SOME NEW GENERALIZATIONS OF HARDY'S INTEGRAL INEQUALITY

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We have studied some new generalizations of Hardy's integral inequality using the generalized Holder's inequality.

1. Introduction

The classical Hardy's inequality [2] states that for $p > 1$, $1/p + 1/q = 1$, $f \geq 0$, and $0 < \int_0^\infty f^p(t)dt < \infty$, then

$$\int_0^\infty \left[\frac{1}{x}\int_0^x f(t)dt\right]^p dx < q^p \int_0^\infty f^p(t)dt, \tag{1.1}$$

where $q^p = (p/(p-1))^p$ is the best possible. This inequality plays important role in analysis. It is obvious that, for parameters a and b such that $0 < a < b < \infty$, the following inequality is also valid:

$$\int_a^b \left[\frac{1}{x}\int_a^x f(t)dt\right]^p dx < q^p \int_a^b f^p(t)dt, \tag{1.2}$$

where $0 < \int_a^b f^p(t)dt < \infty$.

Bicheng et al. [1] have given some improvements of (1.1) and (1.2) as follows.

Let $0 < a < b < \infty$, $p > 1$, $1/p + 1/q = 1$, $f \geq 0$, and $0 < \int_a^b f^p(t)dt < \infty$, then

$$\begin{aligned}\int_a^b \left[\frac{1}{x}\int_a^x f(t)dt\right]^p dx \\ < q^p\left[1-\left(\frac{a}{b}\right)^{1/q}\right]^p \int_a^b f^p(t)dt.\end{aligned} \tag{1.3}$$

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Let $a > 0$, $p > 1$, $1/p + 1/q = 1$, $f \geq 0$, and $0 < \int_a^\infty f^p(t)dt < \infty$, then

$$\int_a^\infty \left[\frac{1}{x}\int_a^x f(t)dt\right]^p dx < q^p \int_a^\infty \left[1 - \theta_p(t)\right] f^p(t)dt, \tag{1.4}$$

where $\theta_p(t) = 1/p\sum_{k=1}^\infty \binom{p}{k}(-1)^{k-1}(a/t)^{k/q} > 0$ for $t > a$, and $\theta_p(a) = 1/q$.

Oguntuase and Imoru [3] generalized (1.3) and (1.4) as follows.

Let $0 < a < b < \infty$, $p > 1$, $1/p + 1/q = 1 - 1/r$, $f \geq 0$, $r > 1$, and $0 < \int_a^b f^p(t)dt < \infty$, then

$$\begin{aligned}&\int_a^b \left[\frac{1}{x^{(1-1/r)}}\int_a^x f(t)dt\right]^p dx \\ &\quad < q^{(1-1/r)p}\left(1 - \frac{1}{r}\right)^{(1-1/r)p}\left[1 - \left(\frac{a}{b}\right)^{1/q}\right]^{(1-1/r)p}\int_a^b f^p(t)dt.\end{aligned} \tag{1.5}$$

Let $a > 0$, $p > 1$, $1/p + 1/q = 1 - 1/r$, $f \geq 0$, $r > 1$, and $0 < \int_a^\infty f^p(t)dt < \infty$, then

$$\int_a^\infty \left[\frac{1}{x^{(1-1/r)}}\int_a^x f(t)dt\right]^p dx < q^{(1-1/r)p}\left(1 - \frac{1}{r}\right)^{(1-1/r)p}\int_a^\infty \left[1 - \theta_p(t)\right] f^p(t)dt, \tag{1.6}$$

where $\theta_p(t) = 1/(1-1/r)p\sum_{k=1}^\infty \binom{(1-1/r)p}{k+1}(-1)^{k-1}(a/t)^{k/(1-1/r)q} > 0$ for $t > a$, and $\theta_p(a) = 1/(1-1/r)q$.

Definition 1.1. Let $1 \leq p < \infty$, then the function space L_p is given by

$$L_p = \left\{f : \int_0^\infty |f(x)|^p dx < \infty\right\}. \tag{1.7}$$

The function space L_p has been generalized to $L(p)$ in the following manner.

Definition 1.2. Let p be a bounded measurable function, with $0 < p(x) \leq \sup p(x) = H < \infty$. Define

$$L(p) = \left\{f : \int_0^\infty |f(x)|^{p(x)} dx < \infty\right\}. \tag{1.8}$$

Note that $L(p)$ is a linear topological space paranormed by $d(f)$,

$$d(f) = \left(\int_0^\infty |f(x)|^{p(x)} dx\right)^{1/M}, \tag{1.9}$$

where $M = \max(1, H)$.

In this paper, we have the generalized Holder's inequality in $L(p)$ space and the results of [1, 3].

2. Main results

Lemma 2.1. (a) *Let the functions* p *and* q *be such that* $p(x)^{-1}+q(x)^{-1}=1$ *for all* x. *Let* $f\in L(p)$, $g\in L(q)$. *Let*

$$A=\int_0^\infty |f(x)|^{p(x)}dx,\qquad B=\int_0^\infty |g(x)|^{q(x)}dx. \tag{2.1}$$

Then for $p(x)>1$, $fg\in L_1$ *and*

$$\int_0^\infty |f(x)g(x)|\,dx\le\alpha\beta, \tag{2.2}$$

where $\alpha=\sup_x 1/p(x)+\sup_x 1/q(x)$, $\beta=\sup_x A^{1/p(x)}B^{1/q(x)}$.

(b) *Let* $0<p(x)<1$ *and* $p(x)^{-1}+q(x)^{-1}=1$. *If* $f\in L(p)$ *and* $g\in L(q)$, *then*

$$\int_0^\infty |f(x)|^{p(x)}dx\le\alpha\Big[\sup_x p(x)+\sup_x\big(1-p(x)\big)\Big], \tag{2.3}$$

where $\alpha=\sup_x[(\int|g(x)|^{q(x)}dx)^{1-p(x)}(\int|f(x)g(x)|dx)^{p(x)}]$.

