A DENSITY RESULT IN VECTOR OPTIMIZATION

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We study a class of vector minimization problems on a complete metric space such that all its bounded closed subsets are compact. We show that a subclass of minimization problems with a nonclosed set of minimal values is dense in the whole class of minimization problems.

1. Introduction and the main result

The study of vector optimization problems has recently been a rapidly growing area of research. See, for example, [1–5] and the references mentioned therein. In this paper we study a class of vector minimization problems on a complete metric space such that all its bounded closed subsets are compact. This class of problems is associated with a complete metric space of continuous vector functions $\mathcal{A}$ defined below. For each F from $\mathcal{A}$ we denote by $v(F)$ the set of all minimal elements of the image $F(X) = \{F(x) : x \in X\}$. In this paper we will study the sets $v(F)$ with $F \in \mathcal{A}$. It is clear that for a minimization problem with only one criteria the set of minimal values is a singleton. In the present paper we will show that the subspace of all $F \in \mathcal{A}$ with nonclosed sets $v(F)$ is dense in $\mathcal{A}$. Therefore in general the sets $v(F)$, $F \in \mathcal{A}$ can be rather complicated.

In this paper we use the convention that $\infty/\infty = 1$ and denote by $\text{Card}(E)$ the cardinality of the set E.

Let $\mathbb{R}$ be the set of real numbers and let n be a natural number. Consider the finite-dimensional space R^n with the norm

$$\|x\| = \|(x_1,\dots,x_n)\| = \max\{|x_i| : i = 1,\dots,n\}, \quad x = (x_1,\dots,x_n) \in R^n. \tag{1.1}$$

Let $\{e_1,\dots,e_n\}$ be the standard basis in R^n:

$$e_1 = (1,0,\dots,0),\dots,e_n = (0,\dots,0,1). \tag{1.2}$$

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We equip the space R^n with the natural order. Let $x = (x_1,\dots,x_n), y = (y_1,\dots,y_n) \in R^n$. We say that

$$\begin{aligned} x \geq y &\quad \text{if } x_i \geq y_i \ \forall i \in \{1,\dots,n\}, \\ x > y &\quad \text{if } x \geq y,\ x \neq y, \\ x \gg y &\quad \text{if } x_i > y_i \ \forall i \in \{1,\dots,n\}. \end{aligned} \tag{1.3}$$

We say that $x \ll y$ (resp., $x < y$, $x \leq y$) if $y \gg x$ (resp., $y > x$, $y \geq x$).

Let (X,ρ) be a complete metric space such that each of its bounded closed subsets is compact. Fix $\theta \in X$.

Denote by $\mathcal{A}$ the set of all continuous mappings $F = (f_1,\dots,f_n) : X \to R^n$ such that for $i = 1,\dots,n$

$$\lim_{\rho(x,\theta)\to\infty} f_i(x) = \infty. \tag{1.4}$$

For each $F = (f_1,\dots,f_n), G = (g_1,\dots,g_n) \in \mathcal{A}$ set

$$\begin{aligned} \widetilde{d}(F,G) &= \sup\{ | f_i(x) - g_i(x) | : x \in X,\ i = 1,\dots,n\}, \\ d(F,G) &= \widetilde{d}(F,G)\bigl(1 + \widetilde{d}(F,G)\bigr)^{-1}. \end{aligned} \tag{1.5}$$

Clearly the metric space $(\mathcal{A},d)$ is complete.

Let $A \subset R^n$ be a nonempty set. An element $x \in A$ is called a minimal element of A if there is no $y \in A$ for which $y < x$.

Let $F \in \mathcal{A}$. A point $x \in X$ is called a point of minimum of F if $F(x)$ is a minimal element of $F(X)$. If $x \in X$ is a point of minimum of F, then $F(x)$ is called a minimal value of F. Denote by $M(F)$ the set of all points of minimum of F and put $v(F) = F(M(F))$.

The following proposition is proved in [6].

