SIMULTANEOUS APPROXIMATION FOR THE PHILLIPS OPERATORS

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Received 18 October 2005; Accepted 1 March 2006

We study the simultaneous approximation properties of the well-known Phillips operators. We establish the rate of convergence of the Phillips operators in simultaneous approximation by means of the decomposition technique for functions of bounded variation.

1. Introduction

Phillips operators (see, e.g., [5]) are defined as

$$P_n(f,x) = n\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k-1}(t)f(t)dt + e^{-nx}f(0), \quad n\in N,\ x\in[0,\infty), \tag{1.1}$$

where

$$p_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}. \tag{1.2}$$

May [4] estimated some direct and inverse results for certain combinations of these operators. Very recently Finta and Gupta [1] studied some direct and inverse results for the second-order Ditzian-Totik modulus of smoothness. The rates of convergence in ordinary approximation for function of bounded variation for these operators and similar types of operators were estimated in [2, 3, 7]. Very recently Srivastava and Gupta [6] proposed a general family of linear positive operators, which include the Phillips operators as special case, but they have estimated the rate of convergence in ordinary approximation. In the present paper we investigate and estimate the rate of convergence of the Phillips operators in simultaneous approximation by means of the decomposition technique for functions of bounded variation.

Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2006, Article ID 49094, Pages 1–9
DOI 10.1155/IJMMS/2006/49094

2. Auxiliary results

In this section we give the following lemmas, which will be needed to prove our main result, given in Section 3.

LEMMA 2.1. *For all $x \in (0,\infty)$ and $k \in N \cup \{0\}$,*

$$p_{n,k}(x) \le \frac{1}{\sqrt{2enx}}, \tag{2.1}$$

where the constant $1/\sqrt{2e}$ and the estimation order $n^{-1/2}$ (for $n \to \infty$) are the best possible.

Proof. Following [8], we have

$$p_{n,k}(x) \le \frac{H(j)}{\sqrt{nx}}, \quad k \ge j, \tag{2.2}$$

where $H(j) = (j+1/2)^{j+1/2}e^{-(j+1/2)}/j!$.

Since $\max_{j\ge 0} H(j) = H(0) = 1/\sqrt{2e}$, it follows that

$$p_{n,k}(x) \le \frac{1}{\sqrt{2enx}} \quad \text{for each integer } k \ge 0, \tag{2.3}$$

and Lemma 2.1 is thus proved. □

Remark 2.2. The above lemma can be utilized to give better estimate over the main results of [2, 3, 6].

LEMMA 2.3. *If $f \in L_1[0,\infty)$, $f^{(r-1)} \in A\cdot C\cdot_{\text{loc}}$, $r \in N$, and $f^{(r)} \in L_1[0,\infty)$, then*

$$P_n^{(r)}(f,x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^\infty p_{n,k+r-1}(t)f^{(r)}(t)dt. \tag{2.4}$$

Proof. First by the definition,

$$P_n^{(1)}(f,x) = n\sum_{k=1}^{\infty} p_{n,k}^{(1)}(x)\int_0^\infty p_{n,k-1}(t)f(t)dt - n.e^{-nx}f(0). \tag{2.5}$$

Using the identity $p_{n,k}^{(1)}(x) = n[p_{n,k-1}(x) - p_{n,k}(x)]$, $k \ge 1$, we have

$$\begin{aligned}
P_n^{(1)}(f,x) &= n\sum_{k=1}^{\infty} n[p_{n,k-1}(x) - p_{n,k}(x)]\int_0^\infty p_{n,k-1}(t)f(t)dt - n\cdot e^{-nx}f(0) \\
&= n^2 p_{n,0}(x)\int_0^\infty p_{n,0}(t)f(t)dt - ne^{-nx}f(0) \\
&\quad + n^2\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^\infty [p_{n,k}(t) - p_{n,k-1}(t)]f(t)dt \\
&= n^2e^{-nx}\int_0^\infty e^{-nt}f(t)dt + n^2\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^\infty \frac{-1}{n}p_{n,k}^{(1)}(t)f(t)dt - ne^{-nx}f(0)
\end{aligned}$$

