CLASSICAL 2-ORTHOGONAL POLYNOMIALS AND DIFFERENTIAL EQUATIONS

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We construct the linear differential equations of third order satisfied by the classical 2-orthogonal polynomials. We show that these differential equations have the following form: $R_{4,n}(x)P_{n+3}^{(3)}(x)+R_{3,n}(x)P''_{n+3}(x)+R_{2,n}(x)P'_{n+3}(x)+R_{1,n}(x)P_{n+3}(x)=0$, where the coefficients $\{R_{k,n}(x)\}_{k=1,4}$ are polynomials whose degrees are, respectively, less than or equal to 4, 3, 2, and 1. We also show that the coefficient $R_{4,n}(x)$ can be written as $R_{4,n}(x)=F_{1,n}(x)S_3(x)$, where $S_3(x)$ is a polynomial of degree less than or equal to 3 with coefficients independent of n and $\deg(F_{1,n}(x)) \leq 1$. We derive these equations in some cases and we also quote some classical 2-orthogonal polynomials, which were the subject of a deep study.

1. Introduction

The classical orthogonal polynomials (Hermite, Laguerre, Jacobi, and Bessel) satisfy a hypergeometric-type differential equation of second order [5]:

$$\sigma(x)y''(x)+\tau(x)y'(x)+\lambda_n y(x)=0, \quad \text{where } \deg\sigma \leq 2,\ \deg\tau \leq 1,$$
$$\lambda_n = -\frac{n(n-1)}{2}\sigma'' - n\tau' \neq 0, \quad n \geq 0. \tag{1.1}$$

These polynomials are the unique polynomial solutions of a second-order linear differential equation of hypergeometric type [14].

The aim of this work is to generalize the results obtained in the standard orthogonality to 2-orthogonality. We first look for the differential equations whose the solutions are classical 2-orthogonal polynomials and we explicit them, where it is possible.

First, we recall some basic notions of the d-orthogonality, then we study the nature of coefficients of recurrence relations satisfied by the classical 2-orthogonal polynomials sequences. We show afterwards that these polynomials are solutions of a third-order linear differential equation with polynomial coefficients of degree less than or equal to 4, 3, 2,

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and 1, depending generally on n. The main result is that the coefficient associated with highest derivative can be written as the product of 2 polynomials of which one is of degree ≤ 3 and independent of n. The latter, will allow us not only to enumerate some polynomial solutions, but also to explicit some ODEs. Of course, these equations generalize the Sturm-Liouville equations.

The cases where the polynomial solutions are 2-symmetric orthogonal are completely derived. Finally, we mention some examples of classical 2-orthogonal polynomials with some of their properties.

The final goal being naturally to search for the analog theorem of Bochner, that is, first, to enumerate all sequences of classical 2-orthogonal polynomials and afterwards, to study their properties, in particular the representation of the pair of linear forms in each case.

2. Preliminary notions

First, we recall some definitions and properties of the sequences of d-orthogonal polynomials, without forgetting to mention however, that the d-orthogonal polynomials $P_n(n \geq 0)$ are a special case of type II multiple orthogonal polynomials $R_{\vec{s}(n)}$, where the sequence $\vec{s}(n)$ $(n \geq 0)$ of multi-indices in $\mathbb{N}^d$, with $n = md + \alpha$, $0 \leq \alpha \leq d-1$, $m \geq 0$, is defined by

$$\vec{s}(n) = \Big(\underbrace{m+1, m+1, \dots, m+1}_{\alpha \text{ times}}, m, m, \dots, m\Big), \tag{2.1}$$

and where $P_n(x) = R_{\vec{s}(n)}$ $(n \geq 0)$ [1, 21].

Note that the multiple orthogonal polynomials are narrowly related to simultaneous vectorial Pade approximation, to be more precise as Hermite-Pade approximation. In particular, the type II multiple orthogonal polynomials $R_{\vec{n}}$, $\vec{n} = (n_1, n_2, \dots, n_d)$, for the measures $\{\mu_j\}_{j=1}^{d}$, that is, the monic polynomial $R_{\vec{n}}$ of degree $|\vec{n}| = n_1 + n_2 + \cdots + n_d$ which satisfies the orthogonal conditions

$$\int_{\triangle_k} x^m R_{\vec{n}}(x) d\mu_j = 0, \quad k = 0, 1, \dots, n_j - 1,\ j = 1, 2, \dots, d \tag{2.2}$$

(resp., the d-orthogonal polynomials with respect to the vector linear form $\mathcal{L} = (\mathcal{L}_0, \mathcal{L}_1, \dots, \mathcal{L}_{d-1})^T$) represent the common denominator of rational approximation of the d Stieltjes functions [3, 17, 19, 21]

$$f_j(z) = \int_{\triangle_j} \frac{d\mu_j}{z-x}, \quad z \notin \triangle_j,\ j = 1, 2, \dots, d, \tag{2.3}$$

that is,

$$R_{\vec{n}}(z) f_j(z) - Q_{\vec{n},j}(z) = O(z^{-n_j-1}), \quad |z| \longrightarrow \infty,\ j = 1, 2, \dots, d. \tag{2.4}$$

Definition 2.1. Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials (i.e., $P_n(x) = x^n + \cdots$). Call the dual sequence of the sequence $\{P_n\}_{n\geq 0}$, the sequence of linear forms $\{£_n\}_{n\geq 0}$

defined by

$$\pounds_n(P_m(x)) = \langle \pounds_n, P_m(x) \rangle = \delta_{n,m}, \quad m,n \geq 0, \tag{2.5}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality bracket between the vector space of polynomials $\mathcal{P}$ and its algebraic dual space $\mathcal{P}'$.

Definition 2.2 [16, 21]. A sequence of polynomials $\{P_n\}_{n\geq 0}$ is d-orthogonal with respect to $\pounds = (\pounds_0, \pounds_1, \ldots, \pounds_{d-1})^T$ if it satisfies

$$\begin{aligned} &\pounds_\alpha(x^m P_n(x)) = 0, \quad n \geq md + \alpha + 1,\ m \geq 0, \\ &\pounds_\alpha(x^m P_{md+\alpha}(x)) \neq 0, \quad m \geq 0,\ 0 \leq \alpha \leq d-1. \end{aligned} \tag{2.6}$$

Theorem 2.3 [16, 21]. *Let $\{P_n\}_{n\geq 0}$ be a monic sequence of polynomials, then the following statements are equivalent.*

(a) *The sequence $\{P_n\}_{n\geq 0}$ is d-orthogonal with respect to $\pounds = (\pounds_0, \pounds_1, \ldots, \pounds_{d-1})^T$.*

(b) *The sequence $\{P_n\}_{n\geq 0}$ satisfies a recurrence relation of order $d+1$ ($d \geq 1$):*

$$P_{m+d+1}(x) = (x - \beta_{m+d})P_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} P_{m+d-1-\nu}(x), \quad m \geq 0, \tag{2.7}$$

with the initial data

$$\begin{aligned} &P_0(x) = 1, \qquad P_1(x) = x - \beta_0, \\ &P_m(x) = (x - \beta_{m-1})P_{m-1}(x) - \sum_{\nu=0}^{m-2} \gamma_{m-1-\nu}^{d-1-\nu} P_{m-2-\nu}(x), \quad 2 \leq m \leq d, \end{aligned} \tag{2.8}$$

where $\gamma_{m+1}^0 \neq 0$, $m \geq 0$. (Regularity conditions.)

Remark 2.4 [12, 18]. This result generalizes the Shohat-Favard theorem.

Definition 2.5. The sequence $\{P_n\}_{n\geq 0}$ is said to be d-symmetric if it satisfies

$$P_n(\rho_k x) = \rho_k^n P_n(x), \quad n \geq 0, \text{ where } \rho_k = \exp\left(\frac{2ik\pi}{d+1}\right),\ k = 1, \ldots, d. \tag{2.9}$$

Theorem 2.6 [9–11]. *For each monic d-orthogonal sequence $\{P_n\}_{n\geq 0}$, the following equivalences hold.*

(a) $\{P_n\}_{n\geq 0}$ *is d-symmetric.*
(b) $\{P_n\}_{n\geq 0}$ *satisfies the recurrence relation*

$$\begin{gathered} P_n(x) = x^n, \quad 0 \leq n \leq d, \\ P_{n+d+1}(x) = xP_{n+d}(x) - \gamma_{n+1}^0 P_n(x), \quad n \geq 0. \end{gathered} \tag{2.10}$$

Definition 2.7 [10]. A sequence of polynomials $\{P_n\}_{n\geq 0}$ $(d \geq 1)$ is said to be "classical" if the sequence of the derivatives is also d-orthogonal.

Corollary 2.8 [9, 15]. *When the sequence $\{P_n\}_{n\geq 0}$ is classical d-orthogonal and d-symmetric, then the monic sequence of derivatives $\{Q_n\}_{n\geq 0}$ (i.e., $Q_n(x) = P'_{n+1}(x)/(n+1)$) satisfies the following recurrence relation:*

$$\begin{gathered} Q_n(x) = x^n, \quad 0 \leq n \leq d, \\ Q_{n+d+1}(x) = xQ_{n+d}(x) - \delta_{n+1}^0 Q_n(x) \quad \text{with } \delta_{n+1}^0 \neq 0,\ n \geq 0. \end{gathered} \tag{2.11}$$

3. Classical 2-orthogonal polynomials

Statement of the problem. In this work, we try to answer three main questions.

(i) Which type of differential equations have as solutions classical 2-orthogonal polynomials?
(ii) Can we exhibit these differential equations?
(iii) What are these polynomials solutions?

For this, we consider a monic sequence of classical 2-orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$, such that the recurrence relations satisfied by the polynomials $P_n(x)$ and $P'_n(x)$ $(n \geq 0)$ are given, respectively, by

$$\begin{gathered} P_0(x) = 1, \qquad P_1(x) = x - \beta_0^0, \qquad P_2(x) = (x - \beta_1^0)P_1(x) - \gamma_1^0, \\ P_{n+3}(x) = (x - \beta_{n+2}^0)P_{n+2}(x) - \gamma_{n+2}^0 P_{n+1}(x) - \delta_{n+1}^0 P_n(x), \quad n \geq 0, \end{gathered} \tag{3.1}$$

with the regularity condition $\delta_n^0 \neq 0$, $n \geq 1$, and

$$\begin{gathered} P'_1(x) = 1, \qquad P'_2(x) = 2(x - \beta_1^1), \qquad P'_3(x) = \frac{3}{2}\left[(x - \beta_2^1)P'_2(x) - \gamma_2^1\right], \\ \frac{n+3}{n+4}P'_{n+4}(x) = (x - \beta_{n+3}^1)P'_{n+3}(x) - \gamma_{n+3}^1 P'_{n+2}(x) - \delta_{n+2}^1 P'_{n+1}(x), \quad n \geq 0, \end{gathered} \tag{3.2}$$

with the regularity condition $\delta_{n+1}^1 \neq 0$, $n \geq 1$.

Proposition 3.1 [9]. *The coefficients β_n^0, β_n^1, γ_n^0, γ_n^1, δ_n^0, and δ_n^1 satisfy the following finite difference system:*

$$(n+2)\beta_{n+1}^1 - n\beta_n^1 - (n+1)\beta_{n+1}^0 + (n-1)\beta_n^0 = 0, \quad n \geq 0, \tag{3.3}$$

$$\frac{(n+3)\gamma_{n+2}^1 - (n+2)\gamma_{n+2}^0}{n+2} - \frac{n\gamma_{n+1}^1 - (n-1)\gamma_{n+1}^0}{n+1} = \left(\beta_{n+1}^1 - \beta_{n+1}^0\right)^2, \quad n \geq 0, \tag{3.4}$$

$$\frac{(n+4)\delta^1_{n+2}-(n+3)\delta^0_{n+2}}{n+3}-\frac{n\delta^1_{n+1}-(n-1)\delta^0_{n+1}}{n+1}$$
$$=\gamma^0_{n+2}\left(\beta^0_{n+2}+\beta^0_{n+1}-2\beta^1_{n+1}\right)-\gamma^1_{n+2}\left(2\beta^0_{n+2}-\beta^1_{n+2}-\beta^1_{n+1}\right),\quad n\geq 0, \tag{3.5}$$

$$\delta^0_{n+1}\left(\beta^0_n-\beta^1_n\right)-\delta^1_{n+1}\left(\beta^0_{n+2}-\beta^1_{n+2}\right)+\left(\delta^0_{n+1}-\delta^1_{n+1}\right)\left(\beta^0_{n+2}-\beta^1_n\right)$$
$$=\gamma^1_{n+1}\left(\gamma^0_{n+2}-\gamma^1_{n+2}\right)-\gamma^0_{n+2}\left(\gamma^0_{n+1}-\gamma^1_{n+1}\right),\quad n\geq 1, \tag{3.6}$$

$$\delta^0_{n+2}\left(\gamma^0_{n+1}-\gamma^1_{n+1}\right)-\delta^1_{n+1}\left(\gamma^0_{n+3}-\gamma^1_{n+3}\right)$$
$$=\gamma^1_{n+1}\left(\delta^0_{n+2}-\delta^1_{n+2}\right)-\gamma^0_{n+3}\left(\delta^0_{n+1}-\delta^1_{n+1}\right),\quad n\geq 1, \tag{3.7}$$

$$\delta^0_{n+3}\left(\delta^0_{n+1}-\delta^1_{n+1}\right)=\delta^1_{n+1}\left(\delta^0_{n+3}-\delta^1_{n+3}\right),\quad n\geq 1. \tag{3.8}$$

Proof. From (3.1) and (3.2), we get the relation

$$P_{n+3}(x)=\frac{1}{n+4}P'_{n+4}(x)+\left(\beta^0_{n+3}-\beta^1_{n+3}\right)P'_{n+3}+\left(\gamma^0_{n+3}-\gamma^1_{n+3}\right)P'_{n+2}(x)$$
$$+\left(\delta^0_{n+2}-\delta^1_{n+2}\right)P'_{n+1}(x),\quad n\geq 0. \tag{3.9}$$

Multiplying by x both hand sides of this relation and using once again (3.2), we get the precedent system. □

Remark 3.2. We see that the determination of all the 2-orthogonal sequences goes through the resolution of the system (3.1)–(3.8). Many authors have tried to solve it, but up to now, its resolution is still giving many problems because it is nonlinear as well as the number of unknowns is relatively high (six). Nevertheless, we will analyze the cases where its resolution is complete. In fact, we have the following.