Proposition 1.1. *Let $F = (f_1,\dots,f_n) \in \mathcal{A}$. Then $M(F)$ is a nonempty bounded subset of (X,ρ) and for each $z \in F(X)$ there is $y \in v(F)$ such that $y \leq z$.*

In the sequel we assume that $n \geq 2$ and that the space (X,ρ) has no isolated points.

The following theorem is our main result. It will be proved in Section 2.

Theorem 1.2. *Suppose that the space (X,ρ) is connected. Let $F = (f_1,\dots,f_n) \in \mathcal{A}$ and let $\epsilon > 0$. Then there exists $G \in \mathcal{A}$ such that $\widetilde{d}(F,G) \leq \epsilon$ and the set $v(G)$ is not closed.*

2. Proof of Theorem 1.2

By Proposition 1.1 there exists $x_* \in X$ such that

$$F(x_*) \in v(F). \tag{2.1}$$

There exists $\delta \in (0,1/4)$ such that

$$||F(x) - F(x_*)|| \le \frac{\epsilon}{8} \quad \text{for each } x \in X \text{ such that } \rho(x,x_*) \le 2\delta, \tag{2.2}$$

$$\{z \in X : \rho(z,x_*) = 8\delta\} \ne \emptyset. \tag{2.3}$$

Since the metric space X is connected for each $t \in (0,8\delta]$, there is $z \in X$ such that $\rho(z, x_*) = t$.

It is clear that there exists a continuous function $\psi : X \to [0,1]$ such that

$$\begin{aligned} \psi(x) = 1 \quad &\text{for each } x \in X \text{ satisfying } \rho(x,x_*) \le \delta, \\ \psi(x) = 0 \quad &\text{for each } x \in X \text{ satisfying } \rho(x,x_*) \ge 2\delta. \end{aligned} \tag{2.4}$$

For $x \in X$ and $i = 1,\dots,n$ define

$$f_i^{(1)}(x) = \psi(x) f_i(x_*) + (1 - \psi(x)) f_i(x), \tag{2.5}$$

$$F^{(1)} = (f_1^{(1)},\dots,f_n^{(1)}). \tag{2.6}$$

Clearly, $F^{(1)} \in \mathcal{A}$ and

$$F^{(1)}(x) = F(x_*) \quad \forall x \in X \text{ satisfying } \rho(x,x_*) \le \delta. \tag{2.7}$$

We will show that $\tilde{d}(F,F^{(1)}) \le \epsilon/8$. Let $x \in X$. If $\rho(x,x_*) \ge 2\delta$, then $\psi(x) = 0$ and $F^{(1)}(x) = F(x)$. If $\rho(x,x_*) \le 2\delta$, then by (2.2)

$$||F(x) - F(x_*)|| \le \frac{\epsilon}{8}. \tag{2.8}$$

Combined with (2.5) the inequality above implies that

$$||F^{(1)}(x) - F(x)|| = ||\psi(x)(F(x_*) - F(x))|| \le ||F(x_*) - F(x)|| \le \frac{\epsilon}{8}. \tag{2.9}$$

Therefore

$$\tilde{d}(F,F^{(1)}) \le \frac{\epsilon}{8}. \tag{2.10}$$

We will show that $F^{(1)}(x_*) \in \nu(F^{(1)})$. Assume that $x \in X$ and that

$$F^{(1)}(x) \le F^{(1)}(x_*) = F(x_*). \tag{2.11}$$

By (2.11), (2.5), and (2.6)

$$F(x_*) \ge F^{(1)}(x) = \psi(x)F(x_*) + (1 - \psi(x))F(x). \tag{2.12}$$

If $\psi(x) = 1$, then $F^{(1)}(x) = F(x_*)$. If $\psi(x) < 1$, then by (2.12) and (2.1) $F(x_*) \ge F(x)$ and $F(x_*) = F(x)$ and in view of (2.5) and (2.6) $F^{(1)}(x) = F(x_*)$. Therefore $F(x_*) \in \nu(F^{(1)})$.