$$\begin{aligned}
&= n^2 e^{-nx}\left[f(t)\frac{e^{-nt}}{-n}\bigg|_0^\infty - \int_0^\infty f^{(1)}(t)\frac{e^{-nt}}{-n}\,dt\right] \\
&\quad - n\sum_{k=1}^{\infty} p_{n,k}(x)\left[f(t)p_{n,k}(t)\bigg|_0^\infty - \int_0^\infty p_{n,k}(t)f^{(1)}(t)dt\right] - ne^{-nx}f(0) \\
&= ne^{-nx}\int_0^\infty e^{-nt}f^{(1)}(t)dt + n\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^\infty p_{n,k}(t)f^{(1)}(t)dt \\
&= n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^\infty p_{n,k}(t)f^{(1)}(t)dt.
\end{aligned} \tag{2.6}$$

Thus the result is true for $r = 1$. We prove the result by induction hypothesis, and for this we suppose it is true for $r = i$. Then

$$P_n^{(i)}(f,x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^\infty p_{n,k+i-1}(t)f^{(i)}(t)dt. \tag{2.7}$$

Again using the identity $p_{n,k}^{(1)}(x) = n[p_{n,k-1}(x) - p_{n,k}(x)]$, $k \geq 1$, it follows that

$$\begin{aligned}
P_n^{(i+1)}(f,x) &= n\sum_{k=0}^{\infty} n[p_{n,k-1}(x) - p_{n,k}(x)]\int_0^\infty p_{n,k+i-1}(t)f^{(i)}(t)dt \\
&\quad + n(-ne^{-nx})\int_0^\infty p_{n,i-1}(t)f^{(i)}(t)dt \\
&= n^2 p_{n,0}(x)\int_0^\infty p_{n,i}(t)f^{(i)}(t)dt - n^2 p_{n,0}(x)\int_0^\infty p_{n,i-1}(t)f^{(i)}(t)dt \\
&\quad + n^2\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^\infty [p_{n,k+i}(t) - p_{n,k+i-1}(t)]f^{(i)}(t)dt \\
&= n^2 p_{n,0}(x)\int_0^\infty \left(\frac{-p_{n,i}^{(1)}(t)}{n}\right)f^{(i)}(t)dt + n^2\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^\infty \left(\frac{-p_{n,k+i}^{(1)}(t)}{n}\right)f^{(i)}(t)dt.
\end{aligned} \tag{2.8}$$

Integrating by parts, we obtain

$$P_n^{(i+1)}(f,x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^\infty p_{n,k+i}(t)f^{(i+1)}(t)dt, \tag{2.9}$$

which was proved and this completes the proof of Lemma 2.3. □

Lemma 2.4. *For $m \in N \cup \{0\}$, $r \in N$, if the mth-order moment is defined by*

$$\mu_{r,n,m}(x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^\infty p_{n,k+r-1}(t)(t-x)^m dt, \tag{2.10}$$

then

$$\mu_{r,n,0}(x) = 1, \qquad \mu_{r,n,1}(x) = \frac{r}{n}, \qquad \mu_{r,n,2}(x) = \frac{2nx + r(1+r)}{n^2}. \tag{2.11}$$

Also, there holds the following recurrence relation:

$$n\mu_{r,n,m+1}(x) = x\left[\mu^{(1)}_{r,n,m}(x) + 2m\mu_{r,n,m-1}(x)\right] + (m+r)\mu_{r,n,m}(x). \tag{2.12}$$

Consequently, by the recurrence relation, for all $x \in [0,\infty)$,

$$\mu_{r,n,m}(x) = O\left(n^{-[(m+1)/2]}\right). \tag{2.13}$$

Proof. Using the identity $xp'_{n,k}(x) = (k - nx)p_{n,k}(x)$, we have

$$x\mu^{(1)}_{r,n,m}(x) = n\sum_{k=0}^{\infty} xp'_{n,k}(x)\int_0^{\infty} p_{n,k+r-1}(t)(t-x)^m dt - mx\mu_{r,n,m-1}(x). \tag{2.14}$$