Lemma 3.3. *Equation (3.8) admits the following as a unique set of solutions.*
(A) $\delta^1_{n+1}=\delta^0_{n+1}$, $n\geq 1$.
(B) $\delta^1_{2n}=(n+\rho_2)/(n-1+\rho_2)\delta^0_{2n}$ *and* $\delta^1_{2n+1}=\delta^0_{2n+1}$, $n\geq 1$.
(C) $\delta^1_{2n+1}=(n+\rho_3)/(n-1+\rho_3)\delta^0_{2n+1}$ *and* $\delta^1_{2n}=\delta^0_{2n}$, $n\geq 1$.
(D) $\delta^1_{2n}=(n+\rho_2)/(n-1+\rho_2)\delta^0_{2n}$ *and* $\delta^1_{2n+1}=(n+\rho_3)/(n-1+\rho_3)\delta^0_{2n+1}$, $n\geq 1$, *where* $\rho_2=-\delta^0_2/(\delta^0_2-\delta^1_2)$, $\rho_3=-\delta^0_3/(\delta^0_3-\delta^1_3)$, *and* ($\rho_2$ *and* $\rho_3\notin\mathbb{Z}$).

Remark 3.4. In the last case if we put $\rho=2\rho_2=2\rho_3-1$, then we obtain the important particular case denoted by $D1$ and where

$$\delta^1_{n+1}=\frac{n+1+\rho}{n-1+\rho}\delta^0_{n+1},\quad n\geq 1. \tag{3.10}$$

Proof of Lemma 3.3. $\delta^1_{n+1}=\delta^0_{n+1}$, $n\geq 1$, is a trivial solution of (3.8).

In case, where there exists $n_0\geq 1$ such that $\delta^1_{n_0+1}\neq\delta^0_{n_0+1}$, then for $n_0=2k_0$ (resp., $n_0=2k_0+1$), $k_0\in\mathbb{N}$, (3.8) becomes

$$\delta^0_{2k_0+3}\left(\delta^0_{2k_0+1}-\delta^1_{2k_0+1}\right)=\delta^1_{2k_0+1}\left(\delta^0_{2k_0+3}-\delta^1_{2k_0+3}\right)\neq 0 \tag{3.11}$$

(resp., $\delta^0_{2k_0+4}(\delta^0_{2k_0+2}-\delta^1_{2k_0+2})=\delta^1_{2k_0+2}(\delta^0_{2k_0+4}-\delta^1_{2k_0+4})\neq 0$).

Thus $\delta^1_{2(k_0+1)+1}\neq\delta^0_{2(k_0+1)+1}$, $k_0\geq 1$ (resp., $\delta^1_{2(k_0+1)+2}\neq\delta^0_{2(k_0+1)+2}$, $k_0\geq 0$), and therefore $\delta^1_{2n_0+1}\neq\delta^0_{2n_0+1}$, $n_0\geq 1$ (resp., $\delta^1_{2n_0+2}\neq\delta^0_{2n_0+2}$, $n_0\geq 0$). Equation (3.8) can be written as

$$\begin{gathered}\frac{\delta^0_{2n_0+3}}{\delta^0_{2n_0+3}-\delta^1_{2n_0+3}}-\frac{\delta^0_{2n_0+1}}{\delta^0_{2n_0+1}-\delta^1_{2n_0+1}}=-1\\ \left(\text{resp., }\frac{\delta^0_{2n_0+4}}{\delta^0_{2n_0+4}-\delta^1_{2n_0+4}}-\frac{\delta^0_{2n_0+2}}{\delta^0_{2n_0+2}-\delta^1_{2n_0+2}}=-1\right)\end{gathered} \tag{3.12}$$

then

$$\begin{gathered}\frac{\delta^0_{2n_0+3}}{\delta^0_{2n_0+3}-\delta^1_{2n_0+3}}-\frac{\delta^0_{3}}{\delta^0_{3}-\delta^1_{3}}=-n_0,\quad n_0\geq 0\quad\text{or}\quad \delta^1_{2n_0+1}=\frac{n_0+\rho_3}{n_0-1+\rho_3}\delta^0_{2n_0+1},\quad n_0\geq 1,\\ \left(\text{resp., }\delta^1_{2n_0}=\frac{n_0+\rho_2}{n_0-1+\rho_2}\delta^0_{2n_0},\ n_0\geq 1\right).\end{gathered} \tag{3.13}$$

□

Lemma 3.5. *In case* (A) *(i.e.,* $\delta^1_{n+1}=\delta^0_{n+1}$, $n\geq 1$*), (3.7) admits the following four solutions.*

(A1) $\gamma^0_{n+1}=\gamma^1_{n+1}$, $n\geq 1$.

(A2) $\gamma^0_{2n}=\gamma^1_{2n}$ *and* $\gamma^0_{2n+1}-\gamma^1_{2n+1}=(\gamma^0_3-\gamma^1_3)(\delta^0_1/\delta^0_2)\prod_{\nu=1}^{n}(\delta^0_{2\nu}/\delta^0_{2\nu-1})$, $n\geq 1$.

(A3) $\gamma^0_{2n+1}=\gamma^1_{2n+1}$ *and* $\gamma^0_{2n}-\gamma^1_{2n}=(\gamma^0_2-\gamma^1_2)(1/\delta^0_1)\prod_{\nu=1}^{n-1}(\delta^0_{2\nu+1}/\delta^0_{2\nu})$, $(\delta^0_0=1)$, $n\geq 1$.

(A4) $\gamma^0_{2n}-\gamma^1_{2n}=(\gamma^0_2-\gamma^1_2)(1/\delta^0_1)\prod_{\nu=1}^{n-1}(\delta^0_{2\nu+1}/\delta^0_{2\nu})$ *and* $\gamma^0_{2n+1}-\gamma^1_{2n+1}=(\gamma^0_3-\gamma^1_3)(\delta^0_1/\delta^0_2)\times\prod_{\nu=1}^{n}(\delta^0_{2\nu}/\delta^0_{2\nu-1})$, $n\geq 1$.

Lemma 3.6 [9]. *In case* (A1) *(i.e.,* $\gamma^0_{n+1}=\gamma^1_{n+1}$ *and* $\delta^1_{n+1}=\delta^0_{n+1}$, $n\geq 1$*),*

$$\begin{gathered}\beta^0_{2n}=\beta^0_0+n(b_1+3b_2),\quad n\geq 0,\\ \beta^0_{2n+1}=\beta^0_1+n(3b_1+b_2),\quad n\geq 0,\end{gathered} \tag{3.14}$$

$$\begin{gathered}\gamma^0_{2n+1}=(2n+1)\left[\gamma^0_1+n\left(b_1^2+b_2^2\right)\right],\quad n\geq 0,\\ \gamma^0_{2n+2}=2(n+1)\left[\gamma^0_1+(n+1)b_1^2+nb_2^2\right],\quad n\geq 0,\end{gathered} \tag{3.15}$$

$$\begin{gathered}\delta^0_{2n+1}=(n+1)(2n+1)\left[\delta^0_1+2nb_2^2(b_2-b_1)\right],\quad n\geq 0,\\ \delta^0_{2n+2}=(n+1)(2n+3)\left\{\delta^0_1+2(b_1-b_2)\left[\gamma^0_1+(n+1)b_1^2\right]\right\},\quad n\geq 0,\end{gathered} \tag{3.16}$$

Table 3.1

Case	$\beta_n^0,\ n \geq 0$	$\gamma_{n+1}^0,\ n \geq 0$
(A1.1)	$\beta_n^0 = 0$	$\gamma_{n+1}^0 = (n+1)\gamma_1^0$
(A1.2)	$\beta_n^0 = 2nb_1$	$\gamma_{n+1}^0 = (n+1)(\gamma_1^0 + nb_1^2)$
(A1.3)	$\beta_{2n+1}^0 = n$	$\gamma_{2n+1}^0 = (2n+1)\left(n + \dfrac{\delta_1^0 - k_1}{2}\right)$
	$\beta_{2n}^0 = 3n$	$\gamma_{2n+2}^0 = (n+1)(2n + \delta_1^0 - k_1)$
(A1.4)	$\beta_{2n+1}^0 = 3n+2$	$\gamma_{2n+1}^0 = (2n+1)\left(n + \dfrac{k_2 - 2 - \delta_1^0}{2}\right)$
	$\beta_{2n}^0 = n$	$\gamma_{2n+2}^0 = (n+1)(2n + k_2 - \delta_1^0)$
(A1.5)	See (3.14)	See (3.15)

where β_0^0, β_1^0, γ_1^0, and δ_1^0 are arbitrary and b_1 and b_2 are constants defined by

$$b_1 := \beta_1^0 - \beta_1^1 = \frac{1}{2}(\beta_1^0 - \beta_0^0), \qquad b_2 := \beta_2^0 - \beta_2^1 = \frac{1}{6}(2\beta_2^0 - \beta_1^0 - \beta_0^0). \tag{3.17}$$

Proof. From (3.6) we have $(\beta_n^0 - \beta_n^1) = (\beta_{n+2}^0 - \beta_{n+2}^1)$, $n \geq 0$.

In particular for $n = 2k$,

$$(\beta_{2k+2}^0 - \beta_{2k+2}^1) = (\beta_{2k}^0 - \beta_{2k}^1) = \cdots = \beta_2^0 - \beta_2^1 = \frac{1}{6}(2\beta_2^0 - \beta_1^0 - \beta_0^0) = b_2, \tag{3.18}$$

and for $n = 2k+1$,

$$(\beta_{2k+3}^0 - \beta_{2k+3}^1) = (\beta_{2k+1}^0 - \beta_{2k+1}^1) = \cdots = \beta_1^0 - \beta_1^1 = \frac{1}{2}(\beta_1^0 - \beta_0^0) = b_1. \tag{3.19}$$

Using (3.3), we get

$$\begin{gathered} 2[(k+1)b_1 - kb_2] = \beta_{2k+1}^0 - \beta_{2k}^0, \quad n \geq 0, \\ (2k+3)b_2 - (2k+1)b_1 = \beta_{2k+2}^0 - \beta_{2k+1}^0, \quad n \geq 0. \end{gathered} \tag{3.20}$$

By adding up term by term these last 2 relations and summing this last result, we obtain the first relation of (3.14). The second relation of (3.14) is obtained in the same way.

Equations (3.15) and (3.16) are obtained similarly by using, respectively, (3.4) and (3.5). □

Proposition 3.7 [9]. *The case (A1) is constituted by the following five canonical classical 2-orthogonal polynomials.*

(A1.1) $b_1 = b_2 = 0$.

(A1.2) $b_1 = b_2 \neq 0$.

Table 3.2

Case	δ^0_{n+1}, $n \geq 0$ ($\delta^0_2 = 2$)	Initials parameters
(A1.1)	$\delta^0_{n+1} = (n+1)(n+2)\dfrac{\delta^0_2}{2}$	$\beta^0_0 = 0$; γ^0_1 and δ^0_1 arbitrary
(A1.2)	$\delta^0_{n+1}(n+1)(n+2)\dfrac{\delta^0_2}{2}$	$\beta^0_0 = 0$; $b_1\delta^0_1$ and γ^0_1 arbitrary
(A1.3)	$\delta^0_{2n+1} = (n+1)(2n+1)(2n+\delta^0_1)$ $\delta^0_{2n+2} = k_1(n+1)(2n+3)$	$b_2 = 1, \beta^0_0 = 0$ δ^0_1 and $k_1 = \delta^0_1 - 2\gamma^0_1 \neq 0$ arbitrary
(A1.4)	$\delta^0_{2n+1} = (n+1)(2n+1)\delta^0_1$ $\delta^0_{2n+2} = (n+1)(2n+3)(2n+k_2)$	$b_1 = 1, \beta^0_0 = 0$ δ^0_1 and $k_2 = \delta^0_1 + 2\gamma^0_1 + 2 \neq 0$ arbitrary
(A1.5)	See (3.16)	β^0_0, b_1, γ^0_1, and δ^0_1 arbitrary

(A1.3) $b_1 = 0$ *and* $b_2 \neq 0$.
(A1.4) $b_2 = 0$ *and* $b_1 \neq 0$.
(A1.5) $b_1 \neq b_2$ *and* $b_1 b_2 \neq 0$.

Remark 3.8. In the precedent case (i.e., (A1)) the coefficients $\beta^0_n, \gamma^0_{n+1}$, and δ^0_{n+1} can be written, respectively, in the simplified forms in Tables 3.1 and 3.2.

Proposition 3.9 [9]. *There exist only four sequences of classical 2-symmetric 2-orthogonal polynomials. The coefficients δ^0_{n+1} and δ^1_{n+1} ($n \geq 0$) are explicit in Table 4.1.*

4. Main results

4.1. Differential equations. In this section, we will construct the differential equations, whose solutions are classical 2-orthogonal polynomials, afterwards, we will give the nature of these equations by the study of the coefficient associated with highest derivative.

Let us note that the polynomials enumerated in Proposition 3.7 and the 2-symmetric solution polynomials will be completely exhibited (perfectly identified).

An analysis of a particular case (already studied) is done at the end of this section, as well as the citation of some classical 2-orthogonal polynomials, which were the subject of a deep study.

