A NOTE ON COMPREHENSIVE BACKWARD BIORTHOGONALIZATION

DAVID K. RUCH

Received 16 February 2006; Revised 21 July 2006; Accepted 25 July 2006

We present a backward biorthogonalization technique for giving an orthogonal projection of a biorthogonal expansion onto a smaller subspace, reducing the dimension of the initial space by dropping d basis functions. We also determine which basis functions should be dropped to minimize the L^2 distance between a given function and its projection. This generalizes some recent results of Rebollo-Neira.

In [3], Rebollo-Neira gives a backward biorthogonalization technique for projecting a biorthogonal expansion onto a subspace, reducing the dimension N of the initial space by dropping $d = 1$ basis function. In this note, we generalize this method to reduce the space by an arbitrary number d of basis functions, $d < N$. Proposition 3.4 in [3] indicates which single basis function is to be removed in order to minimize the L^2 distance between a function f and its orthogonal projection into the reduced space. We will also generalize this result in Proposition 7. If more than one basis function is to be dropped, Rebollo-Neira recommends iterating the $d = 1$ process. We show via Example 8 that in some circumstances iterating the $d = 1$ process k times leads to results inferior to using Proposition 7 and dropping $k = d$ basis functions *simultaneously*.

We begin with a Hilbert space H and an N-dimensional subspace V. Assume biorthogonal bases of V given by $\{x_i'\}_{i=1}^N$ and $\{x_i\}_{i=1}^N$ such that $\langle x_i', x_j\rangle = \delta_{ij}$. Now drop d basis elements from each set, without loss of generality the first d elements for notational purposes, and form the reduced subspaces $\widetilde{V} = \operatorname{span}\{x_i\}_{i=d+1}^N$ and $\widetilde{V}' = \operatorname{span}\{x_i'\}_{i=1}^d$. We wish to modify the x_i' so that the projection from V to $\widetilde{V}$ is orthogonal. We next recursively construct the sequence $\{v_i'\}_{i=1}^d \subset \widetilde{V}'$ by

$$v_1' = x_1', \qquad v_i' = x_i' - \sum_{\ell=1}^{i-1} \frac{\langle x_i', v_\ell'\rangle}{\langle v_\ell', v_\ell'\rangle} v_\ell', \quad i \le d. \tag{1}$$

Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2006, Article ID 87392, Pages 1–5
DOI 10.1155/IJMMS/2006/87392

We observe that the set $\{v'_i\}_{i=1}^d$ forms an orthogonal basis of $\widetilde{V}'$ by construction. We then construct the sequence $\{\widetilde{x}'_i\}_{i=d+1}^N$ by

$$\widetilde{x}'_i = x'_i - \sum_{\ell=1}^{d} \frac{\langle x'_i, v'_\ell\rangle}{\langle v'_\ell, v'_\ell\rangle} v'_\ell \tag{2}$$

and set $U = \operatorname{span}\{\widetilde{x}'_i\}_{i=d+1}^N$. We will see that this formula generalizes the dual modification of [3, Theorem 3.1] for $d \geq 1$. Note that each $\widetilde{x}'_i$ is created to be orthogonal to $\widetilde{V}'$ by subtracting from x'_i its projection onto $\widetilde{V}'$.

Proposition 1. *The spaces U and $\widetilde{V}'$ are orthogonal complements in V, $V = \widetilde{V} \oplus \widetilde{V}'$.*

Proof. Choose i, j such that $j \leq d < i$ and use the definition of $\widetilde{x}'_i$ and the orthogonality of $\{v'_i\}$,

$$\langle \widetilde{x}'_i, v'_j\rangle = \langle x'_i, v'_j\rangle - \sum_{\ell=1}^{d} \frac{\langle x'_i, v'_\ell\rangle}{\langle v'_\ell, v'_\ell\rangle}\langle v'_\ell, v'_j\rangle = \langle x'_i, v'_j\rangle - \langle x'_i, v'_j\rangle = 0. \tag{3}$$

Thus U and $\widetilde{V}'$ are orthogonal subspaces of V, and their dimensions add to N. □

We next verify that U and $\widetilde{V}$ are actually the same space.

Lemma 2. *The spaces U and $\widetilde{V}'$ are orthogonal complements in V, and $U = \widetilde{V}$.*

Proof. By (1), we can write $v'_j = \sum_{n=1}^{j} a_n x'_n$ for some constants a_n, so the original biorthogonality condition $\langle x'_i, x_j\rangle = \delta_{ij}$ says that, for $j < i$, $\langle v'_j, x_i\rangle = \sum_{n=1}^{j} a_n\langle x'_n, x_i\rangle = 0$. Thus $\widetilde{V}$ and $\widetilde{V}'$ are orthogonal subspaces of V, and their dimensions add to N. By the previous proposition, $U = \widetilde{V}$. □

Next we give the desired biorthogonal bases of the reduced subspace $\widetilde{V}$.

