ON THE LATTICE THEORY OF FUNCTION SEMI-NORMS

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ABSTRACT. We consider the lattice of function semi-norms over a measure space, as well as certain distinguished subsets. We determine which subsets are sublattices and, in turn, which of these are Dedekind complete. We also investigate the extent to which the distributive and DeMorgan Laws are valid in this setting.

KEY WORDS AND PHRASES. *Function semi-norm, associate semi-norm, infimum, supremum, Riesz-Fisher property, strong and weak Fatou properties, infinite triangle inequality, absolutely continuous norm, (sub-) lattice, Dedekind completeness, distributive laws, DeMorgan's Laws.*

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1. INTRODUCTION.

In this paper, we investigate the sublattice structure of the lattice of semi-norms on a fixed measure space.

Section 2 is devoted to establishing the necessary preliminaries. In Section 3, we distinguish the semi-norms of interest, namely those having either the Riesz-Fisher, weak or strong Fatou properties, those satisfying the infinite triangle inequality and those which are of absolutely continuous norm. In determining the sublattice status of each of these collections, we are also interested in Dedekind completeness for each of these, as well as for the space of all semi-norms. In particular, when a subset is closed under arbitrary suprema or infima, we wish to know the specific description of these extrema. The answers to these questions in the case of the supremum are essentially known.

For the sake of completeness, these are summarized in Section 4. However, the answers to these questions in the case of the infimum are hardly known at all. In fact, the notion of infimum is not unique; it depends on which sublattice is being considered. The main difficulty here involves the infimum of an arbitrary collection of function semi-norms. There are three candidates for this notion which we define and study in Section 5. Here we also determine which infimum goes with which sublattice. This is the main section of this paper. In Section 6, we are then able to determine (1) whether a given subset of semi-norms is a sublattice and (2) if it is complete.

In Section 7, we turn our attention to the DeMorgan Laws. As we shall see, the validity of these laws depends on which of the two laws is being considered--as well as its interpretation.

In concluding, Section 8 observes that the sublattices under consideration are not distributive in general and also shows that there is a connection between the lattice theory of function semi-norms and finite sums of such semi-norms.

2.   PRELIMINARIES.

Here we recall the preliminary results we will require throughout the remainder of this paper. A general reference for this section is [5].

Let (X, S, μ) be a fixed sigma-finite measure space and $[0, \infty]$ the extended, non-negative real numbers. As usual, the real numbers and the natural numbers will be denoted by $\mathbb{R}$ and $\mathbb{N}$, equipped with their usual measure structures. If C is the complex numbers, define

$$M = \{f : X \to C : f \text{ is } \mu\text{-measurable}\}$$

and

$$M^+ = \{f : X \to [0, \infty] : f \text{ is } \mu\text{-measurable}\} .$$

In general, we will not distinguish between elements of M or M^+ which are equal almost everywhere.

In will also be convenient to denote $x \in X : f(x) \neq 0$ by $supp(f)$.

DEFINITION 2.1. A (function) semi-norm on M^+ is a mapping $\rho : M^+ \to [0, \infty]$ satisfying:

(i) $\rho(0) = 0$.

(ii) $\rho(af) = a\,\rho\,(f)$, $a \geq 0$, $f \in M^+$.

(iii) $\rho(f + g) \leq \rho(f) + \rho(g)$, $f,g \in M^+$.

(iv) $\rho(f) \leq \rho(g)$, whenever $f \leq g$, $f,g \in M^+$.

The semi-norm ρ is a <u>norm</u> if $\rho(f) = 0$ implies $f = 0$. Let P denote the

set of all semi-norms and P_o the subset of all norms (never empty). Of course,

$\rho = \|\cdot\|_p$ is a norm for each $1 \leq p \leq \infty$.

Observe that P is canonically partially ordered by

$$\rho_1 \geq \rho_2 \quad \text{if} \quad \rho_1(f) \geq \rho_2(f) \text{ , } f \in M^+ \text{ .}$$

EXAMPLE 2.2. For f in M^+ , define $\theta(f) = 0$. Then $\theta \in P - P_o$.

Also, for f in M^+ , define $\omega(f) = \infty$, $f \neq 0$, $\omega(0) = 0$. Then $\omega \in P_o$.

Clearly, θ and ω are the smallest and largest elements of P .

LEMMA 2.3. Let $\rho \in P$ and $g \in M^+$. Define $\rho_g(f) = \rho(fg)$, $f \in M^+$.

Then ρ_g is a semi-norm. Of particular interest are the cases where

(i) g is the characteristic function χ_E of a measurable set E or

(ii) ρ is the L^1-norm $\|\cdot\|_1$.

Now define

$$L^\rho = \{f \in M : \rho(|f|) < \infty\}$$

and

$$N^\rho = \{f \in M : \rho(|f|) = 0\}$$

Then L^ρ is a semi-normed linear space with N^ρ a closed subspace. Thus,

$L^\rho = L^\rho/N^\rho$ is a normed linear space. If $\rho = \|\cdot\|_p$, then L^ρ is the familiar

L^p-space .