First, let us note

$$d_{n+2} := \delta^1_{n+2} - \delta^0_{n+2}, \quad n \geq 0, \qquad B_n := (n+4)\beta^1_{n+3} - (n+3)\beta^0_{n+3}, \quad n \geq 0,$$
$$h_n = \delta^1_{n+2}\beta^0_{n+3} - \delta^0_{n+2}\beta^1_{n+3}, \quad n \geq 0, \qquad G_n := (n+4)\gamma^1_{n+3} - (n+3)\gamma^0_{n+3}, \quad n \geq 0,$$
$$C_n := \delta^1_{n+2}\gamma^0_{n+3} - \delta^0_{n+2}\gamma^1_{n+3}, \quad n \geq 0, \qquad D_n := \frac{[(n+4)\delta^1_{n+2} - (n+3)\delta^0_{n+2}]}{n+4}, \quad n \geq 0. \tag{4.1}$$

Then, we have the following result.

Table 4.1

Case	$\delta^0_{n+1},\ n \geq 0$
(A)	$\delta^0_{n+1} = (n+1)(n+2)\dfrac{\delta^0_1}{2}$
(B)	$\delta^0_{2n+1} = \dfrac{(n+1)(2n+1)}{3n+1+\rho_2}(\rho_2+1)\delta^0_1$ $\delta^0_{2n+2} = \dfrac{(n+1)(2n+3)(n+\rho_2)}{(3n+1+\rho_2)(3n+4+\rho_2)}(\rho_2+1)\delta^0_1$
(C)	$\delta^0_{2n+1} = \dfrac{(n+1)(2n+1)(n-1+\rho_3)}{(3n-1+\rho_3)(3n+2+\rho_3)}(\rho_3+2)\delta^0_1$ $\delta^0_{2n+2} = \dfrac{(n+1)(2n+3)}{3n+2+\rho_3}(\rho_3+1)\delta^0_1$
(D)	$\delta^0_{2n+1} = \dfrac{(n+1)(2n+1)(n-1+\rho_3)}{(3n-1+\rho_3)(3n+2+\rho_3)(3n+1+\rho_2)}(\rho_2+1)(\rho_3+2)\dfrac{\delta^0_1}{2}$ $\delta^0_{2n+2} = \dfrac{(n+1)(2n+3)(n+\rho_2)}{(3n+1+\rho_2)(3n+4+\rho_2)(3n+2+\rho_3)}(\rho_2+1)(\rho_3+2)\dfrac{\delta^0_1}{2}$
(D1)	$\delta^0_{n+1} = \dfrac{(n+1)(n+2)(n-1+\rho)}{(3n-1+\rho)(3n+2+\rho)(3n+5+\rho)}(\rho+2)(\rho+5)\dfrac{\delta^0_1}{2}$

Theorem 4.1. *When*

$$C_n d_{n+2} \neq 0, \quad n \geq 0, \tag{4.2}$$

the classical 2-orthogonal polynomials $P_{n+3}(x)$ $(n \geq 0)$ which satisfy a differential equation are solutions of a third-order linear differential equation with polynomial coefficients of the form

$$R_{4,n}(x)P^{(3)}_{n+3}(x) + R_{3,n}(x)P''_{n+3}(x) + R_{2,n}(x)P'_{n+3}(x) + R_{1,n}(x)P_{n+3}(x) = 0, \quad n \geq 0, \tag{4.3}$$

with

$$\begin{aligned}
R_{4,n}(x) &:= F_{1,n}(x)S_{3,n}(x),\\
R_{3,n}(x) &:= F_{1,n}(x)V_{2,n}(x) - F'_{1,n}(x)S_{3,n}(x),\\
R_{2,n}(x) &:= F_{1,n}(x)W_{1,n}(x) - F'_{1,n}(x)T_{2,n}(x),\\
R_{1,n}(x) &:= (n+3)\frac{\delta^1_{n+2}}{d_{n+2}}\left\{\left[\left(x - \frac{h_{n-1}}{d_{n+1}}\right) + \frac{C_{n-1}G_n}{(n+4)D_n d_{n+1}}\right]F'_{1,n}(x) - \left(\frac{\delta^1_{n+1}}{d_{n+1}} + 2\right)F_{1,n}(x)\right\},
\end{aligned} \tag{4.4}$$

and where

$$\begin{aligned}
\frac{F_{1,n}(x)}{(n+4)D_nd_{n+1}} :=&\ (n+3)C_n\left[\left(x-\frac{h_{n-1}}{d_{n+1}}\right)+\frac{C_{n-1}G_n}{(n+4)D_nd_{n+1}}\right]\\
&+\left(\frac{\delta^1_{n+1}}{d_{n+1}}+1\right)\left[C_n(x-B_{n+1})+(n+5)D_nD_{n+1}\right]\\
=&\left[\frac{\delta^1_{n+1}}{d_{n+1}}+(n+4)\right]C_nx+(n+3)\frac{C_n}{d_{n+1}}\left[\frac{C_{n-1}G_n}{(n+4)D_n}-h_{n-1}\right]\\
&+\left(\frac{\delta^1_{n+1}}{d_{n+1}}+1\right)\left[(n+5)D_nD_{n+1}-C_nB_{n+1}\right]:=F^{(1)}_{1,n}x+F^{(0)}_{1,n},
\end{aligned}\tag{4.5}$$

$$\begin{aligned}
S_{3,n}(x):=&\left[\left(x-\frac{h_{n-1}}{d_{n+1}}\right)+\frac{C_{n-1}G_n}{(n+4)D_nd_{n+1}}\right]\left[(x-B_{n+1})\left(x-\frac{h_n}{d_{n+2}}\right)-\frac{D_nG_{n+1}}{d_{n+2}}\right]\\
&-\frac{1}{(n+4)d_{n+1}d_{n+2}}\left[\frac{C_n}{D_n}(x-B_{n+1})+(n+5)D_{n+1}\right]\\
&\times\left[C_{n-1}(x-B_n)+(n+4)D_nD_{n-1}\right],
\end{aligned}\tag{4.6}$$

$$\begin{aligned}
V_{2,n}(x)=&\left[\left(x-\frac{h_{n-1}}{d_{n+1}}\right)+\frac{C_{n-1}G_n}{(n+4)D_nd_{n+1}}\right]\left[\left(\frac{\delta^1_{n+2}}{d_{n+2}}+2\right)(x-B_{n+1})-(n+2)\left(x-\frac{h_n}{d_{n+2}}\right)\right]\\
&+\left(\frac{\delta^1_{n+1}}{d_{n+1}}+2\right)\left[(x-B_{n+1})\left(x-\frac{h_n}{d_{n+2}}\right)-\frac{D_nG_{n+1}}{d_{n+2}}\right]\\
&+\frac{1}{(n+4)d_{n+1}d_{n+2}}\left\{(n+1)C_{n-1}\left[\frac{C_n}{D_n}(x-B_{n+1})+(n+5)D_{n+1}\right]\right.\\
&\left.+(n+2)C_n\left[\frac{C_{n-1}}{D_n}(x-B_n)+(n+4)D_{n-1}\right]\right\},
\end{aligned}\tag{4.7}$$

$$\begin{aligned}
T_{2,n}(x):=&\left[\left(x-\frac{h_{n-1}}{d_{n+1}}\right)+\frac{C_{n-1}G_n}{(n+4)D_nd_{n+1}}\right]\left[\left(\frac{\delta^1_{n+2}}{d_{n+2}}+1\right)(x-B_{n+1})\right.\\
&\left.-(n+3)\left(x-\frac{h_n}{d_{n+2}}\right)\right]\\
&+\frac{1}{(n+4)d_{n+1}d_{n+2}}(n+2)C_{n-1}\left[\frac{C_n}{D_n}(x-B_{n+1})+(n+5)D_{n+1}\right],
\end{aligned}\tag{4.8}$$

$$W_{1,n}(x) := \left(\frac{\delta^1_{n+1}}{d_{n+1}} + 2\right)\left[\left(\frac{\delta^1_{n+2}}{d_{n+2}} + 1\right)(x - B_{n+1}) - (n+3)\left(x - \frac{h_n}{d_{n+2}}\right)\right]$$
$$- (n+2)\left(\frac{\delta^1_{n+2}}{d_{n+2}} + 1\right)\left[\left(x - \frac{h_{n-1}}{d_{n+1}}\right) + \frac{C_{n-1}G_n}{(n+4)D_n d_{n+1}}\right] \tag{4.9}$$
$$- \frac{(n+2)^2 C_{n-1}C_n}{(n+4)D_n d_{n+1} d_{n+2}}.$$

Proof. Differentiating (3.1) with $n \to n+1$ and eliminating successively $P'_{n+1}(x)$ and $P'_{n+4}(x)$ by substitution in (3.2), we get, respectively,

$$D_n P'_{n+4}(x) - (d_{n+2}x - h_n)P'_{n+3}(x) - \delta^1_{n+2}P_{n+3}(x) + C_n P'_{n+2}(x) = 0, \tag{4.10}$$

$$(x - B_{n+1})P'_{n+4}(x) - (n+4)P_{n+4}(x) - G_{n+1}P'_{n+3}(x) - (n+5)D_{n+1}P'_{n+2}(x) = 0. \tag{4.11}$$

Eliminating successively $P'_{n+2}(x)$ and $P'_{n+4}(x)$ by substitution between (4.10) and (4.11) (because $C_n \neq 0$ by hypothesis and $D_n \neq 0$), we obtain

$$[(n+5)D_{n+1}(d_{n+2}x - h_n) + C_n G_{n+1}]P'_{n+3}(x) + (n+5)\delta^1_{n+2}D_{n+1}P_{n+3}(x)$$
$$- [C_n(x - B_{n+1}) + (n+5)D_n D_{n+1}]P'_{n+4}(x) + (n+4)C_n P_{n+4}(x) = 0, \tag{4.12}$$

$$-(n+4)D_n P_{n+4}(x) + [(x - B_{n+1})(d_{n+2}x - h_n) - D_n G_{n+1}]P'_{n+3}(x)$$
$$+ \delta^1_{n+2}(x - B_{n+1})P_{n+3}(x) - [C_n(x - B_{n+1}) + (n+5)D_n D_{n+1}]P'_{n+2}(x) = 0. \tag{4.13}$$

Differentiating (4.11), (4.12), and (4.13), and eliminating successively $P''_{n+4}(x)$ and $P'_{n+4}(x)$ we get, respectively,

$$(\delta^1_{n+2} + d_{n+2})(x - B_{n+1})P'_{n+3}(x) + [(x - B_{n+1})(d_{n+2}x - h_n) - D_n G_{n+1}]P''_{n+3}(x)$$
$$- [C_n(x - B_{n+1}) + (n+5)D_n D_{n+1}]P''_{n+2}(x) - (n+3)D_n P'_{n+4}(x) = 0, \tag{4.14}$$

$$[(x - B_{n+1})(d_{n+2}x - h_n) - D_n G_{n+1}]P''_{n+3}(x) - (n+3)\delta^1_{n+2}P_{n+3}(x)$$
$$+ [(\delta^1_{n+2} + d_{n+2})(x - B_{n+1}) - (n+3)(d_{n+2}x - h_n)]P'_{n+3}(x) \tag{4.15}$$
$$+ (n+3)C_n P'_{n+2}(x) - [C_n(x - B_{n+1}) + (n+5)D_n D_{n+1}]P''_{n+2}(x) = 0.$$

We replace n by $n-1$ and differentiate (4.12), that is,

$$\begin{aligned}&\left[(n+4)D_n\left(d_{n+1}x-h_{n-1}\right)+C_{n-1}G_n\right]P''_{n+2}(x)+(n+4)D_n\left(d_{n+1}+\delta^1_{n+1}\right)P'_{n+2}(x)\\&\quad-\left[C_{n-1}\left(x-B_n\right)+(n+4)D_nD_{n-1}\right]P''_{n+3}(x)+(n+2)C_{n-1}P'_{n+3}(x)=0.\end{aligned}\tag{4.16}$$

Taking into account $d_{n+1}\neq 0$, then eliminating successively $P'_{n+2}(x)$ and $P''_{n+2}(x)$ by substitution between (4.16) and (4.15), we get, respectively,

$$\begin{aligned}&-F_{1,n}(x)P''_{n+2}(x)+\left\{(n+4)D_n\left(\delta^1_{n+1}+d_{n+1}\right)\left[\left(x-B_{n+1}\right)\left(d_{n+2}x-h_n\right)-D_nG_{n+1}\right]\right.\\&\quad\left.+(n+3)C_n\left[C_{n-1}\left(x-B_n\right)+(n+4)D_{n-1}D_n\right]\right\}P''_{n+3}(x)\\&\quad+\left\{(n+4)D_n\left(\delta^1_{n+1}+d_{n+1}\right)\left[\left(\delta^1_{n+2}+d_{n+2}\right)\left(x-B_{n+1}\right)-(n+3)\left(d_{n+2}x-h_n\right)\right]\right.\\&\quad\left.-(n+2)(n+3)C_{n-1}C_n\right\}P'_{n+3}(x)-(n+3)(n+4)D_n\delta^1_{n+2}\left(\delta^1_{n+1}+d_{n+1}\right)P_{n+3}(x)=0,\end{aligned}\tag{4.17}$$

$$\begin{aligned}&S_{3,n}(x)P''_{n+3}(x)+T_{2,n}(x)P'_{n+3}(x)+\frac{F_{1,n}(x)}{(n+4)D_nd_{n+1}d_{n+2}}P'_{n+2}(x)\\&\quad-(n+3)\frac{\delta^1_{n+2}}{d_{n+2}}\left[\left(x-\frac{h_{n-1}}{d_{n+1}}\right)+\frac{C_{n-1}G_n}{(n+4)D_nd_{n+1}}\right]P_{n+3}(x)=0.\end{aligned}\tag{4.18}$$

Then differentiating (4.18) and eliminating $P''_{n+2}(x)$ by substitution in (4.17) we get

$$\begin{aligned}&S_{3,n}(x)P^{(3)}_{n+3}(x)+V_{2,n}(x)P''_{n+3}(x)+W_{1,n}(x)P'_{n+3}(x)+\frac{F'_{1,n}(x)}{(n+4)D_nd_{n+1}d_{n+2}}P'_{n+2}(x)\\&\quad-(n+3)\frac{\delta^1_{n+2}}{d_{n+2}}\left(\frac{\delta^1_{n+1}}{d_{n+1}}+2\right)P_{n+3}(x)=0.\end{aligned}\tag{4.19}$$

Finally, (4.3) is obtained by eliminating $P'_{n+2}(x)$ by substitution between (4.18) and (4.19). □

Before giving the main result of this work whose proof contains cumbersome calculations, we give the following lemmas.

