EXACT CATEGORY OF HYPERMODULES

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It is shown, among other things, that the category of hypermodules is an exact category, thus generalizing the classical case.

1. Introduction

The theory of hyperstructures has been introduced by Marty in 1934 during the 8th Congress of the Scandinavian Mathematicians [4]. Marty introduced the notion of a hypergroup and since then many researchers have worked on this new topic of modern algebra and developed it. The notion of a hyperfield and a hyperring was studied first by Krasner [2] and then some authors followed him, for example, see [8].

The canonical hypergroups are a special type of hypergroup. Initially they were derived from the additive part of the hyperfield and hyperring. The name canonical has been given to these hypergroups by Mittas, who is the first one that studied them extensively [7]. Again in the context of canonical hypergroups some mathematicians, for example, [5] studied hypermodules whose additive structure is just a canonical hypergroup.

Considering the class of hypermodules over a fixed hyperring R and the class of all homomorphisms among hypermodules together with the composition of the mappings, knowing that the composite of two homomorphisms is again a homomorphism and that for any hypermodules A over the hyperring R, $\mathrm{id}_A : A \to A$, $\mathrm{id}(a) = a$, is a homomorphism among hypermodules, we can construct a category which is denoted by Hmod.

In this paper some aspects of hypermodules are studied. We will show that the category Hmod is exact in the sense that it is normal and conormal with kernels and cokernels in which every arrow f has a factorization $f = \nu q$, with ν being a monomorphism and q an epimorphism (see [6]). Two of the most used results of the paper are those which state that the monomorphisms of Hmod (in the categorical sense) are the one- to one-homomorphisms and the epimorphisms of Hmod are the onto homomorphisms.

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2. Preliminaries and notations

In this section we recall some definitions from the theory of hyperstructures and fix our notations. For other notions concerning this theory which may be used in this paper we refer the reader to [1, 8]. The principal notions of category theory can be found in [3, 6].

A nonvoid set H endowed with a hyperoperation $+$ is called a canonical hypergroup if the following axioms are satisfied:

(i) $(x+y)+z = x+(y+z)$;

(ii) $x+y = y+x$;

(iii) there is an element $0 \in H$ such that for every $x \in H$ there is a unique element $x' \in H$ such that $0 \in (x+x') \cap (x'+x)$. We will denote x' by $-x$ and we will call it the opposite of x; instead of $x+(-y)$ we will write $x-y$;

(iv) $z \in x+y$ implies $x \in z-y$.

Remark 2.1. It can be easily seen that 0 is unique and that $x+0 = 0+x = x$ for every $x \in H$. (For a study of canonical hypergroups we refer the reader to [7].)

Following [2] a nonvoid set R endowed with a hyperoperation $(+)$ and with an (internal) operation $(\cdot)$ is called a hyperring if

(i) $(R,+)$ is a canonical hypergroup;

(ii) $(R,\cdot)$ is a multiplicative semigroup having 0 as a bilaterally absorbing element; that is, $x \cdot 0 = 0 \cdot x = 0$ for every $x \in R$;

(iii) $z \cdot (x+y) = z \cdot x + z \cdot y$, $(x+y) \cdot z = x \cdot z + y \cdot z$ (distributivity).

Following [1] a (left) hypermodule over a hyperring R is a canonical hypergroup $(A,+)$ together with a scaler single-valued operation $(\cdot)$, that is, a function which associates to any pair $(r,x) \in R \times A$ an element $r \cdot x \in A$ such that for all $(x,y) \in A^2$ and $(r,s) \in R^2$,

(i) $r \cdot (x+y) = r \cdot x + r \cdot y$;

(ii) $(r+s) \cdot x = r \cdot x + s \cdot x$;

(iii) $(rs) \cdot x = r \cdot (s \cdot x)$;

(iv) $0 \cdot x = x$.

Definition 2.2. If A and B are two hypermodules over a fixed hyperring R, then a function $f : A \to B$ is called a homomorphism if

(1) $f(x+y) = f(x) + f(y)$ for all $x, y \in A$,

(2) $f(rx) = rf(x)$ for all $r \in R$ and $x \in A$.

Throughout the following sections we will be concerned almost exclusively with hypermodules.