EXAMPLE 2.4. Let $g = \chi_E$ as in 2.3, with $\rho \in P$. Then $\rho_E = \rho_g \leq \rho$,

so that $L^\rho \subseteq L^{\rho_E}$. On the other hand, for $E = X$, we have $L^\rho = L^{\rho_E}$. Hence,

in general,

$$L^\rho = \cap \{L^{\rho_E} : E \in S\} \text{ .}$$

REMARK 2.5. If ρ_2, $\rho_2 \, \varepsilon \, P$, then the pointwise supremum $\sup(\rho_1, \rho_2)$ of ρ_1, ρ_2 belongs to P [5,p.442]. This is not the case for the pointwise infimum.

DEFINITION 2.6. Let ρ_1, $\rho_2 \, \varepsilon \, P$. For f in M^+ , define

$$\inf(\rho_1, \rho_2)(f) = \inf\{\rho_1(f_1) + \rho_2(f_2) \; : \; f_1, f_2 \, \varepsilon \, M^+, f_1 + f_2 = f\} \; .$$

The function $\inf(\rho_1, \rho_2)$ is a semi-norm and is the infimum of ρ_1, ρ_2 in P [5,Ex.71].

In what follows, it will be convenient to write $\rho_1 \vee \rho_2$ for $\sup(\rho_1, \rho_2)$ and $\rho_1 \wedge \rho_2$ for $\inf(\rho_1, \rho_2)$. Now let

$$F = \{\rho : M^+ \to [0, \infty] \; : \; \rho(0) = 0\}$$

(Note that an element of F need not be a semi-norm.) For such ρ , we may define the associate ρ' by

$$\rho'(f) = \mathrm{Sup}\{\int_X (fg)\,d\mu \; ; \; \rho(g) \le 1\} \; , \; f \, \varepsilon \, M^+ \; .$$

Observe that the supremum is taken over a non-empty set. The function ρ' is a semi-norm, i.e., $\rho' \, \varepsilon \, P$. In particular, $\theta' = \omega, \omega' = \theta$ and $(\rho_p)' = \rho_{p'}$, where $1/p + 1/p' = 1$, $1 \le p \le \infty$.

LEMMA 2.7. (i) If $\rho_1, \rho_2 \, \varepsilon \, P$ and $\rho_1 \le \rho_2$, then $\rho_1' \ge \rho_2'$. (ii) If $\rho \, \varepsilon \, P$, then $\rho'' \le \rho$ and $p^{(n)} = p^{(n+2)}$, $n = 1,2,3,\ldots$.

LEMMA 2.8. (Finite DeMorgan Laws) If $\rho_1, \rho_2 \, \varepsilon \, P$, then $(\rho_1 \wedge \rho_2)' = \rho_1' \vee \rho_2'$ and $(\rho_1 \vee \rho_2)' = \rho_1' \wedge \rho_2'$.

PROOF. The first law is well-known [5,Ex.71.2]. The second law is much less well-known; in fact, there appears to be no proof of it in the literature. The proof we give next is due to Luxemburg [3]. We are grateful to Professor Luxemburg for permitting its inclusion here.

For $\rho_1, \rho_2 \, \varepsilon \, P$, we have $\rho_1, \rho_2 < \rho_1 \vee \rho_2$, so that $\rho_1' \, \rho_2' \ge (\rho_1 \vee \rho_2)'$ (1.14), i.e., $(\rho_1' \wedge \rho_2') \ge (\rho_1 \vee \rho_2)'$. To prove the other inequality, let $\rho = \rho_1 \vee \rho_2$ and suppose f is an element of M^+ such that $\rho'(f) < \infty$ (otherwise, equality is trivial). Since $f \, \varepsilon \, L^{\rho'}$, it corresponds to a bounded linear functional

φ on L^ρ given by

$$\varphi(g) = \int_X fgd\mu \ , \ g \ \varepsilon \ L^\rho \ .$$

Also, the Banach dual norm $\rho*$ of ρ is equal to ρ' on $L^{\rho'}$. It then follows

from [2, p. 247] that there exist non-negative linear functionals φ_1 , φ_2 on

L^{ρ_1} , L^{ρ_2} such that $\varphi = \varphi_1 + \varphi_2$ on L^ρ and

$$\rho'(f) = \rho^*(f) = \rho^*(\varphi) = \rho_1^*(\varphi_1) + \rho_2^*(\varphi_2) \ .$$

Since $\varphi_1, \varphi_2 \leq \varphi$ on L^ρ , it follows from the Radon-Nikodym Theorem that

φ_1, φ_2 are integrals, i.e., there exist functions $f_1, f_2 \geq 0$ in $L^{\rho_1'}$, $L^{\rho_2'}$

respectively such that

$$\varphi_i(g) = \int_X gf_i d\mu \ , \ g \ \varepsilon \ L^\rho \ .$$

Thus, $f_i \ \varepsilon \ L^{\rho_i'}$ and $\rho_i^*(\varphi_i) = \rho_i'(f_i)$, $i = 1,2$. Since $\varphi = \varphi_1 + \varphi_2$, it

follows that $f = f_1 + f_2$; also, $\rho'(f) = \rho_1'(f_1) + \rho_2'(f_2)$. Consequently,

$\rho'(f) \geq (\rho_1' \wedge \rho_2')(f)$, i.e., $(\rho_1 \vee \rho_2)' \geq \rho_1' \wedge \rho_2'$.

