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Research Article

Limit Properties of Solutions of Singular Second-Order Differential Equations

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We discuss the properties of the differential equation $u''(t) = (a/t)u'(t) + f(t, u(t), u'(t))$, a.e. on $(0, T]$, where $a \in \mathbb{R}\backslash\{0\}$, and f satisfies the L_p-Carathéodory conditions on $[0, T] \times \mathbb{R}^2$ for some $p > 1$. A full description of the asymptotic behavior for $t \to 0+$ of functions u satisfying the equation a.e. on $(0, T]$ is given. We also describe the structure of boundary conditions which are necessary and sufficient for u to be at least in $C^1[0, T]$. As an application of the theory, new existence and/or uniqueness results for solutions of periodic boundary value problems are shown.

1. Motivation

In this paper, we study the analytical properties of the differential equation

$$u''(t) = \frac{a}{t}u'(t) + f\big(t, u(t), u'(t)\big), \quad \text{a.e. on } (0, T], \tag{1.1}$$

where $a \in \mathbb{R} \setminus \{0\}$, $u : [0, T] \to \mathbb{R}$, and the function f is defined for a.e. $t \in [0, T]$ and for all $(x, y) \in \mathfrak{D} \subset \mathbb{R} \times \mathbb{R}$. The above equation is singular at $t = 0$ because of the first term in the right-hand side, which is in general unbounded for $t \to 0$. In this paper, we will also alow the function f to be unbounded or bounded but discontinuous for certain values of the time variable $t \in [0, T]$. This form of f is motivated by a variety of initial and boundary value problems known from applications and having nonlinear, discontinuous forcing terms, such as electronic devices which are often driven by square waves or more complicated discontinuous inputs. Typically, such problems are modelled by differential equations where f has jump discontinuities at a discrete set of points in $(0,T)$, compare [1].

This study serves as a first step toward analysis of more involved nonlinearities, where typically, f has singular points also in u and u'. Many applications, compare [2–12], showing these structural difficulties are our main motivation to develop a framework on existence and uniqueness of solutions, their smoothness properties, and the structure of boundary conditions necessary for u to have at least continuous first derivative on $[0,T]$. Moreover, using new techniques presented in this paper, we would like to extend results from [13, 14] (based on ideas presented in [15]) where problems of the above form but with *appropriately smooth data function* f have been discussed.

Here, we aim at the generalization of the existence and uniqueness assertions derived in those papers for the case of smooth f. We are especially interested in studying the limit properties of u for $t \to 0$ and the structure of boundary conditions which are necessary and sufficient for u to be at least in $C^1[0,T]$.

To clarify the aims of this paper and to show that it is necessary to develop a new technique to treat the nonstandard equation given above, let us consider a model problem which we designed using the structure of the boundary value problem describing a membrane arising in the theory of shallow membrane caps and studied in [10]; see also [6, 9],

$$\left(t^3u'(t)\right)' + t^3\left(\frac{1}{8u^2(t)} - \frac{a_0}{u(t)} + b_0t^{2\gamma-4}\right) = 0, \quad 0 < t < 1, \tag{1.2}$$

subject to boundary conditions

$$\lim_{t\to 0+} t^3u'(t) = 0, \qquad u(1) = 0, \tag{1.3}$$

where $a_0 \ge 0$, $b_0 < 0$, $\gamma > 1$. Note that (1.2) can be written in the form

$$u''(t) = -\frac{3}{t}u'(t) - \left(\frac{1}{8u^2(t)} - \frac{a_0}{u(t)} + b_0t^{2\gamma-4}\right) = 0, \quad 0 < t < 1, \tag{1.4}$$

which is of form (1.1) with

$$T = 1, \qquad a = -3, \quad f(t,u,u') = -\left(\frac{1}{8u^2} - \frac{a_0}{u} + b_0t^{2\gamma-4}\right). \tag{1.5}$$

Function f is not defined for $u = 0$ and for $t = 0$ if $\gamma \in (1,2)$. We now briefly discuss a simplified linear model of (1.4),

$$u''(t) = -\frac{3}{t}u'(t) - b_0t^{\beta}, \quad 0 < t < 1, \tag{1.6}$$

where $\beta = 2\gamma - 4$ and $\gamma > 1$. Clearly, this means that $\beta > -2$.

The question which we now pose is the role of the boundary conditions (1.3), more precisely, are these boundary conditions *necessary and sufficient* for the solution u of (1.6) to be unique and at least continuously differentiable, $u \in C^1[0,1]$? To answer this question, we can use techniques developed in the classical framework dealing with boundary value problems, exhibiting a singularity of the first and second kind; see [15, 16], respectively. However, in these papers, the analytical properties of the solution u are derived for nonhomogeneous terms being at least continuous. Clearly, we need to rewrite problem (1.6) first and obtain its new form stated as,

$$\left(t^3 u'(t)\right)' + t^3\left(b_0 t^\beta\right) = 0, \quad 0 < t < 1, \tag{1.7}$$

which suggest to introduce a new variable, $v(t) := t^3 u'(t)$. In a general situation, especially for the nonlinear case, it is not straightforward to provide such a transformation, however. We now introduce $z(t) := (u(t), v(t))^T$ and immediately obtain the following system of ordinary differential equations:

$$z'(t) = \frac{1}{t^3}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z(t) - \begin{pmatrix} 0 \\ b_0 t^{\beta+3} \end{pmatrix}, \quad 0 < t < 1, \tag{1.8}$$

where $\beta + 3 > 1$, or equivalently,

$$z'(t) = \frac{1}{t^3} M z(t) + g(t), \quad M := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; g(t) := -\begin{pmatrix} 0 \\ b_0 t^{\beta+3} \end{pmatrix}, \tag{1.9}$$

where $g \in C[0,1]$. According to [16], the latter system of equations has a continuous solution if and only if the regularity condition $Mz(0) = 0$ holds. This results in

$$v(0) = 0 \Longleftrightarrow \lim_{t \to 0+} t^3 u'(t) = 0, \tag{1.10}$$

compare conditions (1.3). Note that the Euler transformation, $\zeta(t) := (u(t), tu'(t))^T$ which is usually used to transform (1.6) to the first-order form would have resulted in the following system:

$$\zeta'(t) = \frac{1}{t} N \zeta(t) + w(t), \quad N := \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}, \; w(t) := -\begin{pmatrix} 0 \\ b_0 t^{\beta+1} \end{pmatrix}. \tag{1.11}$$

Here, w may become unbounded for $t \to 0$, the condition $N\zeta(0) = 0$, or equivalently $\lim_{t \to 0+} tu'(t) = 0$ is not the correct condition for the solution u to be continuous on $[0,1]$.

From the above remarks, we draw the conclusion that a new approach is necessary to study the analytical properties of (1.1).

2. Introduction

The following notation will be used throughout the paper. Let $J \subset \mathbb{R}$ be an interval. Then, we denote by $L_1(J)$ the set of functions which are (Lebesgue) integrable on J. The corresponding norm is $\|u\|_1 := \int_J |u(t)|\mathrm{d}t$. Let $p > 1$. By $L_p(J)$, we denote the set of functions whose pth powers of modulus are integrable on J with the corresponding norm given by $\|u\|_p := (\int_J |u(t)|^p \mathrm{d}t)^{1/p}$.

