CLASSIFYING TWO-DIMENSIONAL HYPOREDUCTIVE TRIPLE ALGEBRAS

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Two-dimensional real hyporeductive triple algebras (h.t.a.) are investigated. A classification of such algebras is presented. As a consequence, a classification of two-dimensional real Lie triple algebras (i.e., generalized Lie triple systems) and two-dimensional real Bol algebras is given.

1. Introduction

Hyporeductive algebras were introduced by Sabinin [8, 9] as an infinitesimal tool for the study of smooth hyporeductive loops which are a generalization both of smooth Bol loops and smooth reductive loops (i.e., smooth A-loops with monoalternative property [8]). It is shown that the fundamental vector fields of any smooth hyporeductive loop constitute an algebra called a hyporeductive algebra of vector fields. Further (see [1, 2, 9]) this notion has been extended to the one of an abstract hyporeductive triple algebra (h.t.a. for short) meaning a finite-dimensional linear space with two binary and one ternary operations satisfying some specific identities. It turns out that hyporeductive algebras generalize Bol algebras and Lie triple algebras (see [6, 12] about Bol and Lie triple algebras).

In this paper we consider 2-dimensional h.t.a. over the field of real numbers (i.e., 2-dimensional real h.t.a.) and the search of clear expressions of operations for such algebras led us to their classification (such a classification includes the one of 2-dimensional real Lie triple algebras, Lie triple systems, and Bol algebras).

Petersson [7] solved the classification problem for 2-dimensional nonassociative algebras over arbitrary base fields, and in his approach structure constants or multiplication tables almost never play a significant role. Underlying this classification is the use of an isomorphism theorem and the principal Albert isotopes. One observes that the algebras considered in that paper are binary algebras. It seems that the Petersson approach is not applicable to the case of 2-dimensional h.t.a. because of the following reasons. First, h.t.a. are contained in the class of tangent structures *with one (or more) binary operations and a ternary operation* (this is why they are usually called *binary-ternary algebras*) satisfying

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some compatibility conditions. Thus, if the Petersson approach could be applied to the binary operations of h.t.a. (under some conditions), it does not work, in general, for the ternary operation of h.t.a. (e.g., we still do not know what the principal Albert isotope of a ternary operation of an algebra is). Next, even for Bol algebras which are a very particular instance of h.t.a., almost no classification results are known (the classification of 2-dimensional real Bol algebras given in the present paper seems to be, to our knowledge, the first one so far). Because of the nature of h.t.a., their classification over arbitrary base fields should generalize, for example, the one of Bol algebras over arbitrary base fields but, unfortunately, the latter is still not available in litterature. The other reason for considering in this paper only real h.t.a. is related to the correspondence between h.t.a. and smooth hyporeductive loops [10] (this problem is solved by Kuz'min [5] for *real* finite-dimensional Malcev algebras and smooth Moufang loops and by Miheev and Sabinin [6] for *real* finite-dimensional Bol algebras and smooth Bol loops).

In Section 2 some results on hyporeductive algebras are recalled and the classification theorem is stated. Section 3 deals with its proof (this proof gives the classification strategy).

2. Background and results

Hyporeductive algebras were originally introduced [8, 9] as algebras of vector fields on a smooth finite-dimensional manifold, satisfying a specific condition. More exactly it was given the following definition.

Definition 2.1 [8, 9]. A linear space V of vector fields on a real n-dimensional manifold M with a singled out point e, satisfying

$$[X,[Y,Z]] = [X,a(Y,Z)] + r(X;Y,Z) \tag{2.1}$$

is called a *hyporeductive algebra* of vector fields with determining operations a and r if $\dim\{X(e) : X \in V\} = n$.

Obviously $a(Y,Z)$ is a bilinear skew-symmetric operation and $r(X;Y,Z)$ a trilinear operation on V, skew-symmetric in the last two variables. We called the relation (2.1) the *hyporeductive condition* for algebras of vector fields (see [1, 2]). Considering a hyporeductive algebra as a tangent algebra at the identity e of a smooth hyporeductive loop it is shown [9, 10] that a hyporeductive algebra may be viewed as an algebra with two binary operations $a(X,Y)(e)$, $T_e(X,Y) = [X,Y](e)$ and one ternary operation $r(Z;X,Y)(e)$ and then, working out the Jacobi identities in the corresponding enveloping Lie algebra, one can get the full system of identities linking the operations a, T_e, r. A similar construction is carried out in [1, 2], where instead of $T_e(X,Y)$ the operation $b(X,Y) = [X,Y](e) - a(X,Y)(e)$ is introduced (this is made in connection with a more suitable differential geometric interpretation of a hyporeductive algebra of vector fields and then the system of identities mentioned above constitutes the integrability conditions of the structure equations of the affinely connected smooth manifold associated with a local smooth hyporeductive loop). Define the operations $X \cdot Y = a(X,Y)(e)$, $X * Y = b(X,Y)$, and $\langle Z;X,Y\rangle = r(Z;X,Y)(e)$, then $(T_eM, \cdot, *, \langle \cdot;\cdot,\cdot\rangle)$ is an algebra satisfying the system of identities mentioned above. This led us to introduce the notion of an abstract hyporeductive triple algebra (h.t.a.).