3.   THE SUBSETS.

In addition to the subset P of F , we will be interested in studying a

chain of subsets of P having certain desirable properties. In this section we

recall these properties, show their relationships to each other and consider some

general examples. References for this section are [4,5]. With the exception of

the subset satisfying the infinite triangle inequality, these subsets have been

studied before.

DEFINITION 3.1. Let $\rho \ \varepsilon \ P$. Then:

(i) ρ has the Riesz-Fisher Property (RFP) if for each sequence $\{f_n\}$ in M^+

such that $\Sigma\rho(f_n) < \infty$, it follows that there exists a sequence $\{g_n\}$ in M^+

such that $\rho(f_n - g_n) = 0$ and $\rho(\Sigma g_n) < \infty$.

(ii) ρ satisfies the Infinite Triangle Inequality (ITI) if for each sequence

$\{f_n\}$ in M^+ it follows that $\rho(\Sigma f_n) \leq \Sigma\rho(f_n)$.

(iii) ρ has the Weak Fatou Property (WFP) if for each sequence $\{f_n\}$ in M^+

such that $f_n \uparrow f \in M^+$, and $\lim \rho(f_n) < \infty$, it follows that $\rho(f) < \infty$.

(iv) ρ has the Strong Fatou Property (SFP) if for each sequence $\{f_n\}$ in M^+

such that $f_n \uparrow f$, $f \in M^+$, it follows that $\lim \rho(f_n) = \rho(f)$.

(v) ρ is of Absolutely Continuous Norm (ACN) if $L_a^\rho = L^\rho$, where L_a is defined

to be the subspace of f in L^ρ having absolutely continuous norm. Recall that

f in L^ρ has this property if for each sequence $\{f_n\}$ in M^+ such that

$|f| \geq f_n \downarrow 0$, the sequence $\{\rho(f_n)\}$ converges to 0 .

Let R (resp.I,W,S) denote the subset of P consisting of the RFP

(resp. ITI, WFP, SFP) semi-norms. Also, let A denote the subset of S

(equivalently W) consisting of the ACN semi-norms. Finally, if B is any

subset of P , let B_o denote the set $B \cap P_o$ of norms in B . Thus, we obtain

the subsets R_o, I_o, W_o, S_o, A_o of P_o , where $I_o = R_o$ [4, Thm.4.2]. We then have

the following sequences of proper inclusions:

$A \subset S \subset W \subset I \subset R \subset P$ and $A_o \subset S_o \subset W_o \subset R_o \subset P_o$.

Moreover, $B_o \subset B$, for each choice of $B = A,S,W,I,R,P$. (Note that $\theta \in A$

and $\omega \in A_o$.) These inclusions, as well as the counter-examples for properness,

can be found amongst the results and examples of [4.5].

LEMMA 3.2. If ρ_g is as in 2.3, then ρ_g belongs to S (resp.A) if

ρ does.

Recall that if $\rho \in F$, then $\rho' \in P$; actually, more is true.

LEMMA 3.3. If $\rho \in F$, then $\rho' \in S$. Also, if $\rho \in S$, then $\rho'' = \rho$.

PROOF. Let $\sigma = \|\cdot\|_1$. Then $\sigma \in S$ and hence, $\sigma_g \in S$, for all g in

L^ρ such that $\rho(g) \leq 1 (3.2)$. Moreover, $\rho' = \sup\{\sigma_g : \rho(g) \leq 1\}$. Hence,

$\rho' \in S$, since P and S are closed under pointwise suprema (left to the reader).

The second part of the lemma is well-known.

QUESTION 3.4. Recall that $A = \{\rho \in W : \rho$ is ACN$\}$. If $\rho \in P$ is ACN,

is it automatically WFP, i.e., in W ?

4. SUPREMA.

We begin our study of the lattice theory of P by asking if the subsets of

P introduced in Section 2 are closed under the formation of arbitrary pointwise

suprema. We have already seen that P and S have this property' (proof of 2.3).

Our objective here is to summarize (for the sake of completeness) the known answers

to this question for the other subsets.

4.1. P_o is closed under the formation of arbitrary pointwise suprema. (In

fact, if $\{\rho_j : j \in J\} \subseteq P$ and ρ_k is a norm, for some $j = k$, then $\sup(\rho_j)$

is a norm.) The converse is false, i.e., it is possible for $\sup(\rho_j)$ to be a

norm while all the ρ_j are just semi-norms [4]. All that is required of the

ρ_j for their pointwise supremum to be a norm is that they be total, i.e.,

$\rho_j(f) = 0$, all $j \in J$, implies $f = 0$.

4.2. R_o and I are closed under the formation of arbitrary pointwise suprema.

This is not the case for R [4,p.147].