Moreover, let us by $C(J)$ and $C^1(J)$ denote the sets of functions being continuous on J and having continuous first derivatives on J, respectively. The norm on $C[0,T]$ is defined as $\|u\|_\infty := \max_{t\in[0,T]}\{|u(t)|\}$.

Finally, we denote by $\mathrm{AC}(J)$ and $\mathrm{AC}^1(J)$ the sets of functions which are absolutely continuous on J and which have absolutely continuous first derivatives on J, respectively. Analogously, $\mathrm{AC}_{\mathrm{loc}}(J)$ and $\mathrm{AC}^1_{\mathrm{loc}}(J)$ are the sets of functions being absolutely continuous on each compact subinterval $I \subset J$ and having absolutely continuous first derivatives on each compact subinterval $I \subset J$, respectively.

As already said in the previous section, we investigate differential equations of the form

$$u''(t) = \frac{a}{t}u'(t) + f\left(t, u(t), u'(t)\right), \quad \text{a.e. on } (0,T], \tag{2.1}$$

where $a \in \mathbb{R} \setminus \{0\}$. For the subsequent analysis we assume that

$$f \text{ satisfies the } L_p\text{-Carathéodory conditions on } [0,T]\times\mathbb{R}\times\mathbb{R}, \quad \text{for some } p > 1 \tag{2.2}$$

specified in the following definition.

Definition 2.1. Let $p > 1$. A function f satisfies *the L_p-Carathéodory conditions* on the set $[0,T] \times \mathbb{R} \times \mathbb{R}$ if

(i) $f(\cdot, x, y) : [0,T] \to \mathbb{R}$ is measurable for all $(x,y) \in \mathbb{R}\times\mathbb{R}$,

(ii) $f(t,\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous for a.e. $t \in [0,T]$,

(iii) for each compact set $\mathcal{K} \subset \mathbb{R}\times\mathbb{R}$ there exists a function $m_{\mathcal{K}}(t) \in L_p[0,T]$ such that $|f(t,x,y)| \le m_{\mathcal{K}}(t)$ for a.e. $t \in [0,T]$ and all $(x,y) \in \mathcal{K}$.

We will provide a full description of the asymptotical behavior for $t \to 0+$ of functions u satisfying (2.1) a.e. on $(0,T]$. Such functions u will be called solutions of (2.1) if they additionally satisfy the smoothness requirement $u \in \mathrm{AC}^1[0,T]$; see next definition.

Definition 2.2. A function $u : [0,T] \to \mathbb{R}$ is called *a solution of* (2.1) if $u \in \mathrm{AC}^1[0,T]$ and satisfies

$$u''(t) = \frac{a}{t}u'(t) + f\left(t, u(t), u'(t)\right) \quad \text{a.e. on } (0,T]. \tag{2.3}$$

In Section 3, we consider linear problems and characterize the structure of boundary conditions necessary for the solution to be at least continuous on $[0,1]$. These results are modified for nonlinear problems in Section 4. In Section 5, by applying the theory developed in Section 4, we provide new existence and/or uniqueness results for solutions of singular boundary value problems (2.1) with periodic boundary conditions.

3. Linear Singular Equation

First, we consider the linear equation, $a \in \mathbb{R} \setminus \{0\}$,

$$u''(t) = \frac{a}{t}u'(t) + h(t), \quad \text{a.e. on } (0,T], \tag{3.1}$$

where $h \in L_p[0,T]$ and $p > 1$.

As a first step in the analysis of (3.1), we derive the necessary auxiliary estimates used in the discussion of the solution behavior. For $c \in [0,T]$, let us denote by

$$\varphi_a(c,t) := t^a \int_t^c \frac{h(s)}{s^a} \mathrm{d}s, \quad t \in (0,T]. \tag{3.2}$$

Assume that $a < 0$. Then

$$0 < \left(\int_0^t \frac{\mathrm{d}s}{s^{aq}} \right)^{1/q} = \left(\frac{t^{1-aq}}{1-aq} \right)^{1/q}, \quad t \in (0,T]. \tag{3.3}$$

Now, let $a > 0$, $c > 0$. Without loss of generality, we may assume that $1/p \neq 1-a$. For $1/p = 1-a$, we choose $p^* \in (1,p)$, and we have $h \in L_{p^*}[0,T]$ and $1/p^* > 1-a$.

First, let $a \in (0, 1-1/p)$. Then $1/q = 1-1/p > a$, $1-aq > 0$, and

$$0 < \left| \int_t^c \frac{\mathrm{d}s}{s^{aq}} \right|^{1/q} = \left| \frac{c^{1-aq} - t^{1-aq}}{1-aq} \right|^{1/q} < \begin{cases} \left(\dfrac{c^{1-aq}}{1-aq} \right)^{1/q}, & \text{if } c \geq t > 0, \\ \left(\dfrac{t^{1-aq}}{1-aq} \right)^{1/q}, & \text{if } c < t \leq T. \end{cases} \tag{3.4}$$

Now, let $a > 1-1/p$. Then $1/q = 1-1/p < a$, $1-aq < 0$, and

$$0 \leq \left| \int_t^c \frac{\mathrm{d}s}{s^{aq}} \right|^{1/q} = \left| \frac{c^{1-aq} - t^{1-aq}}{1-aq} \right|^{1/q} < \begin{cases} \left(\dfrac{c^{1-aq}}{aq-1} \right)^{1/q}, & \text{if } c < t \leq T, \\ \left(\dfrac{t^{1-aq}}{aq-1} \right)^{1/q}, & \text{if } c \geq t > 0. \end{cases} \tag{3.5}$$

Hence, for $a > 0$, $c > 0$,

$$0 \leq \left| \int_t^c \frac{\mathrm{d}s}{s^{aq}} \right|^{1/q} < |1-aq|^{-1/q} \left(c^{1/q-a} + t^{1/q-a} \right), \quad t \in (0,T]. \tag{3.6}$$

Consequently, (3.3), (3.6), and the Hölder inequality yield, $t \in (0,T]$,

$$\begin{aligned} \left|\varphi_a(c,t)\right| &\le t^a\left(c^{1/q-a}+t^{1/q-a}\right)\left|1-aq\right|^{-1/q}\|h\|_p, \quad \text{if } a>0,\ c>0, \\ \left|\varphi_a(0,t)\right| &\le t^a t^{1/q-a}(1-aq)^{-1/q}\|h\|_p, \quad \text{if } a<0. \end{aligned} \tag{3.7}$$

Therefore

$$\varphi_a(c,t) \in C(0,T], \quad \lim_{t\to 0+}\varphi_a(c,t)=0, \quad \text{if } a>0,\ c>0, \tag{3.8}$$

$$\varphi_a(0,t) \in C(0,T], \quad \lim_{t\to 0+}\varphi_a(0,t)=0, \quad \text{if } a<0, \tag{3.9}$$

which means that $\varphi_a \in C[0,1]$. We now use the properties of φ_a to represent all functions $u \in \mathrm{AC}^1_{\mathrm{loc}}(0,T]$ satisfying (3.1) a.e. on $[0,T]$. Remember that such function u does not need to be a solution of (3.1) in the sense of Definition 2.2.