Definition 2.2 [1–3]. Let $\mathcal{V}$ be a finite-dimensional linear space. Assume that on $\mathcal{V}$ two binary skew-symmetric operations "$\cdot$," "$*$" and one ternary operation "$\langle\ ;\ ,\ \rangle$" skew-symmetric in the last two variables are defined. Say that the algebra $(\mathcal{V},\cdot,*,\langle\ ;\ ,\ \rangle)$ is an *abstract* h.t.a. if for any ξ, η, ζ, κ, χ, θ in $\mathcal{V}$ the following identities hold:

$$\sigma\{\xi\cdot(\eta\cdot\zeta)-\langle\xi;\eta,\zeta\rangle\}=0, \tag{2.2}$$

$$\sigma\{\zeta*(\xi\cdot\eta)\}=0, \tag{2.3}$$

$$\sigma\{\langle\theta;\zeta,\xi\cdot\eta\rangle\}=0, \tag{2.4}$$

$$\begin{aligned}\kappa\cdot\langle\zeta;\xi,\eta\rangle-\zeta\cdot\langle\kappa;\xi,\eta\rangle+\langle\zeta\cdot\kappa;\xi,\eta\rangle\\ =\langle\xi*\eta;\zeta,\kappa\rangle-\langle\zeta*\kappa;\xi,\eta\rangle+\zeta*\langle\kappa;\xi,\eta\rangle-\kappa*\langle\zeta;\xi,\eta\rangle\\ +(\xi*\eta)*(\zeta*\kappa)+(\xi*\eta)\cdot(\zeta*\kappa),\end{aligned} \tag{2.5}$$

$$\begin{aligned}\chi\cdot(\kappa\cdot\langle\zeta;\xi,\eta\rangle-\zeta\cdot\langle\kappa;\xi,\eta\rangle+\langle\zeta\cdot\kappa;\xi,\eta\rangle)\\ +\langle\langle\chi;\xi,\eta\rangle;\zeta,\kappa\rangle-\langle\langle\chi;\zeta,\kappa\rangle;\xi,\eta\rangle+\langle\chi;\zeta,\langle\kappa;\xi,\eta\rangle\rangle-\langle\chi;\kappa,\langle\zeta;\xi,\eta\rangle\rangle=0,\end{aligned} \tag{2.6}$$

$$\chi*(\kappa\cdot\langle\zeta;\xi,\eta\rangle-\zeta\cdot\langle\kappa;\xi,\eta\rangle+\langle\zeta\cdot\kappa;\xi,\eta\rangle)=0, \tag{2.7}$$

$$\langle\theta;\chi,\kappa\cdot\langle\zeta;\xi,\eta\rangle-\zeta\cdot\langle\kappa;\xi,\eta\rangle+\langle\zeta\cdot\kappa;\xi,\eta\rangle\rangle=0, \tag{2.8}$$

$$\begin{aligned}\kappa\cdot\langle\zeta;\xi,\eta\rangle-\zeta\cdot\langle\kappa;\xi,\eta\rangle+\langle\zeta\cdot\kappa;\xi,\eta\rangle\\ +\eta\cdot\langle\xi;\zeta,\kappa\rangle-\xi\cdot\langle\eta;\zeta,\kappa\rangle+\langle\xi\cdot\eta;\zeta,\kappa\rangle=0,\end{aligned} \tag{2.9}$$

$$\zeta*\langle\kappa;\xi,\eta\rangle-\kappa*\langle\zeta;\xi,\eta\rangle+\xi*\langle\eta;\zeta,\kappa\rangle-\eta*\langle\xi;\zeta,\kappa\rangle=0, \tag{2.10}$$

$$\begin{aligned}\Sigma\{\langle(\langle\xi\cdot\eta;\zeta,\kappa\rangle+\eta\cdot\langle\xi;\zeta,\kappa\rangle-\xi\cdot\langle\eta;\zeta,\kappa\rangle);\lambda,\mu\rangle\\ +\langle\lambda\cdot\mu;\langle\eta;\zeta,\kappa\rangle,\xi\rangle+\mu\cdot\langle\lambda;\langle\eta;\zeta,\kappa\rangle,\xi\rangle-\lambda\cdot\langle\mu;\langle\eta;\zeta,\kappa\rangle,\xi\rangle\\ -(\langle\lambda\cdot\mu;\langle\xi;\zeta,\kappa\rangle,\eta\rangle+\mu\cdot\langle\lambda;\langle\xi;\zeta,\kappa\rangle,\eta\rangle-\lambda\cdot\langle\mu;\langle\xi;\zeta,\kappa\rangle,\eta\rangle)\}=0,\end{aligned} \tag{2.11}$$