For $x \in X$ and $i \in \{1,\dots,n\}$ set

$$f_i^{(2)}(x) = f_i^{(1)}(x) + \min\left\{1, \max\left\{\rho(x,x_*) - \frac{\delta}{2}, 0\right\}\right\}\left(\frac{\epsilon}{8}\right), \tag{2.13}$$

$$F^{(2)} = \left(f_1^{(2)},\dots,f_n^{(2)}\right). \tag{2.14}$$

Clearly

$$F^{(2)} \in \mathcal{A}, \quad F^{(2)}(x) \geq F^{(1)}(x) \quad \forall x \in X. \tag{2.15}$$

By (2.13), and (2.7)

$$F^{(2)}(x) = F^{(1)}(x) = F(x_*) \quad \forall x \in X \text{ satisfying } \rho(x,x_*) \leq \frac{\delta}{2}. \tag{2.16}$$

By (2.13) and (2.7) for each $x \in X$ satisfying$\rho(x,x_*) \in [\delta/2,\delta]$,

$$F^{(2)}(x) = F^{(1)}(x_*) + \left(\frac{\epsilon}{8}\right)\left[\rho(x,x_*) - \frac{\delta}{2}\right](1,1,\dots,1). \tag{2.17}$$

By (2.13)

$$\tilde{d}\left(F^{(2)},F^{(1)}\right) \leq \frac{\epsilon}{8}. \tag{2.18}$$

This inequality and (2.10) imply that

$$\tilde{d}\left(F,F^{(2)}\right) \leq \frac{\epsilon}{4}. \tag{2.19}$$

It is clear that the inclusion $F(x_*) \in \nu(F^{(1)})$, (2.15), and (2.16) imply that

$$F(x_*) \in \nu\left(F^{(2)}\right). \tag{2.20}$$

We will show that

$$\text{if } x \in X \text{ satisfies } F^{(2)}(x) = F(x_*), \text{ then } \rho(x,x_*) \leq \frac{\delta}{2}. \tag{2.21}$$

Assume that

$$x \in X, \quad F^{(2)}(x) = F(x_*). \tag{2.22}$$

In view of (2.15) and (2.22) $F^{(1)}(x) \leq F^{(2)}(x) \leq F(x_*)$. Together with the inclusion $F(x_*) \in \nu(F^{(1)})$ the inequality above implies that $F^{(1)}(x) = F(x_*) \geq F^{(2)}(x)$. Combined with (2.13) this relation implies that

$$\rho(x,x_*) \leq \frac{\delta}{2}. \tag{2.23}$$

Thus (2.21) is proved.

Since $F(x_*) \in \nu(F^{(1)})$, it follows from (2.13) that the following property holds.
(P1) For each $x \in X$ satisfying $\rho(x,x_*) > \delta/2$ there is $i \in \{1,\dots,n\}$ such that

$$f_i^{(1)}(x) \geq f_i^{(1)}(x_*) = f_i(x_*) \tag{2.24}$$

and that

$$\begin{aligned} f_i^{(2)}(x) &= f_i^{(1)}(x) + \left(\frac{\epsilon}{8}\right)\min\left\{1,\rho(x,x_*) - \frac{\delta}{2}\right\} \\ &\geq f_i^{(1)}(x_*) + \left(\frac{\epsilon}{8}\right)\min\left\{1,\rho(x,x_*) - \frac{\delta}{2}\right\} \\ &= f_i(x_*) + \left(\frac{\epsilon}{8}\right)\min\left\{1,\rho(x,x_*) - \frac{\delta}{2}\right\}. \end{aligned} \tag{2.25}$$

Choose

$$\delta_0 \in \left(0,\frac{\delta}{8}\right), \quad \lambda_0 \in \left(0,\frac{\epsilon}{16}\right). \tag{2.26}$$