Lemma 3.1. *Let $a \in \mathbb{R}\setminus\{0\}$, $c \in (0,T]$, and let $\varphi_a(c,t)$ be given by* (3.2).

(i) *If $a \neq -1$, then*

$$\left\{c_1 + c_2 t^{a+1} + \int_t^c \varphi_a(c,s)ds,\ c_1,c_2 \in \mathbb{R},\ t \in (0,T]\right\} \tag{3.10}$$

is the set of all functions $u \in \mathrm{AC}^1_{\mathrm{loc}}(0,T]$ satisfying (3.1) *a.e. on* $(0,T]$.

(ii) *If $a=-1$, then*

$$\left\{c_1 + c_2 \ln t + \int_t^c \varphi_{-1}(c,s)ds,\ c_1,c_2 \in \mathbb{R},\ t \in (0,T]\right\} \tag{3.11}$$

is the set of all functions $u \in \mathrm{AC}^1_{\mathrm{loc}}(0,T]$ satisfying (3.1) *a.e. on* $(0,T]$.

Proof. Let $a \neq -1$. Note that (3.1) is linear and regular on $(0,T]$. Since the functions $u_h^1(t)=1$ and $u_h^2(t)=t^{a+1}$ are linearly independent solutions of the homogeneous equation $u''(t)-(a/t)u'(t)=0$ on $(0,T]$, the general solution of the homogeneous problem is

$$u_h(t) = c_1 + c_2 t^{a+1}, \quad c_1,c_2 \in \mathbb{R}. \tag{3.12}$$

Moreover, the function $u_p(t)=\int_t^c \varphi_a(c,s)\mathrm{d}s$ is a particular solution of (3.1) on $(0,T]$. Therefore, the first statement follows. Analogous argument yields the second assertion. □

We stress that by (3.8), the particular solution $u_p=\int_t^c \varphi_a(c,s)\mathrm{d}s$ of (3.1) belongs to $C^1[0,T]$. For $a<0$, we can see from (3.9) that it is useful to find other solution representations which are equivalent to (3.10) and (3.11), but use $\varphi_a(0,t)$ instead of $\varphi_a(c,t)$, if $c>0$.

Lemma 3.2. *Let $a<0$ and let $\varphi_a(0,t)$ be given by* (3.2).

(i) *If* $a \neq -1$, *then*

$$\left\{c_1 + c_2 t^{a+1} - \int_0^t \varphi_a(0,s)ds,\ c_1, c_2 \in \mathbb{R},\ t \in (0,T]\right\} \tag{3.13}$$

is the set of all functions $u \in \mathrm{AC}^1_{\mathrm{loc}}(0,T]$ *satisfying* (3.1) *a.e. on* $(0,T]$.

(ii) *If* $a = -1$, *then*

$$\left\{c_1 + c_2 \ln t - \int_0^t \varphi_{-1}(0,s)ds,\ c_1, c_2 \in \mathbb{R},\ t \in (0,T]\right\} \tag{3.14}$$

is the set of all functions $u \in \mathrm{AC}^1_{\mathrm{loc}}(0,T]$ *satisfying* (3.1) *a.e. on* $(0,T]$.

Proof. Let us fix $c \in (0,T]$ and define

$$p(t) := \int_t^c \varphi_a(c,s)\mathrm{d}s + \int_0^t \varphi_a(0,s)\mathrm{d}s, \quad t \in (0,T]. \tag{3.15}$$

In order to prove (i) we have to show that $p(t) = d_1 + d_2 t^{a+1}$ for $t \in (0,T]$, where $d_1, d_2 \in \mathbb{R}$. This follows immediately from (3.9), since

$$\begin{aligned} p(c) &= \int_0^c \varphi_a(0,s)\mathrm{d}s, \\ p'(t) &= -\varphi_a(c,t) + \varphi_a(0,t) \\ &= -t^a \int_0^c \frac{h(s)}{s^a}\mathrm{d}s, \quad t \in (0,T], \end{aligned} \tag{3.16}$$

and hence we can define d_i as follows:

$$d_2 := -\frac{1}{a+1}\int_0^c \frac{h(s)}{s^a}\mathrm{d}s, \qquad d_1 := p(c) - d_2 c^{a+1}. \tag{3.17}$$

For $a = -1$ we have

$$d_2 := -\int_0^c s h(s)\mathrm{d}s, \qquad d_1 := \int_0^c \varphi_{-1}(0,s)\mathrm{d}s - d_2 \ln c, \tag{3.18}$$

which completes the proof. □

Again, by (3.9), the particular solution,

$$u_p(t) = -\int_0^t \varphi_a(0,s)\mathrm{d}s, \tag{3.19}$$

of (3.1) for $a < 0$ satisfies $u_p \in C^1[0,1]$. Main results for the linear singular equation (3.1) are now formulated in the following theorems.

Theorem 3.3. *Let $a > 0$ and let $u \in AC^1_{loc}(0,T]$ satisfy equation (3.1) a.e. on $[0,T]$. Then*

$$\lim_{t\to 0+} u(t) \in \mathbb{R}, \qquad \lim_{t\to 0+} u'(t) = 0. \tag{3.20}$$

Moreover, u can be extended to the whole interval $[0,T]$ in such a way that $u \in AC^1[0,T]$.

Proof. Let a function u be given. Then, by (3.10), there exist two constants $c_1, c_2 \in \mathbb{R}$ such that for $t \in (0,T]$,

$$\begin{aligned} u(t) &= c_1 + c_2 t^{a+1} + \int_t^c \varphi_a(c,s)\,ds, \\ u'(t) &= c_2(a+1)t^a - \varphi_a(c,t). \end{aligned} \tag{3.21}$$

Using (3.8), we conclude

$$\lim_{t\to 0+} u(t) = c_1 + \int_0^c \varphi_a(c,s)\,ds =: c_3 \in \mathbb{R}, \qquad \lim_{t\to 0+} u'(t) = 0. \tag{3.22}$$

For $u(0) := c_3$ and $u'(0) = 0$, we have $u \in C^1[0,T]$. Furthermore, for a.e. $t \in (0,T]$,

$$u''(t) = c_2(a+1)at^{a-1} - h(t) + at^{a-1}\int_t^c \frac{h(s)}{s^a}\,ds. \tag{3.23}$$

By the Hölder inequality and (3.6) it follows that

$$|u''(t)| \le c_2(a+1)at^{a-1} + |h(t)| + Mt^{a-1}\left(c^{1/q-a} + t^{1/q-a}\right)\|h\|_p \in L_1[0,T], \tag{3.24}$$

where

$$M = a\left|1 - aq\right|^{-1/q}. \tag{3.25}$$

Therefore $u'' \in L_1[0,T]$, and consequently $u \in AC^1[0,T]$. □

It is clear from the above theorem, that $u \in AC^1[0,T]$ given by (3.21) is a solution of (3.1) for $a > 0$. Let us now consider the associated boundary value problem,

$$u''(t) = \frac{a}{t}u'(t) + h(t), \quad \text{a.e. on } (0,T], \tag{3.26a}$$

$$B_0 U(0) + B_1 U(T) = \beta, \quad U(t) := (u(t), u'(t))^T, \tag{3.26b}$$

where $B_0, B_1 \in \mathbb{R}^{2\times 2}$ are real matrices, and $\beta \in \mathbb{R}^2$ is an arbitrary vector. Then the following result follows immediately from Theorem 3.3.