Proof. To prove (a), for $a,b>0$, we have

$$ab\le\frac{a^{p(x)}}{p(x)}+\frac{b^{q(x)}}{q(x)}\qquad\forall x. \tag{2.4}$$

Using the above inequality, we have

$$\frac{|f(x)|}{A^{1/p(x)}}\frac{|g(x)|}{B^{1/q(x)}}\le\frac{1}{p(x)}\frac{|f(x)|^{p(x)}}{A}+\frac{1}{q(x)}\frac{|g(x)|^{q(x)}}{B}. \tag{2.5}$$

Therefore,

$$\begin{aligned}\int_0^\infty\frac{|f(x)||g(x)|}{A^{1/p(x)}B^{1/q(x)}}dx&\le\frac{1}{A}\int_0^\infty\frac{|f(x)|^{p(x)}}{p(x)}dx+\frac{1}{B}\int_0^\infty\frac{|g(x)|^{q(x)}}{q(x)}dx\\&\le\sup_x\frac{1}{p(x)}+\sup_x\frac{1}{q(x)}.\end{aligned} \tag{2.6}$$

Also

$$\frac{1}{\sup_x A^{1/p(x)}B^{1/q(x)}}\int_0^\infty|f(x)g(x)|\,dx\le\int_0^\infty\frac{|f(x)g(x)|}{A^{1/p(x)}B^{1/q(x)}}dx. \tag{2.7}$$

From (2.6) and (2.7), we get (2.2).

To prove (b), let $p_1(x)=1/p(x)$, so that $p_1(x)>1$ for all x. Let

$$A(x)=|g(x)|^{-1/p_1(x)},\qquad B(x)=|f(x)g(x)|^{1/p_1(x)}. \tag{2.8}$$

So

$$\begin{aligned}\int_0^\infty |f(x)|^{p(x)}dx &= \int_0^\infty |A(x)B(x)|\,dx \\ &\le \sup_x\left[\left(\int_0^\infty |A(x)|^{q_1(x)}dx\right)^{1/q_1(x)}\left(\int_0^\infty |B(x)|^{p_1(x)}dx\right)^{1/p_1(x)}\right] \\ &\quad\times\left(\sup_x\frac{1}{p_1(x)}+\sup_x\frac{1}{q_1(x)}\right)\quad \text{by (2.2).}\end{aligned}\tag{2.9}$$

Since $1/p_1(x)+1/q_1(x)=1$, so $1/q_1(x)=1-p(x)$ and $q_1(x)=1/1-p(x)$. Substituting the values, we get (2.3):

$$\begin{aligned}\int_0^\infty |f(x)|^{p(x)}dx &= \sup_x\left[\left(\int_0^\infty |g(x)|^{q(x)}dx\right)^{1-p(x)}\left(\int_0^\infty |f(x)g(x)|\,dx\right)^{p(x)}\right] \\ &\quad\times\left[\sup_x p(x)+\sup_x(1-p(x))\right]=\alpha\left[\sup_x p(x)+\sup_x(1-p(x))\right].\end{aligned}\tag{2.10}$$

This completes the proof of the lemma. □

Lemma 2.2. *Let* $0<b\le\infty$, *for all* $x\in(0,b)$, $p(x)>1$, $p(x)^{-1}+q(x)^{-1}=1$, $f\ge 0$, *and* $0<\int_0^b f^{p(t)}(t)dt<\infty$. *Then the following inequality holds:*

$$\int_0^x f(t)dt<\alpha\sup_{x\in(0,b)}\left\{\{\inf q(x)\}^{1/q(x)}x^{\sup 1/q(x)2}\left[\int_0^x t^{1/q(t)}f^{p(t)}(t)dt\right]^{1/p(x)}\right\},\tag{2.11}$$

where $\alpha=\sup_x 1/p(x)+\sup_x 1/q(x)$.

Proof. For any $x\in(0,b)$, by the generalized Holder's inequality (2.2), we have

$$\begin{aligned}\int_0^x f(t)dt &= \int_0^x t^{1/p(t)q(t)}f(t)t^{-1/p(t)q(t)}dt \\ &\le\alpha\sup_{x\in(0,b)}\left\{\left(\int_0^x t^{1/q(t)}f^{p(t)}(t)dt\right)^{1/p(x)}\left(\int_0^x t^{-1/p(t)}dt\right)^{1/q(x)}\right\} \\ &=\alpha\sup_{x\in(0,b)}\left\{\{\inf q(x)\}^{1/q(x)}x^{\sup 1/q(x)2}\left[\int_0^x t^{1/q(t)}f^{p(t)}(t)dt\right]^{1/p(x)}\right\}.\end{aligned}\tag{2.12}$$

Strictness follows from [1, Lemma 2.1]. Thus (2.11) is valid. □

Lemma 2.3. *Let* $a\ge 0$, *for all* $x\in(a,\infty)$, $p(x)>1$, $p(x)^{-1}+q(x)^{-1}=1$, $f\ge 0$, *and* $0<\int_a^x f^{p(t)}(t)dt<\infty$. *Then the following inequality is true:*

$$\int_a^x f(t)dt<\alpha\sup_{x\in(a,\infty)}\left\{\{\inf q(x)\}^{1/q(x)}\left(x^{\sup 1/q(x)}-a^{\sup 1/q(x)}\right)\left[\int_a^x t^{1/q(t)}f^{p(t)}(t)dt\right]^{1/p(x)}\right\},\tag{2.13}$$

where $\alpha=\sup_x 1/p(x)+\sup_x 1/q(x)$.