3. Monomorphisms and epimorphisms in Hmod

Let R be a fixed hyperring. As we have explained in the introduction, if we consider the class of all hypermodules over the hyperring R and the class of all homomorphisms among them, we get a category which is denoted by Hmod. In this section we will study some properties of this category.

We start with the following useful theorem.

Theorem 3.1. *Let A, B, C be hypermodules over the hyperring R and $f : A \to B$, $g : A \to C$ homomorphisms. If g is onto, then*

(1) $\ker g \subseteq \ker f$ *implies the existence of a homomorphism $h : C \to B$ such that $f = h \circ g$;*
(2) *f is onto implies that h is onto;*
(3) $\ker f = \ker g$ *implies that h is one to one;*
(4) *f is onto and* $\ker f = \ker g$ *implies h is an isomorphism.*

Proof. (1) Since g is onto for any $c \in C$, there exists $a \in A$ such that $g(a) = c$. Define $h(c) := f(a)$. Then f is well defined, because if there exist $a_1, a_2 \in A$ such that $g(a_1) = g(a_2) = c$, then $0 \in g(a_1) - g(a_2) = g(a_1 - a_2)$ and so there exists $a \in a_1 - a_2$ such that $0 = g(a)$. That is, $a \in \ker g$ so $a \in \ker f$, that is, $0 \in f(a_1 - a_2) = f(a_1) - f(a_2)$. Hence $f(a_1) = f(a_2)$.

(2) It follows easily from definition of h. Statements (3) and (4) are direct applications of (1) and (2). □

Let A be a hypermodule and let B be a subhypermodule of A. Then there is only one hyperstructure of hypermodule on $A/B = \{a + B \mid a \in A\}$ such that the mapping $\pi : A \to A/B$ is a homomorphism given by the hyperoperation

$$(x + B) + (y + B) = \{z + B \mid z \in x + y\} \tag{3.1}$$

and (internal) operation

$$r(x + B) = rx + B. \tag{3.2}$$

The homomorphism π is onto. We call it the canonical projection.

The following fact is an immediate consequence of Theorem 3.1.

Corollary 3.2. *Let A, B be hypermodules over the hyperring R, $f : A \to B$ a homomorphism, and C a subhypermodule of A. Then,*

(1) *if $C \subseteq \ker f$, then there is a unique homomorphism $\overline{f} : A/C \to B$ such that $\overline{f}(a + C) = f(a)$ for all $a \in A$, that is, the diagram*

$$\begin{tikzcd} A \arrow[r, two heads, "\pi"] \arrow[dr, "f"'] & A/C \arrow[d, "\overline{f}"] \\ & B \end{tikzcd} \tag{3.3}$$

commutes,
(2) *if f is onto, then $\overline{f}$ is onto,*
(3) *if $C = \ker f$, then $\overline{f}$ is one to one,*
(4) *if f is onto and $C = \ker f$, then $\overline{f}$ is an isomorphism.*

The next two results characterize monomorphism and epimorphism in the category Hmod.

Theorem 3.3. *Let A, B be hypermodules over a hyperring R. For any homomorphism $f : A \to B$ the following are equivalent:*

(1) *f is one to one;*

(2) $\ker f = \{0\}$*;*

(3) *for any hypermodule C and for any homomorphisms $g,h : C \to A$, if $f \circ g = f \circ h$, then $g = h$.*

Proof. (2)⇒(1). Assume that (2) holds. Then, for any $x, y \in A$, $f(x) = f(y)$ implies that $0 \in f(x) + [-f(y)] = f(x) + f(-y) = f(x - y)$, so there exists $a \in A$ such that $f(a) = 0$. That is, $a \in \ker f$, therefore $a = 0$. This shows that $0 \in x - y$ and hence, $x = y$.

(1)⇒(3) is trivial.