Lemma 4.2. *The system (3.3)–(3.8) is equivalent to*

$$B_n-B_{n-1}=\beta^0_{n+2}-\beta^1_{n+2},\tag{4.20}$$

$$\frac{G_n}{n+3} = (B_n - B_{n-1})^2 + \frac{(n+1)\gamma^0_{n+2} - n\gamma^1_{n+2}}{n+2}, \tag{4.21}$$

$$\begin{aligned}\frac{(n+6)D_{n+2}}{(n+5)d_{n+3}} = \frac{D_{n-1}}{d_{n+1}} + \frac{1}{d_{n+3}}\{\gamma^0_{n+4}(\beta^0_{n+4} + \beta^1_{n+3} \\ - 2\beta^1_{n+3}) - \gamma^1_{n+4}(2\beta^0_{n+4} - \beta^1_{n+4} - 2\beta^1_{n+3})\},\end{aligned} \tag{4.22}$$

$$\begin{aligned}(\beta^0_{n+2} - \beta^1_n) = \frac{\delta^0_{n+1}}{d_{n+1}}(B_{n-2} - B_{n-3}) - \frac{\delta^1_{n+1}}{d_{n+1}}(B_n - B_{n-1}) \\ + \frac{1}{d_{n+1}}\{\gamma^0_{n+2}(\gamma^0_{n+1} - \gamma^1_{n+1}) - \gamma^1_{n+1}(\gamma^0_{n+2} - \gamma^1_{n+2})\},\end{aligned} \tag{4.23}$$

$$\frac{C_{n-1}}{d_{n+1}} = \frac{\delta^1_{n+2}(\gamma^0_{n+4} - \gamma^1_{n+4}) + \gamma^0_{n+4}d_{n+2}}{d_{n+3}}, \tag{4.24}$$

$$\frac{\delta^0_{n+3}}{d_{n+3}} = \frac{\delta^1_{n+1}}{d_{n+1}}. \tag{4.25}$$

Lemma 4.3. *Also, the following relations hold:*

$$\frac{C_n - d_{n+2}G_n}{(n+4)D_n} = \gamma^0_{n+3} - \gamma^1_{n+3}, \tag{4.26}$$

$$\frac{C_nB_n - h_nG_n}{(n+4)D_n} = \gamma^0_{n+3}\beta^1_{n+3} - \gamma^1_{n+3}\beta^0_{n+3}, \tag{4.27}$$

$$\frac{h_{n+1}}{d_{n+3}} - B_{n+1} = \frac{(n+5)D_{n+1}}{d_{n+3}}(B_{n+2} - B_{n+1}), \tag{4.28}$$

$$\begin{aligned}\frac{h_{n+1}}{d_{n+3}} - \frac{h_{n-1}}{d_{n+1}} &= \frac{\gamma^0_{n+4}(\gamma^0_{n+3} - \gamma^1_{n+3}) - \gamma^1_{n+3}(\gamma^0_{n+4} - \gamma^1_{n+4})}{d_{n+3}} - (B_{n+2} - B_{n+1}) \\ &= (\beta^0_{n+4} - \beta^1_{n+4}) + \frac{\delta^0_{n+3}}{d_{n+3}}\{(B_{n+2} - B_{n+1}) - (B_n - B_{n-1})\}.\end{aligned} \tag{4.29}$$

Theorem 4.4. *The polynomial $S_{3,n}(x)$ is of degree 3 and it is independent of n. Henceforth, it will be denoted by $S_3(x)$.*

Proof. We have

$$\begin{aligned}
S_{3,n}(x) = x^3 - \Bigg[&\frac{(\gamma^0_{n+3}-\gamma^1_{n+3})C_{n-1}}{d_{n+2}d_{n+1}} + \frac{h_n}{d_{n+2}} + \frac{h_{n-1}}{d_{n+1}} + B_{n+1}\Bigg]x^2 \\
&+ \Bigg[\frac{B_{n+1}h_n - D_nG_{n+1}}{d_{n+2}} + \frac{h_{n-1}}{d_{n+1}}\left(\frac{h_n}{d_{n+2}} + B_{n+1}\right) + \frac{(\gamma^0_{n+3}\beta^1_{n+3}-\gamma^1_{n+3}\beta^0_{n+3})C_{n-1}}{d_{n+2}d_{n+1}} \\
&+ \frac{(\gamma^0_{n+3}-\gamma^1_{n+3})B_{n+1}C_{n-1}}{d_{n+2}d_{n+1}} - \frac{C_nD_{n-1}}{d_{n+2}d_{n+1}} + \frac{(n+5)D_{n+1}C_{n-1}}{(n+4)d_{n+2}d_{n+1}}\Bigg]x \\
&+ \frac{1}{(n+4)D_nd_{n+2}d_{n+1}}\{[B_{n+1}h_n - D_nG_{n+1}][C_{n+1}G_n - (n+4)D_nh_{n-1}] \\
&\qquad - [(n+4)D_nD_{n-1} - C_{n-1}B_n][(n+5)D_nD_{n+1} - C_nB_{n+1}]\} \\
:= {} & x^3 + s^{(2)}_{3,n}x^2 + s^{(1)}_{3,n}x + s^{(0)}_{3,n}.
\end{aligned} \tag{4.30}$$

Let us prove that $S_{3,n+1}(x) - S_{3,n}(x) = 0$, that is,

$$s^{(2)}_{3,n+1} - s^{(2)}_{3,n} = 0, \qquad s^{(1)}_{3,n+1} - s^{(1)}_{3,n} = 0, \qquad s^{(0)}_{3,n+1} - s^{(0)}_{3,n} = 0. \tag{4.31}$$

Indeed

$$\begin{aligned}
s^{(2)}_{3,n} - s^{(2)}_{3,n+1} = {} & \frac{(\gamma^0_{n+4}-\gamma^1_{n+4})C_n}{d_{n+2}d_{n+3}} - \frac{(\gamma^0_{n+3}-\gamma^1_{n+3})C_{n-1}}{d_{n+2}d_{n+1}} \\
&+ \left(\frac{h_{n+1}}{d_{n+3}} - \frac{h_{n-1}}{d_{n+1}}\right) + (B_{n+2} - B_{n+1}).
\end{aligned} \tag{4.32}$$

Replacing C_{n-1}/d_{n+1} and $h_{n+1}/d_{n+3} - h_{n-1}/d_{n+1}$ by using (4.24) and (4.26), respectively, we obtain

$$\begin{aligned}
s^{(2)}_{3,n} - s^{(2)}_{3,n+1} = {} & \frac{1}{d_{n+3}}[\gamma^0_{n+4}(\gamma^0_{n+3}-\gamma^1_{n+3}) - \gamma^1_{n+3}(\gamma^0_{n+4}-\gamma^1_{n+4}) \\
&+ \delta^0_{n+3}(\beta^0_{n+2}-\beta^1_{n+2}) - \delta^1_{n+3}(\beta^0_{n+4}-\beta^1_{n+4})] \\
&- (n+5)(B_{n+2}-B_{n+1}) - (B_{n+1}-B_n) - (n+2)(B_n - B_{n-1}).
\end{aligned} \tag{4.33}$$

Now, using (4.23), we get

$$s^{(2)}_{3,n} - s^{(2)}_{3,n+1} = 0, \tag{4.34}$$

that is,

$$s_{3,n}^{(2)} = s_{3,n+1}^{(2)} \overset{\text{Denoted}}{:=} s_3^{(2)} = \text{constant}. \tag{4.35}$$

In the same way, we have

$$\begin{aligned} s_{3,n+1}^{(1)} - s_{3,n}^{(1)} = {} & \left[\frac{h_{n+1}}{d_{n+3}} + \frac{h_n}{d_{n+2}} + B_{n+2} + \frac{(\gamma_{n+4}^0 - \gamma_{n+4}^1) C_n}{d_{n+2} d_{n+3}}\right] B_{n+2} - (B_{n+2}^2 - B_{n+1}^2) \\ & - \left[\frac{h_n}{d_{n+2}} + \frac{h_{n-1}}{d_{n+1}} + B_{n+1} + \frac{(\gamma_{n+3}^0 - \gamma_{n+3}^1) C_{n-1}}{d_{n+2} d_{n+1}}\right] B_{n+1} \\ & - \left[\frac{D_{n+1} G_{n+2}}{d_{n+3}} + \frac{D_n (C_{n+1} - d_{n+3} G_{n+1})}{d_{n+2} d_{n+3}} - \frac{(n+5) D_{n+1} C_{n-1}}{(n+4) d_{n+2} d_{n+1}}\right] \\ & - \frac{C_n}{d_{n+2}} \left[\frac{(n+6) D_{n+2}}{(n+5) d_{n+3}} - \frac{D_{n-1}}{d_{n+1}} - \frac{\gamma_{n+4}^0 \beta_{n+4}^1 - \gamma_{n+4}^1 \beta_{n+4}^0}{d_{n+3}}\right] \\ & + \left[\frac{h_n}{d_{n+2}} \left(\frac{h_{n+1}}{d_{n+3}} - \frac{h_{n-1}}{d_{n+1}}\right) - \frac{(\gamma_{n+3}^0 \beta_{n+3}^1 - \gamma_{n+3}^1 \beta_{n+3}^0) C_{n-1}}{d_{n+2} d_{n+1}}\right]. \end{aligned} \tag{4.36}$$

Taking into account

$$s_3^{(2)} = \frac{h_{n+1}}{d_{n+3}} + \frac{h_n}{d_{n+2}} + B_{n+2} + \frac{(\gamma_{n+4}^0 - \gamma_{n+4}^1) C_n}{d_{n+2} d_{n+3}}, \tag{4.37}$$

and from (4.21), (4.26), and (4.24), we have

$$\frac{D_{n+1} G_{n+2}}{d_{n+3}} + \frac{D_n (C_{n+1} - d_{n+3} G_{n+1})}{d_{n+2} d_{n+3}} - \frac{(n+5) D_{n+1} C_{n-1}}{(n+4) d_{n+2} d_{n+3}} = \frac{(n+5) D_{n+1}}{d_{n+3}} (B_{n+2} - B_{n+1})^2. \tag{4.38}$$

From (4.22) and (4.25), we have

$$\begin{aligned} & \left[\frac{(n+6) D_{n+2}}{(n+5) d_{n+3}} - \frac{D_{n-1}}{d_{n+1}} - \frac{\gamma_{n+4}^0 \beta_{n+4}^1 - \gamma_{n+4}^1 \beta_{n+4}^0}{d_{n+3}}\right] \\ & \quad = \frac{1}{d_{n+3}} \{\gamma_{n+4}^0 (\beta_{n+3}^0 - \beta_{n+3}^1) + (\gamma_{n+4}^0 - \gamma_{n+4}^1)(\beta_{n+4}^0 - \beta_{n+4}^1) - \beta_{n+3}^1 (\gamma_{n+4}^0 - \gamma_{n+4}^1)\}, \end{aligned} \tag{4.39}$$

and from (4.29) and (4.24), we have

$$\frac{h_n}{d_{n+2}}\left(\frac{h_{n+1}}{d_{n+3}}-\frac{h_{n-1}}{d_{n+1}}\right)-\frac{(\gamma^0_{n+3}\beta^1_{n+3}-\gamma^1_{n+3}\beta^0_{n+3})}{d_{n+2}}\frac{C_{n-1}}{d_{n+1}}$$
$$=\frac{C_n}{d_{n+2}d_{n+3}}[\gamma^0_{n+4}(\beta^0_{n+3}-\beta^1_{n+3})-\beta^1_{n+3}(\gamma^0_{n+4}-\gamma^1_{n+4})]-\frac{h_n}{d_{n+2}}(B_{n+2}-B_{n+1}). \tag{4.40}$$

Then

$$\begin{aligned}s^{(1)}_{3,n+1}-s^{(1)}_{3,n}&=s^{(2)}_3(B_{n+2}-B_{n+1})-\frac{(n+5)D_{n+1}}{d_{n+3}}(B_{n+2}-B_{n+1})^2-(B^2_{n+2}-B^2_{n+1})\\&\quad-\frac{C_n}{d_{n+2}d_{n+3}}(\gamma^0_{n+4}-\gamma^1_{n+4})(B_{n+2}-B_{n+1})-\frac{h_n}{d_{n+2}}(B_{n+2}-B_{n+1})\\&=(B_{n+2}-B_{n+1})\left\{s^{(2)}_3-\frac{C_n}{d_{n+2}d_{n+3}}(\gamma^0_{n+4}-\gamma^1_{n+4})-\frac{h_n}{d_{n+2}}-B_{n+1}\right.\\&\qquad\left.-B_{n+2}-\frac{(n+5)D_{n+1}}{d_{n+3}}(B_{n+2}-B_{n+1})\right\}=0.\end{aligned} \tag{4.41}$$

That is,

$$s^{(1)}_{3,n+1}=s^{(1)}_{3,n}\overset{\text{Denoted}}{:=}s^{(1)}_3=\text{constant}. \tag{4.42}$$