Theorem 3.4. *Let $a > 0$, $p > 1$. Then for any $h(t) \in L_p[0,T]$ and any $\beta \in \mathbb{R}^2$ there exists a unique solution $u \in \mathrm{AC}^1[0,1]$ of the boundary value problem* (3.26a) *and* (3.26b) *if and only if the following matrix,*

$$B_0\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + B_1\begin{pmatrix}1 & T^{a+1}\\ 0 & (a+1)T^a\end{pmatrix} \in \mathbb{R}^{2\times 2}, \tag{3.27}$$

is nonsingular.

Proof. Let u be a solution of (3.1). Then u satisfies (3.21), and the result follows immediately by substituting the values,

$$\begin{gathered} u(0) = c_1 + \int_0^c \varphi_a(c,s)\mathrm{d}s, \qquad u(T) = c_1 + c_2T^{a+1} + \int_T^c \varphi_a(c,s)\mathrm{d}s, \\ u'(0) = 0, \qquad u'(T) = c_2(a+1)T^a - \varphi_a(c,T), \end{gathered} \tag{3.28}$$

into the boundary conditions (3.26b). □

Theorem 3.5. *Let $a < 0$ and let a function $u \in \mathrm{AC}^1_{\mathrm{loc}}(0,T]$ satisfy equation* (3.1) *a.e. on $(0,T]$. For $a \in (-1,0)$, only one of the following properties holds:*

(i) $\lim_{t\to 0+}u(t) \in \mathbb{R}$, $\lim_{t\to 0+}u'(t) = 0$,

(ii) $\lim_{t\to 0+}u(t) \in \mathbb{R}$, $\lim_{t\to 0+}u'(t) = \pm\infty$.

For $a \in (-\infty,-1]$, u satisfies only one of the following properties:

(i) $\lim_{t\to 0+}u(t) \in \mathbb{R}$, $\lim_{t\to 0+}u'(t) = 0$,

(ii) $\lim_{t\to 0+}u(t) = \mp\infty$, $\lim_{t\to 0+}u'(t) = \pm\infty$.

In particular, u can be extended to the whole interval $[0,T]$ with $u \in \mathrm{AC}^1[0,T]$ if and only if $\lim_{t\to 0+}u'(t) = 0$.

Proof. Let $a \in (-1,0)$, and let u be given. Then, by (3.13), there exist two constants $c_1, c_2 \in \mathbb{R}$ such that

$$u(t) = c_1 + c_2t^{a+1} - \int_0^t \varphi_a(0,s)\mathrm{d}s \quad \text{for } t \in (0,T]. \tag{3.29}$$

Hence

$$u'(t) = c_2(a+1)t^a - \varphi_a(0,t) \quad \text{for } t \in (0,T]. \tag{3.30}$$

Let $c_2 = 0$, then it follows from (3.9) $\lim_{t\to 0+} u'(t) = 0$. Also, by (3.29), $\lim_{t\to 0+} u(t) = c_1 \in \mathbb{R}$. Let $c_2 \neq 0$. Then (3.9), (3.29), and (3.30) imply that

$$\begin{aligned} &\lim_{t\to 0+} u(t) = c_1 \in \mathbb{R}, \quad \lim_{t\to 0+} u'(t) = +\infty, \quad \text{if } c_2 > 0, \\ &\lim_{t\to 0+} u(t) = c_1 \in \mathbb{R}, \quad \lim_{t\to 0+} u'(t) = -\infty, \quad \text{if } c_2 < 0. \end{aligned} \tag{3.31}$$

Let $a = -1$. Then, by (3.14), for any $c_1, c_2 \in \mathbb{R}$,

$$u(t) = c_1 + c_2 \ln t - \int_0^t \varphi_{-1}(0,s)\mathrm{d}s \quad \text{for } t \in (0,T], \tag{3.32}$$

$$u'(t) = c_2 \frac{1}{t} - \varphi_{-1}(0,t) \quad \text{for } t \in (0,T]. \tag{3.33}$$

If $c_2 = 0$, then $\lim_{t\to 0+} u'(t) = 0$ by (3.9), and it follows from (3.32) that $\lim_{t\to 0+} u(t) = c_1 \in \mathbb{R}$. Let $c_2 \neq 0$. Then we deduce from (3.9), (3.32), and (3.33) that

$$\begin{aligned} &\lim_{t\to 0+} u(t) = -\infty, \quad \lim_{t\to 0+} u'(t) = +\infty, \quad \text{if } c_2 > 0, \\ &\lim_{t\to 0+} u(t) = +\infty, \quad \lim_{t\to 0+} u'(t) = -\infty, \quad \text{if } c_2 < 0. \end{aligned} \tag{3.34}$$

Let $a < -1$. Then on $(0,T]$, u satisfies (3.29) and (3.30), with $c_1, c_2 \in \mathbb{R}$. If $c_2 = 0$, then, by (3.9), $\lim_{t\to 0+} u'(t) = 0$ and $\lim_{t\to 0+} u(t) = c_1 \in \mathbb{R}$. Let $c_2 \neq 0$. Then

$$\begin{aligned} &\lim_{t\to 0+} u(t) = +\infty, \quad \lim_{t\to 0+} u'(t) = -\infty, \quad \text{if } c_2 > 0, \\ &\lim_{t\to 0+} u(t) = -\infty, \quad \lim_{t\to 0+} u'(t) = +\infty, \quad \text{if } c_2 < 0. \end{aligned} \tag{3.35}$$

In particular, for $a < 0$, u can be extended to $[0,T]$ in such a way that $u \in C^1[0,T]$ if and only if $c_2 = 0$. Then, the associated boundary conditions read $u(0) = c_1$ and $u'(0) = 0$. Finally, for a.e. $t \in (0,T]$,

$$u''(t) = -h(t) - a t^{a-1} \int_0^t \frac{h(s)}{s^a}\mathrm{d}s, \tag{3.36}$$

and by the Hölder inequality, (3.3), and (3.25),

$$\left|u''(t)\right| \le |h(t)| + M t^{a-1} t^{1/q-a} \|h\|_p \in L_1[0,T]. \tag{3.37}$$

Therefore $u'' \in L_1[0,T]$, and consequently $u \in AC^1[0,T]$. □

Again, it is clear that u given by (3.29) for $a \in (-1, 0)$ and $a < -1$, and u given by (3.32) for $a = -1$ is a solution of (3.1), and $u \in AC^1[0, 1]$ if and only if $u'(0) = 0$. Let us now consider the boundary value problem

$$u''(t) = \frac{a}{t}u'(t) + h(t), \quad \text{a.e. on } (0, T], \tag{3.38a}$$

$$u'(0) = 0, \qquad b_0u(0) + b_1u(T) + b_2u'(T) = \beta, \tag{3.38b}$$

where $b_0, b_1, b_2, \beta \in \mathbb{R}$ are real constants. Then the following result follows immediately from Theorem 3.5.