Thus

$$\begin{aligned}
x&\left[\mu^{(1)}_{r,n,m}(x) + m\mu_{r,n,m-1}(x)\right] \\
&= n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} \left[\{(k+r-1) - nt\} + n(t-x) + 1 - r\right]p_{n,k+r-1}(t)(t-x)^m dt \\
&= n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} tp'_{n,k+r-1}(t)(t-x)^m dt + n\mu_{r,n,m+1}(x) + (1-r)\mu_{r,n,m}(x) \\
&= n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p'_{n,k+r-1}(t)(t-x)^{m+1} dt \\
&\quad + nx\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p'_{n,k+r-1}(t)(t-x)^m dt + n\mu_{r,n,m+1}(x) + (1-r)\mu_{r,n,m}(x).
\end{aligned} \tag{2.15}$$

Integrating by parts, we get the required recurrence relation. The other consequences easily follow from the recurrence relation. □

Remark 2.5. In particular by Lemma 2.4, for given any number $\lambda > 2$ and $0 < x < \infty$, we get for $n \geq 2$,

$$\mu_{r,n,2}(x) \leq \frac{\lambda x}{n}. \tag{2.16}$$

Remark 2.6. We can observe from Lemmas 2.3 and 2.4 that for $r = 0$, the summation over k starts from 1. For $r = 0$, Lemma 2.4 may be defined as in [6, Lemma 2], with $c = 0$.

Lemma 2.7. *Suppose $x \in (0,\infty)$, $r \in N$, and $K_{r,n}(x,t) = n\sum_{k=0}^{\infty} p_{n,k}(x)p_{n,k+r-1}(t)$. Then for $\lambda > 2$ and for $n \geq 2$, there hold*

$$\int_0^y K_{r,n}(x,t)dt \leq \frac{\lambda x}{n(x-y)^2}, \quad 0 \leq y < x, \tag{2.17}$$

$$\int_z^{\infty} K_{r,n}(x,t)dt \leq \frac{\lambda x}{n(z-x)^2}, \quad x < z < \infty. \tag{2.18}$$

Proof. We first prove (2.17) as follows:

$$\begin{aligned}\int_0^y K_{r,n}(x,t)dt &\leq \int_0^y \frac{(x-t)^2}{(x-y)^2} K_{r,n}(x,t)dt \\ &\leq \frac{1}{(x-y)^2} P_n\left((t-x)^2,x\right) \leq \frac{\mu_{r,n,2}(x)}{(x-y)^2} \leq \frac{\lambda x}{n(x-y)^2}\end{aligned} \tag{2.19}$$

by using (2.16). The proof of (2.18) follows along similar lines. □

3. Rate of convergence

We denote the class $B_{r,\alpha}$ by $B_{r,\alpha} = \{f : f^{(r-1)} \in C[0,\infty),\ f_{\pm}^{(r)}(x)$ exist everywhere and are bounded on every finite subinterval of $[0,\infty)$ and $f_{\pm}^{(r)} = O(e^{\alpha t})(t \to \infty)$, for some $\alpha > 0\}$, $r = 1,2,\ldots$. By $f_{\pm}^{(0)}(x)$ we mean $f(x\pm)$.

Now we are ready to prove and state our main theorem.

Theorem 3.1. *Let $f \in B_{r,\alpha}$, $r = 1,2,\ldots$, $\alpha > 0$. Then for every $x \in (0,\infty)$ and $n \geq \max\{2,4\alpha\}$,*

$$\begin{aligned}&\left| P_n^{(r)}(f,x) - \frac{1}{2}\left\{f_+^{(r)}(x) + f_-^{(r)}(x)\right\}\right| \\ &\quad\leq \frac{|2r-1|}{\sqrt{8enx}} \cdot \left| f_+^{(r)}(x) - f_-^{(r)}(x)\right| + \frac{x+2\lambda}{nx}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_{r,x}) + (nx)^{-1/2}\left(\lambda 2^r\right)^{1/2} e^{2\alpha x},\end{aligned} \tag{3.1}$$