Define functions $\phi_1,\phi_2 : [0,\infty) \to R$ as follows:

$$\begin{gathered} \phi_1(x) = x, \quad x \in [0,1], \quad \phi_1(x) = 1, \quad x \in (1,2], \\ \phi_1(x) = x-1, \quad x \in (2,8], \quad \phi_1(x) = 15-x, \quad x \in (8,14], \\ \phi_1(x) = 1, \quad x \in (14,15], \quad \phi_1(x) = 16-x, \quad x \in (15,16], \quad \phi_1(x) = 0, \quad x \in (16,\infty), \end{gathered} \tag{2.27}$$

$$\begin{gathered} \phi_2(x) = -x, \quad x \in [0,2], \quad \phi_2(x) = x-4, \quad x \in (2,8], \\ \phi_2(x) = 12-x, \quad x \in (8,14], \quad \phi_2(x) = -16+x, \quad x \in (14,16], \\ \phi_2(x) = 0, \quad x \in (16,\infty). \end{gathered} \tag{2.28}$$

It is clear that ϕ_1, ϕ_2 are continuous functions and that

$$\sup\{|\phi_i(x)| : x \in R,\ i = 1,2\} \leq 7. \tag{2.29}$$

Define a function $G = (g_1,\dots,g_n)\colon X \to R^n$ as follows:

$$g_1(x) = f_1^{(2)}(x) + \lambda_0\phi_1\left(16\rho(x,x_*)\delta_0^{-1}\right), \quad x \in X; \tag{2.30}$$

for $i \in \{2,\dots,n\}$

$$g_i(x) = f_i^{(2)}(x) + \lambda_0\phi_2\left(16\rho(x,x_*)\delta_0^{-1}\right), \quad x \in X. \tag{2.31}$$

Clearly, $G \in \mathcal{A}$. By (2.31), (2.30), (2.29), and (2.26) $\tilde{d}(G,F^{(2)}) \leq \epsilon/2$. Together with (2.19) this implies that

$$\tilde{d}(G,F) < \epsilon. \tag{2.32}$$

Relations (2.30), (2.31), and (2.28) imply that for each $x \in X$ satisfying $\rho(x,x_*) \geq \delta_0$

$$G(x) = F^{(2)}(x). \tag{2.33}$$

In view of (2.33) and (2.16) for each $x \in X$ satisfying $\rho(x,x_*) \in [\delta_0, \delta/2]$ we have

$$G(x) = F(x_*). \tag{2.34}$$

It follows from (2.33) and (2.17) that for each $x \in X$ satisfying $\rho(x,x_*) \in [\delta/2, \delta]$

$$G(x) = F(x_*) + \left(\frac{\epsilon}{8}\right)\left[\rho(x,x_*) - \frac{\delta}{2}\right](1,1,\dots,1). \tag{2.35}$$

By (2.30), (2.31), (2.16), (2.27), and (2.28) for each $x \in X$ satisfying $\rho(x,x_*) \leq \delta_0/16$ we have

$$g_1(x) = f_1^{(2)}(x) + \lambda_0\rho(x,x_*)\delta_0^{-1}16 = f_1(x_*) + \lambda_0\rho(x,x_*)\delta_0^{-}16; \tag{2.36}$$

for $i = 2,\dots,n$

$$g_i(x) = f_i^{(2)}(x_*) - \lambda_0\rho(x,x_*)\delta_0^{-1}16 = f_i(x_*) - \lambda_0\rho(x,x_*)\delta_0^{-1}16, \tag{2.37}$$

$$G(x) = F(x_*) + \lambda_0\rho(x,x_*)\delta_0^{-1}16(1,-1,\dots,-1). \tag{2.38}$$

Relations (2.27), (2.30), (2.31), (2.16) , and (2.28) imply that for each $x \in X$ satisfying $\rho(x,x_*) = \delta_0/8$ we have

$$g_1(x) = f_1^{(2)}(x) + \lambda_0\phi_1(2) = f_1(x_*) + \lambda_0; \tag{2.39}$$

for $i = 2,\dots,n$

$$g_i(x) = f_i^{(2)}(x) + \lambda_0\phi_2(2) = f_i(x_*) - 2\lambda_0, \tag{2.40}$$

$$G(x) = F(x_*) + \lambda_0(1,-2,-2,\dots,-2). \tag{2.41}$$

We will show that for each $x \in X$ satisfying $\rho(x,x_*) \leq \delta_0$ the following property holds.