Theorem 3.6. *Let $a < 0$, $p > 1$. Then for any $h(t) \in L_p[0, T]$ and any $b_2, \beta \in \mathbb{R}$ there exists a unique solution $u \in AC^1[0, 1]$ of the boundary value problem* (3.38a) *and* (3.38b) *if and only if $b_0 + b_1 \neq 0$.*

Proof. Let u be a solution of (3.1). Then u satisfies (3.29) for $a \in (-1, 0)$ and $a < -1$, and (3.32) for $a = -1$. We first note that, by (3.9), for all $a < 0$,

$$u'(0) = \lim_{t \to 0+} u'(t) = 0 \Longleftrightarrow c_2 = 0. \tag{3.39}$$

Therefore, $c_2 = 0$ in both, (3.29) and (3.32), and the result now follows by substituting the values,

$$u(0) = c_1, \qquad u(T) = c_1 - \int_0^T \varphi_a(0, s)\mathrm{d}s, \qquad u'(T) = -\varphi_a(0, T), \tag{3.40}$$

into the boundary conditions (3.38b). □

To illustrate the solution behaviour, described by Theorems 3.3 and 3.5, we have carried out a series of numerical calculations on a MATLAB software package bvpsuite designed to solve boundary value problems in ordinary differential equations. The solver is based on a collocation method with Gaussian collocation points. A short description of the code can be found in [17]. This software has already been used for a variety of singular boundary value problems relevant for applications; see, for example, [18].

The equations being dealt with are of the form

$$u''(t) = \frac{a}{t}u'(t) + \frac{1}{\sqrt[3]{1 - t}}, \quad t \in (0, 1), \tag{3.41}$$

subject to initial or boundary conditions specified in the following graphs. All solutions were computed on the unit interval $[0, 1]$.

Finally, we expect $\lim_{t \to 0+} u(t) = \pm\infty$, and therefore we solve (3.41) subject to the terminal conditions $u(1) = \alpha$, $u'(1) = \beta$. See Figures 1, 2, and 3.

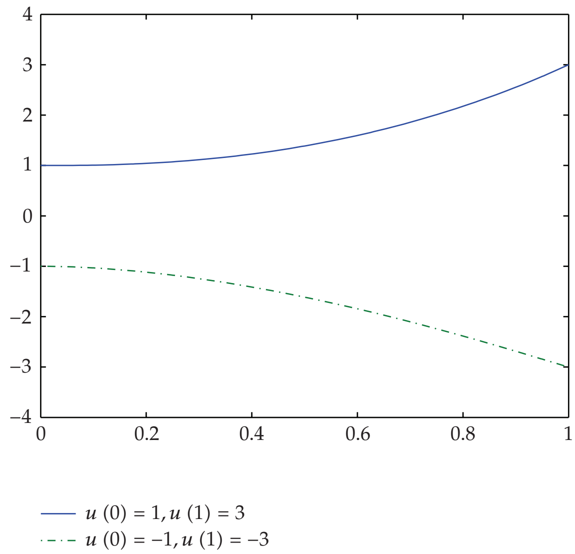


Figure 1: Illustrating Theorem 3.3: solutions of differential equation (3.41) with $a = 1$, subject to boundary conditions $u(0) = \alpha$, $u(1) = \beta$. See graph legend for the values of α and β. According to Theorem 3.3 it holds that $u'(0) = 0$ for each choice of α and β.

4. Limit Properties of Functions Satisfying Nonlinear Singular Equations

In this section we assume that the function $u \in AC^1_{\text{loc}}(0, T]$ satisfying differential equation (2.1) a.e. on $[0, T]$ is given. The first derivative of such a function does not need to be continuous at $t = 0$ and hence, due to the lack of smoothness, u does not need to be a solution of (2.1) in the sense of Definition 2.2. In the following two theorems, we discuss the limit properties of u for $t \to 0$.

Theorem 4.1. *Let us assume that* (2.2) *holds. Let $a > 0$ and let $u \in AC^1_{\text{loc}}(0, T]$ satisfy equation* (2.1) *a.e. on $[0, T]$. Finally, let us assume that that*

$$\sup\{|u(t)| + |u'(t)| : t \in (0, T]\} < \infty. \tag{4.1}$$

Then

$$\lim_{t \to 0+} u(t) \in \mathbb{R}, \qquad \lim_{t \to 0+} u'(t) = 0, \tag{4.2}$$

and u can be extended on $[0, T]$ in such a way that $u \in AC^1[0, T]$.

Proof. Let $h(t) := f(t, u(t), u'(t))$ for a.e. $t \in [0, T]$. By (2.2), there exists a function $m_{\mathcal{K}} \in L_p[0, T]$ such that $|f(t, u(t), u'(t))| \le m_{\mathcal{K}}(t)$ for a.e. $t \in [0, T]$. Therefore, $h \in L_p[0, T]$. Since the equality $u''(t) = (a/t)u'(t) + h(t)$ holds a.e. on $[0, T]$, the result follows immediately due to Theorem 3.3. □

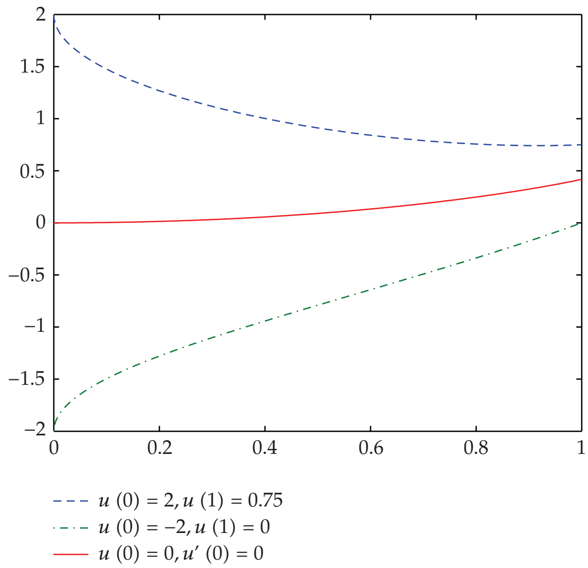


Figure 2: Illustrating Theorem 3.5 for $a \in (-1, 0)$: solutions of differential equation (3.41) with $a = -1/2$, subject to boundary conditions $u(0) = \alpha$, $u(1) = \beta$. See graph legend for the values of α and β. According to Theorem 3.5 a solution u satisfies $u'(0) = +\infty$ or $u'(0) = -\infty$ or $u'(0) = 0$ in dependence of values α and β. In order to precisely recover a solution satisfying $u'(0) = 0$, the respective simulation was carried out as an initial value problem with $u(0) = 0$ and $u'(0) = 0$.