Proposition 3. *The reduced spaces U and $\widetilde{V}$ are identical and have biorthogonal bases $\{\widetilde{x}'_i\}_{i=d+1}^N$ and $\{x_j\}_{j=d+1}^N$.*

Proof. Using Lemma 2 and (2), we have for $i, j > d \geq \ell$,

$$\langle \widetilde{x}'_i, x_j\rangle = \langle x'_i, x_j\rangle - \sum_{\ell=1}^{d} \frac{\langle x'_i, v'_\ell\rangle}{\langle v'_\ell, v'_\ell\rangle}\langle v'_\ell, x_j\rangle = \delta_{ij} - \sum_{\ell=1}^{d} \frac{\langle x'_i, v'_\ell\rangle}{\langle v'_\ell, v'_\ell\rangle} \cdot 0 = \delta_{ij}. \tag{4}$$

□

In order to give an explicit method for determining which basis functions to drop to minimize the residual, we give a formula for the projection operator.

Proposition 4. *The orthogonal projection of V onto $\widetilde{V}$ is $P(\cdot) = \sum_{i=d+1}^N \widetilde{x}'_i(\cdot)x_i$.*

Proof. By Proposition 3, $P(w) = w$ for all $w \in \widetilde{V}$ and Range$(P) = \widetilde{V}$. From Propositions 1 and 3, $\widetilde{V}'$ is the null space of P, and Range(P) and $\widetilde{V}' = $ Null(P) are orthogonal, so P is an *orthogonal* projection. □

The following generalizes [3, Corollary 3.2] to give the coefficients of $P(f)$ for the case $d \geq 1$.

THEOREM 5. *If* $f = \sum_{i=1}^{N} c_i x_i$, *where* $c_i = \langle x_i', f \rangle$, *then*

$$P(f) = \sum_{i=d+1}^{N} c_i' x_i, \quad \text{where } c_i' = c_i - \sum_{\ell=1}^{d} \frac{\langle x_i', v_\ell' \rangle}{\langle v_\ell', v_\ell' \rangle} \langle v_\ell', f \rangle. \tag{5}$$

Proof. We calculate, using (2),

$$P(f) = \sum_{i=d+1}^{N} \tilde{x}_i'(f) x_i = \sum_{i=d+1}^{N} \left(\langle x_i', f \rangle - \sum_{\ell=1}^{d} \frac{\langle x_i', v_\ell' \rangle}{\langle v_\ell', v_\ell' \rangle} \langle v_\ell', f \rangle \right) x_i. \tag{6}$$

so $P(f) = \sum_{i=d+1}^{N} c_i' x_i$, where

$$c_i' = c_i - \sum_{\ell=1}^{d} \frac{\langle x_i', v_\ell' \rangle}{\langle v_\ell', v_\ell' \rangle} \langle v_\ell', f \rangle. \tag{7}$$

□

The following generalizes [3, Corollary 3.3] for the case $d \geq 1$.

COROLLARY 6. *If* $f = \sum_{i=1}^{N} c_i x_i$, *where* $c_i = \langle x_i', f \rangle$, *then*

$$\| f \|^2 = \| P(f) \|^2 + \sum_{i=1}^{d} \frac{1}{\| v_i' \|^2} \left| \sum_{k=1}^{i} c_k \langle v_i', x_k \rangle \right|^2. \tag{8}$$

Proof. Since $V = \tilde{V} \oplus \tilde{V}'$, we can write $f = P(f) \oplus \text{proj}_{\tilde{V}'}(f)$, where $\text{proj}_{\tilde{V}'}(f) = \sum_{i=1}^{d} \langle v_i' / \| v_i' \|, f \rangle (v_i' / \| v_i' \|)$ is the projection of f onto $\tilde{V}'$ using the orthogonal basis $\{v_i'\}$. Thus by Parseval and then Lemma 2, we have

$$\begin{aligned} \| f \|^2 &= \| P(f) \|^2 + \left\| \sum_{i=1}^{d} \left\langle \frac{v_i'}{\| v_i' \|}, f \right\rangle \frac{v_i'}{\| v_i' \|} \right\|^2 = \| P(f) \|^2 + \sum_{i=1}^{d} \frac{1}{\| v_i' \|^2} \left| \langle v_i', f \rangle \right|^2 \\ &= \| P(f) \|^2 + \sum_{i=1}^{d} \frac{1}{\| v_i' \|^2} \left| \sum_{k=1}^{i} c_k \langle v_i, x_k \rangle \right|^2. \end{aligned} \tag{9}$$

□

Next we generalize [3, Proposition 3.4] for the case $d \geq 1$.

