CONVERGENCE OF p-SERIES REVISITED WITH APPLICATIONS

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We construct two adjacent sequences that converge to the sum of a given convergent p-series. In case of a divergent p-series, lower and upper bounds of the (kn)th partial sum are constructed. In either case, we extend the results obtained by Hansheng and Lu (2005) to any integer $k \geq 2$. Some numerical examples are given.

Theorem 2 is the main result in this note. Lemma 1 is key to our main result. It is different from the result in [2] because it does not restrict the number of terms in the partial sum to an even one. Also, our inequalities (2) should be compared with the corresponding result in [2] for $k = 2$.

Lemma 1. *Let $s_n(p)$ be the nth partial sum of the p-series $\sum_{i=1}^{\infty}(1/i^p)$, and let k be any integer greater than 1.*

(a) If $p > 0$, then

$$s_{k-1}(p) - \frac{k-1}{k^p} + \frac{k}{k^p}s_n(p) < s_{kn}(p) < s_{k-1}(p) + \frac{k}{k^p}s_n(p). \tag{1}$$

(b) If $p < 0$, then

$$k + \frac{k}{k^p}s_{n-1}(p) < s_{kn}(p) < 1 - \frac{1}{k^p} + \frac{k}{k^p}s_n(p). \tag{2}$$

Proof. Let us observe that by the definition of $s_n(p)$, we have

$$s_{kn}(p) = 1 + \frac{1}{2^p} + \cdots + \frac{1}{(kn)^p} = \sum_{j=0}^{k-1} s_{kn}^j, \tag{3}$$

where

$$s_{kn}^j = \frac{1}{(k-j)^p} + \frac{1}{(2k-j)^p} + \cdots + \frac{1}{(nk-j)^p}. \tag{4}$$

In particular,

$$s_{kn}^{0} = \frac{1}{k^p} + \frac{1}{(2k)^p} + \cdots + \frac{1}{(nk)^p} = \frac{1}{k^p} s_n(p). \tag{5}$$

(a) Assume that $p > 0$, and k is any integer greater than 1. For $j = 1,\ldots,k-1$,

$$s_{kn}^{j} = \frac{1}{(k-j)^p} + \frac{1}{(2k-j)^p} + \cdots + \frac{1}{(nk-j)^p} > \frac{1}{(k-j)^p} + \frac{1}{(2k)^p} + \cdots + \frac{1}{(nk)^p}. \tag{6}$$

Therefore,

$$s_{kn}^{j} > \frac{1}{(k-j)^p} - \frac{1}{k^p} + \frac{1}{k^p} s_n(p). \tag{7}$$

Finally,

$$\sum_{j=0}^{k-1} s_{kn}^{j} > s_{kn}^{0} + \sum_{j=1}^{k-1} \left(\frac{1}{(k-j)^p} - \frac{1}{k^p} + \frac{1}{k^p} s_n(p) \right) = s_{k-1}(p) - \frac{k-1}{k^p} + \frac{k}{k^p} s_n(p), \tag{8}$$

which concludes the proof of the left inequality of (1).

Now, for $j = 1,\ldots,k-1$, we have

$$s_{kn}^{j} = \frac{1}{(k-j)^p} + \frac{1}{(2k-j)^p} + \cdots + \frac{1}{(nk-j)^p} < \frac{1}{(k-j)^p} + \frac{1}{k^p} + \cdots + \frac{1}{(nk)^p}, \tag{9}$$

so that

$$s_{kn}^{j} < \frac{1}{(k-j)^p} + \frac{1}{k^p} s_n(p). \tag{10}$$

It follows that

$$s_{kn}(p) < s_{kn}^{0} + \sum_{j=1}^{k-1} \left(\frac{1}{(k-j)^p} + \frac{1}{k^p} s_n(p) \right) = s_{k-1}(p) + \frac{k}{k^p} s_n(p), \tag{11}$$

which concludes the proof of the right inequality of (1). The proof of (1) is complete.

(b) Assume now that $p < 0$, and k is any integer greater than 1. We have

$$\begin{aligned} s_{kn}^{j} &= \frac{1}{(k-j)^p} + \frac{1}{(2k-j)^p} + \cdots + \frac{1}{(nk-j)^p} \\ &> 1 + \frac{1}{k^p} + \frac{1}{(2k)^p} + \cdots + \frac{1}{(k(n-1))^p} > 1 + \frac{1}{k^p} s_{n-1}(p) \end{aligned} \tag{12}$$

for $j = 0,\ldots,k-1$.