Proof. For any $x \in (a,\infty)$, by the generalized Holder's inequality (2.2), we have

$$\begin{aligned}\int_a^x f(t)dt &\le \alpha \sup_{x\in(a,\infty)} \left\{ \left(\int_a^x t^{1/q(t)} f^{p(t)}(t)dt \right)^{1/p(x)} \left(\int_a^x t^{-1/p(t)}dt \right)^{1/q(x)} \right\} \\ &\le \alpha \sup_{x\in(a,\infty)} \left\{ \{\inf q(x)\}^{1/q(x)} \left(x^{\sup 1/q(x)} - a^{\sup 1/q(x)}\right) \left[\int_a^x t^{1/q(t)} f^{p(t)}(t)dt \right]^{1/p(x)} \right\}.\end{aligned} \tag{2.14}$$

Strictness follows from [1, Lemma 2.2]. This completes the proof of the lemma. □

Theorem 2.4. *Let* $0 < a < b < \infty$, *for all* $x \in (a,b)$, $p(x) > 1$, $p(x)^{-1} + q(x)^{-1} = 1$, $f \ge 0$, *and* $0 < \int_a^b f^{p(t)}(t)dt < \infty$. *Then*

$$\int_a^b \left(\frac{1}{x} \int_a^x f(t)dt \right)^{p(x)} dx < \alpha^M \{\inf q(t)\}^M \left[1 - \left(\frac{a}{b} \right)^{\sup 1/q(t)} \right]^M \int_a^b f^{p(t)}(t)dt, \tag{2.15}$$

where $M = \sup p(t)$, $\alpha = \sup\{p(t)^{-1}\} + \sup\{q(t)^{-1}\}$.

Proof. Using (2.13), we obtain

$$\begin{aligned}&\int_a^b \left(\frac{1}{x} \int_a^x f(t)dt \right)^{p(x)} dx \\ &\quad < \alpha^{\sup p(x)} \int_a^b x^{-p(x)} \left(x^{\sup 1/q(x)} - a^{\sup 1/q(x)}\right)^{p(x)/q(x)} \{\inf q(x)\}^{p(x)/q(x)} \\ &\quad\quad \times \int_a^x t^{1/q(t)} f^{p(t)}(t)dt\,dx \le \alpha^{\sup p(t)} \{\inf q(t)\}^{\sup p(t)/q(t)} \\ &\quad\quad \times \int_a^b \left\{ \int_t^b x^{-p(x)+\sup(p(x)/q(x)^2)} \left(1 - \left(\frac{a}{x} \right)^{\sup(1/q(x))} \right)^{p(x)/q(x)} dx \right\} t^{1/q(t)} f^{p(t)}(t)dt \\ &\quad \le \alpha^M \{\inf q(t)\}^{\sup(p(t)-1)} \\ &\quad\quad \times \int_a^b \left\{ \int_t^b x^{-1-1/q(x)} \left(1 - \left(\frac{a}{b} \right)^{\sup 1/q(x)} \right)^{p(x)/q(x)} dx \right\} t^{1/q(t)} f^{p(t)}(t)dt \\ &\quad \le \alpha^M \{\inf q(t)\}^{\sup p(t)} \left[1 - \left(\frac{a}{b} \right)^{\sup(1/q(t))} \right]^{\sup p(t)-1} \int_a^b \left[1 - \left(\frac{t}{b} \right)^{\sup(1/q(t))} \right] f^{p(t)}(t)dt \\ &\quad \le \alpha^M \{\inf q(t)\}^M \left[1 - \left(\frac{a}{b} \right)^{\sup(1/q(t))} \right]^M \int_a^b f^{p(t)}(t)dt.\end{aligned} \tag{2.16}$$

□

Theorem 2.5. *Let* $a > 0$, *for all* $x \in (a,\infty)$, $p(x) > 1$, $p(x)^{-1} + q(x)^{-1} = 1$, $f \ge 0$, *and* $0 < \int_a^\infty f^{p(t)}(t)dt < \infty$. *Then*

$$\int_a^\infty \left(\frac{1}{x} \int_a^x f(t)dt \right)^{p(x)} dx < \alpha^M \{\inf q(t)\}^M \int_a^\infty [1 - \theta_M(t)] f^{p(t)}(t)dt, \tag{2.17}$$

where $\theta_M(t) = (1/M)[\sum_{k=1}^{\infty}(\Gamma(M+1)/\Gamma(k+2)\Gamma(M-k))(-1)^{k-1}(a/t)^{(k/\inf q(t))}]$, *for* $t > a > 0$, $M > 1$, *and* $\theta_M(a) = \sup 1/q(t)$, $M = \sup p(t)$, *and* $\alpha = \sup\{p(t)^{-1}\} + \sup\{q(t)^{-1}\}$.

Proof. In view of inequalities (2.13) and (2.15), we find

$$\begin{aligned}
&\int_a^\infty \left(\frac{1}{x}\int_a^x f(t)dt\right)^{p(x)} dx \\
&\quad < \alpha^{\sup p(x)} \int_a^\infty x^{-p(x)} \left(x^{\sup 1/q(x)} - a^{\sup 1/q(x)}\right)^{p(x)/q(x)} \{\inf q(x)\}^{p(x)/q(x)} \\
&\qquad \times \int_a^x t^{1/q(t)} f^{p(t)}(t)dt\,dx \le \alpha^{\sup p(t)}\{\inf q(t)\}^{\sup(p(t)-1)} \\
&\qquad \times \int_a^\infty \left\{\int_t^\infty x^{-1-1/q(x)}\left(1-\left(\frac{a}{b}\right)^{\sup(1/q(x))}\right)^{p(x)/q(x)} dx\right\} t^{1/q(t)} f^{p(t)}(t)dt \\
&\quad \le \alpha^M\{\inf q(t)\}^{\sup(p(t)-1)} \\
&\qquad \times \int_a^\infty \left[\int_t^\infty \left(1-\left(\frac{a}{b}\right)^{\sup(1/q(x))}\right)^{p(x)/q(x)} d\left(1-\left(\frac{a}{b}\right)^{\sup(1/q(x))}\right)\right] \\
&\qquad \times \left(\frac{t}{a}\right)^{\sup(1/q(t))} f^{p(t)}(t)dt \le \alpha^M\{\inf q(t)\}^{\sup p(t)} \\
&\qquad \times \int_a^\infty \frac{1}{\sup p(t)}\left\{1-\left[1-\left(\frac{a}{t}\right)^{\sup(1/q(t))}\right]^{\sup p(t)}\right\}\left(\frac{t}{a}\right)^{\sup(1/q(t))} f^{p(t)}(t)dt \\
&\quad = \alpha^M\{\inf q(t)\}^M \int_a^\infty [1-\theta_M(t)] f^{p(t)}(t)dt,
\end{aligned} \tag{2.18}$$