PROPOSITION 7. *By reindexing the original* x_i *and* x_i' *to examine all possible* $\binom{N}{d}$ *combinations of* d *components dropped from the original basis of* V *and to minimize the* L^2 *distance between* f *and* $P(f)$, *choose the set of* d *basis elements* x_i *that minimizes*

$$\sum_{i=1}^{d} \frac{1}{\| v_i' \|^2} \left| \sum_{k=1}^{i} c_k \langle v_i', x_k \rangle \right|^2. \tag{10}$$

We now give an example demonstrating that iterating the process k times with $d = 1$ may give a projection considerably farther from the original f than reducing by $k = d$ basis functions *simultaneously.*

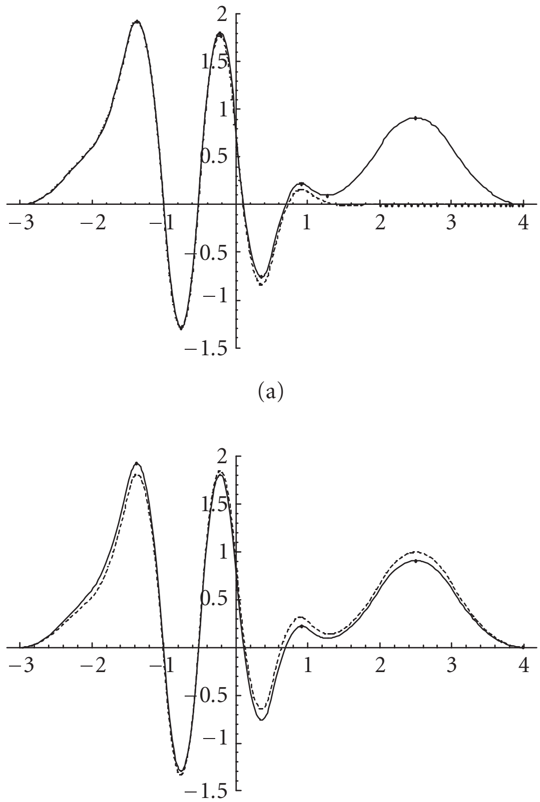

Figure 1. Drop two basis functions: iteratively (a), and simultaneously (b), for Example 8.

Example 8. For simplicity, we consider a function $f(t)$ in the four-dimensional subspace V with basis functions generated from cardinal spline wavelets. Let $B_3(x)$ be the standard quadratic cardinal spline supported on $[-1,2]$ and let $w(t)$ be the standard associated wavelet for the Riesz basis of $L^2(\mathbb{R})$ generated by $B_3(x)$ as mentioned in [1] or [2]. Let $V = \operatorname{span}\{x_1, x_2, x_3, x_4\}$, where $x_1(t) = B_3(t+2)/\|B_3\|$, $x_2(t) = B_3(t-2)/\|B_3\|$, $x_3(t) = (B_3(t-2) + B_3(t+2) + 0.2B_3(t))/\|B_3\|$, $x_4 = w(t)$. The function f can be expressed as $f(t) = 0.7x_1(t) + 0.5x_2(t) + 0.4x_3(t) + x_4(t)$. We wish to drop $d = 2$ basis elements and obtain the best two-dimensional approximation to f. If we iteratively drop one basis element at a time using Proposition 7 with $d = 1$, then we remove x_3 and then x_2 leaving projection $P(f) = 0.9x_1 + x_4$ as shown in Figure 1(a) with residual error $\|f - P(f)\|^2 = 0.82$. However, if we simultaneously drop two elements with $d = 2$, then we instead drop x_1 and x_2 leaving projection $P(f) = 1.1x_3 + x_4$ as shown in Figure 1(b) with residual error $\|f - P(f)\|^2 = 0.03$. As can be seen from these errors and the plots in Figure 1, there is a considerable advantage for $t \geq 1.5$ in removing two basis elements together, rather than dropping them iteratively.

When the value of $\binom{N}{d}$ is large, the computational expense of choosing the optimal set of basis elements to be dropped can be quite large. Investigation of this issue merits further study.

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David K. Ruch: Department of Mathematical and Computer Sciences, Metropolitan State College of Denver, Denver, CO 80217-3362, USA
E-mail address: ruch@mscd.edu