4.3 W and W_o are closed under the formation of finite pointwise suprema.

This is not true for infinite collections. In fact, there exists [4,p.150] a

sequence in W_o whose pointwise supremum is not in W .

Although $I_o = R_o$, we have seen that $W \subset I \subset R$. Furthermore, I is

closed under the formation of arbitrary pointwise suprema, while W and R are

not. This is good reason for distinguishing the ITI property from WFP and RFP in

general.

4.4. S_o (as well as S) is closed under the formation of arbitrary pointwise

suprema.

4.5. A and A_o are closed under the formation of finite pointwise suprema.

This is not true for infinite collections. In fact, there exists a sequence in

A_o whose pointwise supremum is not in A .

. QUESTION 4.6. Is R closed under the formation of finite pointwise suprema?

5.   INFIMA.

In Section 1, we saw that the infimum in P of two semi-norms ρ_1, ρ_2 is

given by

$$(\rho_1 \wedge \rho_2)(f) = \inf\{\rho_1(f_1) + \rho_2(f_2) : f_1, f_2 \in M^+ , f_1 + f_2 = f\}, f \in M^+ .$$

There are two cononical generalizations of this formula to the context of an

arbitrary collection $\{\rho_j : j \in J\}$ of semi-norms. In this regard, it will be

convenient to refer to a collection $(f_j)_J$ in M^+ as a J-decomposition of f

in M^+ (of countable type) if:

(i) $f_j = 0$, for all but countably many j in J.

(ii) $\sum\limits_J f_j = f$, in the almost-everywhere sense.

If $(f_j)_J$ has the property (i') $f_j = 0$, for all but finitely many j in J,

then we will say it is of finite type. Now let $f \in M^+$ and define:

$$\lambda(f) = \inf\{\sum\limits_J \rho_j(f_j) : (f_j)_J \text{ is a J-decomposition of } f\}$$

and

$$\gamma(f) = \inf\{\sum\limits_J \rho_j(f_j) : (f_j)_J \text{ is a J-decomposition of } f \text{ of finite type}\}$$

Note that the sums in each case have at most countably many non-zero terms.

As suggested by Luxemburg in [3], there is another less canonical generalization

of $(\rho_1 \wedge \rho_2)(f)$ motivated by $(\rho_1 \wedge \rho_2)' = \rho_1' \vee \rho_2'$, namely,

$$\sigma = (\sup(\rho_j'))' \quad .$$

In this section, we shall make use of all three of these generalizations.

It is well-known that σ is a semi-norm; in fact, $\sigma \in S$. Moreover:

THEOREM 5.1. Each of the mappings γ, λ is a function semi-norm.

PROOF. The only part of the theorem for λ needing verification is the

monotonicity; we will show this. The proof for γ is similar to that of λ.

Suppose $f, g \in M^+$ with $f \leq g$. Let $\{g_k\}$ be a sequence in M^+ such that

$\sum g_k = g$. Define $f_1 = \min(f, g_1)$. Then $f_1 \leq g_1$ and $f_1 \leq f$, i.e.,

$f_1' = f - f_1 \geq 0$. Suppose we have obtained $f_1, \ldots, f_n$ in M^+ such that

$f_k \leq g_k$, $1 \leq k \leq n$, and $f_1 + \ldots + f_n \leq f$, so that $f_n' = f - f_1 - \ldots - f_n \geq 0$.

Define $f_{n+1} = \min(f_n' g_{n+1})$. Then $f_{n+1} \leq g_{n+1}$ and

$f_{n+1} \leq f_n' = f - f_1 - \ldots - f_n$, i.e., $f_1 + \ldots + f_n + f_{n+1} \leq f$. In this way, we

obtain a sequence $\{f_k\}$ in M^+ such that $f_k \leq f$ and $f_k \leq g_k$, $1 \leq k$. We

next show $\sum f_k = f$ (in the almost-every-where sense).

We have $f(x) \leq g(x)$, for x outside a null set. Let x be a point in

X outside the null set. If $f(x) = g(x)$, then $f_k(x) = g_k(x)$, for all k, so

that

$$\Sigma f_k(x) = \Sigma g_k(x) = g(x) = f(x) \ .$$

If $f(x) \neq g(x)$, then $f(x) < g(x)$ and hence, $f_i(x) \neq g_i(x)$, for some i .
Consequently,

$$f_i(x) = f'_{i-1}(x) = f(x) - f_1(x) - f_1(x) - \ldots - f_{i-1}(x) \ ,$$

i.e.,

$$f(x) = f_1(x) + \ldots + f_{i-1}(x) + f_i(x) \ ,$$

which implies $\Sigma f_k(x) = f(x)$, i.e., $\Sigma f_k = f$.

Therefore, each decomposition of g by a sequence (g_k) in M^+ yields
a sequence decomposition (f_k) of f in M^+ having the properties $f_k \leq g_k$,
$1 \leq k$. Consequently, the same is true for any J-decomposition of g in M^+ ,
since such is essentially a sequence (define $f_j = 0$, for $g_j = 0$) . Hence,
if $(g_j)_J$ is a J-decomposition of g in M^+ , there exists a corresponding
J-decomposition $(f_j)_J$ of f in M^+ for which $f_j \leq g_j$, j ε J . Necessarily,
$\lambda(f) \leq \lambda(g)$, since the infimum for $\lambda(f)$ is over a larger index set.