where

$$\begin{gathered}
\theta_M(t) = 1 - \frac{1}{M}\left\{1-\left[1-\left(\frac{a}{t}\right)^{\sup(1/q(t))}\right]^M\right\}\left(\frac{t}{a}\right)^{\sup(1/q(t))}, \quad t > a > 0,\ M > 1, \\
\theta_M(a) = \sup\frac{1}{q(t)}.
\end{gathered} \tag{2.19}$$

Since

$$\begin{gathered}
\left[1-\left(\frac{a}{t}\right)^{\sup(1/q(t))}\right]^M = \sum_{k=0}^{\infty}\frac{\Gamma(M+1)}{\Gamma(k+1)\Gamma(M-k+1)}(-1)^k\left(\frac{a}{t}\right)^{k/\inf q(t)}, \quad t > a > 0,\ M > 1, \\
\theta_M(t) = \frac{1}{M}\left[\sum_{k=1}^{\infty}\frac{\Gamma(M+1)}{\Gamma(k+2)\Gamma(M-k)}(-1)^{k-1}\left(\frac{a}{t}\right)^{k/\inf q(t)}\right], \quad t > a > 0,\ M > 1,
\end{gathered} \tag{2.20}$$

the proof is complete. □

Note. When $t > a > 0$, by Bernoulli's inequality (see [2, Chapter 2.4]), we obtain

$$1 - M\left(\frac{a}{t}\right)^{\sup 1/q(t)} < \left[1 - \left(\frac{a}{t}\right)^{\sup 1/q(t)}\right]^M,$$
$$\theta_M(t) > 1 - \frac{1}{M}\left[1 - \left\{1 - M\left(\frac{a}{t}\right)^{\sup 1/q(t)}\right\}\right]\left(\frac{t}{a}\right)^{\sup 1/q(t)} = 0. \quad (2.21)$$

Applications

Theorem 2.6. *Let* $0 < b \le \infty$, *for all* $x \in (0,\infty)$, $r \ge p(x) > 1$, $p(x)^{-1} + q(x)^{-1} = 1$, $f \ge 0$, *and* $0 < \int_0^b x^{-r+\sup p(x)} f^{p(x)}(x) < \infty$. *Then*

(i) *for* $b \in (0,\infty)$,

$$\int_0^b x^{-r}\left(\int_0^x f(t)dt\right)^{p(x)} dx < \frac{\alpha^M\{\inf q(t)\}^M}{\inf\{(r-p(t))q(t)+1\}}\int_0^b \left[1 - \left(\frac{t}{b}\right)^{\sup\{r-p(t)+1/q(t)\}}\right] t^{-r+\sup p(t)} f^{p(t)}(t)dt, \quad (2.22)$$

(ii) *for* $b = \infty$,

$$\int_0^\infty x^{-r}\left(\int_0^x f(t)dt\right)^{p(x)} dx < \frac{\alpha^M\{\inf q(t)\}^M}{\inf\{(r-p(t))q(t)+1\}}\int_0^\infty t^{-r+\sup p(t)} f^{p(t)}(t)dt, \quad (2.23)$$

where $M = \sup p(t)$ *and* $\alpha = \sup\{p(t)^{-1}\} + \sup\{q(t)^{-1}\}$.

Proof. For case (i), $b \in (0,\infty)$, we use (2.11) to obtain

$$\begin{aligned}
&\int_0^b x^{-r}\left(\int_0^x f(t)dt\right)^{p(x)} dx \\
&\quad < \alpha^M \int_0^b x^{-r+\sup\{p(x)/q(x)2\}}\{\inf q(x)\}^{p(x)/q(x)}\int_0^x t^{1/q(t)} f^{p(t)}(t)dt\,dx \\
&\quad \le \alpha^M\{\inf q(t)\}^{\sup\{p(t)-1\}}\int_0^b \left(\int_t^b x^{-r+\sup\{p(x)-1-1/q(x)\}}dx\right) t^{1/q(t)} f^{p(t)}(t)dt \\
&\quad \le \frac{\alpha^M\{\inf q(t)\}^M}{-r+\sup\{p(t)-1/q(t)\}}\int_0^b \left(b^{-r+\sup\{p(t)-1/q(t)\}} - t^{-r+\sup\{p(t)-1/q(t)\}}\right) \\
&\qquad\qquad \times t^{1/q(t)} f^{p(t)}(t)dt \\
&\quad \le \frac{\alpha^M\{\inf q(t)\}^M}{\inf\{(r-p(t))q(t)+1\}}\int_0^b \left[1 - \left(\frac{t}{b}\right)^{\sup\{r-p(t)+1/q(t)\}}\right] t^{-r+\sup p(t)} f^{p(t)}(t)dt.
\end{aligned} \quad (2.24)$$

For case (ii), $b=\infty$, we use (2.11) to find

$$\begin{aligned}
&\int_0^\infty x^{-r}\left(\int_0^x f(t)dt\right)^{p(x)}dx\\
&\quad< \alpha^M\{\inf q(t)\}^{\sup(p(t)-1)}\int_0^\infty x^{-r+\sup\{p(x)/q(x)^2\}}\int_0^x t^{1/q(t)}f^{p(t)}(t)dt\,dx\\
&\quad= \alpha^M\{\inf q(t)\}^{\sup(p(t)-1)}\int_0^\infty\left(\int_t^\infty x^{-r+\sup\{p(x)-1-1/q(x)\}}dx\right)t^{1/q(t)}f^{p(t)}(t)dt\\
&\quad= \frac{\alpha^M\{\inf q(t)\}^M}{\inf\{(r-p(t))q(t)+1\}}\int_0^\infty t^{-r+\sup\{p(t)\}}f^{p(t)}(t)dt.
\end{aligned}\tag{2.25}$$

□

Remark 2.7. (a) If $p(x)$ and $q(x)$ are constants in Lemma 2.1, then (1.6) reduces to usual Holder's inequality in L_p space.