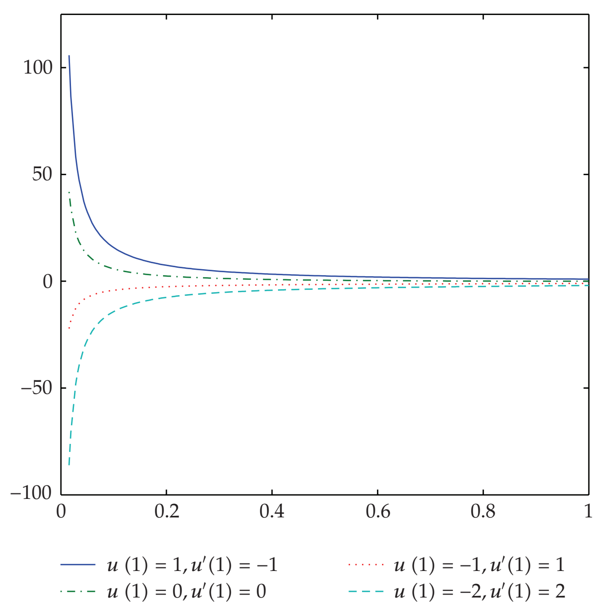


Figure 3: Illustrating Theorem 3.5 for $a \in (-\infty, -1)$: solutions of differential equation (3.41) with $a = -2$, subject to boundary conditions $u(1) = \alpha$, $u'(1) = \beta$. See graph legend for the values of α and β. Here, $\lim_{t\to 0+} u(t) = \pm\infty$, and $\lim_{t\to 0+} u'(t) = \mp\infty$, or $u(0) \in \mathbb{R}$, $u'(0) = 0$.

Theorem 4.2. *Let us assume that condition* (2.2) *holds. Let* $a < 0$ *and let* $u \in \mathrm{AC}^1_{\mathrm{loc}}(0,T]$ *satisfy equation* (2.1) *a.e. on* $(0,T]$. *Let us also assume that* (4.1) *holds. Then*

$$\lim_{t\to 0+} u(t) \in \mathbb{R}, \qquad \lim_{t\to 0+} u'(t) = 0, \tag{4.3}$$

and u *can be extended on* $[0,T]$ *in such a way that* $u \in \mathrm{AC}^1[0,T]$.

Proof. Let $h \in L_p[0,T]$ be as in the proof of Theorem 4.1. According to Theorem 3.5 and (4.1), u satisfies (4.3) both for $a \in (-1,0)$ and $a \in (-\infty,-1]$. □

5. Applications

Results derived in Theorems 4.1 and 4.2 constitute a useful tool when investigating the solvability of nonlinear singular equations subject to different types of boundary conditions. In this section, we utilize Theorem 4.1 to show the existence of solutions for periodic problems. The rest of this section is devoted to the numerical simulation of such problems.

Periodic Problem

We deal with a problem of the following form:

$$u''(t) = \frac{a}{t}u'(t) + f\left(t,u(t),u'(t)\right), \quad \text{a.e. on } (0,T], \tag{5.1a}$$

$$u(0) = u(T), \qquad u'(0) = u'(T). \tag{5.1b}$$

Definition 5.1. A function $u \in \mathrm{AC}^1[0,T]$ is called *a solution of the boundary value problem* (5.1a) and (5.1b), if u satisfies equation (5.1a) for a.e. $t \in [0,T]$ and the periodic boundary conditions (5.1b).

Conditions (5.1b) can be written in the form (3.26b) with $B_0 = I$, $B_1 = -I$, and $\beta = 0$. Then, matrix (3.27) has the form

$$I\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - I\begin{pmatrix} 1 & T^{a+1} \\ 0 & (a+1)T^a \end{pmatrix} = \begin{pmatrix} 0 & -T^{a+1} \\ 0 & -(a+1)T^a \end{pmatrix}, \tag{5.2}$$

and we see that it is singular. Consequently, the assumption of Theorem 3.4 is not satisfied, and the linear periodic problem (3.26b) subject to (5.1b) is not uniquely solvable. However this is not true for nonliner periodic problems. In particular, Theorem 5.6 gives a characterization of a class of nonlinear periodic problems (5.1a) and (5.1b) which have only one solution. We begin the investigation of problem (5.1a) and (5.1b) with a uniqueness result.

Theorem 5.2 (uniqueness). *Let* $a > 0$ *and let us assume that condition* (2.2) *holds. Further, assume that for each compact set* $\mathcal{K} \subset \mathbb{R}\times\mathbb{R}$ *there exists a nonnegative function* $h_{\mathcal{K}} \in L_1[0,T]$ *such that*

$$x_1 > x_2,\ y_1 \ge y_2 \implies f\left(t,x_1,y_1\right) - f\left(t,x_2,y_2\right) > -h_{\mathcal{K}}(t)\left(y_1 - y_2\right) \tag{5.3}$$

for a.e. $t \in [0,T]$ *and all* $(x_1,y_1),(x_2,y_2) \in \mathcal{K}$. *Then problem* (5.1a) *and* (5.1b) *has at most one solution.*

Proof. Let u_1 and u_2 be different solutions of problem (5.1a) and (5.1b). Since $u_1,u_2 \in AC^1[0,T]$, there exists a compact set $\mathcal{K} \subset \mathbb{R}\times\mathbb{R}$ such that $(u_i(t),u_i'(t)) \in \mathcal{K}$ for $t \in [0,T]$. Let us define the difference function $v(t) := u_1(t) - u_2(t)$ for $t \in [0,T]$. Then

$$v(0) = v(T), \qquad v'(0) = v'(T). \tag{5.4}$$

First, we prove that there exists an interval $[\alpha,\beta] \subset (0,T]$ such that

$$v(t) > 0 \quad \text{for } t \in [\alpha,\beta], \qquad v'(t) > 0 \quad \text{for } t \in [\alpha,\beta), \qquad v'(\beta) = 0. \tag{5.5}$$

We consider two cases.

Case 1. Assume that u_1 and u_2 have an intersection point, that is, there exists $t_0 \in [0,T)$ such that $v(t_0) = 0$. Since u_1 and u_2 are different, there exists $t_1 \in [0,T]$, $t_1 \neq t_0$, such that $v(t_1) \neq 0$.

(i) Let $t_1 > t_0$. We can assume that $v(t_1) > 0$. (Otherwise we choose $v := u_2 - u_1$.) Then we can find $a_0 \in (t_0,t_1)$ satisfying $v(t) > 0$ for $t \in [a_0,t_1]$ and $v'(a_0) > 0$. Let $b_0 \in (a_0,T]$ be the first zero of v'. Then, if we set $[\alpha,\beta] := [a_0,b_0]$, we see that $[\alpha,\beta]$ satisfies (5.5). Let v' have no zeros on $[a_0,T]$. Then $v > 0$, $v' > 0$ on $[a_0,T]$, and, due to (5.4), $v(0) > 0$, $v'(0) > 0$. Since $v(t_0) = 0$, we can find $\alpha \in (0,t_0)$ and $\beta \in (a,t_0)$ such that $[\alpha,\beta]$ satisfies (5.5).