(P2) There is $z \in X$ such that $\rho(z,x_*) \in [0,\delta_0/16] \cup \{\delta_0/8\}$ and $G(z) \leq G(x)$.

Let $x \in X$ satisfy

$$\rho(x,x_*) \leq \delta_0. \tag{2.42}$$

Clearly, if $\rho(x,x_*) \le \delta_0/16$, then (P2) holds with $z = x$. We consider the following cases:

$$\rho(x,x_*) \in \left[\frac{\delta_0}{16}, \frac{\delta_0}{8}\right]; \tag{2.43}$$

$$\rho(x,x_*) \in \left(\frac{\delta_0}{8}, \frac{\delta_0}{2}\right]; \tag{2.44}$$

$$\rho(x,x_*) \in \left(\frac{\delta_0}{2}, \frac{7}{8}\delta_0\right]; \tag{2.45}$$

$$\rho(x,x_*) \in \left(\frac{7}{8}\delta_0, \frac{1}{16}\delta_0\right]; \tag{2.46}$$

$$\rho(x,x_*) \in \left(\frac{15}{16}\delta_0, \delta_0\right]. \tag{2.47}$$

Let (2.43) hold. Then by (2.43), (2.30), (2.31), (2.27), (2.28), and (2.16) we have

$$g_1(x) = f_1^{(2)}(x) + \lambda_0\phi_1\left(\rho(x,x_*)\delta_0^{-1}16\right) = f_1(x_*) + \lambda_0; \tag{2.48}$$

for $i = 2,\dots,n$

$$g_i(x) = f_i^{(2)}(x) + \lambda_0\phi_2\left(\rho(x,x_*)\delta_0^{-1}16\right) = f_i(x_*) - \lambda_0\left(\rho(x,x_*)\delta_0^{-1}16\right) \ge f_i(x_*) - 2\lambda_0. \tag{2.49}$$

Together with (2.41) these relations imply that

$$G(x) \ge G(z) \quad \text{if } z \in X \text{ satisfies } \rho(z,x_*) = \frac{\delta_0}{8}. \tag{2.50}$$

Thus property (P2) holds if (2.43) is valid.

Assume that (2.44) is true. By (2.44), (2.30), (2.31), (2.27), (2.28), and (2.16) we have

$$g_1(x) = f_1(x_*) + \lambda_0\left(\rho(x,x_*)\delta_0^{-1}16 - 1\right) \ge f_1(x_*) + \lambda_0; \tag{2.51}$$

for $i = 2,\dots,n$

$$g_i(x) = f_i(x_*) + \lambda_0\left(\rho(x,x_*)\delta_0^{-1}16 - 4\right) \ge f_i(x_*) + \lambda_0(-2). \tag{2.52}$$

Together with (2.41) these relations imply that for each $z \in X$ satisfying $\rho(z,x_*) = \delta_0/8$ we have $G(z) \le G(x)$. Thus property (P2) holds if (2.44) is valid.

Assume that (2.45) holds. By (2.45), (2.30), (2.31), (2.27), (2.28), and (2.16) we have

$$g_1(x) = f_1(x_*) + \lambda_0\left(15 - \rho(x,x_*)\delta_0^{-1}16\right) \ge f_1(x_*) + \lambda_0; \tag{2.53}$$

for $i = 2,\dots,n$

$$g_i(x) = f_i(x_*) + \lambda_0\left(12 - \rho(x,x_*)\delta_0^{-1}16\right) \ge f_i(x_*) - 2\lambda_0. \tag{2.54}$$

Together with (2.41) these relations imply that for each $z \in X$ satisfying $\rho(z,x_*) = \delta_0/8$ we have $G(z) \le G(x)$. Thus property (P2) holds if (2.45) is valid.