(b) If we take $p(x)$ and $q(x)$ constants in Lemmas 2.2 and 2.3 and Theorems 2.4 and 2.5, then our results reduce to the corresponding Lemmas 2.1 and 2.2 and Theorems 2.4 and 2.5 obtained in [1].

(c) When $p(x)=r$ and $q(x)$ is constant, inequality (2.23) reduces to (1.1).

3. Some more generalized results

In this section, we have generalized the results of [3]. We use the generalized form of Holder's inequality with $p(x),q(x),r(x)>1$. The normalization $1/p(x)+1/q(x)=1$ in Holder's inequality is replaced by relation of the form $1/p(x)+1/q(x)=1-1/r(x)$.

LEMMA 3.1. *Let* $0<b\le\infty$, *for all* $x\in(0,b)$, $p(x)>1$, $p(x)^{-1}+q(x)^{-1}=1-r(x)^{-1}$, $f\ge 0$, $r(x)>1$, *and* $0<\int_0^b f^{p(t)}(t)dt<\infty$. *Then the following inequality holds:*

$$\begin{aligned}
\int_0^x f(t)dt<\alpha\sup_{x\in(0,b)}&\left\{\left\{\inf\left\{\left(1-\frac{1}{r(x)}\right)q(x)\right\}\right\}^{1/q(x)}x^{\sup 1/\{(1-1/r(x))q(x)^2\}}\right.\\
&\left.\times\left[\int_0^x t^{1/\{(1-1/r(t))q(t)\}}f^{p(t)}(t)dt\right]^{1/p(x)}\right\},
\end{aligned}\tag{3.1}$$

where $\alpha=\sup_x 1/p(x)+\sup_x 1/q(x)+\sup_x 1/r(x)$.

Proof. For any $x\in(0,b)$, by the generalized Holder's inequality (2.2), we have

$$\begin{aligned}
\int_0^x f(t)dt&=\int_0^x t^{1/\{(1-1/r(t))p(t)q(t)\}}f(t)t^{-1/\{(1-1/r(t))p(t)q(t)\}}dt\\
&\le\alpha\sup_{x\in(0,b)}\left\{\left(\int_0^x t^{1/\{(1-1/r(t))q(t)\}}f^{p(t)}(t)dt\right)^{1/p(x)}\left(\int_0^x t^{-1/\{(1-1/r(t))p(t)\}}dt\right)^{1/q(x)}\right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha \sup_{x\in(0,b)} \left\{ \left(\int_0^x t^{\sup\{-1/\{(1-1/r(t))p(t)\}\}} dt \right)^{1/q(x)} \right. \\
&\qquad \left. \times \left(\int_0^x t^{1/\{(1-1/r(t))q(t)\}} f^{p(t)}(t) dt \right)^{1/p(x)} \right\} \\
&= \alpha \sup_{x\in(0,b)} \left\{ \left\{ \inf \left\{ \left(1 - \frac{1}{r(x)}\right) q(x) \right\} \right\}^{1/q(x)} x^{\sup 1/\{(1-1/r(x))q(x)^2\}} \right. \\
&\qquad \left. \times \left[\int_0^x t^{\{1/(1-1/r(t))q(t)\}} f^{p(t)}(t) dt \right]^{1/p(x)} \right\}.
\end{aligned}
\tag{3.2}
$$

Strictness follows from [3, Lemma 2.1]. Thus (3.1) is valid. □

LEMMA 3.2. *Let $a \geq 0$, for all $x \in (a,\infty)$, $p(x) > 1$, $p(x)^{-1} + q(x)^{-1} = 1 - r(x)^{-1}$, $f \geq 0$, $r(x) > 1$, and $0 < \int_a^x f^{p(t)}(t)dt < \infty$. Then the following inequality holds:*

$$
\begin{aligned}
\int_a^x f(t)dt < \alpha \sup_{x\in(a,\infty)} &\left\{ \left\{ \inf \left\{ \left(1 - \frac{1}{r(x)}\right) q(x) \right\} \right\}^{1/q(x)} \right. \\
&\times \left(x^{\sup 1/\{(1-1/r(x))q(x)\}} - a^{\sup 1/\{(1-1/r(x))q(x)\}}\right)^{1/q(x)} \\
&\left. \times \left[\int_a^x t^{1/\{(1-1/r(t))q(t)\}} f^{p(t)}(t) dt \right]^{1/p(x)} \right\},
\end{aligned}
\tag{3.3}
$$

where $\alpha = \sup_x 1/p(x) + \sup_x 1/q(x) + \sup_x 1/r(x)$.