It now follows that

$$s_{kn}(p)=\sum_{j=0}^{k-1}s_{kn}^{j}>\sum_{j=0}^{k-1}\left(1+\frac{1}{k^{p}}s_{n-1}(p)\right)>k+\frac{k}{k^{p}}s_{n-1}(p). \tag{13}$$

This completes the proof of the left inequality of (2).

The proof of the right inequality of (2) follows from the inequalities below:

$$\begin{aligned} s_{kn}^{k-1}&=1+\frac{1}{(k+1)^{p}}+\cdots+\frac{1}{(k(n-1)+1)^{p}}<1+\frac{1}{(2k)^{p}}+\cdots+\frac{1}{(nk)^{p}}=1-\frac{1}{k^{p}}+\frac{1}{k^{p}}s_{n}(p),\\ s_{kn}^{j}&=\frac{1}{(k-j)^{p}}+\frac{1}{(2k-j)^{p}}+\cdots+\frac{1}{(nk-j)^{p}}<\frac{1}{k^{p}}+\frac{1}{(2k)^{p}}+\cdots+\frac{1}{(nk)^{p}}=\frac{1}{k^{p}}s_{n}(p) \end{aligned} \tag{14}$$

for $j=1,\ldots,k-2$.

It is now clear that

$$s_{kn}(p)=s_{kn}^{0}+s_{kn}^{k-1}+\sum_{j=1}^{k-2}s_{kn}^{j}<\frac{1}{k^{p}}s_{n}(p)+1-\frac{1}{k^{p}}+\frac{1}{k^{p}}s_{n}(p)+\sum_{j=1}^{k-2}\frac{1}{k^{p}}s_{n}(p)<1-\frac{1}{k^{p}}+\frac{k}{k^{p}}s_{n}(p). \tag{15}$$

The proof of the right inequality of (2) is complete; so are the proofs of (2) and Lemma 1. □

Let us now state and prove our main theorem. Like Lemma 1, Theorem 2 generalizes the results contained in [2] to any integer $k\geq 2$. In addition to that, it extends the results in [2] from the computational point of view.

Theorem 2. *Let k be any integer greater than 1.*

(a) *For $p\leq 1$, the p-series is divergent and*

$$\lim_{n\to\infty}\frac{s_{kn}(p)}{s_{n}(p)}=k^{1-p}. \tag{16}$$

(b) *For $p>1$, the p-series converges and*

$$l_{k}(p)\leq\lim_{n\to\infty}s_{n}(p)\leq u_{k}(p), \tag{17}$$

where

$$l_{k}(p)=\frac{k^{p}}{k^{p}-k}\left(s_{k-1}(p)-\frac{k-1}{k^{p}}\right),\qquad u_{k}(p)=\frac{k^{p}}{k^{p}-k}s_{k-1}(p). \tag{18}$$

Proof. (a) Assume $p \le 1$.

For $p \le 0$, the general term of the p-series does not go to 0 as n goes to ∞ and thus, the p-series diverges.

For $0 < p \le 1$, let us assume that the p-series converges to $S(p)$. By taking the limit as n goes to ∞ of (1) and solving for $S(p)$ the left inequality, one obtains

$$0 < s_{k-1}(p) - \frac{k-1}{k^p} \le \frac{k^p - k}{k^p} S(p) \le 0. \tag{19}$$

The contradiction displayed by (19) shows that the p-series diverges for $0 < p \le 1$. This completes the proof that the p-series is divergent for $p \le 1$.

Now, assume that the p-series diverges. By dividing (1) and (2) by $s_n(p)$, and taking the limit as n goes to ∞ of the newly obtained inequalities, the squeeze theorem shows that

$$\lim_{n\to\infty} \frac{s_{kn}(p)}{s_n(p)} = \frac{k}{k^p} = k^{1-p}. \tag{20}$$

This proves (16).

(b) Assume now that $p > 1$.