In the same way, using (4.27), we can write

$$\begin{aligned}s^{(0)}_{3,n+1}-s^{(0)}_{3,n}&=\frac{h_n}{d_{n+2}}\left[\frac{D_{n+1}}{d_{n+3}}G_{n+2}-\frac{B_{n+2}}{d_{n+3}}h_{n+1}+\frac{B_{n+1}}{d_{n+1}}h_{n-1}\right]\\&\quad+\frac{C_nB_{n+1}}{d_{n+2}}\left[\frac{(n+6)D_{n+2}}{(n+5)d_{n+3}}-\frac{D_{n-1}}{d_{n+1}}\right]\\&\quad-\frac{D_{n+1}D_n}{d_{n+2}}\left[\frac{(n+6)D_{n+2}}{d_{n+3}}-\frac{(n+5)D_{n-1}}{d_{n+1}}\right]\\&\quad+\frac{B_{n+1}C_{n-1}}{d_{n+2}d_{n+1}}[\gamma^0_{n+3}\beta^1_{n+3}-\gamma^1_{n+3}\beta^0_{n+3}]-\frac{C_nG_{n+1}G_{n+2}}{(n+5)d_{n+2}d_{n+3}}\\&\quad-\frac{B_{n+2}C_n}{d_{n+2}d_{n+3}}[\gamma^0_{n+4}\beta^1_{n+4}-\gamma^1_{n+4}\beta^0_{n+4}]+\frac{D_nC_{n+1}B_{n+2}}{d_{n+2}d_{n+3}}\\&\quad+\frac{C_{n-1}G_nG_{n+1}}{(n+4)d_{n+1}d_{n+2}}-\frac{h_{n-1}D_nG_{n+1}}{d_{n+2}d_{n+1}}-\frac{(n+5)D_{n+1}C_{n-1}B_n}{(n+4)d_{n+1}d_{n+2}}.\end{aligned} \tag{4.43}$$

By using (4.22), (4.23), (4.24), (4.25), (4.26), (4.27), and (4.29), we get

$$
\begin{aligned}
s_{3,n+1}^{(0)} - s_{3,n}^{(0)} = {} & \frac{(B_{n+2}-B_{n+1})h_n}{d_{n+2}}\left[B_{n+2}-\frac{h_{n-1}}{d_{n+1}}+\frac{(n+5)D_{n+1}}{d_{n+3}}(B_{n+2}-B_{n+1})\right] \\
& - \frac{h_nB_{n+2}}{d_{n+2}d_{n+3}}[\gamma_{n+4}^0(\gamma_{n+3}^0-\gamma_{n+3}^1)-\gamma_{n+3}^1(\gamma_{n+4}^0-\gamma_{n+4}^1)] \\
& + \frac{(n+5)D_{n+1}h_n}{(n+4)d_{n+2}d_{n+3}}[(n+3)\gamma_{n+4}^1-(n+2)\gamma_{n+4}^0] \\
& + \frac{B_{n+1}C_n}{d_{n+2}d_{n+3}}[\gamma_{n+4}^0(\beta_{n+4}^0+\beta_{n+3}^0-2\beta_{n+3}^1)-\gamma_{n+4}^1(2\beta_{n+4}^0-\beta_{n+4}^1-\beta_{n+3}^1)] \\
& - \frac{(n+5)D_{n+1}D_n}{d_{n+2}d_{n+3}}[\gamma_{n+4}^0(\beta_{n+4}^0+\beta_{n+3}^0-2\beta_{n+3}^1)-\gamma_{n+4}^1(2\beta_{n+4}^0-\beta_{n+4}^1-\beta_{n+3}^1)] \\
& + \frac{B_{n+1}}{d_{n+2}d_{n+3}}[\gamma_{n+3}^0\beta_{n+3}^1-\gamma_{n+3}^1\beta_{n+3}^0][\delta_{n+2}^1(\gamma_{n+4}^0-\gamma_{n+4}^1)+d_{n+2}\gamma_{n+4}^0] \\
& - \frac{C_nG_{n+1}}{(n+4)d_{n+2}d_{n+3}}[(n+3)\gamma_{n+4}^1-(n+2)\gamma_{n+4}^0] - \frac{C_nG_{n+1}}{d_{n+2}d_{n+3}}(B_{n+2}-B_{n+1})^2 \\
& - \frac{B_{n+2}C_n}{d_{n+2}d_{n+3}}[\gamma_{n+4}^0\beta_{n+4}^1-\gamma_{n+4}^1\beta_{n+4}^0] + \frac{D_nC_{n+1}B_{n+2}}{d_{n+2}d_{n+3}} \\
& + \frac{G_{n+1}G_n}{(n+4)d_{n+2}}\frac{C_{n-1}}{d_{n+1}} - \frac{G_{n+1}D_n}{d_{n+2}}\frac{h_{n-1}}{d_{n+1}} - \frac{(n+5)D_{n+1}B_n}{(n+4)d_{n+2}}\frac{C_{n-1}}{d_{n+1}} \\
:= {} & Q_{n,1}+Q_{n,2}+Q_{n,3}+Q_{n,4}+Q_{n,5},
\end{aligned}
\tag{4.44}
$$

where

$$
\begin{aligned}
Q_{n,1} &:= \frac{(B_{n+2}-B_{n+1})h_n}{d_{n+2}}\left[B_{n+2}-\frac{h_{n-1}}{d_{n+1}}+\frac{(n+5)D_{n+1}}{d_{n+3}}(B_{n+2}-B_{n+1})\right] \\
&= \frac{(B_{n+2}-B_{n+1})h_n}{d_{n+2}d_{n+3}}[\gamma_{n+4}^0(\gamma_{n+3}^0-\gamma_{n+3}^1)-\gamma_{n+3}^1(\gamma_{n+4}^0-\gamma_{n+4}^1)], \\
Q_{n,2} &:= -\frac{B_{n+2}h_n}{d_{n+2}d_{n+3}}[\gamma_{n+4}^0(\gamma_{n+3}^0-\gamma_{n+3}^1)-\gamma_{n+3}^1(\gamma_{n+4}^0-\gamma_{n+4}^1)], \\
Q_{n,3} &:= \frac{B_{n+1}C_n}{d_{n+2}d_{n+3}}[\gamma_{n+4}^0(+\beta_{n+3}^0-2\beta_{n+3}^1)-\gamma_{n+4}^1(2\beta_{n+4}^0-\beta_{n+4}^1-\beta_{n+3}^1)] \\
&\quad - \frac{B_{n+2}C_n}{d_{n+2}d_{n+3}}[\gamma_{n+4}^0\beta_{n+4}^1-\gamma_{n+4}^1\beta_{n+4}^0] - \frac{C_nG_{n+1}}{d_{n+2}d_{n+3}}(\beta_{n+4}^0-\beta_{n+4}^1)^2 \\
&\quad + \frac{B_{n+1}}{d_{n+2}d_{n+3}}[\gamma_{n+3}^0\beta_{n+3}^1-\gamma_{n+3}^1\beta_{n+3}^0][\delta_{n+2}^1(\gamma_{n+4}^0-\gamma_{n+4}^1)+d_{n+2}\gamma_{n+4}^0] \\
&= \frac{B_{n+1}h_n}{d_{n+2}d_{n+3}}[\gamma_{n+4}^0(\gamma_{n+3}^0-\gamma_{n+3}^1)-\gamma_{n+3}^1(\gamma_{n+4}^0-\gamma_{n+4}^1)],
\end{aligned}
\tag{4.45}
$$

$$
\begin{aligned}
Q_{n,4} :=& \frac{(n+5)D_{n+1}h_n}{(n+4)d_{n+2}d_{n+3}}\left[(n+3)\gamma^1_{n+4}-(n+2)\gamma^0_{n+4}\right]-\frac{(n+5)D_{n+1}B_n}{(n+4)d_{n+2}}\frac{C_{n-1}}{d_{n+1}}\\
&-\frac{(n+5)D_{n+1}D_n}{d_{n+2}d_{n+3}}\left[\gamma^0_{n+4}\left(\beta^0_{n+4}+\beta^0_{n+3}-2\beta^1_{n+3}\right)-\gamma^1_{n+4}\left(2\beta^0_{n+4}-\beta^1_{n+4}-\beta^1_{n+3}\right)\right]\\
&-\frac{D_nG_{n+1}}{d_{n+2}}\frac{h_{n-1}}{d_{n+1}}+\frac{D_nB_{n+2}C_{n+1}}{d_{n+2}d_{n+3}}\\
=& \frac{D_nG_{n+1}}{d_{n+2}d_{n+3}}\left[\gamma^0_{n+4}\left(\gamma^0_{n+3}-\gamma^1_{n+3}\right)-\gamma^1_{n+3}\left(\gamma^0_{n+4}-\gamma^1_{n+4}\right)\right],
\end{aligned}
\tag{4.46}
$$

$$
\begin{aligned}
Q_{n,5} :=& \frac{G_{n+1}C_n}{(n+4)d_{n+2}}\frac{C_{n-1}}{d_{n+1}}-\frac{G_{n+1}G_n}{(n+4)d_{n+2}d_{n+3}}\left[(n+3)\gamma^1_{n+4}-(n+2)\gamma^0_{n+4}\right]\\
=& -\frac{D_nG_{n+1}}{d_{n+2}d_{n+3}}\left[\gamma^0_{n+4}\left(\gamma^0_{n+3}-\gamma^1_{n+3}\right)-\gamma^1_{n+3}\left(\gamma^0_{n+4}-\gamma^1_{n+4}\right)\right],
\end{aligned}
$$

then

$$
s^{(0)}_{3,n+1}-s^{(0)}_{3,n}=Q_{n,1}+Q_{n,2}+Q_{n,3}+Q_{n,4}+Q_{n,5}=0, \tag{4.47}
$$

that is,

$$
s^{(0)}_{3,n}\overset{\text{Denoted}}{:=}s^{(0)}_{3}. \tag{4.48}
$$

□

Now, we are going to study the case $d_{n+2}=0$.

Theorem 4.5. *When*

$$
\begin{aligned}
&d_{n+2}=0\quad\left(\text{i.e., } \delta^0_{n+2}=\delta^1_{n+2}\overset{\text{Denoted}}{:=}\delta_{n+2}\right),\qquad C_n\neq 0\quad\left(\text{i.e., } \gamma^0_{n+3}-\gamma^1_{n+3}\neq 0\right),\\
&E_{n+2}:=\frac{1}{\delta_{n+1}}\left[\frac{C_{n-1}G_n}{(n+4)D_n}-h_{n-1}\right]\neq 0\quad\left(\text{i.e., } \beta^0_{n+2}-\beta^1_{n+2}-\frac{\gamma^0_{n+3}-\gamma^1_{n+3}}{\delta_{n+2}}G_n\neq 0\right),\quad n\geq 0.
\end{aligned}
\tag{4.49}
$$

The polynomials $P_{n+3}(x)$ $(n\geq 0)$ satisfy a third-order linear differential equation with polynomial coefficients of the form

$$
\begin{aligned}
&\hat{F}_{1,n}(x)\hat{S}_{2,n}(x)P^{(3)}_{n+3}(x)+\left[\hat{F}_{1,n}(x)\hat{V}_{1,n}(x)-\hat{F}'_{1,n}(x)\hat{S}_{2,n}(x)\right]P''_{n+3}(x)\\
&\quad+\left[\hat{F}_{1,n}(x)\widehat{W}_{1,n}(x)-\hat{F}'_{1,n}(x)\hat{T}_{1,n}(x)\right]P'_{n+3}(x)\\
&\quad-(n+3)\left[\hat{F}_{1,n}(x)+E_{n+2}\hat{F}'_{1,n}(x)\right]P_{n+3}(x)=0,\quad n\geq 0,
\end{aligned}
\tag{4.50}
$$

where

$$\widehat{F}_{1,n}(x) := (\gamma^0_{n+3} - \gamma^1_{n+3})\left[(x - B_{n+1}) - (n+3)E_{n+2}\right] + \frac{\delta_{n+3}}{n+4}, \tag{4.51}$$

$$\begin{aligned}\widehat{S}_{2,n}(x) := {}& E_{n+2}\left[(\beta^0_{n+3} - \beta^1_{n+3})(x - B_{n+1}) + \frac{G_{n+1}}{n+4}\right] \\ & - \left[(\gamma^0_{n+3} - \gamma^1_{n+3})(x - B_{n+1}) + \frac{\delta_{n+3}}{n+4}\right]\left[\frac{\gamma^0_{n+2} - \gamma^1_{n+2}}{\delta_{n+2}}(x - B_n) + \frac{1}{n+3}\right],\end{aligned} \tag{4.52}$$

$$\begin{aligned}\widehat{V}_{1,n}(x) := {}& (n+1)\left[(\gamma^0_{n+3} - \gamma^1_{n+3})(x - B_{n+1}) + \frac{\delta_{n+3}}{n+4}\right] \\ & + (n+3)(\gamma^0_{n+3} - \gamma^1_{n+3})\left[\frac{\gamma^0_{n+2} - \gamma^1_{n+2}}{\delta_{n+2}}(x - B_n) + \frac{1}{n+3}\right] \\ & - E_{n+2}\left[(x - B_{n+1}) + (n+2)(\beta^0_{n+3} - \beta^1_{n+3})\right] - \frac{G_{n+1}}{n+4} \\ & - (\beta^0_{n+3} - \beta^1_{n+3})(x - B_{n+1}),\end{aligned} \tag{4.53}$$

$$\begin{aligned}\widehat{T}_{1,n}(x) := {}& (n+2)\frac{\gamma^0_{n+2} - \gamma^1_{n+2}}{\delta_{n+2}}\left[(\gamma^0_{n+3} - \gamma^1_{n+3})(x - B_{n+1}) + \frac{\delta_{n+3}}{n+4}\right] \\ & - E_{n+2}\left[(x - B_{n+1}) + (n+3)(\beta^0_{n+3} - \beta^1_{n+3})\right],\end{aligned} \tag{4.54}$$

$$\begin{aligned}\widehat{W}_{1,n}(x) := {}& (x - B_{n+1}) + (n+3)(\beta^0_{n+3} - \beta^1_{n+3}) + (n+2)E_{n+2} \\ & - (n+2)^2\frac{(\gamma^0_{n+2} - \gamma^1_{n+2})(\gamma^0_{n+3} - \gamma^1_{n+3})}{\delta_{n+2}}.\end{aligned} \tag{4.55}$$

Remark 4.6. (i) The proof of this theorem is analogous to Theorem 4.1.

(ii) The condition $E_{n+2} \neq 0$ is natural condition which appears in the construction of (4.50).