where $g_{r,x}$ is the auxiliary function defined by

$$g_{r,x}(t) = \begin{cases} f^{(r)}(t) - f_-^{(r)}(x), & 0 \leq t < x, \\ 0, & t = x, \\ f^{(r)}(t) - f_+^{(r)}(x), & x < t < \infty, \end{cases} \tag{3.2}$$

and $V_a^b(g_{r,x}(t))$ is the total variation of $g_{r,x}(t)$ on [a, b]. In particular $g_{0,x}(t) \equiv g_x(t)$ as defined in [3].

Proof. Clearly

$$\begin{aligned} &\left| P_n^{(r)}(f,x) - \frac{1}{2}\left\{ f_+^{(r)}(x) + f_-^{(r)}(x) \right\} \right| \\ &\quad \le \left| P_n^{(r)}(g_{r,x}(t),x) \right| + \frac{1}{2}\left| f_+^{(r)}(x) - f_-^{(r)}(x) \right| \cdot \left| P_n^{(r)}(\operatorname{sign}(t-x),x) \right|. \end{aligned} \tag{3.3}$$

In order to prove the result we need the estimates for $P_n^{(r)}(g_{r,x},x)$ and $P_n^{(r)}(\operatorname{sign}(t-x),x)$.

We first estimate $P_n^{(r)}(\operatorname{sign}(t-x),x)$ as follows:

$$\begin{aligned} P_n^{(r)}(\operatorname{sign}(t-x),x) &= \int_x^\infty K_{r,n}(x,t)dt - \int_0^x K_{r,n}(x,t)dt \\ &= A_{r,n}(x) - B_{r,n}(x), \quad \text{say}. \end{aligned} \tag{3.4}$$

It is easily verified that $A_{r,n}(x) + B_{r,n}(x) = 1$. Thus $P_n^{(r)}(\operatorname{sign}(t-x),x) = 1 - 2A_{r,n}(x)$.

Using the fact that $\sum_{j=0}^{k+r-1} p_{n,j}(x) = n\int_x^\infty p_{n,k+r-1}(t)dt$, we get

$$\begin{aligned} A_{r,n}(x) &= n\sum_{k=0}^\infty p_{n,k}(x)\int_x^\infty p_{n,k+r-1}(t)dt = \sum_{k=0}^\infty p_{n,k}(x)\sum_{j=0}^{k+r-1} p_{n,j}(x) \\ &= \sum_{k=0}^\infty p_{n,k}(x)\left(\sum_{j=0}^{k} p_{n,j}(x) + \sum_{j=k+1}^{k+r-1} p_{n,j}(x)\right) \\ &\le \sum_{k=0}^\infty p_{n,k}(x)\sum_{j=0}^{k} p_{n,j}(x) + \frac{r-1}{\sqrt{2enx}}. \end{aligned} \tag{3.5}$$

Proceeding along similar lines as in [3], we get

$$\left| A_{r,n}(x) - B_{r,n}(x) \right| = \left| 2A_{r,n}(x) - 1 \right| \le \frac{|2r-1|}{\sqrt{2enx}}. \tag{3.6}$$

Next we estimate $P_n^{(r)}(g_{r,x},x)$, and for this, note that by Lebesgue-Stieltjes integral representation, we have

$$\begin{aligned} P_n^{(r)}(g_{r,x},x) &= \int_0^\infty g_{r,x}(t)K_{r,n}(x,t)dt = \left(\int_{I_1} + \int_{I_2} + \int_{I_3} + \int_{I_4}\right)K_{r,n}(x,t)g_{r,x}(t)dt \\ &= R_1 + R_2 + R_3 + R_4, \quad \text{say}, \end{aligned} \tag{3.7}$$

where $I_1 = [0, x - x/\sqrt{n}]$, $I_2 = [x - x/\sqrt{n}, x + x/\sqrt{n}]$, $I_3 = [x + x/\sqrt{n}, 2x]$, and $I_4 = [2x, \infty)$. Let us define $\eta_{r,n}(x,t) = \int_0^t K_{r,n}(x,u)du$.