From (1) and the fact that $s_{k-1}(p) \le k-1$ for each $k \ge 2$, one can write

$$s_n(p) \le k - 1 + \frac{k}{k^p} s_n(p). \tag{21}$$

Solving (21) for $s_n(p)$, one obtains

$$s_n(p) \le \frac{(k-1)k^{p-1}}{k^{p-1} - 1}, \tag{22}$$

which shows that the sequence of the partial sums of the p-series is bounded above. Since it is also increasing as the sum of positive numbers, it is convergent. This concludes the proof of the convergence of the p-series for $p > 1$.

Now, let $S(p)$ be the sum of the p-series. By taking the limit as n goes to ∞ of the inequalities (1), one obtains

$$s_{k-1}(p) - \frac{k-1}{k^p} + \frac{k}{k^p} S(p) < S(p) < s_{k-1}(p) + \frac{k}{k^p} S(p) \tag{23}$$

or equivalently

$$\frac{k^p}{k^p - k}\left(s_{k-1}(p) - \frac{k-1}{k^p}\right) \le \lim_{n\to\infty} s_n(p) \le \frac{k^p}{k^p - k} s_{k-1}(p). \tag{24}$$

The proof of Theorem 2 is complete. □

Corollary 3. *For a divergent p-series,*

$$\lim_{n\to\infty}\frac{s_{kn}(p)}{s_{mn}(p)}=\left(\frac{k}{m}\right)^{1-p}. \tag{25}$$

Proof.

$$\lim_{n\to\infty}\frac{s_{kn}(p)}{s_{mn}(p)}=\lim_{n\to\infty}\frac{s_{kn}(p)}{s_n(p)}\times\lim_{n\to\infty}\frac{s_n(p)}{s_{mn}(p)}=k^{1-p}\times m^{-(1-p)}=\left(\frac{k}{m}\right)^{1-p}. \tag{26}$$

□

Example 4.

$$\lim_{n\to\infty}\frac{1+1/\sqrt[5]{2}+1/\sqrt[5]{3}+\cdots+1/\sqrt[5]{7n}}{1+1/\sqrt[5]{2}+1/\sqrt[5]{3}+\cdots+1\sqrt[5]{3n}}=\lim_{n\to\infty}\frac{s_{7n}(1/5)}{s_{3n}(1/5)}=\left(\frac{7}{3}\right)^{4/5}. \tag{27}$$

Lemma 5. *For $x\geq 2$ and $p>1$, the function*

$$f(x)=\frac{x^p}{x^p-x} \tag{28}$$

is decreasing.

Proof. Assume $x\geq 2$ and $p>1$. Then

$$f'(x)=\frac{(1-p)x^p}{(x^p-x)^2}<0. \tag{29}$$

□

Lemma 6. *For $x\geq 2$ and $p>1$,*

$$\left(\frac{(x+1)^p}{(x+1)^p-(x+1)}-\frac{x^p}{x^p-x}\right)+\left(\frac{(x+1)^p}{(x+1)^p-(x+1)}\right)\frac{1}{x^p}\leq 0. \tag{30}$$

Proof. Assume $x\geq 2$ and $p>1$; let

$$g(x)=\left(\frac{(x+1)^p}{(x+1)^p-(x+1)}-\frac{x^p}{x^p-x}\right)+\left(\frac{(x+1)^p}{(x+1)^p-(x+1)}\right)\frac{1}{x^p}. \tag{31}$$

Then

$$\lim_{x\to\infty}g(x)=0. \tag{32}$$

As in Lemma 5, one can show that $g(x)$ is increasing. This shows that

$$g(x)\leq 0. \tag{33}$$

□

We would like to point out that $l_k(p)$ and $u_k(p)$ as defined in (18) are, respectively, a lower estimate and an upper estimate of the sum of a convergent p-series. They are the general terms of sequences that enjoy some interesting properties from the computational point of view. We would like to study some of these properties in the next theorem and its corollary.

Theorem 7. *The sequences* $(l_k(p))_{k=2}^{\infty}$ *and* $(u_k(p))_{k=2}^{\infty}$ *are adjacent.*

Proof. (i) By construction,

$$l_k(p) \leqslant u_k(p), \tag{34}$$

for each $k \geqslant 2$.