Observe that each λ, γ, σ is a lower bound for $\{\rho_j : j \ \varepsilon \ J\}$.

Next we wish to compare γ, λ, σ . Obviously, $\lambda \leq \gamma$. Moreover, if J is
finite, then $\gamma = \lambda$.

LEMMA 5.2. If ρ satisfies the ITI (i.e., $\rho \ \varepsilon \ I$) and ρ is a lower
bound for $\{\rho_j : j \ \varepsilon \ J\}$, then $\rho \leq \lambda$.

PROOF. Let $f \ \varepsilon \ M^+$ and let $(f_j)_J$ be a J-decomposition of f in M^+ .
By our hypothesis, we have

$$\rho(f) = \rho(\underset{J}{\Sigma} f_j) \leq \underset{J}{\Sigma} \rho(f_j) \leq \underset{J}{\Sigma} \rho_j(f_j) \ ,$$

from which follows that $\rho(f) \leq \lambda(f)$.

THEOREM 5.3. Let $\{\rho_j : j \ \varepsilon \ J\}$ be a subset of P with γ, λ, σ as above.
Then $\sigma < \lambda < \gamma$ and $\sigma = \gamma'' = \lambda'' = \sigma''$.

PROOF. Since $\sigma \ \varepsilon \ S \subset I$ and $\sigma \leq \rho_j$, j ε J , we have $\sigma \leq \lambda \leq \gamma$ (5.2) .
Hence, $\sigma = \sigma'' \leq \lambda'' \leq \gamma''$. Now $\gamma \leq \rho_j$ implies $\gamma' \geq \rho'_j$, j ε J , i.e., γ'
is an upper bound for $\{\rho'_j : j \ \varepsilon \ J\}$. Thus, $\gamma' \geq \sigma'$, since $\sigma' = \sup(\rho'_j)$.
Consequently, $\gamma'' \leq \sigma'' = \sigma$ and $\sigma = \gamma''$. This establishes the equalities. The

following examples establish the inequalities.

EXAMPLE 5.4. Let $X = \mathbb{N}$. Define

$$\alpha(f) = \Sigma f(n)/2^n \ , \ \beta(f) = \lim \sup f(n)$$

and

$$\rho_j(f) = \alpha(f) + \beta(f) \ , \ f \ \varepsilon \ M^+ \ , \ j = 1,2,\ldots,$$

so that $\{\rho_j\}$ is a constant sequence of norms. We leave to the reader the task

of verifying that $\lambda < \gamma$ for this choice of $\{\rho_j\}$.

EXAMPLE 5.5. Let ρ be a non-trivial element of P such that $\rho \notin S$,

i.e., $\rho'' \neq \rho$, in which case $\rho'' < \rho$. For the collection $\{\rho_j\} = \{\rho, 2\rho\}$,

we have $\sigma = \rho'' < \rho = \lambda$, i.e., $\sigma < \lambda$.

LEMMA 5.6. For finite collections in S , we have $\sigma = \lambda = \gamma$ (use 2.8) .

For the remainder of this section, we investigate the status of the infimum

in P, R, I, W, S, A . Once again, for $\{\rho_j : j \ \varepsilon \ J\} \subseteq P$, let σ, λ, γ be as above.

LEMMA 5.7. The infimum of the ρ_j in P is γ ; notationally,

$\gamma = \inf_P(\rho_j)$. In particular, $\rho_1 \wedge \rho_2 = \inf_P(\rho_1, \rho_2)$. In this sense, P is

closed under the formation of arbitrary infima.

LEMMA 5.8. The infimum of a family in P_o which is bounded below in P_o

also belongs to P_o . This is not true for arbitrary families in P_o . (For

example, consider $\{(1/n)\rho\}$, for $\rho \ \varepsilon \ P_o$.)

THEOREM 5.9. If $\{\rho_j\} \subseteq I$, then the infimum of the ρ_j in I is λ ;

notationally, $\inf_I(\rho_j) = \lambda$. In particular, $\rho_1 \wedge \rho_2 = \inf_I(\rho_1, \rho_2)$. In this

sense, I is closed under the formation of arbitrary infima.