$$\begin{aligned}\Sigma\{(\langle\mu;\langle\eta;\zeta,\kappa\rangle,\xi\rangle-\langle\mu;\langle\xi;\zeta,\kappa\rangle,\eta\rangle)*\lambda\\ +(\langle\lambda;\langle\xi;\zeta,\kappa\rangle,\eta\rangle-\langle\lambda;\langle\eta;\zeta,\kappa\rangle,\xi\rangle)*\mu\}=0,\end{aligned} \tag{2.12}$$

$$\begin{aligned}\Sigma\{\langle\theta;(\langle\mu;\langle\eta;\zeta,\kappa\rangle,\xi\rangle-\langle\mu;\langle\xi;\zeta,\kappa\rangle,\eta\rangle),\lambda\rangle\\ +\langle\theta;(\langle\lambda;\langle\xi;\zeta,\kappa\rangle,\eta\rangle-\langle\lambda;\langle\eta;\zeta,\kappa\rangle,\xi\rangle),\mu\rangle\}=0,\end{aligned} \tag{2.13}$$

where σ denotes the sum over cyclic permutations of ξ, η, ζ and Σ the one on pairs (ξ,η), (ζ,κ), (λ,μ).

Remark 2.3. The study of h.t.a. is more tractable if they are given in terms of identities as in the definition above. For instance, we observe that if in (2.2)–(2.13) we set $\xi\cdot\eta=0$ for any ξ, η of $\mathcal{V}$, then we get the defining identities of a *Bol algebra* $(\mathcal{V}, *, \langle\ ;\ ,\ \rangle)$:

$$\begin{gathered}
\sigma\{\langle \xi;\eta,\zeta\rangle\} = 0,\\
\langle \xi * \eta;\zeta,\kappa\rangle - \langle \zeta * \kappa;\xi,\eta\rangle + \zeta * \langle \kappa;\xi,\eta\rangle - \kappa * \langle \zeta;\xi,\eta\rangle + (\xi * \eta) * (\zeta * \kappa) = 0,\\
\langle\langle \chi;\xi,\eta\rangle;\zeta,\kappa\rangle - \langle\langle \chi;\zeta,\kappa\rangle;\xi,\eta\rangle + \langle \chi;\zeta,\langle \kappa;\xi,\eta\rangle\rangle - \langle \chi;\kappa,\langle \zeta;\xi,\eta\rangle\rangle = 0.
\end{gathered} \tag{2.14}$$

On the other hand, setting $\xi * \eta = 0$, we get a *Lie triple algebra* (i.e., a *generalized Lie triple system*) $(\mathcal{V}, \cdot, \langle\ ;\ ,\ \rangle)$:

$$\begin{gathered}
\sigma\{\xi \cdot (\eta \cdot \zeta) - \langle \xi;\eta,\zeta\rangle\} = 0,\\
\sigma\{\langle \theta;\zeta,\xi \cdot \eta\rangle\} = 0,\\
\kappa \cdot \langle \zeta;\xi,\eta\rangle - \zeta \cdot \langle \kappa;\xi,\eta\rangle + \langle \zeta \cdot \kappa;\xi,\eta\rangle = 0,\\
\langle\langle \chi;\xi,\eta\rangle;\zeta,\kappa\rangle - \langle\langle \chi;\zeta,\kappa\rangle;\xi,\eta\rangle + \langle \chi;\zeta,\langle \kappa;\xi,\eta\rangle\rangle - \langle \chi;\kappa,\langle \zeta;\xi,\eta\rangle\rangle = 0
\end{gathered} \tag{2.15}$$

and if, moreover, we put $\xi \cdot \eta = 0$ then we obtain a *Lie triple system (L.t.s.)* (see Yamaguti [11]). Note that for $\xi * \eta = 0$ and $\xi \cdot \eta = 0$, the identities (2.9)–(2.13) hold trivially.

The question naturally arises whether there exist proper abstract h.t.a. The answer to this problem is easier to seek among low-dimensional h.t.a. because of the specific properties of operations "$\cdot$," "$*$," and "$\langle\ ;\ ,\ \rangle$." Thus we are led to the study of two-dimensional real h.t.a., that is, to find the clear expressions of their defining operations. The following classification theorem describes, up to isomorphisms, all 2-dimensional real h.t.a. Such a classification includes the one of 2-dimensional real Bol algebras, Lie triple algebras and Lie triple systems.