(ii) Let $v = 0$ on $[t_0,T]$. By (5.4), $v(0) = 0$, $v'(0) = 0$ and $t_1 \in (0,t_0)$. We may again assume that $v(t_1) > 0$. It is possible to find $\alpha \in (0,t_1)$ such that $v(\alpha) > 0$, $v'(\alpha) > 0$, $v(t) > 0$ on $[\alpha,t_1]$. Since $v(t_0) = 0$, v' has at least one zero in (α,t_0). If $\beta \in (\alpha,t_0)$ is the first zero of v', then $[\alpha,\beta]$ satisfies (5.5).

Case 2. Assume that u_1 and u_2 have no common point, that is, $v(t) \neq 0$ on $[0,T]$. We may assume that $v > 0$ on $[0,T]$. By (5.4), there exists a point $t_0 \in (0,T)$ satisfying $v'(t_0) = 0$.

(i) Let $v' = 0$ on $[0,t_0]$. Then, by (5.1a) and (5.3),

$$v''(t) = \frac{a}{t}v'(t) + f(t,u_1(t),u_1'(t)) - f(t,u_2(t),u_2'(t)) > \left(\frac{a}{t} - h_{\mathcal{K}}(t)\right)v'(t) = 0 \tag{5.6}$$

for a.e. $t \in [0,t_0]$, which is a contradiction to $v'' = 0$ on $[0,t_0]$.

(ii) Let $v'(t_1) \neq 0$ for some $t_1 \in [0,t_0)$. If $v'(t_1) > 0$, then we can find an interval $[\alpha,\beta] \subset (t_1,t_0]$ satisfying (5.5). If $v'(t_1) < 0$ and $v'(t) \le 0$ on $[0,t_0]$, then $v(0) > v(t_0)$ and, by (5.4), $v(T) > v(t_0)$, $v'(T) \le 0$. Hence, there exists an interval $[\alpha,\beta] \subset (t_0,T]$ satisfying (5.5).

To summarize, we have shown that in both, the case of intersecting solutions u_1 and u_2 and the case of separated u_1 and u_2, there exists an interval $[\alpha,\beta] \subset (0,T]$ satisfying (5.5).

Now, by (5.1a), (5.3), and (5.5), we obtain

$$v''(t) > \left(\frac{a}{t} - h_{\mathcal{K}}(t)\right)v'(t) \quad \text{for a.e. } t \in [\alpha,\beta]. \tag{5.7}$$

Denote by $h^*(t) := a/t - h_{\mathcal{K}}(t)$. Then $h^* \in L_1[\alpha,\beta]$, and $v''(t) - h^*(t)v'(t) > 0$ for a.e. $t \in [\alpha,\beta]$. Consequently,

$$\left(v'(t)\exp\left(-\int_{\alpha}^{t} h^*(s)ds\right)\right)' > 0 \quad \text{for a.e. } t \in [\alpha,\beta]. \tag{5.8}$$

Integrating the last inequality in $[\alpha,\beta]$, we obtain

$$v'(\beta)\exp\left(-\int_{\alpha}^{\beta} h^*(s)ds\right) > v'(\alpha) > 0, \tag{5.9}$$

which contradicts $v'(\beta) = 0$. Consequently, we have shown that $u_1 \equiv u_2$, and the result follows. □

In the following theorem we formulate sufficient conditions for the existence of at least one solution of problem (5.1a) and (5.1b) with $a > 0$. In the proof of this theorem, we work also with auxiliary two-point boundary conditions:

$$u(0) = u(T), \qquad u'(T) = 0. \tag{5.10}$$

Under the assumptions of Theorem 4.1 any solution of (5.1a) satisfies $u'(0) = 0$. Therefore, we can investigate (5.1a) subject to the auxiliary conditions (5.10) instead of the equivalent original problem (5.1a) and (5.1b). This change of the problem setting is useful for obtaining of a priori estimates necessary for the application of the Fredholm-type existence theorem (Lemma 5.5) during the proof.

Theorem 5.3 (existence). *Let $a > 0$ and let* (2.2) *hold. Further, assume that there exist $A, B \in \mathbb{R}$, $c > 0$, $\omega \in C[0,\infty)$, and $\varphi \in L_1[0,T]$ such that $A \le B$,*

$$f(t,A,0) \le 0, \qquad f(t,B,0) \ge 0 \tag{5.11}$$

for a.e. $t \in [0,T]$,

$$f(t,x,y)\operatorname{sign} y \ge -\omega(|y|)(|y| + \varphi(t)) \tag{5.12}$$

for a.e. $t \in [0,T]$ and all $x \in [A,B]$, $y \in \mathbb{R}$, where

$$\omega(x) \ge c, \quad x \in [0,\infty), \qquad \int_0^{\infty} \frac{ds}{\omega(s)} = \infty. \tag{5.13}$$

Then problem (5.1a) *and* (5.1b) *has a solution u such that*

$$A \le u(t) \le B, \quad t \in [0,T]. \tag{5.14}$$

Proof.
Step 1 (existence of auxiliary solutions u_n). By (5.13), there exists $\rho^* > 0$ such that

$$\int_0^{\rho^*} \frac{\mathrm{d}s}{\omega(s)} > \|\psi\|_1 + \left(1 + \frac{T}{c}\right)(B - A) =: r. \tag{5.15}$$

For $y \in \mathbb{R}$, let

$$\chi(y) := \begin{cases} 1, & \text{if } |y| \le \rho^*, \\ 2 - \dfrac{|y|}{\rho^*}, & \text{if } \rho^* < |y| < 2\rho^*, \\ 0, & \text{if } |y| \ge 2\rho^*. \end{cases} \tag{5.16}$$

Motivated by [19], we choose $n \in \mathbb{N}$, $n > 1/T$, and, for a.e. $t \in [0,T]$, all $(x,y) \in \mathbb{R} \times \mathbb{R}$, and $\varepsilon \in [0,1]$, we define the following functions:

$$h_n(t,x,y) = \begin{cases} \chi(y)\left(\dfrac{a}{t}y + f(t,x,y)\right) - \dfrac{A}{n}, & \text{if } t \ge \dfrac{1}{n}, \\ -\dfrac{A}{n}, & \text{if } t < \dfrac{1}{n}, \end{cases} \tag{5.17}$$

$$\begin{aligned} w_A(t,\varepsilon) &= \sup\{|h_n(t,A,0) - h_n(t,A,y)| : |y| \le \varepsilon\}, \\ w_B(t,\varepsilon) &= \sup\{|h_n(t,B,0) - h_n(t,B,y)| : |y| \le \varepsilon\}, \end{aligned} \tag{5.18}$$

$$f_n(t,x,y) = \begin{cases} h_n(t,B,y) + w_B\left(t, \dfrac{x-B}{x-B+1}\right), & \text{if } x > B, \\ h_n(t,x,y), & \text{if } A \le x \le B, \\ h_n(t,A,y) - w_A\left(t, \dfrac{A-x}{A-x+1}\right), & \text{if } x < A. \end{cases} \tag{5.19}$$