PROOF. Fix $\varepsilon > 0$ and let $\{g_n\}$ be a sequence in M^+ for which

$\Sigma\lambda(g_n) < \infty$. By the definition of λ , for each n , there exists a J^n-decomposition

$(g_{n,j})_{J^n}$ of g_n in M^+ such that

$$\sum_{J^n} \rho_j(g_{n,j}) < \lambda(g_n) + \varepsilon/2^n \ .$$

Let $J_n = \{j \ \varepsilon \ J^n : g_{n,j} \neq 0\}$. Then J_n is a countable subset of J^n and

$$\sum_{J^n} \rho_j(g_{n,j}) = \sum_{J_n} \rho_j(g_{n,j}) \ , \ n = 1,2,\ldots \ .$$

Let $J = \cup J_n$, which is also a countable subset of $\cup J^n$. Furthermore, if $j \notin J$, then $g_{n,j} = 0$, all n . Observe that

$$\sum_n \sum_{J_n} \rho_j(g_{n,j}) < \sum_n [\lambda(g_n) + \varepsilon/2^n]$$

$$\leq \sum_n \lambda(g_n) + \varepsilon$$

$$< \infty .$$

Therefore, rearranging the double series, we obtain

$$\sum_n \sum_{J_n} \rho_j(g_{n,j}) = \sum_n \sum_J \rho_j(g_{n,j})$$

$$= \sum_J \sum_n \rho_j(g_{n,j}) \, ,$$

Moreover, $\{g_{n,j} : j \in J , n \in \mathbb{N}\}$ is a countable subset of M^+ satisfying

$$\sum_n \sum_J g_{n,j} = \sum_n g_n .$$

Thus, $\{g_{n,j}\}$ is a subset of M^+ which is indexed by $\mathbb{N} \times J$, has countably many elements and satisfies

$$\sum_J \sum_n g_{n,j} = \sum_n \sum_J g_{n,j} = \sum_n g_n .$$

Thus, the family $(\sum_n g_{n,j})_J$ is a J-decomposition of $\sum_n g_n$ is M^+ .

Consequently,

$$\lambda(\sum_n g_n) \leq \sum_J \rho_j(\sum_n g_{n,j})$$

$$\leq \sum_J \sum_n \rho_j(g_{n,j}) \quad (\rho_j \in I)$$

$$\leq \sum_n \sum_{J_n} \rho_j(g_{n,j})$$

$$< \sum_n \lambda(g_n) + \varepsilon , \text{ for arbitrary } \varepsilon > 0 .$$

Hence,

$$\lambda(\underset{n}{\Sigma}\ g_n) \leq \underset{n}{\Sigma}\ \lambda(g_n)\ ,$$

which implies that $\lambda\ \varepsilon\ I$.

If ρ is in I and ρ is a lower bound for $\{\rho_j\}$, then by 5.2 $\rho \leq \lambda$. Hence, $\lambda = \inf_I(\rho_j)$.

COROLLARY 5.10. If $\{\rho_j\} \subseteq R_o = I_o$ and λ is a norm, then λ is the infimum of the ρ_j if R_o .

COROLLARY 5.11. If $\rho_1, \rho_2\ \varepsilon\ R_o$ and $\rho_1 \wedge \rho_2$ is a norm, then $\rho_1 \wedge \rho_2$ is the infimum of ρ_1, ρ_2 in R_o .

PROPOSITION 5.12. If $\rho_1, \rho_2\ \varepsilon\ W$, then $\rho_1 \wedge \rho_2$ is the infimum of ρ_1, ρ_2 is W . Thus, W is closed under the formation of finite infima.

PROOF. Recall that τ belongs to W if and only if τ is equivalent to τ'' [5,p.472], i.e., there exists $0 < a \leq 1$ such that $a\tau \leq \tau'' \leq \tau$. Thus, for $\rho_1, \rho_2\ \varepsilon\ W$, there exists $0 < a_i \leq 1$ such that $a_i\rho_i \leq \rho_i'' \leq \rho_i$, $i = 1, 2$. Letting $a = \min(a_1, a_2)$, we have $0 < a \leq 1$ and $a\rho_i \leq \rho_i'' \leq \rho_i$, $i = 1, 2$. From this it follows that

$$a(\rho_1 \wedge \rho_2) \leq \rho_1'' \wedge \rho_2'' \leq \rho_1 \wedge \rho_2\ ,$$

i.e., $\rho_1 \wedge \rho_2$ is equivalent to $\rho_1'' \wedge \rho_2''$. However, by 2.8 we have that $\rho_1'' \wedge \rho_2'' = (\rho_1 \wedge \rho_2)''$. Hence, $\rho_1 \wedge \rho_2\ \varepsilon\ W$.

COROLLARY 5.13. If $\rho_1, \rho_2\ \varepsilon\ W_o$ and $\rho_1 \wedge \rho_2\ \varepsilon\ P_o$, then $\rho_1 \wedge \rho_2\ \varepsilon\ W_o$.

PROPOSITION 5.14. If $\{\rho_j\} \subseteq S$, then the infimum of the ρ_j in S is σ; notationally, $\sigma = \inf_S(\rho_j)$. In particular, $\rho_1 \wedge \rho_2 = \inf_S(\rho_1, \rho_2)$ (recall 5.6) . In this sense, S is closed under the formation of arbitrary infima.

PROOF. We have seen that σ is a lower bound for $\{\rho_j\}$. If ρ in S is any lower bound for $\{\rho_j\}$, then $\rho' \geq \rho_j'$, $j\ \varepsilon\ J$, so that $\rho' \geq \sup(\rho_j')$. Consequently, $\rho'' \leq (\sup(\rho_j'))'$, i.e., $\rho \leq \sigma$.