(3)⇒(2) will be proved by showing that the negation of (2) leads us to the negation of (3). Indeed, if $\ker f \neq \{0\}$, then there exists $a \in A$ such that $a \neq 0$, $f(a) = 0$. Let $g : \ker f \to A$ be the inclusion mapping and consider $h : \ker f \to A$ as the zero mapping, that is, $h(a) = 0$ for all $a \in \ker f$. Clearly g, h are homomorphisms and we have also

$$\begin{aligned} f \circ g(a) &= f\big(g(a)\big) = f(a) = 0; \\ f \circ h(a) &= f\big(h(a)\big) = f(0) = 0; \end{aligned} \tag{3.4}$$

thus $f \circ g = f \circ h$. □

Theorem 3.4. *Let A, B be hypermodules over a hyperring R. For any homomorphism $f : A \to B$ the following are equivalent:*

(1) *f is onto;*

(2) *for any hypermodule C and for any homomorphisms $g,h : B \to C$, if $g \circ f = h \circ f$, then $g = h$.*

Proof. (1)⇒(2) is trivial.

(2)⇒(1) will be proved by showing that the negation of (1) leads us to the negation of (2). Indeed, if $\operatorname{Im} f \neq B$, then $B/\operatorname{Im} f$ is a hypermodule with zero $\operatorname{Im} f$ and $|B/\operatorname{Im} f| \geq 2$. Consider the homomorphisms $g = \pi_{\operatorname{Im} f} : B \to B/\operatorname{Im} f$ and $h : B \to B/\operatorname{Im} f$ given by $h(b) = \operatorname{Im} f$ for any $b \in B$. They are obviously different, but we have for any $a \in A$,

$$\begin{aligned} g \circ f(a) = g\big(f(a)\big) &= f(a) + \operatorname{Im} f, \\ h \circ f(a) = h\big(f(a)\big) &= \operatorname{Im} f. \end{aligned} \tag{3.5}$$

Thus $g \circ f = h \circ f$. □

The following theorem follows immediately from Theorems 3.3 and 3.4.

Theorem 3.5. (1) *In the category Hmod the monomorphisms are the homomorphisms which are one to one.*

(2) *In the category Hmod the epimorphisms are the homomorphisms which are onto.*

Corollary 3.6. *The category Hmod is a balanced category.*

4. Categorical properties of Hmod

In this final section we give some properties of the category Hmod. First of all we characterized subobjects and quotient objects in this category.

Theorem 4.1. *Let A be a hypermodule over the hyperring R. Then,*
(1) *there exists a bijection between the subhypermodules of A and the subobjects of A in Hmod;*
(2) *there exists a bijection between the quotient hypermodules of A over subhypermodules and the quotient objects of A in Hmod.*

Proof. (1) Let S_0 be the class of subhypermodules of A and S the class of subobjects of A in Hmod and let us consider the function $f : S_0 \to S$, $f(B) = [B, i_B]$. ($i_B : B \to A$, $i_B(b) = b$, is the inclusion homomorphism).

First of all we prove that f is one to one. To do so, considering $B_1, B_2 \in S_0$, $f(B_1) = f(B_2)$ implies that $[B_1, i_{B_1}] = [B_2, i_{B_2}]$, that is, $(B_1, i_{B_1}) \sim (B_2, i_{B_2})$, thus there exists an isomorphism $\phi : B_1 \to B_2$ such that $i_{B_1} = i_{B_2} \circ \phi$ and so for any $b \in B_1$, $b = \phi(b) \in B_2$. Hence, $B_1 \subseteq B_2$.

But we also have $i_{B_1} \circ \phi^{-1} = i_{B_2}$, it follows that $B_2 \subseteq B_1$ and that is why $B_1 = B_2$.

The function f is also onto. In fact, for any $[B, h] \in S$ knowing that h is a homomorphism it follows that $h(B) \in S_0$. We will show that $f(h(B)) = [B, h]$, that is, $(h(B), i_{h(B)}) \sim (B, h)$.

The mapping $\phi : B \to h(B)$, $\phi(b) = h(b)$ is a bijection one and it is a homomorphism, thus ϕ is an isomorphism and we have $h = i_{h(B)} \circ \phi$ which means that $(h(B), i_{h(B)}) \sim (B, h)$:

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[d, dotted, "\phi"'] & A \\ h(B) \arrow[ur, "i_{h(B)}"'] & \end{tikzcd} \tag{4.1}$$

The first part of the theorem is proved now.