THEOREM 4.7. *The polynomial $\widehat{S}_{2,n}(x)$ is of degree 2 and it is independent of n. Henceforth, it will be denoted by $\widehat{S}_2(x)$.*

Proof. From (3.7), note that

$$\frac{(\gamma^0_{n+2} - \gamma^1_{n+2})(\gamma^0_{n+3} - \gamma^1_{n+3})}{\delta_{n+2}} \overset{\text{Denoted}}{:=} \gamma = \text{constant} \neq 0, \quad n \geq 0, \tag{4.56}$$

that is,

$$\frac{1}{(\gamma^0_{n+3} - \gamma^1_{n+3})} = \frac{(\gamma^0_{n+2} - \gamma^1_{n+2})}{\gamma\delta_{n+2}}, \quad n \geq 0, \tag{4.57}$$

then

$$
\begin{aligned}
&\widehat{S}_{2,n+1}(x)-\widehat{S}_{2,n}(x)\\
&\quad=(x-B_{n+1})\left\{\gamma\left[\frac{\delta_{n+4}}{(n+5)(\gamma^0_{n+4}-\gamma^1_{n+4})}-\frac{\delta_{n+2}}{(n+3)(\gamma^0_{n+2}-\gamma^1_{n+2})}-B_{n+2}+B_n\right]\right.\\
&\qquad\qquad+(B_{n+1}-B_{n+n})\left[(B_n-B_{n-1})-(B_{n+2}-B_{n+1})-\frac{(\gamma^0_{n+2}-\gamma^1_{n+2})G_n}{\delta_{n+2}}\right]\\
&\qquad\qquad\left.+(B_{n+2}-B_{n+1})\frac{(\gamma^0_{n+3}-\gamma^1_{n+3})G_{n+1}}{\delta_{n+3}}\right\}\\
&\qquad+\frac{G_{n+1}}{n+4}\left[(\beta^0_{n+2}-\beta^1_{n+2})-\frac{(\gamma^0_{n+2}-\gamma^1_{n+2})G_n}{\delta_{n+2}}\right]\\
&\qquad+\frac{K\delta_{n+3}}{(n+3)(\gamma^0_{n+3}-\gamma^1_{n+3})}\left[\frac{\delta_{n+4}}{(n+5)(\gamma^0_{n+4}-\gamma^1_{n+4})}-\frac{\delta_{n+2}}{(n+3)(\gamma^0_{n+2}-\gamma^1_{n+2})}+B_n-B_{n+2}\right]\\
&\qquad+\left[(B_{n+2}-B_{n+1})^2-\frac{G_{n+2}}{n+5}\right]\left[(\beta^0_{n+3}-\beta^1_{n+3})-\frac{(\gamma^0_{n+3}-\gamma^1_{n+3})G_{n+1}}{\delta_{n+3}}\right]\\
&\quad=\left(\widehat{s}^{(1)}_{2,n+1}-\widehat{s}^{(1)}_{2,n}\right)(x-B_{n+1})+\left(\widehat{s}^{(0)}_{2,n+1}-\widehat{s}^{(0)}_{2,n}\right).
\end{aligned}
\tag{4.58}
$$

Since (from (3.5))

$$
\begin{aligned}
\frac{\delta_{n+4}}{(n+5)(\gamma^0_{n+4}-\gamma^1_{n+4})}&=\frac{\delta_{n+3}}{(n+3)(\gamma^0_{n+4}-\gamma^1_{n+4})}+(B_{n+2}-B_{n+1})\\
&\quad+3(B_{n+1}-B_n)+\frac{(B_{n+2}-2B_{n+1}+B_n)G_{n+1}}{(\gamma^0_{n+4}-\gamma^1_{n+4})},
\end{aligned}
\tag{4.59}
$$

and from (3.6)

$$
(B_n-B_{n-1})-(B_{n+2}-B_{n+1})-\frac{(\gamma^0_{n+2}-\gamma^1_{n+2})G_n}{\delta_{n+2}}=-2\gamma-\frac{(\gamma^0_{n+3}-\gamma^1_{n+3})G_{n+1}}{\delta_{n+3}},\tag{4.60}
$$

then by using (3.7) (i.e., $\delta_{n+2}/(\gamma^0_{n+2}-\gamma^1_{n+2})=\delta_{n+3}/(\gamma^0_{n+4}-\gamma^1_{n+4})$) we get

$$
\widehat{s}^{(1)}_{2,n+1}-\widehat{s}^{(1)}_{2,n}=0\quad\left(\text{i.e., }\widehat{s}^{(1)}_{2,n}\overset{\text{Denoted}}{:=}\widehat{s}^{(1)}_2\right).\tag{4.61}
$$

In the same way, by using (4.59), (4.60), and the following relation:

$$
\frac{G_{n+2}}{n+5}-(B_{n+2}-B_{n+1})^2=\frac{G_{n+1}}{n+4}+2\frac{(\gamma^0_{n+4}-\gamma^1_{n+4})}{n+4},\tag{4.62}
$$

we obtain

$$
\widehat{s}^{(0)}_{2,n+1}-\widehat{s}^{(0)}_{2,n}=0,\tag{4.63}
$$

that is,

$$\widehat{s}_{2,n}^{(0)} \overset{\text{Denoted}}{:=} \widehat{s}_{2}^{(0)}. \tag{4.64}$$

□

Now, we are going to study the case $d_{n+2} = 0$ and $E_{n+2} = 0$. We start with the following lemma.

LEMMA 4.8. *When $d_{n+2} = 0$, $E_{n+2} = 0$ and $C_n \neq 0$, $n \geq 0$, that is,*

$$\begin{gathered}\delta_{n+2}^0 = \delta_{n+2}^1 \overset{\text{Denoted}}{:=} \delta_{n+2}, \qquad \gamma_{n+2}^0 - \gamma_{n+2}^1 \neq 0, \\ (\beta_{n+2}^0 - \beta_{n+2}^1) - \frac{(\gamma_{n+2}^0 - \gamma_{n+2}^1)}{\delta_{n+2}^0}\left[(n+4)\gamma_{n+3}^1 - (n+3)\gamma_{n+3}^0\right] = 0, \quad n \geq 0,\end{gathered} \tag{4.65}$$

then

$$\beta_{n+3}^0 = n\left[(3n+7)\gamma + 2b_3\right] + \beta_3^0, \quad n \geq 0, \tag{4.66}$$

$$\gamma_{n+3}^0 = (n+3)\left[(3n+2)\gamma + b_3\right]\left[n\left[(n-1)\gamma + b_3\right] + \frac{\gamma_3^0}{3(2\gamma + b_3)}\right], \quad n \geq 0, \tag{4.67}$$

$$\begin{aligned}\delta_{n+3} = (n+3)(n+4)\gamma\Bigg[n\left[(n-1)\gamma + b_3\right] \\ + \frac{\gamma_3^0}{3(2\gamma + b_3)}\Bigg]\left[(n+1)\left[n\gamma + b_3\right] + \frac{\gamma_3^0}{3(2\gamma + b_3)}\right], \quad n \geq 0,\end{aligned} \tag{4.68}$$

where

$$\gamma = \frac{(\gamma_2^0 - \gamma_2^1)(\gamma_3^0 - \gamma_3^1)}{\delta_2^0} \neq 0, \qquad b_3 = (\beta_3^0 - \beta_3^1). \tag{4.69}$$

Assume also that $[(3n+1)\gamma + b_3][(3n+2)\gamma + b_3] \neq 0$, $n \geq 0$, as well as that the initial conditions are given by

$$\begin{gathered}\beta_1^1 = \frac{1}{2}(\beta_1^0 + \beta_0^0), \qquad \gamma_2^1 = \frac{2}{3}(\gamma_2^0 + \gamma_1^2 + b_1^2), \\ \beta_1^2 = \frac{1}{6}(4\beta_2^1 + \beta_1^0 + \beta_0^0), \qquad \gamma_3^1 = \frac{1}{4}(3\gamma_3^0 + \gamma_2^0 + \gamma_1^2 + 3b_2^2 + b_1^2), \\ \beta_1^3 = \frac{1}{12}(9\beta_3^0 + 7\beta_2^1 + \beta_1^0 + \beta_0^0), \qquad \delta_2 = 3\left[\delta_1 + \gamma_2^0(\beta_1^0 - \beta_0^1) - \frac{2}{3}\gamma_2^1(2\beta_2^0 - \beta_1^0 - \beta_0^0)\right],\end{gathered} \tag{4.70}$$

where β_0^0, β_1^0, β_2^0, β_3^0, γ_2^0, γ_2^1, γ_3^0, and δ_1 are arbitrary.

Proof. From (3.7), we get

$$(\gamma_{n+1}^0 - \gamma_{n+1}^1)(\gamma_{n+2}^0 - \gamma_{n+2}^1) = \gamma\delta_{n+1} \neq 0, \quad n \geq 1. \tag{4.71}$$

Knowing that (4.65) can be written in the following form:

$$(n+3)(\gamma^0_{n+2}-\gamma^1_{n+2})(\gamma^1_{n+3}-\gamma^0_{n+3})+\gamma^1_{n+3}(\gamma^0_{n+2}-\gamma^1_{n+2})=\delta_{n+2}(\beta^0_{n+2}-\beta^1_{n+2}),\quad n\geq 0, \tag{4.72}$$

then, from (4.71) we obtain

$$\begin{aligned}\gamma^0_{n+3}(\gamma^0_{n+2}-\gamma^1_{n+2})&=\delta_{n+2}[(\beta^0_{n+2}-\beta^1_{n+2})+(n+4)\gamma],\quad n\geq 0,\\ \gamma^1_{n+3}(\gamma^0_{n+2}-\gamma^1_{n+2})&=\delta^0_{n+2}[(\beta^0_{n+2}-\beta^1_{n+2})+(n+3)\gamma],\quad n\geq 0,\end{aligned} \tag{4.73}$$

that is,

$$\gamma^0_{n+3}[(\beta^0_{n+2}-\beta^1_{n+2})+(n+3)\gamma]=\gamma^1_{n+3}[(\beta^0_{n+2}-\beta^1_{n+2})+(n+4)\gamma],\quad n\geq 0. \tag{4.74}$$

Then (3.6) is written as

$$\delta_{n+2}[(\beta^0_{n+1}-\beta^1_{n+1})-(\beta^0_{n+3}-\beta^1_{n+3})]=\gamma^1_{n+2}(\gamma^0_{n+3}-\gamma^1_{n+3})-\gamma^0_{n+3}(\gamma^0_{n+2}-\gamma^1_{n+2}),\quad n\geq 0. \tag{4.75}$$

Thus, from (3.6), (3.7), and (4.73) we have

$$\begin{aligned}&\delta_{n+2}[(\beta^0_{n+1}-\beta^1_{n+1})-(\beta^0_{n+3}-\beta^1_{n+3})]\\ &\quad=\gamma^1_{n+2}\frac{\delta_{n+2}}{\delta_{n+1}}(\gamma^0_{n+1}-\gamma^1_{n+1})-\gamma^0_{n+3}(\gamma^0_{n+2}-\gamma^1_{n+2})\\ &\quad=\delta_{n+2}\{[(\beta^0_{n+1}-\beta^1_{n+1})+(n+2)\gamma]-[(\beta^0_{n+2}-\beta^1_{n+2})+(n+4)\gamma]\},\quad n\geq 1,\end{aligned} \tag{4.76}$$

then

$$(\beta^0_{n+3}-\beta^1_{n+3})-(\beta^0_{n+2}-\beta^1_{n+2})=2\gamma, \tag{4.77}$$

that is,

$$\beta^0_{n+2}-\beta^1_{n+2}=2(n-1)\gamma+b_3,\quad n\geq 1. \tag{4.78}$$

Equation (3.3) is written as

$$\beta^0_{n+2}-\beta^0_{n+1}=2(3n-1)\gamma+2b_3,\quad n\geq 2, \tag{4.79}$$

then

$$\beta^0_{n+3}=n[(3n+7)\gamma+2b_3]+\beta^0_3,\quad n\geq 0. \tag{4.80}$$

In this case (4.74) is written as

$$[(3n+1)\gamma+b_3]\gamma^0_{n+3}=[(3n+2)\gamma+b_3]\gamma^1_{n+3},\quad n\geq 0. \tag{4.81}$$

Taking into account $[(3n+1)\gamma+b_3][(3n+2)\gamma+b_3]\neq 0$, (3.4) gives

$$\frac{\gamma_{n+3}^0}{(n+3)[(3n+2)\gamma+b_3]}-\frac{\gamma_{n+2}^0}{(n+2)[(3n-1)\gamma+b_3]}=2(n-1)\gamma+b_3, \tag{4.82}$$

then we get (4.67), and from (4.71) we obtain (4.68). □

Remark 4.9. According to the lemma above, it easy to see that the coefficients β_{n+3}^0, γ_{n+3}^0, and δ_{n+2} are, respectively, polynomials in n with degrees exactly 2, 4, and 6. So, we conclude that the case of $d_{n+2}=0$, $E_{n+2}=0$, and $C_n\neq 0$, $(n\geq 0)$ is constituted by one sequence of polynomials, which we can consider as the canonical sequence.

THEOREM 4.10. *When $d_{n+2}=0$, $E_{n+2}=0$, and $C_n\neq 0$, $n\geq 0$, the polynomials $P_{n+3}(x)$ $(n\geq 0)$ are solutions of the following third-order linear differential equation:*

$$\begin{aligned}&\gamma A^2(x)P_{n+3}^{(3)}(x)+(b_3-4\gamma)A(x)P_{n+3}''(x)\\&\quad-\left[x-\beta_3^0+6(b_3-\gamma)\right]P_{n+3}'(x)+(n+3)P_{n+3}(x)=0,\quad n\geq 0,\end{aligned}\tag{4.83}$$

where $A(x):=x-\beta_2^1+b_2+\gamma_3^0/(3(2\gamma+b_3))$, $(2\gamma+b_3\neq 0)$.