Proof. For any $x \in (a,\infty)$, by the generalized Holder's inequality (2.2), we have

$$
\begin{aligned}
\int_a^x f(t)dt &\leq \alpha \sup_{x\in(a,\infty)} \left\{ \left(\int_a^x t^{\sup\{-1/\{(1-1/r(t))p(t)\}\}} dt \right)^{1/q(x)} \right. \\
&\qquad \left. \times \left(\int_a^x t^{1/\{(1-1/r(t))q(t)\}} f^{p(t)}(t) dt \right)^{1/p(x)} \right\} \\
&\leq \alpha \sup_{x\in(a,\infty)} \left\{ \left\{ \inf \left\{ \left(1 - \frac{1}{r(x)}\right) q(x) \right\} \right\}^{1/q(x)} \right. \\
&\qquad \times \left(x^{\sup 1/\{(1-1/r(x))q(x)\}} - a^{\sup 1/\{(1-1/r(x))q(x)\}}\right)^{1/q(x)} \\
&\qquad \left. \times \left(\int_a^x t^{1/\{(1-1/r(t))q(t)\}} f^{p(t)}(t) dt \right)^{1/p(x)} \right\}.
\end{aligned}
\tag{3.4}
$$

Strictness follows from [3, Lemma 2.2]. This completes the proof of the lemma. □

Theorem 3.3. *Let* $0 < a < b < \infty$, *for all* $x \in (a,b)$, $p(x) > 1$, $p(x)^{-1} + q(x)^{-1} = 1 - r(x)^{-1}$, $f \geq 0$, $r(x) > 1$, *and* $0 < \int_a^b f^{p(t)}(t)dt < \infty$. *Then*

$$\int_a^b \left(\frac{1}{x^{(1-1/r(x))}} \int_a^x f(t)dt \right)^{p(x)} dx < \alpha^M \left\{ \inf \left\{ \left(1 - \frac{1}{r(t)} \right) q(t) \right\} \right\}^N \left[1 - \left(\frac{a}{b} \right)^{\sup 1/\{(1-1/r(t))q(t)\}} \right]^N \int_a^b f^{p(t)}(t)dt, \tag{3.5}$$

where $M = \sup p(t)$, $\alpha = \sup 1/p(t) + \sup 1/q(t) + \sup 1/r(t)$, *and* $N = \sup\{p(t)(1 - 1/r(t))\}$.

Proof. Using (3.3), we obtain

$$\begin{aligned}
&\int_a^b \left(\frac{1}{x^{(1-1/r(x))}} \int_a^x f(t)dt \right)^{p(x)} dx \\
&\quad < \int_a^b \left[\frac{1}{x^{(1-1/r(x))}} \alpha \sup \left\{ \left\{ \inf \left\{ \left(1 - \frac{1}{r(x)} \right) qx \right\} \right\}^{1/q(x)} \right. \right. \\
&\qquad\qquad \times \left(x^{\sup 1/\{(1-1/r(x))q(x)\}} - a^{\sup 1/\{(1-1/r(x))q(x)\}} \right)^{1/q(x)} \\
&\qquad\qquad \left. \left. \times \left\{ \int_a^x t^{1/\{(1-1/r(t))q(t)\}} f^{p(t)}(t)dt \right\}^{1/p(x)} \right\}^{p(x)} dx \right] \\
&\quad \leq \alpha^{\sup p(x)} \left\{ \inf \left\{ \left(1 - \frac{1}{r(x)} \right) qx \right\} \right\}^{\sup\{p(x)/q(x)\}} \\
&\qquad \times \int_a^b x^{-(1-1/r(x))p(x)} \left(x^{\sup 1/\{(1-1/r(x))q(x)\}} - a^{\sup 1/\{(1-1/r(x))q(x)\}} \right)^{p(x)/q(x)} \\
&\qquad \times \int_a^x t^{1/q(t)} f^{p(t)}(t)dt\,dx \\
&\quad \leq \alpha^{\sup p(t)} \left\{ \inf \left\{ \left(1 - \frac{1}{r(t)} \right) q(t) \right\} \right\}^{\sup p(t)/q(t)} \\
&\qquad \times \int_a^b \left\{ \int_t^b x^{-(1-1/r(x))p(x)+\sup(p(x)/(1-1/r(x))q(x)^2)} \right. \\
&\qquad\qquad \left. \times \left(1 - \left(\frac{a}{x} \right)^{\sup(1/(1-1/r(x))q(x))} \right)^{p(x)/q(x)} dx \right\} \\
&\qquad \times t^{1/\{(1-1/r(t))q(t)\}} f^{p(t)}(t)dt \leq \alpha^M \left\{ \inf \left\{ \left(1 - \frac{1}{r(t)} \right) q(t) \right\} \right\}^{\sup(p(t)-1)}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_a^b \left\{ \int_t^b x^{-1-1/\{(1-1/r(x))q(x)\}} \left(1-\left(\frac{a}{x}\right)^{\sup(1/(1-1/r(x))q(x))}\right)^{p(x)/q(x)} dx \right\} \\
&\times t^{1/\{(1-1/r(t))q(t)\}} f^{p(t)}(t)dt \\
&\leq \alpha^M \left\{ \inf \left\{ \left(1-\frac{1}{r(t)}\right) q(t) \right\} \right\}^{\sup\{p(t)(1-1/r(t))\}} \\
&\times \left[1-\left(\frac{a}{b}\right)^{\sup(1/(1-1/r(t))q(t))} \right]^{\sup\{p(t)(1-1/r(t))-1\}} \\
&\times \left[1-\left(\frac{t}{b}\right)^{\sup(1/(1-1/r(t))q(t))} \right] f^{p(t)}(t)dt \\
&\leq \alpha^M \left\{ \inf \left\{ \left(1-\frac{1}{r(t)}\right) q(t) \right\} \right\}^N \left[1-\left(\frac{a}{b}\right)^{\sup(1/(1-1/r(t))q(t))} \right]^N \int_a^b f^{p(t)}(t)dt.
\end{aligned}
\tag{3.6}
$$

