak{g}^*$ using ideas of dual pairs (cf. [4, 13]). The level set of these Casimirs, defined for any momentum map value $(K_1, K_2) \in U$, where $U \equiv J(\mathbb{C}^3) \subset \mathfrak{t}^2$, by the set $\{C_1 = 0, C_2 = 0\}$ determines the corresponding symplectic leaf $P_{(K_1,K_2)}$ in the reduced space.

The reduced Poisson bracket on $\mathbb{C}^3/T^2 \cong (X, Y, Z_1, Z_2)$ is given for any two functions f, k by

$$\{f,k\} = \det\left(\nabla C_2 \nabla C_1 \nabla f \nabla k\right). \tag{5.11}$$

The reduced Hamiltonian equations of motion have the following form:

$$\begin{aligned} &\dot{X} = 0, \qquad \dot{Y} = -\frac{\partial C_2}{\partial Z_2}\frac{\partial C_1}{\partial Z_1} + \frac{\partial C_2}{\partial Z_1}\frac{\partial C_1}{\partial Z_2}, \\ &\dot{Z}_1 = \frac{\partial C_2}{\partial Z_2}\frac{\partial C_1}{\partial Y}, \qquad \dot{Z}_1 = -\frac{\partial C_2}{\partial Z_1}\frac{\partial C_1}{\partial Y}. \end{aligned} \tag{5.12}$$

Notice that the second Casimir C_2 establishes a linear dependence between Z_1 and Z_2; hence, we can solve for one of them, say Z_1 and restrict ourselves to the consideration of three-dimensional subspace in $\mathbb{C}^3/T^2$ defined by (X, Y, Z_2). The dynamics in Z_1 can then be trivially reconstructed. In this case, the first Casimir C_1 can be rewritten as

$$\phi = \left(s_2\gamma_2 + s_3\gamma_3\right)\left[\left(X^2+Y^2\right) - \kappa_3\left(\delta - Z_2\right)\left(2s_3\gamma_3K_2 + Z_2\right)\left(2s_2\gamma_2K_2 - Z_2\right)\right], \tag{5.13}$$

where $\kappa_3 = (s_1\gamma_1 s_2\gamma_2 s_3\gamma_3)/(s_2\gamma_2 + s_3\gamma_3)^3$ and $\delta = 2s_2\gamma_2K_1 + 2s_3\gamma_3(K_1 - K_2)$. This relation defines two-dimensional (perhaps singular) surfaces in (X, Y, Z_2) space, with Z_1 determined by the values of the invariants and conserved quantities. These surfaces are called *three-wave surfaces*.

The reduced Poisson bracket in (X, Y, Z_2) space is given by

$$\{f,k\} = \nabla\phi \cdot (\nabla f \times \nabla k) \tag{5.14}$$

for any functions f, k. For a nonsingular point y on a symplectic leaf $P_{(K_1,K_2)}$ the induced symplectic form is then given by

$$\omega_{(K_1,K_2)}(v,w) = -\frac{\nabla\phi}{\|\nabla\phi\|^2}\cdot(v \times w), \tag{5.15}$$

where $v, w \in T_y P_{(K_1,K_2)}$; here (K_1, K_2) is the momentum value which determines a particular symplectic leaf $P_{(K_1,K_2)}$. Thus, for a function f on the orbit space (X, Y, Z_2), the corresponding Hamiltonian vector field has the form $X_f = -\nabla\phi \times \nabla f$.

The reduced Hamiltonian for the three-wave interaction is given by $h = -X$ and produces the following reduced equations of motion:

$$\dot{X} = 0, \qquad \dot{Y} = \frac{\partial\phi}{\partial Z_2}, \qquad \dot{Z}_2 = -2(s_2\gamma_2 + s_3\gamma_3)Y, \tag{5.16}$$

which otherwise can be obtained by the restriction of the equations of motion in $\mathbb{C}^3/T^2 \cong (X,Y,Z_1,Z_2)$ to three-wave surfaces.