Proof. In this case (4.16) and (4.17) are, respectively, written as

$$\begin{aligned}&\left[(\gamma_{n+2}^0-\gamma_{n+2}^1)(x-B_n)+\frac{\delta_{n+2}}{n+3}\right]P_{n+3}''(x)\\&\quad-(n+2)(\gamma_{n+12}^0-\gamma_{n+2}^1)P_{n+3}'(x)-\delta_{n+2}P_{n+2}'(x)=0,\end{aligned}\tag{4.84}$$

$$\begin{aligned}&(n+3)P_{n+3}(x)+\Bigg\{\delta_{n+2}\left[(\beta_{n+3}^0-\beta_{n+3}^1)(x-B_{n+1})+\frac{G_{n+1}}{n+4}\right]\\&\qquad\qquad-(n+3)(\gamma_{n+3}^0-\gamma_{n+3}^1)\left[(\gamma_{n+2}^0-\gamma_{n+2}^1)(x-B_n)+\frac{\delta_{n+2}}{n+3}\right]\Bigg\}P_{n+3}''(x)\\&\quad+\delta_{n+2}\left[(\gamma_{n+3}^0-\gamma_{n+3}^1)(x-B_{n+1})+\frac{\delta_{n+3}}{n+4}\right]P_{n+2}''(x)\\&\quad-(n+3)\Bigg\{\delta_{n+2}\left[\frac{x-B_{n+1}}{n+3}+(\beta_{n+3}^0-\beta_{n+3}^1)\right]\\&\qquad\qquad-(n+2)(\gamma_{n+2}^0-\gamma_{n+2}^1)(\gamma_{n+3}^0-\gamma_{n+3}^1)\Bigg\}P_{n+3}'(x)=0.\end{aligned}\tag{4.85}$$

Differentiating (4.84) and eliminating $P''_{n+2}(x)$ by substitution between this relation and (4.85), we obtain

$$\begin{aligned}
&\left[(\gamma^0_{n+3}-\gamma^1_{n+3})(x-B_{n+1})+\frac{\delta_{n+3}}{n+4}\right]\left[(\gamma^0_{n+2}-\gamma^1_{n+2})(x-B_n)+\frac{\delta_{n+2}}{n+3}\right]P^{(3)}_{n+3}(x)\\
&\quad+\left\{\delta_{n+2}(\beta^0_{n+3}-\beta^1_{n+3})\left[x-B_{n+1}+\frac{\delta_{n+3}}{(n+4)(\gamma^0_{n+3}-\gamma^1_{n+3})}\right]\right.\\
&\qquad-(\gamma^0_{n+2}-\gamma^1_{n+2})(\gamma^0_{n+3}-\gamma^1_{n+3})\left\{(n+3)\left[(x-B_n)+\frac{\delta_{n+2}}{(n+3)(\gamma^0_{n+2}-\gamma^1_{n+2})}\right]\right.\\
&\qquad\qquad\left.\left.-(n+1)\left[x-B_{n+1}+\frac{\delta_{n+3}}{(n+4)(\gamma^0_{n+3}-\gamma^1_{n+3})}\right]\right\}\right\}P''_{n+3}(x)\\
&\quad-\{\delta_{n+2}[x-B_{n+1}+(n+3)(\beta^0_{n+3}-\beta^1_{n+3})]\\
&\qquad-(n+2)(n+3)(\gamma^0_{n+2}-\gamma^1_{n+2})(\gamma^0_{n+3}-\gamma^1_{n+3})\}P'_{n+3}(x)+(n+3)\delta_{n+2}P_{n+3}(x)=0.
\end{aligned}\tag{4.86}$$

Taking into account

$$(\gamma^0_{n+2}-\gamma^1_{n+2})(\gamma^0_{n+3}-\gamma^1_{n+3})=\gamma\delta_{n+2},\tag{4.87}$$

then this last equation is written as

$$\begin{aligned}
&\left[x-B_{n+1}+\frac{(\gamma^0_{n+4}-\gamma^1_{n+4})}{(n+4)\gamma}\right]\left[x-B_n+\frac{(\gamma^0_{n+3}-\gamma^1_{n+3})}{(n+3)\gamma}\right]P^{(3)}_{n+3}(x)\\
&\quad+\left\{\frac{(\beta^0_{n+3}-\beta^1_{n+3})}{\gamma}\left[x-B_{n+1}+\frac{(\gamma^0_{n+4}-\gamma^1_{n+4})}{(n+4)\gamma}\right]-(n+3)\left[x-B_n+\frac{(\gamma^0_{n+3}-\gamma^1_{n+3})}{(n+3)\gamma}\right]\right.\\
&\qquad\left.-(n+1)\left[x-B_{n+1}+\frac{(\gamma^0_{n+4}-\gamma^1_{n+4})}{(n+4)\gamma}\right]\right\}P''_{n+3}(x)\\
&\quad-\left\{\frac{1}{K}[x-B_{n+1}+(n+3)(\beta^0_{n+3}-\beta^1_{n+3})]-(n+2)(n+3)\right\}P'_{n+3}(x)\\
&\quad+\frac{(n+3)}{\gamma}\delta_{n+2}P_{n+3}(x)=0,
\end{aligned}\tag{4.88}$$

taking into account

$$\begin{aligned}\frac{(\gamma_{n+4}^{0}-\gamma_{n+4}^{1})}{(n+4)\gamma}-B_{n+1}&=4b_3-\beta_3^0+\frac{\gamma_3^0}{3(2\gamma+b_3)},\\ \frac{(\beta_{n+3}^{0}-\beta_{n+3}^{1})}{\gamma}-2(n+2)&=\frac{b_3}{\gamma}-4,\\ \frac{(n+3)(\beta_{n+3}^{0}-\beta_{n+3}^{1})-B_{n+1}}{\gamma}-(n+2)(n+3)&=\frac{6(b_3-K)-\beta_3^0}{\gamma},\end{aligned}\tag{4.89}$$

then we get (4.83). □

Remark 4.11. (a) When $2\gamma+b_3=0$, that is, $\gamma_3^0=0$ and $b_4=\beta_4^0-\beta_4^1=0$, the coefficients of the recurrence relation are, respectively, given by

$$\begin{aligned}\beta_{n+3}^{0}&=3n(n+1)\gamma+\beta_3^0,\quad n\geq 0,\\ \gamma_{n+4}^{0}&=3(n+1)(n+4)\left[n(n-1)\gamma^2+\frac{\gamma_4^0}{12}\right],\quad n\geq 0,\\ \delta_{n+4}&=\frac{(n+4)(n+5)}{\gamma}\left[n(n-1)\gamma^2+\frac{\gamma_4^0}{12}\right]\left[n(n+1)\gamma^2+\frac{\gamma_4^0}{12}\right],\quad n\geq 0,\end{aligned}\tag{4.90}$$

and then the differential equation (4.83) becomes

$$\begin{aligned}&\gamma A^2(x)P_{n+3}^{(3)}(x)-6\gamma A(x)P_{n+3}''(x)\\ &\quad-\left[x-\beta_3^0-18\gamma\right]P_{n+3}'(x)+(n+3)P_{n+3}(x)=0,\quad n\geq 0,\end{aligned}\tag{4.91}$$

where $A(x):=x-\beta_3^0-6\gamma+\gamma_4^0/12$.

(b) Equation (4.83) admits one singularity of order 2 at finite distance.

Now, we are going to study the case $C_n=0$.

Proposition 4.12. *When $C_n=0$, then necessarily $\gamma_{n+2}^0=\gamma_{n+2}^1$, $n\geq 0$. Moreover, $\gamma_{n+2}^0=\gamma_{n+2}^1=0$, $n\geq 0$, when $d_{n+2}\neq 0$.*

Proof. Indeed, $C_n=0\Leftrightarrow\gamma_{n+3}^0=(\delta_{n+2}^0/\delta_{n+2}^1)\gamma_{n+3}^1$.

In case (A) (i.e., $d_{n+2}=0$), we have $\gamma_{n+3}^0=\gamma_{n+3}^1$, $n\geq 0$.

In case (B), we have $\delta_{n+2}^0/\delta_{n+2}^1=(n+\rho+2)/(n+\rho)$, then $\gamma_{n+3}^0-\gamma_{n+3}^1=-2\gamma_{n+3}^0/(n+\rho)$. But, from (3.7) we have

$$\left[\frac{n+\rho+1}{(n+\rho)(n+\rho-1)}-\frac{1}{n+\rho-1}\right]\gamma_{n+3}^0\delta_{n+1}^0=0,\quad\text{then }\gamma_{n+3}^0=0.\tag{4.92}$$

In case (C), we have $\delta_{2n+1}^1/\delta_{2n+1}^0=\gamma_{2n+2}^1/\gamma_{2n+2}^0=1$ and $\delta_{2n+2}^1/\delta_{2n+2}^0=(n+\rho_2+1)/(n+\rho_2)$, then $\gamma_{2n+3}^1-\gamma_{2n+3}^0=-\gamma_{2n+3}^0/(n+\rho_2)$. But, from (3.7), we have, for n even ($n=2k$),

$$\frac{\delta_{2k+2}^0\gamma_{2k+3}^0}{k+\rho_2}=0,\quad\text{then }\gamma_{2k+3}^0=0,\tag{4.93}$$

and for n odd $(n = 2k+1)$,

$$\gamma^0_{2k+4}\left(\delta^0_{2n+2} - \delta^1_{2n+2}\right) = 0, \quad \text{then } \gamma^0_{2k+4} = 0. \tag{4.94}$$

That is, $\gamma^0_{n+2} = 0$, $n \geq 0$.

Similarly, in case (D), we show that $\gamma^0_{n+2} = 0$, $n \geq 0$.

In case (E), we have $\delta^0_{2n+2} - \delta^1_{2n+2} = -\delta^0_{2n+2}/(n+\rho_2)$ and $\delta^0_{2n+1} - \delta^1_{2n+1} = -\delta^0_{2n+1}/(n+\rho_3-1)$.

Then $\gamma^0_{2n+3} - \gamma^1_{2n+3} = -\gamma^0_{2n+3}/(n+\rho_2)$ and $\gamma^0_{2n+4} - \gamma^1_{2n+4} = -\gamma^0_{2n+4}/(n+\rho_3)$. But, from (3.7), for n even $(n = 2k)$ we have

$$\frac{\rho_3 - \rho_2}{(k+\rho_2)(k+\rho_3-1)}\delta^0_{2k+1}\gamma^0_{2k+3} = 0 \quad \left(\text{i.e., } (\rho_3-\rho_2)\gamma^0_{2k+3} = 0\right), \tag{4.95}$$

and for n odd $(n = 2k+1)$ we have

$$\frac{\rho_2 - \rho_3 + 1}{(k+\rho_2)(k+\rho_3)}\delta^0_{2k+2}\gamma^0_{2k+4} = 0 \quad \left(\text{i.e., } (\rho_2-\rho_3+1)\gamma^0_{2k+4} = 0\right). \tag{4.96}$$

Therefore we get $\gamma^0_{2k+3} = \gamma^0_{2k+4} = 0$, or $\rho_2 = \rho_3$ and $\gamma^0_{2k+4} = 0$, or $\rho_2 + 1 = \rho_3$ and $\gamma^0_{2k+3} = 0$.

In the first case the proposition is true.

In the case where $\rho_2 = \rho_3$ and $\gamma^0_{2k+4} = 0$, (3.4) gives for n even $(n = 2k+2)$,

$$-2(k+1)\gamma^1_{2k+3} + (2k+1)\gamma^0_{2k+3} = (2k+3)\left(\beta^1_{2k+3} - \beta^0_{2k+3}\right)^2 \tag{4.97}$$

knowing that $\gamma^1_{2k+3} = (k+\rho_2+1)/(k+\rho_2)\gamma^0_{2k+3}$, then this last (4.52) can be written as

$$-(3k+\rho_2+2)\gamma^0_{2k+3} = (2k+3)\left(\beta^1_{2k+3} - \beta^0_{2k+3}\right)^2, \tag{4.98}$$

and for n odd $(n = 2k+1)$, (3.4) gives

$$(3k+\rho_2+4)\gamma^0_{2k+3} = (2k+3)\left(\beta^1_{2k+2} - \beta^0_{2k+2}\right)^2, \tag{4.99}$$

thus

$$(3k+\rho_2+4)\left(\beta^1_{2k+2} - \beta^0_{2k+2}\right)^2 + (3k+\rho_2+2)\left(\beta^1_{2k+3} - \beta^0_{2k+3}\right)^2 = 0, \quad k \geq 0, \tag{4.100}$$

and this last relation is satisfied:

$$\left(\beta^1_{2k+2} - \beta^0_{2k+2}\right) = \left(\beta^1_{2k+3} - \beta^0_{2k+3}\right) = 0, \quad k \geq 0, \tag{4.101}$$

thus $\gamma^0_{2k+3} = 0$, and consequently $\gamma^0_{n+2} = 0$, $n \geq 0$.

In the same way, we show that if $\rho_2 + 1 = \rho_3$ and $\gamma_{2k+3}^0 = 0$, then $\gamma_{2k+4}^0 = 0$. Thus $\gamma_{n+2}^0 = 0$, $n \geq 0$. □

PROPOSITION 4.13. *The 2-classical polynomials sequences such that $\gamma_{n+3}^0 = \gamma_{n+3}^1 = 0$, for $n \geq 0$, are 2-symmetric if $\beta_0^0 = 0$ (i.e., $\beta_n^0 = \beta_n^1 = 0$, $n \geq 1$).*

Proof. Indeed, (3.4) gives us $\beta_{n+1}^0 = \beta_{n+1}^1$ and consequently (3.3) gives $\beta_{n+1}^0 = \beta_n^0$, $n \geq 0$. That is, $\beta_{n+1}^0 = \beta_0^0$, $n \geq 0$. □

Remark 4.14. (a) When $C_n = 0$ and $d_{n+2} \neq 0$, the sequences of polynomials are 2-symmetric (if $\beta_0^0 = 0$).

(b) The case $C_n = 0$ is constituted by five canonical sequences described in Proposition 3.7 ($d_{n+2} = 0$) and the four 2-symmetric sequences, respectively, denoted (A1.1) (with $\gamma_{n+3}^0 = \gamma_{n+3}^1 = 0$), (B), (C), and (D).