Assume that (2.46) holds. By (2.46), (2.30), (2.31), (2.27), (2.28) , and (2.16) we have

$$g_1(x) = f_1(x_*) + \lambda_0; \tag{2.55}$$

for $i = 2,\dots,n$

$$g_i(x) = f_i(x_*) + \lambda_0\big(-16 + \rho(x,x_*)\delta_0^{-1}16\big) \ge f_i(x_*) - 2\lambda_0. \tag{2.56}$$

Together with (2.41) these relations imply that for each $z \in X$ satisfying $\rho(z,x_*) = \delta_0/8$ we have $G(z) \le G(x)$. Thus property (P2) holds if (2.46) is valid.

Assume that (2.47) holds. By (2.47), (2.30), (2.31), (2.27), (2.28) , and (2.16) we have

$$g_1(x) = f_1(x_*) + \lambda_0\big(16 - \rho(x,x_*)\delta_0^{-1}16\big); \tag{2.57}$$

for $i = 2,\dots,n$

$$g_i(x) = f_i(x_*) + \lambda_0\big(\rho(x,x_*)\delta_0^{-1}16 - 16\big). \tag{2.58}$$

Since the space X is connected, it follows from (2.3) and (2.47) that there is $z \in X$ such that

$$\rho(z,x_*) = \delta_0 - \rho(x,x_*) \in \left[0, \frac{\delta_0}{16}\right]. \tag{2.59}$$

In view of (2.59), (2.38), (2.57), and (2.58)

$$\begin{aligned} G(z) &= F(x_*) + \lambda_0\rho(z,x_*)\delta_0^{-1}16(1,-1,\dots,-1) \\ &= F(x_*) + \lambda_0\big(\delta_0 - \rho(x,x_*)\big)\delta_0^{-1}16(1,-1,-1,\dots,-1) = G(x). \end{aligned} \tag{2.60}$$

Thus property (P2) holds if (2.47) is valid.

We have shown that (P2) holds in all the cases. We have also shown that the following property holds.

(P3) For each $x \in X$ satisfing $\rho(x,x_*) \le \delta_0$ there is $z \in X$ such that

$$\rho(z,x_*) \in \left[0, \frac{\delta_0}{16}\right] \cup \left\{\frac{\delta_0}{8}\right\}, \quad G(z) \le G(x). \tag{2.61}$$

We will show that the following property holds.

(P4) If $x \in X$ satisfies $\rho(x,x_*) \ge \delta/2$ and $z \in X$ satisfies $\rho(z,x_*) \in \{\delta_0/8\} \cup [0,\delta_0/16]$, then the inequality $G(x) \le G(z)$ does not hold.

Assume that

$$x \in X, \quad z \in X, \quad \rho(x,x_*) > \frac{\delta}{2}, \quad \rho(z,x_*) \in \left\{\frac{\delta_0}{8}\right\} \cup \left[0, \frac{\delta_0}{16}\right]. \tag{2.62}$$

By property (P1) and (2.62) there is $j \in \{1,\dots,n\}$ such that

$$f_j^{(2)}(x) \ge f_j(x_*) + \left(\frac{\epsilon}{8}\right)\min\left\{1, \rho(x,x_*) - \frac{\delta}{2}\right\}. \tag{2.63}$$

By (2.13) and (2.62)

$$f_i^{(2)}(x) > f_i(x_*), \quad i = 1,\dots,n. \tag{2.64}$$

Together with (2.33) and (2.26) this implies that

$$g_i(x) > f_i(x_*), \quad i = 1,\dots,n. \tag{2.65}$$

It follows from (2.62) that

$$\begin{aligned} G(z) \in \{&F(x_*) + \lambda_0(1,-1,-1,\dots,-1)t : t \in [0,1]\} \\ &\cup \{F(x_*) + \lambda_0(1,-2,-2,\dots,-2)\}. \end{aligned} \tag{2.66}$$

It follows from this inclusion and (2.65) that $G(x) \le G(z)$ does not hold. Therefore property (P4) holds.