Theorem 2.4. *Any 2-dimensional real h.t.a. is isomorphic to one of the h.t.a. of the following types:*

(I) $u * v = 0$, $u \cdot v = 0$, $\langle u;u,v\rangle = eu + fv$, $\langle v;u,v\rangle = ku - ev$,
(II) $u * v = 0$, $u \cdot v = au$, $\langle u;u,v\rangle = 0$, $\langle v;u,v\rangle = ku$, $(a \neq 0)$,
(III) $u * v = 0$, $u \cdot v = au + bv$, $\langle u;u,v\rangle = 0$, $\langle v;u,v\rangle = 0$, $(a \neq 0,\ b \neq 0)$,
(IV) $u * v = 0$, $u \cdot v = au + bv$, $\langle u;u,v\rangle = eu + fv$, $\langle v;u,v\rangle = ku - ev$, $(a \neq 0,\ b \neq 0,\ e \neq 0,\ f \neq 0,\ k \neq -e,\ af - be = 0 = bk + ae)$,
(V) $u * v = cu + dv$, $u \cdot v = 0$, $\langle u;u,v\rangle = eu + fv$, $\langle v;u,v\rangle = ku - ev$, $((c,d) \neq (0,0))$,
(VI) $u * v = cu + dv$, $u \cdot v = au$, $\langle u;u,v\rangle = 0$, $\langle v;u,v\rangle = ku$, $(a \neq 0,\ (c,d) \neq (0,0))$,
(VII) $u * v = cu + dv$, $u \cdot v = au + bv$, $\langle u;u,v\rangle = eu + fv$, $\langle v;u,v\rangle = ku - ev$, $(a \neq 0,\ b \neq 0,\ e \neq 0,\ f \neq 0,\ k \neq 0,\ (c,d) \neq (0,0),\ af - be = 0 = bk + ae)$,
(VIII) $u * v = cu + dv$, $u \cdot v = au + bv$, $\langle u;u,v\rangle = 0$, $\langle v;u,v\rangle = 0$, $(a \neq 0,\ b \neq 0,\ (c,d) \neq (0,0))$,

where a, b, c, d, e, f, k are real numbers.

Remark 2.5. The algebras of types (I), (II), (III), and (IV) are 2-dimensional real Lie triple algebras, the zero algebra and 2-dimensional Lie triple systems are contained in type (I). A comprehensive classification of 2-dimensional complex Lie triple systems is given by Yamaguti [11] (indeed, type (I) above is just [11, Lemma 5.1] when the base field is the one of real numbers); see also Jacobson [4]. Note that types (II), (IV) are nontrivial real Lie triple algebras. The algebras of type (V) constitute nontrivial 2-dimensional real Bol algebras (see Corollary 2.7 below).

Corollary 2.6. *There exist nontrivial 2-dimensional real h.t.a. Moreover, any such an algebra is isomorphic to an algebra of type (VI), (VII), or (VIII).*

At this point we note that the example of a 2-dimensional real h.t.a. that we gave in [3] is isomorphic to the algebra of type (VI) given by $u * v = dv$, $u \cdot v = u$, $\langle u;u,v\rangle = 0$, $\langle v;u,v\rangle = -u$, $(d \neq 0)$.

The subject of this paper is originally motivated by the need of showing concrete nontrivial h.t.a. Besides, an affine connection space locally permitting a structure of h.t.a. of vector fields is already described in [2] and, conversely, the structure equations of such an affine connection space give rise to a h.t.a. structure on the tangent space at a given point of the manifold. In relation with this, we consider here an example of such an affine connection space with a local loop structure [6] with the sole condition that is given a skew-symmetric bilinear function on the space of certain vector fields.

Let $(U,\circ,e)$ be a smooth local loop so that U is a sufficiently small neighborhood of the fixed point e of a real n-dimensional manifold M. We may consider on U the so-called right fundamental vector fields $\{X_\sigma\}$ of the loop $(U,\circ,e)$ (see, e.g., [10] and references therein), $[X_\sigma(x)]^\tau = X_\sigma^\tau(x)$, $x \in U$. Since $X_\sigma^\tau(e) = \delta_\sigma^\tau$ and e is a two-sided identity of $(U,\circ,e)$, it follows that $X_1,\dots,X_n$ define a basis of vector fields linearly independent at each point of U and thus U is parallelizable. The Lie bracket of two basis vector fields X_α, X_β is $[X_\alpha,X_\beta](x) = C_{\alpha\beta}^\gamma(x)X_\gamma(x)$ (observe that, in contrast of the case of left-invariant vector fields of a Lie group, the $C_{\alpha\beta}^\gamma$ are functions of point [9, 10]). Now define on U the $(-)$-connection $\nabla_Z Y = 0$ obtained from the parallelization, for any vector fields Y, Z on U. Assume that on the space of all right fundamental vector fields on U is given a skew-symmetric bilinear function $a(Y,Z)$. The torsion T of the connection defined above has the expression $T(Y,Z) = -[Y,Z]$. The vector field $[Y,Z] - a(Y,Z)$ is defined on U and so is the vector field $[W,[Y,Z] - a(Y,Z)]$, where W,Y,Z are right fundamental vector fields on U. Therefore, with respect to the basis $\{X_1,\dots,X_n\}$, we have the representation