5.5. Abstract mechanical connection

First of all, we compute the locked inertia tensor $\mathbb{I}(q) : \mathbb{R}^2 \mapsto \mathbb{R}^2$ using its definition (4.2). It is an isomorphism for regular points q and is given by the following expression:

$$\mathbb{I}(q) = \begin{pmatrix} 2K_1 & \dfrac{|q_2|^2}{s_2\gamma_2} \\ \dfrac{|q_2|^2}{s_2\gamma_2} & 2K_2 \end{pmatrix}, \tag{5.17}$$

where $K_1(q_1,q_2,q_3)$ and $K_2(q_1,q_2,q_3)$ are components of the momentum map given by (5.5). Notice that $\mathbb{I}$ can be dropped to the quotient space:

$$\mathbb{I}(X,Y,Z_1,Z_2) = \begin{pmatrix} 2K_1 & \dfrac{2s_3\gamma_3K_2+Z_2}{s_2\gamma_2+s_3\gamma_3} \\ \dfrac{2s_3\gamma_3K_2+Z_2}{s_2\gamma_2+s_3\gamma_3} & 2K_2 \end{pmatrix}, \tag{5.18}$$

where K_1, K_2 are now functions of (X,Y,Z_1,Z_2) as the momentum map factors through the quotient and are constant on each symplectic leaf, or a three-wave surface of thereof, in the reduced space.

Using Corollary 4.5, we can explicitly construct the corresponding abstract mechanical connection one-form

$$\mathcal{A}(x)\cdot w = -\mathbb{I}^{-1}(x)\cdot \operatorname{Im}\begin{pmatrix} \dfrac{\bar{q}_1w_1}{s_1\gamma_1} + \dfrac{\bar{q}_2w_2}{s_2\gamma_2} \\ \dfrac{\bar{q}_2w_2}{s_2\gamma_2} + \dfrac{\bar{q}_3w_3}{s_3\gamma_3} \end{pmatrix}. \tag{5.19}$$

Comparing expression (5.19) with the definition of the Riemannian structure (5.2), we can conclude that

$$\begin{aligned} w \in \operatorname{hor}(q) &\Longleftrightarrow \begin{cases} \operatorname{Im}\left(\dfrac{\bar{q}_1w_1}{s_1\gamma_1} + \dfrac{\bar{q}_2w_2}{s_2\gamma_2}\right) = 0 \\ \operatorname{Im}\left(\dfrac{\bar{q}_2w_2}{s_2\gamma_2} + \dfrac{\bar{q}_3w_3}{s_3\gamma_3}\right) = 0 \end{cases} \\ &\Longleftrightarrow \begin{cases} \mathfrak{s}(\xi_P^1(q),w) = 0 \\ \mathfrak{s}(\xi_P^2(q),w) = 0, \end{cases} \end{aligned} \tag{5.20}$$

that is, the horizontal space is precisely the metric orthogonal to the group orbits. Similar computations show that $(\operatorname{hor}(q))^\omega$ is determined by the span of vectors $\mathcal{J}(\xi_P^1(q))$, $\mathcal{J}(\xi_P^2(q))$, and

$$\big(\mathrm{hor}(q)\big)^{\omega} = \mathrm{span}\big\{i\xi_P^1(q), i\xi_P^2(q)\big\} = \big(\mathrm{Ker}\, TJ(q)\big)^{\perp}. \tag{5.21}$$

The distribution $D = \widehat{A}$ on the symplectic stratification $j : P/G \to \mathfrak{g}^*$ is obtained by applying the tangent map $T\pi$ to the space $(\mathrm{hor}(q))^{\omega}$. Define vectors $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2$ tangent to the quotient space at the point (X, Y, Z_1, Z_2) to be the images of a basis in $(\mathrm{hor}(q))^{\omega}$ under this tangent map:

$$\begin{aligned} \boldsymbol{\nu}_1 &= T\pi(q)\big(i\xi_P^1\big) = 2\xi^1 \cdot \left(X, Y, Z_1, \frac{s_2\gamma_2}{s_2\gamma_2 + s_3\gamma_3}\big(2s_3\gamma_3K_2 + Z_2\big)\right), \\ \boldsymbol{\nu}_2 &= T\pi(q)\big(i\xi_P^2\big) = 2\xi^2 \cdot \left(X, Y, -\frac{s_2\gamma_2}{s_1\gamma_1 + s_2\gamma_2}\big(2s_1\gamma_1K_1 - Z_1\big), Z_2\right). \end{aligned} \tag{5.22}$$