We first estimate R_1, and for this if we write $y = x - x/\sqrt{n}$ and use integration by parts, we get

$$R_1 = \int_0^y g_{r,x}(t)d_t(\eta_{r,n}(x,t)) = g_{r,x}(y)\eta_{r,n}(x,t) - \int_0^y \eta_{r,n}(x,t)d_t(g_{r,x}(t)). \tag{3.8}$$

By Remark 2.5, it follows that

$$\begin{aligned} |R_1| &\leq V_y^x(g_{r,x})\eta_{r,n}(x,y)+\int_0^y \eta_{r,n}(x,t)d_t(-V_t^x(g_{r,x})) \\ &\leq V_y^x(g_{r,x})\frac{\lambda x}{n(x-y)^2}+\frac{\lambda x}{n}\int_0^y \frac{1}{(x-t)^2}d_t(-V_t^x(g_{r,x})). \end{aligned} \tag{3.9}$$

Integrating by parts the last term, we get after simple computation

$$|R_1| \leq \frac{\lambda x}{n}\left[\frac{V_0^x(g_{r,x})}{x^2}+2\int_0^y \frac{V_t^x(g_{r,x})}{(x-t)^3}dt\right]. \tag{3.10}$$

Now replacing the variable y in the last integral by $x-x/\sqrt{t}$, we get

$$|R_1| \leq \frac{2\lambda}{nx}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x}(g_{r,x}). \tag{3.11}$$

Next, we estimate R_2, and for this, note that for $t\in[x-x/\sqrt{n},x+x/\sqrt{n}]$, we have

$$|g_{r,x}(t)| = |g_{r,x}(t)-g_{r,x}(x)| \leq V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_{r,x}). \tag{3.12}$$

Also, by the fact that $\int_a^b d_t(\eta_{r,n}(x,t))\leq 1$ for $(a,b)\subset[0,\infty)$, we get

$$|R_2| \leq V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_{r,x}) \leq \frac{1}{n}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_{r,x}). \tag{3.13}$$

Now to estimate R_3, write $z=x+x/\sqrt{n}$. Then

$$\begin{aligned} R_3 &= \int_z^{2x} K_{r,n}(x,t)g_{r,x}(t)dt = -\int_z^{2x} g_{r,x}(t)d_t(1-\eta_{r,n}(x,t)) \\ &= -g_{r,x}(2x)(1-\eta_{r,n}(x,2x))+g_{r,x}(z)(1-\eta_{r,n}(x,z))+\int_z^{2x}(1-\eta_{r,n}(x,t))d_tg_{r,x}(t). \end{aligned} \tag{3.14}$$

Since $|g_{r,x}(t)|=|g_{r,x}(t)-g_{r,x}(x)|\leq V_x^t(g_{r,x})$, it follows by Lemma 2.7 that

$$|R_3| \leq \frac{\lambda x}{n}\left\{x^{-2}V_x^{2x}(g_{r,x})+(z-x)^{-2}V_x^z(g_{r,x})+\int_z^{2x}(t-x)^{-2}d_tV_x^t(g_{r,x})\right\}. \tag{3.15}$$

Again integrating by parts, we get

$$|R_3| \leq \frac{\lambda x}{n}\left\{2x^{-2}V_x^{2x}(g_{r,x})+2\int_z^{2x}V_x^t(g_{r,x})(t-x)^{-3}dt\right\}. \tag{3.16}$$

Arguing similarly as in the estimate of R_1, we obtain

$$|R_3| \leq \frac{2\lambda}{nx}\sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_{r,x}). \tag{3.17}$$