$$[X_l,[X_j,X_k] - a(X_j,X_k)] = r_{l,jk}^i X_i, \tag{$*$}$$

which means that a structure of a h.t.a. of vector fields is locally defined (this is the original definition of a hyporeductive algebra of vector fields [8–10]). The relation $(*)$ may be written as

$$\left(\nabla_l T_{jk}^i - T_{ls}^i\left(T_{jk}^s + a_{jk}^s\right)\right)(x) = -r_{l,jk}^i(x) \tag{$**$}$$

for any $x \in U$, where the skew-symmetric tensor (a_{jk}^s) is defined by $a(X_j,X_k) = a_{jk}^s X_s$, (T_{jk}^s) is the torsion tensor of the connection ∇ and $\nabla_l T_{jk}^i$ denotes the covariant derivative of the function T_{jk}^i by the vector field X_l. Since $\{X_1,\dots,X_n\}$ is a parallelization, the $r_{l,jk}^i$ are constants and the relation $(**)$ means that $\nabla_m(\nabla_l T_{jk}^i - T_{ls}^i(T_{jk}^s + a_{jk}^s)) = 0$ at each point of U. The structure equations of (U,∇) in terms of the basis $\{X_1,\dots,X_n\}$ are then

$$\begin{aligned} d\omega^i &= \frac{1}{2}T^i_{jk}\omega^j\wedge\omega^k,\\ dT^i_{jk} &= \nabla_l T^i_{jk}\omega^l. \end{aligned} \tag{2.16}$$

The integrability conditions for these equations (at the point e) are precisely the defining identities, written in terms of structure constants, of an abstract h.t.a. and so the tangent space T_eM is provided with a h.t.a. structure (see [2] for the general case of an affine connection space related with a smooth local hyporeductive loop).

Suppose now that $\dim M = 2$ and choose the basis vector fields X_1, X_2 such that $[X_1,X_2](e) = 2X_1(e)+X_2(e)$, $(X_1T^1_{12})(e) = 1$, $(X_1T^2_{12})(e) = -1$, $(X_2T^1_{12})(e) = 1$, $(X_2T^2_{12})(e) = 0$. Moreover, choose the skew-symmetric function $a(Y,Z)$ such that $a(X_1,X_2)(e) = X_1(e)+X_2(e)$. Then, as indicated in the beginning of this section, we may define on T_eM two binary operations $\widetilde{X}_1\cdot\widetilde{X}_2 = \widetilde{X}_1+\widetilde{X}_2$, $\widetilde{X}_1 * \widetilde{X}_2 = \widetilde{X}_1$ and, using $(**)$, a ternary operation $\langle\widetilde{X}_1;\widetilde{X}_1,\widetilde{X}_2\rangle = -\widetilde{X}_1+\widetilde{X}_2$, $\langle\widetilde{X}_2;\widetilde{X}_1,\widetilde{X}_2\rangle = \widetilde{X}_1-\widetilde{X}_2$, where $\widetilde{X}_1 := X_1(e)$ and $\widetilde{X}_2 := X_2(e)$. It is easy to see that the space T_eM along with these operations constitutes a h.t.a. of type (VII).

According to the remarks above we have also the following corollary.

