SUBORDINATION PROPERTIES OF p-VALENT FUNCTIONS DEFINED BY INTEGRAL OPERATORS

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By applying certain integral operators to p-valent functions we define a comprehensive family of analytic functins. The subordinations properties of this family is studied, which in certain special cases yield some of the previously obtained results.

1. Introduction

For the natural numbers p let $A(p)$ denote the class of functions of the form $f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots$, which are analytic in the open unit disk $U = \{z : |z| < 1\}$. For $f(z) \in A(p)$ we define

$$\begin{aligned} I^\sigma f(z) &= \frac{(p+1)^\sigma}{z\Gamma(\sigma)} \int_0^z \left(\log\frac{z}{t}\right)^{\sigma-1} f(t)dt \\ &= z^p + \sum_{n=p+1}^{\infty} \left(\frac{p+1}{n+1}\right)^\sigma a_n z^n, \quad \sigma > 0. \end{aligned} \tag{1.1}$$

Also, for $-1 \leq B < A \leq 1$ and $\lambda \geq 0$, let $\Omega_p^\sigma(A,B,\lambda)$ be the class of functions $f \in A(p)$ so that

$$\frac{\lambda}{p}\frac{I^{\sigma-1}f(z)}{z^p} + \frac{p-\lambda}{p}\frac{I^\sigma f(z)}{z^p} \prec \frac{1+Az}{1+Bz}, \quad \lambda \geq 0, \tag{1.2}$$

where "$\prec$" denotes the usual subordination. See [2].

The family $\Omega_p^\sigma(A,B,\lambda)$ is a comprehensive family containing various well-known as well as new classes of analytic functions. For example, for $\sigma = 0$ and $\lambda = p+1$ we obtain the class $\Omega_p^0(A,B,p+1)$ studied by Patel and Mohanty [3] or for nonzero σ see Liu [1].

2. Main results

Our first theorem examins the containment properties of the family $\Omega_p^\sigma(A,B,\lambda)$.

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Theorem 2.1. *For $f \in A(p)$ suppose that $f \in \Omega_p^\sigma(A,B,\lambda)$ and $0 \le \lambda \le p(p+1)$. Then $f \in \Omega_p^\sigma(A,B,0)$.*

To prove our theorem we will need the following lemma which is due to Miller and Mocanu [2].

Lemma 2.2. *Let $g(z)$ be analytic and convex univalent in U and $g(0) = 1$. Also let $p(z)$ be analytic in U with $p(0) = 1$. If $p(z) + (zp'(z))/\gamma \prec g(z)$, where $\gamma \neq 0$ and $\operatorname{Re}\gamma \ge 0$, then $p(z) \prec \gamma z^{-\gamma}\int_0^z t^{\gamma-1}g(t)dt$.*

Proof of Theorem 2.1. First, we note that

$$z(I^\sigma f(z))' = (p+1)I^{\sigma-1}f(z) - I^\sigma f(z). \tag{2.1}$$

Setting $p(z) = (I^\sigma f(z))/z^p$ we also observe that

$$\begin{aligned} \frac{(I^\sigma f(z))'}{pz^{p-1}} &= p(z) + \frac{zp'(z)}{p}, \\ \frac{I^{\sigma-1}f(z)}{z^p} &= p(z) + \frac{zp'(z)}{p+1}. \end{aligned} \tag{2.2}$$

Therefore, for $f \in \Omega_p^\sigma(A,B,\lambda)$, we conclude that

$$p(z) + \frac{\lambda}{p(p+1)}zp'(z) \prec \frac{1+Az}{1+Bz}. \tag{2.3}$$

Now from Lemma 2.2 for $\gamma = p(p+1)/\lambda$ it follows that

$$\frac{I^\sigma f(z)}{z^p} \prec \frac{p(p+1)}{\lambda}z^{-p(p+1)/\lambda}\int_0^z t^{p(p+1)/\lambda-1}\frac{1+At}{1+Bt}dt = q(z) \prec \frac{1+Az}{1+Bz}. \tag{2.4}$$

Thus $f \in \Omega_p^\sigma(A,B,0)$. □

As a special case to Theorem 2.1, we obtain the following.

Corollary 2.3. *Let $f \in A(p)$. Then $(1/(p+1))[(zf'(z)+f(z))/z^p] \prec (1+Az)/(1+Bz)$, implies $f(z)/z^p \prec (1+Az)/(1+Bz)$.*

Theorem 2.4. *For $f \in A(p)$ suppose that $f \in \Omega_p^\sigma(A,B,\lambda)$. If $0 \le \lambda \le p(p+1)$, then*

$$\operatorname{Re}\left(\frac{I^\sigma f(z)}{z^p}\right) \ge \frac{p(p+1)}{\lambda}\int_0^1 u^{p(p+1)/\lambda-1}\frac{1-Au}{1-Bu}du. \tag{2.5}$$

The result is sharp.

Proof. Set $p(z) = I^\sigma f(z)/z^p$. Then, by Theorem 2.1, we have

$$p(z) \prec \frac{p(p+1)}{\lambda}z^{-p(p+1)/\lambda}\int_0^z t^{p(p+1)/\lambda-1}\frac{1+At}{1+Bt}dt \prec \frac{1+Az}{1+Bz}. \tag{2.6}$$

This is equivalent to

$$\frac{I^{\sigma} f(z)}{z^p} = \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)/\lambda - 1} \frac{1 + uAw(z)}{1 + uBw(z)} du, \tag{2.7}$$

where $w(z)$ is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ in U. Therefore

$$\begin{aligned} \operatorname{Re}\left(\frac{I^{\sigma} f(z)}{z^p}\right) &= \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)/\lambda - 1} \operatorname{Re}\left\{\frac{1 + uAw(z)}{1 + uBw(z)}\right\} du \\ &\geq \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)/\lambda - 1} \frac{1 - Au}{1 - Bu} du. \end{aligned} \tag{2.8}$$

Therefore

$$\frac{I^{\sigma} f(z)}{z^p} = \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)/\lambda - 1} \frac{1 + Auz}{1 + Buz} du, \tag{2.9}$$

such that for this function we have

$$\frac{\lambda}{p} \frac{I^{\sigma-1} f(z)}{z^p} + \frac{p - \lambda}{p} \frac{I^{\sigma} f(z)}{z^p} = \frac{1 + Az}{1 + Bz}. \tag{2.10}$$

Letting $z \to -1$ yields

$$\frac{I^{\sigma} f(z)}{z^p} \longrightarrow \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)/\lambda - 1} \frac{1 - Au}{1 - Bu} du. \tag{2.11}$$

□

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