This completes the proof of our theorem. □

Theorem 3.4. *Let* $a > 0$, *for all* $x \in (a, \infty)$, $p(x) > 1$, $p(x)^{-1} + q(x)^{-1} = 1 - r(x)^{-1}$, $f \geq 0$, $r(x) > 1$, *and* $0 < \int_a^\infty f^{p(t)}(t)dt < \infty$. *Then*

$$
\begin{aligned}
&\int_a^\infty \left(\frac{1}{x^{(1-1/r(x))}} \int_a^x f(t)dt \right)^{p(x)} dx \\
&\qquad < \alpha^M \left\{ \inf \left\{ \left(1-\frac{1}{r(t)}\right) q(t) \right\} \right\}^N \int_a^\infty [1-\theta_N(t)] f^{p(t)}(t)dt,
\end{aligned}
\tag{3.7}
$$

where $\theta_N(t) = (1/N)[\sum_{k=1}^{\infty}(\Gamma(N+1)/\Gamma(k+2)\Gamma(N-k))(-1)^{k-1}(a/t)^{k/(1-1/r(t))\inf q(t)}]$, *for* $t > a$, $M = \sup p(t)$, $\alpha = \sup p(t)^{-1} + \sup q(t)^{-1} + \sup r(t)^{-1}$, *and* $N = \sup p(t)(1 - 1/r(t))$.

Proof. In view of inequalities (3.3) and (3.5), we find

$$
\begin{aligned}
&\int_a^\infty \left(\frac{1}{x^{1-1/r(x)}} \int_a^x f(t)dt \right)^{p(x)} dx \\
&\quad < \alpha^{\sup p(x)} \left\{ \inf \left(1-\frac{1}{r(x)}\right) q(x) \right\}^{\sup\{p(x)/q(x)\}} \\
&\quad\times \int_a^\infty x^{-(1-1/r(x))p(x)} \left(x^{\sup 1/(1-1/r(x))q(x)} - a^{\sup(1/(1-1/r(x))q(x))} \right)^{p(x)/q(x)} \\
&\quad\times \int_a^\infty t^{1/(1-1/r(t))q(t)} f^{p(t)}(t)dt\,dx \\
&\quad \leq \alpha^{\sup p(t)} \left\{ \inf \left(1-\frac{1}{r(t)}\right) q(t) \right\}^{\sup\{p(t)-p(t)/r(t)-1\}}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_a^\infty \left\{ \int_t^\infty x^{-(1-1/r(x))p(x)+p(x)/(1-1/r(x))q(x)^2} \right.\\
&\qquad \left. \times \left(1-\left(\frac{a}{x}\right)^{\sup(1/(1-1/r(x))q(x))}\right)^{p(x)/q(x)} dx \right\} t^{1/(1-1/r(t))q(t)} f^{p(t)}(t)dt\\
&\le \alpha^M \left\{ \inf\left(1-\frac{1}{r(t)}\right)q(t)\right\}^{\sup\{p(t)(1-1/r(t)\}}\\
&\times \int_a^\infty \left[\int_t^\infty \left(1-\left(\frac{a}{x}\right)^{\sup 1/(1-1/r(x))q(x)}\right)^{\sup p(x)/q(x)} \right.\\
&\qquad \left. \times d\left(1-\left(\frac{a}{x}\right)^{\sup 1/(1-1/r(x))q(x)}\right)\right]\left(\frac{t}{a}\right)^{\sup 1/(1-1/r(t))q(t)} f^{p(t)}(t)dt\\
&\le \alpha^M \left\{ \inf\left(1-\frac{1}{r(t)}\right)q(t)\right\}^{\sup\{p(t)(1-1/r(t))\}}\\
&\times \int_a^\infty \frac{1}{\sup(1-1/r(t))p(t)}\left\{1-\left[1-\left(\frac{a}{t}\right)^{\sup 1/(1-1/r(t))q(t)}\right]^{\sup(1-1/r(t))p(t)}\right\}\\
&\times \left(\frac{t}{a}\right)^{\sup 1/(1-1/r(t))q(t)} f^{p(t)}(t)dt\\
&= \alpha^M \left\{ \inf\left(1-\frac{1}{r(t)}\right)q(t)\right\}^N \int_a^\infty [1-\theta_N(t)] f^{p(t)}(t)dt,
\end{aligned}
\tag{3.8}
$$

where $N = \sup 1/(1-1/r(t))p(t) > 1$,

$$
\begin{gathered}
\theta_N(t) = 1-\frac{1}{N}\left\{1-\left[1-\left(\frac{a}{t}\right)^{\sup(1/(1-1/r(t))q(t))}\right]^N\right\}\left(\frac{t}{a}\right)^{\sup(1/(1-1/r(t))q(t))}, \quad t>a>0,\\
\theta_N(a) = \sup\frac{1}{(1-1/r(t))}q(t),
\end{gathered}
\tag{3.9}
$$

since

$$
\begin{gathered}
\left[1-\left(\frac{a}{t}\right)^{\sup 1/(1-1/r(t))q(t)}\right]^N = \sum_{k=0}^\infty \frac{\Gamma(N+1)}{\Gamma(k+1)\Gamma(N-k+1)}(-1)^k\left(\frac{a}{t}\right)^{k/\inf 1/(1-1/r(t))q(t)},\\
\theta_N(t) = \frac{1}{N}\sum_{k=0}^\infty \frac{\Gamma(N+1)}{\Gamma(k+2)\Gamma(N-k)}(-1)^{k-1}\left(\frac{a}{t}\right)^{k/\inf(1-1/r(t))q(t)}, \quad t>a>0.
\end{gathered}
\tag{3.10}
$$

This completes the proof. □

Note. When $t > a > 0$, by Bernoulli's inequality (see [2, Chapter 2.4]), we obtain

$$1 - N\left(\frac{a}{t}\right)^{\sup 1/(1-1/r(t))q(t)} < \left[1 - \left(\frac{a}{t}\right)^{\sup 1/(1-1/r(t))q(t)}\right]^N,$$
$$\theta_N(t) > 1 - \frac{1}{N}\left[1 - \left\{1 - N\left(\frac{a}{t}\right)^{\sup 1/(1-1/r(t))q(t)}\right\}\right]\left(\frac{t}{a}\right)^{\sup 1/(1-1/r(t))q(t)} = 0. \tag{3.11}$$