Let $t \in [0,1)$. We show that $F(x_*) + \lambda_0 t(1,-1,-1,\dots,-1) \in \nu(G)$. Since the space X is connected, it follows from (2.3) that there is $z \in X$ such that

$$\rho(z,x_*) = \frac{\delta_0}{16}t. \tag{2.67}$$

By (2.67) and (2.38)

$$\begin{aligned} G(z) &= F(x_*) + \lambda_0\rho(z,x_*)\delta_0^{-1}16(1,-1,-1,\dots,-1) \\ &= F(x_*) + t\lambda_0(1,-1,-1,\dots,-1) \in G(X). \end{aligned} \tag{2.68}$$

Assume that

$$x \in X, \quad G(x) \le G(z). \tag{2.69}$$

We will show that $G(x) = G(z)$. If $\rho(x,x_*) > \delta/2$, then by (2.13), (2.33) , and (2.26) the relation (2.65) is true and together with (2.69) this implies that $G(z) \gg F(x_*)$. This contradicts (2.68). Therefore

$$\rho(x,x_*) \le \frac{\delta}{2}. \tag{2.70}$$

If $\rho(x,x_*) \in [\delta_0,\delta/2]$, then by (2.34), (2.69), (2.68), and (2.67)

$$G(x) = F(x_*), \quad t = 0, \quad z = x_*, \quad G(z) = G(x). \tag{2.71}$$

Assume that

$$\rho(x,x_*) \le \delta_0. \tag{2.72}$$

By (2.72) and property (P3) there is $y \in X$ such that

$$G(y) \le G(x), \quad \rho(y,x_*) \in \left[0,\frac{\delta_0}{16}\right] \cup \left\{\frac{\delta_0}{8}\right\}. \tag{2.73}$$

In view of (2.73), (2.69) , and (2.68)

$$G(y) \le G(x) \le G(z) = F(x_*) + \lambda_0 t(1,-1,-1,\dots,-1). \tag{2.74}$$

If $\rho(y,x_*) = \delta_0/8$, then by (2.41)

$$G(y) = F(x_*) + \lambda_0(1,-2,-2,\dots,-2) \tag{2.75}$$

and since $t \in [0,1)$, the equality above contradicts (2.74). Therefore in view of (2.73)

$$\rho(y,x_*) \in \left[0, \frac{\delta_0}{16}\right]. \tag{2.76}$$

By (2.76) and (2.38)

$$G(y) = F(x_*) + \lambda_0 \rho(x,x_*)\delta_0^{-1}(16)(1,-1,-1,\dots,-1). \tag{2.77}$$

Together with (2.74) this equality implies that $t = \rho(x,x_*)\delta_0^{-1}16$ and $G(y) = G(x) = G(z)$. Thus we have shown that (2.69) implies that $G(x) = G(z)$. Therefore

$$F(x_*) + \lambda_0 t(1,-1,-1,\dots,-1) \in v(G) \quad \forall t \in [0,1). \tag{2.78}$$

Let $x \in X$ satisfy $\rho(x,x_*) = \delta_0/8$. (Note that by (2.3) such an x exists.) In view of (2.41)

$$G(x) = F(x_*) + \lambda_0(1,-2,-2,\dots,-2) < F(x_*) + \lambda_0(1,-1,-1,\dots,-1). \tag{2.79}$$

Thus

$$F(x_*) + \lambda_0(1,-1,-1,\dots,-1) \notin v(G). \tag{2.80}$$

Together with (2.78) this implies that $v(G)$ is not closed. Theorem 1.2 is proved.

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