Corollary 2.7. *Any 2-dimensional real Lie triple algebra is isomorphic to one of the algebras of the following types:*
(T1) $u\cdot v = 0$, $\langle u;u,v\rangle = \alpha u+\beta v$, $\langle v;u,v\rangle = \gamma u-\alpha v$,
(T2) $u\cdot v = u$, $\langle u;u,v\rangle = 0$, $\langle v;u,v\rangle = ku$,
(T3) $u\cdot v = u+v$, $\langle u;u,v\rangle = 0$, $\langle v;u,v\rangle = 0$,
(T4) $u\cdot v = au+bv$, $\langle u;u,v\rangle = eu+fv$, $\langle v;u,v\rangle = ku-ev$, ($a\neq 0$, $b\neq 0$, $e\neq 0$, $f\neq 0$, $k\neq 0$, $af-be = 0 = bk+ae$).
Any 2-dimensional real Bol algebra is isomorphic to one of the algebras of the following types:
(B1) $u*v = 0$, $\langle u;u,v\rangle = \alpha u+\beta v$, $\langle v;u,v\rangle = \gamma u-\alpha v$,
(B2) $u*v = cu+dv$, $\langle u;u,v\rangle = eu+fv$, $\langle v;u,v\rangle = ku-ev$,
(B3) $u*v = cu+dv$, $\langle u;u,v\rangle = eu+fv$, $\langle v;u,v\rangle = -ev$,
(B4) $u*v = cu+dv$, $\langle u;u,v\rangle = eu$, $\langle v;u,v\rangle = ku-ev$,
(B5) $u*v = cu+dv$, $\langle u;u,v\rangle = eu$, $\langle v;u,v\rangle = -ev$,
(B6) $u*v = cu+dv$, $\langle u;u,v\rangle = fv$, $\langle v;u,v\rangle = ku$,
(B7) $u*v = cu+dv$, $\langle u;u,v\rangle = fv$, $\langle v;u,v\rangle = 0$,
(B8) $u*v = cu+dv$, $\langle u;u,v\rangle = 0$, $\langle v;u,v\rangle = 0$,
(B9) $u*v = cu$, $\langle u;u,v\rangle = eu+fv$, $\langle v;u,v\rangle = ku-ev$,
(B10) $u*v = cu$, $\langle u;u,v\rangle = eu+fv$, $\langle v;u,v\rangle = -ev$,
(B11) $u*v = cu$, $\langle u;u,v\rangle = eu$, $\langle v;u,v\rangle = ku-ev$,
(B12) $u*v = cu$, $\langle u;u,v\rangle = eu$, $\langle v;u,v\rangle = -ev$,
(B13) $u*v = cu$, $\langle u;u,v\rangle = fv$, $\langle v;u,v\rangle = ku$,
(B14) $u*v = cu$, $\langle u;u,v\rangle = fv$, $\langle v;u,v\rangle = 0$,
(B15) $u*v = cu$, $\langle u;u,v\rangle = 0$, $\langle v;u,v\rangle = ku$,
(B16) $u*v = cu$, $\langle u;u,v\rangle = 0$, $\langle v;u,v\rangle = 0$,
(B17) $u*v = dv$, $\langle u;u,v\rangle = eu+fv$, $\langle v;u,v\rangle = -ev$,

(B18) $u * v = dv$, $\langle u;u,v\rangle = eu$, $\langle v;u,v\rangle = ku - ev$,
(B19) $u * v = dv$, $\langle u;u,v\rangle = eu$, $\langle v;u,v\rangle = -ev$, *where* $c \neq 0$, $d \neq 0$, $e \neq 0$, $f \neq 0$, $k \neq 0$.

Types (B2)–(B19) constitute just the developed form of type (V).

3. Proof of Theorem 2.4

First we will prove the following lemma.

Lemma 3.1. *If $\{x_1,x_2\}$ is a basis of a 2-dimensional real h.t.a. $\mathcal{V}$, then the identities (2.2)–(2.13) of abstract h.t.a. have the following form:*

$$J(x_1,x_2) - \langle x_1 \cdot x_2;x_1,x_2\rangle + x_1 \cdot \langle x_2;x_1,x_2\rangle - x_2 \cdot \langle x_1;x_1,x_2\rangle = 0, \tag{3.1}$$

$$x_i \cdot J(x_1,x_2) - \langle x_i;x_1,\langle x_2;x_1,x_2\rangle\rangle + \langle x_i;x_2,\langle x_1;x_1,x_2\rangle\rangle = 0, \tag{3.2}$$

$$x_i * J(x_1,x_2) = 0, \tag{3.3}$$

$$\langle x_j;x_i,J(x_1,x_2)\rangle = 0, \tag{3.4}$$

$$\langle x_1 \cdot x_2;x_1,x_2\rangle - x_1 \cdot \langle x_2;x_1,x_2\rangle + x_2 \cdot \langle x_1;x_1,x_2\rangle = 0, \tag{3.5}$$

$$J(x_1,x_2) = 0, \tag{3.6}$$

where $J(x_1,x_2) = x_1 * \langle x_2;x_1,x_2\rangle - x_2 * \langle x_1;x_1,x_2\rangle$ *and* $i,j = 1,2$.

Proof. With respect to the basis $\{x_1,x_2\}$, (2.2), (2.3), and (2.4) are clearly satisfied trivially. Next the left-hand side of (2.5) now reads $x_i \cdot \langle x_j;x_1,x_2\rangle - x_j \cdot \langle x_i;x_1,x_2\rangle + \langle x_j \cdot x_i;x_1,x_2\rangle$ while the right-hand side reads $\langle x_1 * x_2;x_j,x_i\rangle - \langle x_j * x_i;x_1,x_2\rangle + x_j * \langle x_i;x_1,x_2\rangle - x_i * \langle x_j;x_1,x_2\rangle + (x_1 * x_2) * (x_j * x_i) + (x_1 * x_2) \cdot (x_j * x_i)$, with $i,j = 1,2$. Furthermore, because of the skew-symmetry of operations "$\cdot$," "$*$," "$\langle\ ;\ ,\ \rangle$" one observes that the identity (2.5) gets the form $x_1 * \langle x_2;x_1,x_2\rangle - x_2 * \langle x_1;x_1,x_2\rangle = \langle x_1 \cdot x_2;x_1,x_2\rangle - x_1 \cdot \langle x_2;x_1,x_2\rangle + x_2 \cdot \langle x_1;x_1,x_2\rangle$, so we obtain (3.1). In view of (3.1), the identities (2.7) and (2.8) are straightforwardly transformed into (3.3) and (3.4), respectively.