PROPOSITION 5.15. The infimum of a family in S_o which is bounded below S_o is also in P_o . This is not true for arbitrary families in S_o (5.8).

PROOF. By 5.14, $\sigma = \inf_S(\rho_j)$ belongs to S , for $\{\rho_j\} \subseteq S_o$ and

bounded below in S_o . In order that $\sigma \varepsilon S_o$, it is necessary and sufficient

that $\sup(\rho_j')$ be saturated [5,p.458]. By hypothesis, there exists ρ in S_o

such that $\rho \leq \rho_j$, $j \varepsilon J$. Hence, $\rho' \geq \rho_j'$, $j \varepsilon J$, so that $\rho' \geq \sup(\rho_j')$,

where ρ' is saturated [5,p.458]. Therefore, $\sup(\rho_j')$ is also saturated

[5,p.454], i.e., σ is a norm.

PROPOSITION 5.16. If $\rho_1,\rho_2 \varepsilon A$, then $\rho_1 \wedge \rho_2$ is the infimum of ρ_1,ρ_2

in A . Thus, A is closed under the formation of finite infima.

QUESTION 5.17. Are P_o,R closed under the formation of finite infima?

6. SUBLATTICES AND COMPLETENESS

We are now prepared to determine (as well as we can) which of the subsets

introduced in Section 3 are sublattices of P . For those which are, we are also

interested in whether or not they are complete. This section is essentially a

lattice-theoretic summary of the previous three sections.

With regard to completeness, it is worthwhile recalling the following. A

lattice is complete if it is closed under arbitrary suprema and infima. A

lattice is Dedekind complete if it is closed under arbitrary suprema and infima

of collections which are suitably bounded above for suprema and below for infima.

Since $\theta \varepsilon A$ and $\omega \varepsilon A_o$, these two notions of completeness are usually the

same - but not always.

As a consequence of the results of Section 5, we see that it is possible

for a subset of a sublattice to have an infimum in the sublattice which is

different from its infimum in the whole lattice. Therefore, it is possible for

a sublattice to be a complete lattice, but not a complete sublattice!

THEOREM 6.1. P is a complete lattice with supremum given pointwise and

infimum given by γ of Section 5. (See 2.5, 2.6, 4.1, and 5.7.)

REMARKS 6.2. It is not clear whether P_o is a sublattice of P , since we

do not know if P_o is closed under finite infima. For this reason, it is also

difficult to determine the sublattice status of R_o,W_o,S_o,A_o . However, P_o is

Dedekind complete in the sense that: (i) it is closed under the formation of

arbitrary pointwise infima and (ii) it is closed under the formation of

γ-infima of arbitrary collections bounded below in P_o .

THEOREM 6.3. I is a sublattice of P with supremum given pointwise and

infimum given by λ of Section 5. Thus, I is also a complete lattice

(4.2, 5.9), but not a complete sublattice of P (next example).

EXAMPLE 6.4. Let $X = \mathbb{N}$ and define

$$\rho_j(f) = \sup_{n \geq j} f(n) \text{ , } f \in M^+ \text{ , } j = 1,2,\ldots,$$

so that $\rho_j \in S$, all j . For the collection $\{\rho_j : j \in \mathbb{N}\}$, the corresponding

γ is given by

$$\gamma(f) = \lim \sup(f(n)) \text{ , } f \in M^+ \text{ .}$$

We may verify that $\gamma \notin I$, i.e., $\inf_p(\rho_j) \notin I$.

THEOREM 6.5. W is a sublattice of I which is not (Dedekind) complete

in general (4.3, 5.12).

THEOREM 6.6. S is a sublattice of W with supremum given pointwise and

infimum given by σ of Section 5. Thus, S is also a complete lattice, but not

a complete sublattice of W .

PROOF. See 4.4 and 5.15. That S is not complete in W can be seen from

the example in 7.3. For this choice of $\{\rho_n\}$, we may verify that

$$\lambda(f) = \gamma(f) = \sup(f(n)) + \lim \sup(f(n)) \text{ , } f \in M^+ \text{ .}$$

Then $\rho_n \in S_o$, all n , $\lambda \in W$ [4,p.150], $\lambda \notin S$, while $\sigma \in S$. Thus we

have another example of $\sigma \neq \lambda$ (recall 5.5).

THEOREM 6.7. A is a sublattice of S which is not (Dedekind) complete.

PROOF. See 4.6 and 5.16.

7. DEMORGAN'S LAWS

In view of 2.8, it is natural to ask if DeMorgan's Laws hold in general.

Once again, let $\{\rho_j : j \in J\} \subseteq P$. The next theorem shows in what sense one

of the two DeMorgan Laws is valid.

THEOREM 7.1. In general, $[\inf_p(\rho_j)]' = \sup(\rho_j')$.