Finally, we estimate R_4 as follows:

$$
\begin{aligned}
|R_4| &= \left| \int_{2x}^{\infty} K_{r,n}(x,t) g_{r,x}(t)\,dt \right| \le n \sum_{k=0}^{\infty} p_{n,k}(x) \int_{2x}^{\infty} p_{n,k+r-1}(t) e^{\alpha \cdot t}\,dt \\
&\le \frac{n}{x} \sum_{k=0}^{\infty} p_{n,k}(x) \int_{0}^{\infty} p_{n,k+r-1}(t) e^{\alpha \cdot t} |t-x|\,dt \\
&\le \frac{1}{x} \left(n \sum_{k=0}^{\infty} p_{n,k}(x) \int_{0}^{\infty} p_{n,k+r-1}(t)(t-x)^2\,dt \right)^{1/2} \left(n \sum_{k=0}^{\infty} p_{n,k}(x) \int_{0}^{\infty} p_{n,k+r-1}(t) e^{2\alpha \cdot t}\,dt \right)^{1/2}.
\end{aligned}
\tag{3.18}
$$

To estimate the above first expression we will use Remark 2.5, and to evaluate the second expression, we note that

$$
\begin{aligned}
& n \sum_{k=0}^{\infty} p_{n,k}(x) \int_{0}^{\infty} p_{n,k+r-1}(t) e^{2\alpha \cdot t}\,dt \\
&\qquad = n \sum_{k=0}^{\infty} p_{n,k}(x) \frac{n^{k+r-1}}{(k+r-1)!} \int_{0}^{\infty} t^{k+r-1} e^{-(n-2\alpha)t}\,dt \\
&\qquad = n \sum_{k=0}^{\infty} p_{n,k}(x) \frac{n^{k+r-1}}{(k+r-1)!} \frac{\Gamma(k+r)}{(n-2\alpha)^{k+r}} = \frac{n^r}{(n-2\alpha)^r} \sum_{k=0}^{\infty} \left(\frac{n}{n-2\alpha} \right)^k p_{n,k}(x) \\
&\qquad = \frac{n^r}{(n-2\alpha)^r} e^{-nx} \sum_{k=0}^{\infty} \left(\frac{n^2 x}{n-2\alpha} \right)^k \frac{1}{k!} = \frac{n^r}{(n-2\alpha)^r} e^{2nx\alpha/(n-2\alpha)} \le 2^r e^{4\alpha x} \quad \text{for } n > 4\alpha.
\end{aligned}
\tag{3.19}
$$

If we now combine the above estimate with Remark 2.5, we get

$$
\begin{aligned}
|R_4| &\le \frac{1}{x} \left(n \sum_{k=0}^{\infty} p_{n,k}(x) \int_{0}^{\infty} p_{n,k+r-1}(t)(t-x)^2\,dt \right)^{1/2} \left(n \sum_{k=0}^{\infty} p_{n,k}(x) \int_{0}^{\infty} p_{n,k+r-1}(t) e^{2\alpha \cdot t}\,dt \right)^{1/2} \\
&\le (nx)^{-1/2} \left(\lambda 2^r \right)^{1/2} e^{2\alpha x}.
\end{aligned}
\tag{3.20}
$$

Finally, combining the estimates obtained in (3.3)–(3.20), we get the required result, and the proof of the theorem is thus complete. □

Remark 3.2. Since the Bézier variant of the Phillips operators for $\beta \ge 1$ is defined by

$$
P_{n,\beta}(f,x) = n \sum_{k=1}^{\infty} Q_{n,k}^{(\beta)}(x) \int_{0}^{\infty} p_{n,k-1}(t) f(t)\,dt + Q_{n,0}^{(\beta)}(x) f(0),
\tag{3.21}
$$

where $Q_{n,k}^{(\beta)}(x) = J_{n,k}^{\beta}(x) - J_{n,k+1}^{\beta}(x)$, $J_{n,k}(x) = \sum_{i=k}^{\infty} p_{n,i}(x)$, the rate of convergence in simultaneous approximation for the above Bézier variant of Phillips operators can be obtained along similar lines. We omit the details.

Acknowledgment

This research was carried out while the second author was visiting Auburn University, USA, during the fall of 2005.

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