PROPOSITION 4.15. *Each of the five canonical sequences of polynomials described in Proposition 3.7 (i.e., when $C_n = 0$ and $d_{n+2} = 0$), satisfies a third-order linear differential equation with polynomial coefficients of degree less than or equal to 1, where the coefficients of $P_{n+3}^{(3)}(x)$ and $P_{n+3}''(x)$ are independent of n:*

$$\begin{aligned}&\left(b_1 b_2 x + \gamma_1^0 b_2 - \frac{1}{2}\delta_1\right) P_{n+3}^{(3)}(x)\\ &\quad - \left[(b_1 + b_2)x - b_1 b_2 + \gamma_1^0\right] P_{n+3}''(x) \left\{x + \left[\frac{n+3}{2}\right] b_1 + \left[\frac{n+4}{2}\right] b_2\right\} P_{n+3}'(x)\\ &\quad - (n+3) P_{n+3}(x) = 0,\end{aligned} \tag{4.102}$$

where $[n/2]$ is the integer part of $n/2$.

Proof. When $C_n = 0$, the coefficients of the equation $R_{4,n}(x)$, $R_{3,n}(x)$, $R_{2,n}(x)$, and $R_{1,n}(x)$ simplify, respectively, to

$$\begin{aligned}R_{4,n}(x) &:= \left(\beta_{n+2}^0 - \beta_{n+2}^1\right)\left[\left(\beta_{n+2}^0 - \beta_{n+2}^1\right)\left(x - B_{n+1}\right) + \frac{\gamma_{n+4}^0}{n+4}\right] - \frac{\delta_{n+3}}{(n+3)(n+4)},\\ R_{3,n}(x) &:= -\left[\left(\beta_{n+3}^0 - \beta_{n+3}^1\right)\left(x - B_{n+1}\right) + \frac{\gamma_{n+4}^0}{n+4}\right]\\ &\qquad - \left(\beta_{n+2}^0 - \beta_{n+2}^1\right)\left[\left(x - B_{n+1}\right) + (n+2)\left(\beta_{n+3}^0 - \beta_{n+3}^1\right)\right],\\ R_{2,n}(x) &:= \left(x - B_{n+1}\right) + (n+3)\left[\left(\beta_{n+3}^0 - \beta_{n+3}^1\right) + \left(\beta_{n+2}^0 - \beta_{n+2}^1\right)\right],\\ R_{1,n}(x) &:= -(n+3), \quad \text{where } B_n = (n+4)\beta_{n+3}^1 - (n+3)\beta_{n+3}^0.\end{aligned} \tag{4.103}$$

Table 4.2

Case	Equations
(A1.1)	$P^{(3)}_{n+3}(x)+\gamma^0_1 P''_{n+3}(x)-xP'_{n+3}(x)+(n+3)P_{n+3}(x)=0$
(A1.2)	$(b_1x+b_1^2\gamma_1^0-1)P^{(3)}_{n+3}(x)-(2b_1x-b_1^2+\gamma_1^0)P''_{n+3}(x)+[x-(n+3)b_1]P'_{n+3}(x)$ $-(n+3)P_{n+3}(x)=0$
(A1.3)	$\frac{k_1}{2}P^{(3)}_{n+3}(x)+(x+\gamma_1^0)P''_{n+3}(x)-\left(x+\left[\frac{n+4}{2}\right]\right)P'_{n+3}(x)+(n+3)P_{n+3}(x)=0$
(A1.4)	$P^{(3)}_{n+3}(x)+(x+\gamma_1^0)P''_{n+3}(x)-\left(x+\left[\frac{n+3}{2}\right]\right)P'_{n+3}(x)+(n+3)P_{n+3}(x)=0$

Taking into account

$$b_1=(\beta^0_{2n+1}-\beta^1_{2n+1}),\qquad b_2=(\beta^0_{2n+2}-\beta^1_{2n+2}),\qquad \gamma^0_{n+2}=\gamma^1_{n+2}, \tag{4.104}$$

then using (3.14), (3.15), and (3.16) we obtain (4.102). □

Remark 4.16. Equation (4.102) is written, respectively, as shown in Table 4.2.

4.2. 2-symmetric solutions

Proposition 4.17. *Each of the four sequences of* 2*-symmetric polynomials (i.e., when* $C_n = 0$ *and* $d_{n+2}\neq 0$*) satisfies a third-order linear differential equation with polynomial coefficients. In each case, we give this equation* ($\delta^0_1=2$)*.*

(i) *In case* (A.1) ($\gamma^0_{n+3}=\gamma^1_{n+3}=0$)*, the equation is written as*

$$P^{(3)}_{n+3}(x)-xP'_{n+3}(x)+(n+3)P_{n+3}(x)=0,\quad n\geq 0. \tag{4.105}$$

(ii) *In case* (B)*, the equation is written as*

$$\begin{aligned}(\rho_2+1)P^{(3)}_{n+3}(x)-x^2P''_{n+3}(x)+\left\{\rho_2-(-1)^{n+1}-\left[\frac{n+1}{2}\right]\right\}xP'_{n+3}(x)\\ +(n+3)\left\{\rho_2+(-1)^n+\left[\frac{n+4}{2}\right]\right\}P_{n+3}(x)=0,\quad n\geq 0.\end{aligned} \tag{4.106}$$

(iii) *In case* (C)*, the equation is written as*

$$\begin{aligned}(\rho_3+2)P^{(3)}_{n+3}(x)-x^2P''_{n+3}(x)-\left\{\rho_3+(-1)^n-\left[\frac{n+2}{2}\right]\right\}P'_{n+3}(x)\\ +(n+3)\left\{\rho_3+(-1)^n+\left[\frac{n+3}{2}\right]\right\}P_{n+3}(x)=0,\quad n\geq 0.\end{aligned} \tag{4.107}$$

(iv) *In case* (D), *the equation is written as*

$$\begin{aligned}
&[x^3-(\rho_2+1)(\rho_3+2)]P^{(3)}_{n+3}(x)+(\rho_2+\rho_3+3)x^2P''_{n+3}(x)\\
&\quad-\Big\{\Big(\Big[\frac{n+5}{2}\Big]+\frac{1+(-1)^{n+1}}{2}\rho_2+\frac{1+(-1)^n}{2}\rho_3\Big)\\
&\qquad\times\Big(\Big[\frac{n+3}{2}\Big]-\frac{1+(-1)^{n}}{2}\rho_2-\frac{1+(-1)^{n+1}}{2}\rho_3\Big)(n+2)\\
&\qquad\times\Big(\Big[\frac{n+4}{2}\Big]+\frac{1+(-1)^{n}}{2}\rho_2+\frac{1+(-1)^{n+1}}{2}\rho_3\Big)\Big\}xP'_{n+3}(x)\\
&\quad-(n+3)\Big(\Big[\frac{n+2}{2}\Big]+\rho_3\Big)\Big(\Big[\frac{n+3}{2}\Big]+\rho_2\Big)P_{n+3}(x)=0,\quad n\geq 0.
\end{aligned}\tag{4.108}$$

Remark 4.18. In particular case (D.1), the equation is written as

$$\begin{aligned}
&[4x^3-(\rho+2)(\rho+5)]P^{(3)}_{n+3}(x)-2(2\rho+7)x^2P''_{n+3}(x)-[2n^2-6n-8+\rho^2+3\rho]xP'_{n+3}(x)\\
&\quad-(n+3)(n+2+\rho)(n+5+\rho)P_{n+3}(x)=0,\quad n\geq 0.
\end{aligned}\tag{4.109}$$

Corollary 4.19. *From the above propositions (i.e., $C_n=0$), the coefficient of $P^{(3)}_{n+3}(x)$ is independent of n (a fortiori $S_{3,n}(x)$ is independent of n).*

4.3. Particular cases

Remark 4.20. The particular case $\beta^0_n=\beta=$ constant, $n\geq 0$ (i.e., $h_n=\beta(\delta^1_{n+2}-\delta^0_{n+2})$, $B_n=$ constant) is not a natural condition, and has been studied in detail in [13]. We conclude the analysis concerning this case by saying that the latter not only contains the *four* 2-symmetric sequences (if $\beta=0$) but also the new no 2-symmetric sequence that follows from (D.1), where the coefficients γ^0_{n+2} and δ^0_{n+2}, $n\geq 0$ are given by [7]

$$\begin{aligned}
\gamma^0_{n+2}&=\frac{(n+2)(n+1+2\alpha)}{(n+1+\alpha)(n+2+\alpha)}\gamma,\quad n\geq 0,\\
\delta^0_{n+2}&=\frac{(n+1)(n+2)(n+3\alpha)}{(n+\alpha)(n+1+\alpha)(n+2+\alpha)}\delta^0_2,\quad n\geq 0,
\end{aligned}\tag{4.110}$$

when

$$\alpha=\frac{\rho-1}{3}=\frac{\lambda-1}{2},\qquad \rho=2\rho_2=2\rho_3-1,\qquad \lambda=\frac{\gamma^0_2}{\gamma^1_2-\gamma^0_2},\tag{4.111}$$

and where we put

$$\gamma = \frac{(\alpha+1)(\alpha+2)}{2(1+2\alpha)}\gamma_2^0,$$
$$\delta = \frac{\alpha(\alpha+1)(\alpha+2)}{6(3\alpha-1)}\delta_2^0. \tag{4.112}$$

Proposition 4.21 [7]. *When* β_n^0 = constant $=\beta$ *and* γ_{n+2}^0 *and* δ_{n+2}^0, $(n\geq 0)$ *are given by relation (4.110), the coefficients* $F_{1,n}(x)$, $S_3(x)$, $R_{1,n}(x)$, $V_{2,n}(x)$, $T_{2,n}(x)$, *and* $W_{2,n}(x)$ *of (4.3) are*

$$F_{1,n}(x) := 3(3n+3\alpha+10)\delta\gamma x+27(n+4+3\alpha)\delta^2+2(n+3)\gamma^3,$$
$$S_3(x) := 3\big[-4(x-\beta)^3\delta-(x-\beta)^2\gamma^2+18(x-\beta)\delta\gamma+27\delta^2+4\gamma^3\big],$$
$$R_{1,n}(x) := (n+3)(n+3+3\alpha)\big[3(3n+6+3\alpha)\delta F_{1,n}(x)-(6\delta x+2\gamma^2)F'_{1,n}(x)\big],$$
$$V_{2,n}(x) := \frac{2\alpha+3}{2}S'_3(x),$$
$$T_{2,n}(x) := 6(n+1-3\alpha)\delta x^2-(n-6\alpha)\gamma^2x+9(n+2)\delta\gamma,$$
$$W_{2,n}(x) := 3\big\{\big[(n+1-3\alpha)(n+8+3\alpha)+2(n+3)(n+3+3\alpha)\big]\delta x+(n+2)(n+4+2\alpha)\gamma^2\big\}, \tag{4.113}$$

and the degree of $R_{4,n}(x)$ *is exactly* 4.

In conclusion, we have just shown that there are *four* types of linear third-order differential equations

$$R_{4,n}(x)P^{(3)}_{n+3}(x)+R_{3,n}(x)P''_{n+3}(x)+R_{2,n}(x)P'_{n+3}(x)+R_{1,n}(x)P_{n+3}(x)=0, \quad n\geq 0, \tag{4.114}$$

having as solutions classical 2-orthogonal polynomials, namely,

(i) equation (4.3), when $C_nd_{n+2}\neq 0$, together with $R_{4,n}(x)=F_{1,n}(x)S_3(x)$,
(ii) equation (4.50), when $d_{n+2}=0$ and $C_nE_{n+2}\neq 0$, together with $R_{4,n}(x)=\widehat{F}_{1,n}(x)\times\widehat{S}_2(x)$,
(iii) equation (4.83), when $d_{n+2}=0$, $E_{n+2}=0$, and $C_n\neq 0$, together with $R_{4,n}(x)=A^2(x)$ $(\deg A=1)$,
(iv) equation (4.102), when $d_{n+2}=0$ and $C_n=0$, together with $R_{4,n}(x)=B(x)$ $(\deg B\leq 1)$,
(v) equations (4.105), (4.106), and (4.107), together with $R_{4,n}(x)=$ constant, and (4.108), together with $R_{4,n}(x)=\widetilde{S_3(x)}$, $(\deg\widetilde{S}_3=3)$.

Furthermore, the coefficients of (4.83) and (4.102) and the coefficients of the *four*-term recurrence relations associated with the solutions of these equations are derived. Note that the 2-symmetric cases have been completely exhibited.

5. Examples

Several deep works were devoted to classical d-orthogonal polynomials and to type II multiple orthogonal polynomials and many properties concerning theses polynomials have been established.

For the type II multiple orthogonal polynomials, there is a rich bibliography [2–4, 20].

Here, we quote some classical 2-orthogonal polynomial sequences which were a subject of a deep study and whose generating functions and integral representations of the linear forms $\pounds_0$ and $\pounds_1$ have been established [6–11].

Indeed, the sequence (A1.1) is the Hermite 2-orthogonal sequence [7]; the sequence (A1.2) is the Laguerre 2-orthogonal sequence [8]; the sequence (D.1) is the Gegenbauer 2-orthogonal sequence [6]; the sequence (D.1) (where $\alpha = 1$) is the first kind Tchebychev 2-orthogonal sequence [11]; and the sequence (D.1) (where $\rho = 4$) is the second kind Tchebychev 2-orthogonal sequence [10]

6. Conclusion

First, we enumerated *ten* classical 2-orthogonal sequences and derived the coefficients of their recurrences (*nine* sequences for $C_n = 0$ and *one* for $d_{n+2} = 0$, $C_n \neq 0$, and $E_{n+2} = 0$). It remains to do the same thing for ($d_{n+2} = 0$ and $C_n \neq 0$) and ($C_n d_{n+2} \neq 0$), which constitutes the generalization Bochner's result. This enumeration is probably realized by using not only the system (3.3)–(3.8), but also by using the fact that the coefficient $S_{3,n}(x)$ of $P_{n+3}^{(3)}(x)$ is independent of n. This topic will be studied in the near future.

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