PROOF. For convenience, let $\gamma = \inf_P(\rho_j)$ and $\beta = \sup(\rho_j')$. We then have

$$\gamma'(f) = \sup \{ \int_X |fg|d\mu \ : \ \gamma(g) \leq 1 \} \ , \ f \ \varepsilon \ M^+ \ .$$

To show $\gamma' = \beta$, first observe that $\gamma' \geq \beta$ since $\gamma \leq \rho_j$, i.e., $\gamma' \geq \rho_j'$, $j \ \varepsilon \ J$. Suppose next that $f \ \varepsilon \ M^+$, $\gamma(g) \leq 1$ and (g_j) is a J-decomposition of $|g|$ in M^+ of finite type. Since $\gamma(g) \leq 1$, we may assume $\Sigma\rho_j(g_j) < \infty$, i.e., $\rho_j(g_j) < \infty$, $j \ \varepsilon \ J$. If $\beta(f) = \infty$, then $\gamma'(f) = \infty$ also. If $\beta(f) < \infty$, then $\rho_j'(f) < \infty$, $j \ \varepsilon \ J$, and hence

$$\int_X |fg|d\mu = \underset{J}{\Sigma} \int_X fg_j d\mu \qquad \text{(finite sum)}$$

$$\leq \underset{J}{\Sigma} \ \rho_j'(f) \ \rho_j(g_j) \qquad \text{(Hölder's Inequality)}$$

$$\leq \beta(f) \ \underset{J}{\Sigma} \ \rho_j(g_j) \ .$$

Thus, from the definition of γ , it follows that

$$\int_X |fg|d\mu \leq \beta(f)\gamma(g)$$

$$\leq \beta(f) \ .$$

Thus,

$$\gamma'(f) \leq \beta(f) \ , \ \text{i.e.,} \ \gamma' \leq \beta \ .$$

This theorem yields an alternate proof of the equality in 5.3. Specifically, we have:

COROLLARY 7.2. Let the notation be as in 5.3. Then $\sigma = \gamma''$.

PROOF. Since $\gamma = \inf_P(\rho_j)$, we have $\gamma' = \sup(\rho_j')$ by the theorem, so that $\gamma'' = (\sup(\rho_j'))' = \sigma$.

In general, as the next example verifies, the other DeMorgan Law is false.

EXAMPLE 7.3. Let $X = \mathbb{N}$. Define

$$\rho_n(f) = \sup_k(f(k)) + \sup_{k \geq n}(f(k)) \ , \ f \ \varepsilon \ M^+ \ , \ n = 1,2,\ldots \ .$$

We may verify that:

(i)   each ρ_n is an SFP norm, i.e., $\rho_n \varepsilon S_o$, n = 1,2,... ,

(ii) $\inf_P(\rho_n)(f) = \sup_k(f(k)) + \lim\sup_k(f(k))$, $f \varepsilon M^+$.

(iii) $\inf_P(\rho_n)$ does not have the SFP.

Consequently, letting $\sigma_n = \rho_n'$, we see that $\inf_P(\sigma_n') = \inf_P(\rho_n)$ does not

have the SFP while $(\sup(\sigma_n))'$ does. Therefore, $(\sup(\sigma_n))'$ can't equal

$\inf_P(\sigma_n')$.

Despite the previous example, there is a partial result of interest.

LEMMA 7.4. If $\{\rho_j : j \varepsilon J\} \subseteq P$, then $(\sup(\rho_j))' \leq \inf_S(\rho_j')$ (apply 2.7).

However, $\inf_S(\rho_j)' = \sup(\rho_j')$.

Of course, for finite families $\{\rho_j\}$, it is true that $(\sup(\rho_j))' = \inf_P(\rho_j)$

$\inf_S(\rho_j)$ (recall 2.8).

8.   CONCLUDING REMARKS.

A lattice-theoretic question which we have not yet considered is whether P

(or any of its sublattices) is distributive. As indicated by Luxemburg in [3],

the answer is "no" in general. More specifically, there exist norms ρ_1, ρ_2, ρ_3

in A_o for which

$$(\rho_1 \vee \rho_2) \wedge \rho_3 > (\rho_1 \wedge \rho_3) \vee (\rho_2 \wedge \rho_3) ,$$

thus violating one of the distributive laws. As a consequence of this, the

other distributive law is also violated, in general, because

$$(\rho_1' \wedge \rho_2') \vee \rho_3' < (\rho_1' \vee \rho_3') \wedge (\rho_2' \vee \rho_3') .$$

Therefore, both distributive laws fail in A_o and hence, in P and each of

its sublattices of interest.

Finally, observe that at first glance, finite sums seem to have nothing to

do with lattice theory of function semi-norms. However, quite the opposite is

true. If $\{\rho_1,...,\rho_n\} \subseteq P$, then $\rho_1 + ... + \rho_n \varepsilon P$ and is equivalent to

$\sup(\rho_1,...,\rho_n)$. Consequently, its associate $(\rho_1 + ... + \rho_n)'$ is equivalent to

$\inf_P(\rho_1',...,\rho_n')$.

In view of the previous, it is of interest to know which of the subsets of

P (introduced in Section 2) are closed under equivalence. The spaces

P_o, R, I, W are closed under equivalence; S isn't. Is A ?

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