Then,

$$D\big(X, Y, Z_1, Z_2\big) = T\pi\big(\mathrm{hor}(q)\big)^{\omega} = \mathrm{span}\big\{\boldsymbol{\nu}_1, \boldsymbol{\nu}_2\big\}. \tag{5.23}$$

To finish the construction of the map L given by Lemma 4.11, we need to substitute $\mathbb{I}^{-1} \cdot \nu$, $\forall \nu \in (\mathfrak{t}^2)^*$ for the ξ. Then, for any $\nu = \sum \nu_k e^k \in (\mathfrak{t}^2)^*$, where e^k is the dual basis of $(\mathfrak{t}^2)^*$, the map L is given by

$$L : \nu \longmapsto L(\nu) = \sum \nu_k T\pi\Big(i\big(\mathbb{I}^{-1} \cdot \nu\big)_P^k\Big) = \sum \nu_k \boldsymbol{\nu}_k, \tag{5.24}$$

where in the expressions for $\boldsymbol{\nu}_k$ we take $\xi^k = (\mathbb{I}^{-1} \cdot \nu)^k$.

5.6. Phases for the three-wave interaction

Recall that the reduced Hamiltonian on P/G is $h = -X$. Applying Theorem 4.12, we can immediately obtain the ν-component the associated dynamic phase by computing directional derivatives of the reduced Hamiltonian in the directions $\boldsymbol{\nu} = \boldsymbol{\nu}_1 + \boldsymbol{\nu}_2$ in the transverse distribution D:

$$\langle \nu, \xi_{\mathrm{dyn}} \rangle = dh(\boldsymbol{\nu}) = \frac{2h}{\det \mathbb{I}}\Big(2K_1\nu_2 + 2K_2\nu_1 - \frac{2s_3\gamma_3K_2 + Z_2}{s_2\gamma_2 + s_3\gamma_3}\big(\nu_1 + \nu_2\big)\Big), \tag{5.25}$$

where (K_1, K_2) are the momentum values at (X, Y, Z_1, Z_2) along the reduced trajectory y_t. To get the dynamic phase g_{dyn}, one integrates the exponent of this expression along the reduced trajectory y_t on a three-wave surface.

The infinitesimal geometric phase ξ_{geom}, as a two-form on the reduced space, can be computed using (4.48), so that its ν-component is given by

$$\big\langle \nu, \xi_{\mathrm{geom}}(y) \big\rangle = \big\langle \nu, D_{\mu}\omega'(y) \big\rangle. \tag{5.26}$$

This expression should be computed using standard formulas for the differentials of p-forms. We omit here the calculations of the dynamic phase as they crucially depend on the area over which the two-form is integrated.

6. Concluding remarks

If the phase space P has an almost Kähler structure, a preferred connection can be defined by declaring horizontal spaces at each point to be metric orthogonal to the tangent to the group orbit. We call it *abstract mechanical connection*. Then, explicit formulas for the corresponding $\mathfrak{g}^*$-valued one-form $\mathcal{A}$ in terms of the momentum map, symplectic and complex structures can be derived. Also, we show that in this case the horizontal spaces for the induced connections are metric orthogonal to the corresponding natural vertical spaces for each foliation.

These results are applied to the resonant three-wave interaction problem (cf. [2]). The corresponding horizontal spaces are constructed and a formula for the dynamic phase is obtained. The associated geometric phase is given by the integral of a two-form which is defined by the reduced symplectic structure.

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