(2) Let Q_0 be the class of quotient hypermodules of A over the subhypermodules and Q the class of quotient objects of A in Hmod and let us consider the function $g : Q_0 \to Q$, $g(A/B) = [\pi_B, A/B]$, where $\pi_B : A \to A/B$ is the canonical projection.

First we will prove that g is one to one. To this end, let us consider A/B_1 and $A/B_2 \in Q_0$ such that $g(A/B_1) = g(A/B_2)$. Then, $[\pi_{B_1}, A/B_1] = [\pi_{B_2}, A/B_2]$, that is, $(\pi_{B_1}, A/B_1) \sim (\pi_{B_2}, A/B_2)$ thus there exists an isomorphism $\psi : A/B_2 \to A/B_1$ such that $\pi_{B_1} = \psi \circ \pi_{B_2}$, that is, the diagram

$$\begin{tikzcd} A \arrow[r, "\pi_{B_2}"] \arrow[dr, "\pi_{B_1}"'] & A/B_2 \arrow[d, dotted, "\psi"] \\ & A/B_1 \end{tikzcd} \tag{4.2}$$

commutes. We have

$$B_1 = \{a \in A \mid \pi_{B_1}(a) = B_1\} = \{a \in A \mid \psi(\pi_{B_1}(a)) = B_1\} = \{a \in A \mid \pi_{B_2}(a) = B_2\} = B_2, \tag{4.3}$$

which leads to $A/B_1 = A/B_2$.

The function g is also onto. In fact, for any $[\gamma, C] \in Q$ knowing that γ is a homomorphism which is onto (i.e., an epimorphism in Hmod by Theorem 3.4), if we set $B = \ker\gamma$, in view of Theorem 3.1 there exists an isomorphism $\alpha : A/B \to C$ such that $\alpha \circ \pi_B = \gamma$ (indeed, α is an isomorphism because α is onto and $\ker\gamma = B = \ker\pi_B$). This means that $[\gamma, C] = [\pi_B, A/B] = g(A/B)$ which completes the bijectivity of g. □

In the next theorem we show that the category Hmod has images and coimages too.

Theorem 4.2. (1) *The category Hmod is a category with images.*

(2) *The category Hmod is a category with coimages.*

Proof. (1) For any $f : A \to B$ among hypermodules, $[f(A), i_{f(A)}] \in S(B)$, where $S(B)$ is the class of all subobjects of B and $i_{f(A)} : f(A) \to B$ is the inclusion mapping.

We will prove that $\mathrm{Im}(f) = [f(A), i_{f(A)}]$. If we consider $\theta : A \to f(A)$, $\theta(a) = f(a)$ for every $a \in A$, then θ is a homomorphism and we have $f = i_{f(A)} \circ \theta$. Next, let $[K, u] \in S(B)$ and there exists homomorphism $\theta' : A \to K$ in such a way that $f = u \circ \theta'$. Then u is clearly a monomorphism, and $f(a) = u(\theta'(a))$, for any $a \in A$, implies that $f(A) \subseteq u(K)$. But the mapping $u_1 : K \to u(K)$, $u_1(a) = u(a)$, is an isomorphism, hence $u_1^{-1} : u(K) \to K$ is also an isomorphism, therefore $\gamma = u_1|_{f(A)}$ is a homomorphism and $i_{f(A)} = u \circ \gamma$. Thus, $[f(A), i_{f(A)}] \leq [K, u]$ and this results in that $\mathrm{Im}(f) = [f(A), i_{f(A)}]$:

$$\begin{tikzcd} f(A) \arrow[r, "i_{f(A)}"] \arrow[d, dotted, "\gamma"'] & B \arrow[d, "id_B"] \\ K \arrow[r, "u"] & B \end{tikzcd} \tag{4.4}$$