Applications

Theorem 3.5. *Let* $0 < b \leq \infty$, *for all* $x \in (0,\infty)$, $s \geq N > 1$, $p(x)^{-1} + q(x)^{-1} = 1 - r(x)^{-1}$, $f \geq 0$, $r(x) > 1$, *and* $0 < \int_0^b x^{-s+N} f^{p(x)}(x) < \infty$. *Then*

(i) *for* $b \in (0,\infty)$,

$$\int_0^b x^{-s}\left(\int_0^x f(t)dt\right)^{p(x)} dx < \frac{\alpha^M\{\inf(1-1/r(t))q(t)\}^N}{\{\inf(1-1/r(t))q(t)\}(s-N)+1}\int_0^b\left[1-\left(\frac{t}{b}\right)^{\{s-N+\inf 1/(1-1/r(t))q(t)\}}\right] \times t^{-s+(1-1/r(t))p(t)} f^{p(t)}(t)dt, \tag{3.12}$$

(ii) *for* $b = \infty$,

$$\int_0^\infty x^{-s}\left(\int_0^x f(t)dt\right)^{p(x)} dx < \frac{\alpha^M\{\inf(1-1/r(t))q(t)\}^N}{\{\inf(1-1/r(t))q(t)\}(s-N)+1}\int_0^\infty t^{-s+N} f^{p(t)}(t)dt,$$
$$M = \sup p(t), \quad \alpha = \sup p(t)^{-1} + \sup q(t)^{-1} + \sup r(t), \quad N = \sup p(t)\left(1 - \frac{1}{r(t)}\right). \tag{3.13}$$

Proof. For case (i), $b \in (0,\infty)$, we use (3.1) to obtain

$$\int_0^b x^{-r}\left(\int_0^x f(t)dt\right)^{p(x)} dx$$
$$< \alpha^M\left\{\inf\left(1-\frac{1}{r(x)}\right)q(x)\right\}^{\sup p(x)/q(x)}\int_0^b x^{-s+\sup\{1/((1-1/r(x))q(x)2)\}} \times \int_0^x t^{1/(1-1/r(t))q(t)} f^{p(t)}(t)dt\,dx$$
$$\leq \alpha^M\left\{\inf\left(1-\frac{1}{r(t)}\right)q(t)\right\}^{\sup\{p(t)/q(t)\}} \times \int_0^b\left(\int_t^b x^{-s+\sup\{1/(1-1/r(x))p(x)-1/(1-1/r(x))q(x)-1\}}dx\right)t^{1/(1-1/r(t))q(t)} f^{p(t)}(t)dt$$

$$\leq \frac{\alpha^M \{\inf(1-1/r(t))q(t)\}^{\sup(1-1/r(t))p(t)-1}}{-s+\sup\{(1-1/r(t))p(t)-1/(1-1/r(t))q(t)\}}$$
$$\times \int_0^b \left(b^{-s+\sup\{(1-1/r(t))p(t)-1/(1-1/r(t))q(t)\}} - t^{-s+\sup\{(1-1/r(t))p(t)-1/(1-1/r(t))q(t)\}}\right)$$
$$\times t^{1/(1-1/r(t))q(t)} f^{p(t)}(t)dt$$
$$\leq \frac{\alpha^M \{\inf(1-1/r(t))q(t)\}^N}{\{\inf(1-1/r(t))q(t)\}(s-N)+1} \int_0^b \left[1-\left(\frac{t}{b}\right)^{s-N+\inf\{(1-1/r(t))q(t)\}}\right]$$
$$\times t^{-s+\sup(1-1/r(t))p(t)} f^{p(t)}(t)dt. \tag{3.14}$$

For case (ii), $b=\infty$, we use (3.1) to find

$$\int_0^\infty x^{-s}\left(\int_0^x f(t)dt\right)^{p(x)} dx$$
$$< \alpha^M \left\{\inf\left(1-\frac{1}{r(x)}\right)q(x)\right\}^{\sup p(x)/q(x)} \int_0^\infty x^{-s+\sup\{1/(1-1/r(x))q(x)^2\}}$$
$$\times \int_0^x t^{1/(1-1/r(t))q(t)} f^{p(t)}(t)dt\,dx$$
$$= \alpha^M \left\{\inf\left(1-\frac{1}{r(t)}\right)q(t)\right\}^{\sup p(t)/q(t)} \tag{3.15}$$
$$\times \int_0^\infty \left(\int_t^\infty x^{-s+\sup\{1/(1-1/r(x))p(x)-1/(1-1/r(x))q(x)-1\}} dx\right)$$
$$\times t^{1/(1-1/r(t))q(t)} f^{p(t)}(t)dt$$
$$= \frac{\alpha^M \{\inf(1-1/r(t))q(t)\}^N}{\{\inf(1-1/r(t))q(t)\}(s-N)+1} \int_0^\infty t^{-s+N} f^{p(t)}(t)dt.$$

□

Remark 3.6. (a) If $p(x)$, $q(x)$, and $r(x)$ are constants in Lemmas 3.1 and 3.2 and Theorems 3.3 and 3.4, then our results reduce to the corresponding Lemmas 2.1 and 2.2 and Theorems 2.4 and 2.5 obtained in [3].

(b) If we take $r(x) \to \infty$ in Theorems 3.3 and 3.4, then it reduces to corresponding Theorems 2.4 and 2.5.

(c) In the limits $a \to 0$, $b \to \infty$, $r(x) \to \infty$, $p(x)$ and $q(x)$ constants, (3.5) reduces to (1.1). Hence (3.5) is the generalization of (1.1).

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