(ii) Let us show that $(u_k(p))_{k=2}^{\infty}$ is nonincreasing. For each $k \geqslant 2$, we have

$$\begin{aligned} u_{k+1}(p) - u_k(p) &= \frac{(k+1)^p}{(k+1)^p - (k+1)} s_k(p) - \frac{k^p}{k^p - k} s_{k-1}(p) \\ &= \left(\frac{(k+1)^p}{(k+1)^p - (k+1)} - \frac{k^p}{k^p - k} \right) s_{k-1}(p) + \left(\frac{(k+1)^p}{(k+1)^p - (k+1)} \right) \frac{1}{k^p}. \end{aligned} \tag{35}$$

Lemma 5 implies that

$$\left(\frac{(k+1)^p}{(k+1)^p - (k+1)} - \frac{k^p}{k^p - k} \right) s_{k-1}(p) + \left(\frac{(k+1)^p}{(k+1)^p - (k+1)} \right) \frac{1}{k^p} \leq 0, \tag{36}$$

which shows that $(u_k(p))_{k=2}^{\infty}$ is a nonincreasing sequence.

(iii) The proof that $(l_k(p))_{k=2}^{\infty}$ is nondecreasing is similar to the one given in part (ii).

(iv) From (18), we have

$$\lim_{k \to \infty} (u_k(p) - l_k(p)) = \lim_{k \to \infty} \frac{k^p}{k^p - k} \left(\frac{k-1}{k^p} \right) = \lim_{k \to \infty} \frac{k-1}{k^p - k} = 0. \tag{37}$$

The proof of Theorem 7 is complete. □

The following corollary shows the computational importance of Theorem 7. It will be used to illustrate the results obtained in this note.

Corollary 8. *If the p-series converges to $S(p)$, then*

$$\lim_{k \to \infty} u_k(p) = \lim_{k \to \infty} l_k(p) = S(p). \tag{38}$$

Proof. The convergence of $(u_k(p))_{k=2}^{\infty}$ follows from the fact that it is nonincreasing and bounded below by any term of the sequence $(l_k(p))_{k=2}^{\infty}$. Likewise, the convergence of $(l_k(p))_{k=2}^{\infty}$ follows from the fact that it is nondecreasing and bounded above by any term of $(u_k(p))_{k=2}^{\infty}$. Now, from (37), one concludes that $(u_k(p))_{k=2}^{\infty}$ and $(l_k(p))_{k=2}^{\infty}$ have the same limit $S(p)$. □

Example 9. The p-series $\sum_{i=1}^{\infty} 1/i^2$ with $p = 2$ is known to converge to $\pi^2/6$. Let us sum it correct to one decimal place using the sequences $(l_k(p))_{k=2}^{\infty}$ and $(u_k(p))_{k=2}^{\infty}$.

Table 1

k	$l_k(2)$	$u_k(2)$	$u_k(2) - l_k(2)$
2	1.50	2.00	0.50
3	1.54	1.88	0.33
4	1.56	1.81	0.25
5	1.58	1.78	0.20
6	1.59	1.76	0.17
7	1.60	1.74	0.14
8	1.60	1.73	0.13
9	1.61	1.72	0.11
10	1.61	1.71	0.10
11	1.61	1.70	0.09

Solving the inequality obtained from (37) for $p = 2$,

$$\frac{k-1}{k^2-k} < 10^{-1}, \tag{39}$$

we have

$$k > 10. \tag{40}$$

Table 1 shows the details of the computation as obtained using *Microsoft Excel*.

By averaging $l_{11}(2)$ and $u_{11}(2)$, we obtain

$$\sum_{i=1}^{\infty} \frac{1}{i^2} \approx 1.655, \tag{41}$$

while $\pi^2/6 \approx 1.645$. It is important to realize that the values of $l_2(2)$ and $u_2(2)$ are the same as those obtained in [2]; however, unlike in [2], our algorithm allows any given accuracy, thanks to the adjacent sequences $(l_k(p))_{k=2}^{\infty}$ and $(u_k(p))_{k=2}^{\infty}$ that it generates.

Conclusion. While the convergence of p-series has been extensively studied in the literature with different levels of sophistication (see, e.g., [1, 3, 4]), the generalization of the elegant approach developed in [2] has led us to two types of results that translate into the following applications: the estimation of the sum of a convergent p-series as the limit of adjacent sequences and the limit of the ratio of partial sums containing different multiples of n terms.

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