(2) For any homomorphism $f : A \to B$ let $C = \ker f$. Then $[\pi_C, A/C] \in Q(A)$ where $Q(A)$ denotes the class of all quotient objects of A in the sense of previous theorem. We claim that $\mathrm{coim}(f) = [\pi_C, A/C]$. To prove the claim, let $\theta : A/C \to B$ be defined by the rule $\theta(a + C) = f(a)$ for any $a \in A$. Then θ is a homomorphism and $\theta \circ \pi_C = f$. Next let $[\nu, D] \in Q(A)$ and there exists a homomorphism $\theta : D \to B$ such that $f = \theta \circ \nu$, then, according to the fact that ν is a homomorphism which is onto and $\ker\nu \subseteq \ker f = C$, we can see that there exists a homomorphism $\gamma : D \to A/C$ such that the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "\nu"] \arrow[dr, "\pi_C"'] & D \arrow[r, "\theta'"] \arrow[d, dotted, "\gamma"] & B \\ & A/C \arrow[ur, "\theta"'] & \end{tikzcd} \tag{4.5}$$

thus $[\pi_C, A/C] \leq [\nu, D]$ which leads us to the equality $\mathrm{coim}(f) = [\pi_C, A/C]$. □

Now we come to the concepts of kernel and cokernel.

Theorem 4.3. (1) *The category Hmod is a category with kernels and cokernels.*
(2) *The category Hmod is a normal and conormal category.*

Proof

Part 1. (i) *Hmod is a category with kernels.* Let $f : A \to B$ be a morphism in Hmod and let $C = \ker f$. We know that $i_C : C \to A$ is a monomorphism in Hmod. We will prove that $\ker f = [C, i_C]$.

Let 0_{CB} be the zero morphism from C into B, $0_{CB} : C \to B$, $0_{CB}(c) = 0$. For any $c \in C$ we have $f \circ i_C(c) = f(c) = 0 = 0_{CB}(c)$, thus $f \circ i_C = 0_{CB}$. If $D \in Obj$ Hmod and $\mu \in \hom(D,A)$ such that $f \circ \mu = 0_{DB}$, then $f(\mu(d)) = 0$, for all $d \in D$, hence $\mu(D) \subseteq C$. It follows that the mapping $\gamma : D \to C$, $\gamma(d) = \mu(d)$, is a homomorphism and it verifies the equality $\mu = i_C \circ \gamma$:

$$\begin{tikzcd} D \arrow[d, dotted] \arrow[dr, "\mu"] & & \\ C \arrow[r, hook, "i_C"'] & A \arrow[r, "f"'] & B \end{tikzcd} \tag{4.6}$$

(ii) Let $f : A \to B$ be a morphism in Hmod and let $\pi : B \to B/f(A)$ be the projection. We will show that Coker $f = [\pi, B/f(A)]$. We have, $\pi f(a) = \pi(f(a)) = f(a) + f(A) = f(A) = 0_{A,B/f(A)}$. For any $C \in Obj$ Hmod and for any $\mu \in \hom(B,C)$ such that $\mu \circ f = 0_{AC}$ we have $(\mu \circ f)(a) = \mu(f(a)) = 0$, for all $a \in A$, therefore, $f(A) \subseteq \ker\mu$. So, there exists a unique homomorphism $\gamma : B/f(A) \to C$ such that $\gamma \circ \pi = \mu$:

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, two heads, "\pi"] \arrow[dr, "\mu"'] & B/f(A) \arrow[d, dotted, "\gamma"] \\ & & C \end{tikzcd} \tag{4.7}$$

Part 2. (i) *Hmod is a normal category.* Let A be an object in Hmod and $[B, f]$ a subobject of A. Then we have $[B, f] = [f(B), i_{f(B)}]$ and if $\pi_{f(B)} : A \to A/f(B)$ is the canonical projection, then, according to (i) of Part 1, $\ker\pi_{f(B)} = [f(B), i_{f(B)}] = [B, f]$.

(ii) *Hmod is a conormal category.* Let A be an object in Hmod and $[g, B]$ a quotient object of A. Then $[g, B] = [\pi_C, A/C]$ where $C = \ker g, \pi_C$ is the canonical projection. According to (ii) of Part 1 we have coker $i_C = [\pi_c, A/C] = [g, B]$. □

Recall from [6] that normal and conormal categories with kernels and cokernels are exact if every morphism $\alpha : A \to B$ can be written as a composition $A \xrightarrow{q} I \xrightarrow{\nu} B$ where q is an epimorphism and ν is a monomorphism. Now we can easily prove the following theorem from Theorem 4.3.

Theorem 4.4. *The category Hmod is an exact category.*

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