Finally, and again with (3.1) in mind, we work the identity (2.6) as follows: we replace ξ, η, ζ, κ, χ by x_1, x_2, x_k, x_j, x_i, respectively, where $i,j,k = 1,2$ and then by (3.1), we see that (2.6) gets the form $x_i \cdot (x_1 * \langle x_2;x_1,x_2\rangle - x_2 * \langle x_1;x_1,x_2\rangle) + \langle x_i;x_2,\langle x_1;x_1,x_2\rangle\rangle - \langle x_i;x_1,\langle x_2;x_1,x_2\rangle\rangle = 0$, that is, we get (3.2). The equation (2.10) implies (3.6) and hence (3.5), in view of (3.1). The equalities (2.11)–(2.13) hold trivially in view of (3.6) and (3.2). □

One observes that (3.5) and (3.6) are actually equivalent and accordingly the system (3.1)–(3.6) takes a simpler form (see the theorem's proof below). We keep the system (3.1)–(3.6) as above in order to follow the step-by-step transformation of the system (2.2)–(2.13) in the 2-dimensional case. We now turn to the proof of the theorem.

Proof of Theorem 2.4. Let $(\mathcal{V},\cdot,*,\langle\ ;\ ,\ \rangle)$ be a 2-dimensional real h.t.a. with basis $\{u,v\}$. Put $u \cdot v = au + bv$, $u * v = cu + dv$, $\langle u;u,v\rangle = eu + fv$, $\langle v;u,v\rangle = ku + lv$. Then, by the lemma, the identities (2.2)–(2.13) reduce to (3.1)–(3.6) and a careful reading of these identities reveals that the expression $N = \langle u \cdot v;u,v\rangle + v \cdot \langle u;u,v\rangle - u \cdot \langle v;u,v\rangle$ can be conclusive for the study of h.t.a. (at least in the 2-dimensional case). Therefore we will discuss the case $N = 0$ (see (3.5)).

Thus $0 = N = \langle u \cdot v; u, v\rangle + v \cdot \langle u; u, v\rangle - u \cdot \langle v; u, v\rangle = \langle au + bv; u, v\rangle + v \cdot (eu + fv) - u \cdot (ku + lv) = (bk - al)u + (af - be)v$ implies

$$\begin{aligned} bk - al &= 0, \\ af - be &= 0. \end{aligned} \tag{3.7}$$

Discussing the solutions of the system (3.7), we see that the following essential situations occur (any other situation is either one of those enumerated below or is included in some of them):

(1) $a \neq 0$, $b \neq 0$, $e \neq 0$, $f \neq 0$, $k \neq 0$, $l \neq 0$,
(2) $a \neq 0$, $b \neq 0$, $e \neq 0$, $f \neq 0$, $k = 0$, $l = 0$,
(3) $a \neq 0$, $b \neq 0$, $e = 0$, $f = 0$, $k \neq 0$, $l \neq 0$,
(4) $a \neq 0$, $b \neq 0$, $e = 0$, $f = 0$, $k = 0$, $l = 0$,
(5) $a \neq 0$, $b = 0$, e any , $f = 0$, $k = 0$, $l = 0$,
(6) $a = 0$, $b \neq 0$, $e = 0$, f any, $k \neq 0$, l any,
(7) $a = 0$, $b = 0$, e any, f any, k any, l any.

Now each of the cases (1)–(7) must be discussed in connection with the identities (3.1)–(3.6). We observe that with the condition $N = 0$ (i.e., (3.5)) only (3.1) and (3.2) are of interest here.

The identity (3.1) implies $u * \langle v; u, v\rangle - v * \langle u; u, v\rangle = 0$ which means that

$$(e + l)u * v = 0. \tag{3.8}$$

The equation (3.8) yields the following cases:

(8) $u * v = 0$ and $l \neq -e$,
(9) $u * v = 0$ and $l = -e$,
(10) $u * v = cu + dv$ ($(c,d) \neq (0,0)$) and $l = -e$.

Then considering each of the cases (1)–(7) in connection with the conditions (8)–(10), we are led to the following types of algebras:

(A1) $u * v = 0$, $u \cdot v = au + bv$, $\langle u; u, v\rangle = eu + fv$, $\langle v; u, v\rangle = ku + lv$, ($a \neq 0$, $b \neq 0$, $e \neq 0$, $f \neq 0$, $k \neq 0$, $l \neq 0$, $l \neq -e$, $af - be = 0 = bk - al$),
(A2) $u * v = 0$, $u \cdot v = au + bv$, $\langle u; u, v\rangle = eu + fv$, $\langle v; u, v\rangle = ku - ev$, ($a \neq 0$, $b \neq 0$, $e \neq 0$, $f \neq 0$, $k \neq 0$, $af - be = 0 = bk + ae$),
(A3) $u * v = cu + dv$, $u \cdot v = au + bv$, $\langle u; u, v\rangle = eu + fv$, $\langle v; u, v\rangle = ku - ev$, ($a \neq 0$, $b \neq 0$, $(c,d) \neq (0,0)$, $e \neq 0$, $f \neq 0$, $k \neq 0$, $af - be = 0 = bk + ae$),
(A4) $u * v = 0$, $u \cdot v = au + bv$, $\langle u; u, v\rangle = eu + fv$, $\langle v; u, v\rangle = 0$, ($a \neq 0$, $b \neq 0$, $e \neq 0$, $f \neq 0$, $k \neq 0$, $af - be = 0$),
(A5) $u * v = 0$, $u \cdot v = au + bv$, $\langle u; u, v\rangle = 0$, $\langle v; u, v\rangle = ku + lv$, ($a \neq 0$, $b \neq 0$, $k \neq 0$, $l \neq 0$, $bk - al = 0$),
(A6) $u * v = 0$, $u \cdot v = au + bv$, $\langle u; u, v\rangle = 0$, $\langle v; u, v\rangle = 0$, ($a \neq 0$, $b \neq 0$),
(A7) $u * v = cu + dv$, $u \cdot v = au + bv$, $\langle u; u, v\rangle = 0$, $\langle v; u, v\rangle = 0$, ($a \neq 0$, $b \neq 0$, $(c,d) \neq (0,0)$),
(A8) $u * v = cu + dv$, $u \cdot v = au$, $\langle u; u, v\rangle = 0$, $\langle v; u, v\rangle = ku$, ($a \neq 0$, $(c,d) \neq (0,0)$),

(A9) $u * v = 0$, $u \cdot v = au$, $\langle u;u,v\rangle = 0$, $\langle v;u,v\rangle = ku$, $(a \neq 0)$,
(A10) $u * v = 0$, $u \cdot v = au$, $\langle u;u,v\rangle = eu$, $\langle v;u,v\rangle = ku$, $(a \neq 0,\, e \neq 0)$,
(A11) $u * v = cu + dv$, $u \cdot v = bv$, $\langle u;u,v\rangle = fv$, $\langle v;u,v\rangle = 0$, $(b \neq 0,\, (c,d) \neq (0,0))$,
(A12) $u * v = 0$, $u \cdot v = bv$, $\langle u;u,v\rangle = fv$, $\langle v;u,v\rangle = 0$, $(b \neq 0)$,
(A13) $u * v = 0$, $u \cdot v = bv$, $\langle u;u,v\rangle = fv$, $\langle v;u,v\rangle = lv$, $(b \neq 0,\, l \neq 0)$,
(A14) $u * v = cu + dv$, $u \cdot v = 0$, $\langle u;u,v\rangle = eu + fv$, $\langle v;u,v\rangle = ku - ev$, $((c,d) \neq (0,0))$,
(A15) $u * v = 0$, $u \cdot v = 0$, $\langle u;u,v\rangle = eu + fv$, $\langle v;u,v\rangle = ku - ev$,
(A16) $u * v = 0$, $u \cdot v = 0$, $\langle u;u,v\rangle = eu + fv$, $\langle v;u,v\rangle = ku + lv$, $(l \neq -e)$.

Furthermore, the identity (3.2) implies $\langle x;\langle u;u,v\rangle,v\rangle + \langle x;u,\langle v;u,v\rangle\rangle = 0$ with $x = u$ or v, that is,

$$(e+l)\langle x;u,v\rangle = 0, \tag{3.9}$$

$x = u$ or v. The equation (3.9) gives the following cases:
(11) $l = -e$ and $\langle x;u,v\rangle \neq 0$, $x = u$ or v,
(12) $l = -e$ and $\langle u;u,v\rangle \neq 0$, $\langle v;u,v\rangle = 0$,
(13) $l = -e$ and $\langle u;u,v\rangle = 0$, $\langle v;u,v\rangle \neq 0$,
(14) $l = -e$ and $\langle x;u,v\rangle = 0$, $x = u$ or v,
(15) $l \neq -e$ and $\langle x;u,v\rangle = 0$, $x = u$ or v.
Therefore, in view of constraints (11)–(15), the algebras of types (A1), (A4), (A5), (A10), (A13), and (A16) must be cancelled out. Now, observing that an algebra of type (A8) is isomorphic to the one of type (A11) and an algebra of type (A9) is isomorphic to the one of type (A12), we are left with the algebras of types (A2), (A3), (A6), (A7), (A8), (A9), (A14), (A15). These are precisely the ones enumerated in our classification theorem. □

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