Due to (5.11),

$$\frac{A}{n} + h_n(t,A,0) \le 0, \qquad \frac{B}{n} + h_n(t,B,0) \ge 0 \tag{5.20}$$

for a.e. $t \in [0,T]$. It can be shown that w_A and w_B which satisfy the L_p-Carathéodory conditions on $[0,T] \times [0,1]$ are nondecreasing in their second argument and $w_A(t,0) = w_B(t,0) = 0$ a.e. on $[0,T]$; see [19]. Therefore, f_n also satisfies the L_p-Carathéodory conditions on $[0,T] \times \mathbb{R} \times \mathbb{R}$, and there exists a function $m_n \in L_p[0,T]$ such that $|f_n(t,x,y)| \le m_n(t)$ for a.e. $t \in [0,T]$ and all $(x,y) \in \mathbb{R} \times \mathbb{R}$.

We now investigate the auxiliary problem

$$u''(t) = \frac{u(t)}{n} + f_n(t, u(t), u'(t)), \qquad u(0) = u(T), \qquad u'(T) = 0. \tag{5.21}$$

Since the homogeneous problem $u''(t) = (1/n)u(t)$, $u(0) = u(T)$, $u'(T) = 0$, has only the trivial solution, we conclude by the Fredholm-type Existence Theorem (see Lemma 5.5) that there exists a solution $u_n \in AC^1[0,T]$ of problem (5.21).

Step 2 (estimates of u_n). We now show that

$$A \le u_n(t) \le B, \quad t \in [0,T],\ n \in \mathbb{N},\ n > \frac{1}{T}. \tag{5.22}$$

Let us define $v(t) := A - u_n(t)$ for $t \in [0,T]$ and assume

$$\max\{v(t) : t \in [0,T]\} = v(t_0) > 0. \tag{5.23}$$

By (5.21), we can assume that $t_0 \in (0,T]$. Since $v'(t_0) = 0$, we can find $\delta > 0$ such that

$$v(t) > 0, \quad |v'(t)| = |u_n'(t)| < \frac{v(t)}{v(t)+1} < 1 \quad \text{on } (t_0 - \delta, t_0] \subset (0,T]. \tag{5.24}$$

Then, by (5.19), (5.20), and (5.21), we have

$$\begin{aligned} u_n'' &= f_n(t, u_n(t), u_n'(t)) + \frac{u_n(t)}{n} \\ &= h_n(t, A, u_n'(t)) - w_A\left(t, \frac{v(t)}{v(t)+1}\right) + \frac{u_n(t)}{n} \\ &\le h_n(t, A, 0) + h_n(t, A, u_n'(t)) - h_n(t, A, 0) - w_A(t, |u_n'(t)|) + \frac{u_n(t)}{n} \\ &\le h_n(t, A, 0) + \frac{A}{n} - \frac{v(t)}{n} < 0 \end{aligned} \tag{5.25}$$

for a.e. $t \in (t_0 - \delta, t_0]$. Hence,

$$0 > \int_t^{t_0} u_n''(s)\mathrm{d}s = -u_n'(t) = v'(t), \quad t \in (t_0 - \delta, t_0), \tag{5.26}$$

which contradicts (5.23), and thus $A \le u_n(t)$ on $[0,T]$. The inequality $u_n(t) \le B$ on $[0,T]$ can be proved in a very similar way.

Step 3 (estimates of u_n'). We now show that

$$|u_n'(t)| \le \rho^*, \quad t \in [0,T],\ n \in \mathbb{N},\ n > \frac{1}{T}. \tag{5.27}$$

By (5.19) and (5.22) we have $f_n(t, u_n(t), u_n'(t)) = h_n(t, u_n(t), u_n'(t))$ for a.e. $t \in [0, T]$, and so, due to (5.17) and (5.21), we have for a.e. $t \in [0, T]$,

$$\left(u_n''(t) - \frac{1}{n}(u_n(t) - A)\right) \operatorname{sign} u_n'(t) = \begin{cases} \chi(u_n'(t))\left(\frac{a}{t}u_n'(t) + f(t, u_n(t), u_n'(t))\right) \operatorname{sign} u_n'(t), & \text{if } t \geq \frac{1}{n}, \\ 0, & \text{if } t < \frac{1}{n}. \end{cases} \tag{5.28}$$

Denote $\rho := \|u_n'\|_\infty = |u_n'(t_0)|$. If $\rho > 0$, then $t_0 \in [0, T)$.

Case 1. Let $u_n'(t_0) = \rho$. Then there exists $t_1 \in (t_0, T]$ such that $u_n'(t) > 0$ on $[t_0, t_1)$, $u_n'(t_1) = 0$. By (5.12), (5.22), (5.28), and $a > 0$, it follows for a.e. $t \in [t_0, t_1]$,

$$\begin{aligned} u_n''(t) &\geq \chi(u_n'(t)) f(t, u_n(t), u_n'(t)) \operatorname{sign} u_n'(t) \\ &\geq -\chi(u_n'(t))\omega(u_n'(t))(u_n'(t) + \psi(t)) \\ &\geq -\omega(u_n'(t))(u_n'(t) + \psi(t)). \end{aligned} \tag{5.29}$$

Consequently,

$$\begin{aligned} &\int_{t_0}^{t_1} \frac{u_n''(t)}{\omega(u_n'(t))} \mathrm{d}t \geq -\int_{t_0}^{t_1} (u_n'(t) + \psi(t)) \mathrm{d}t, \\ &\int_0^\rho \frac{\mathrm{d}s}{\omega(s)} \leq u_n(t_1) - u_n(t_0) + \|\psi\|_1 < r, \end{aligned} \tag{5.30}$$

where r is given by (5.15). Therefore $\rho < \rho^*$.

Case 2. Let $u_n'(t_0) = -\rho$. Then there exists $t_1 \in (t_0, T]$ such that $u_n'(t) < 0$ on $[t_0, t_1)$, $u_n'(t_1) = 0$. By (5.12), (5.13), (5.22), (5.28), and $a > 0$, we obtain for a.e. $t \in [t_0, t_1]$

$$\begin{aligned} -u_n''(t) &\geq -\chi(u_n'(t)) f(t, u_n(t), u_n'(t)) \operatorname{sign} u_n'(t) - \frac{1}{n}(u_n(t) - A) \\ &\geq -\chi(u_n'(t))\omega(|u_n'(t)|)(|u_n'(t)| + \psi(t)) - \frac{1}{n}(B - A) \\ &\geq -\omega(|u_n'(t)|)\left(|u_n'(t)| + \psi(t) + \frac{1}{c}(B - A)\right). \end{aligned} \tag{5.31}$$

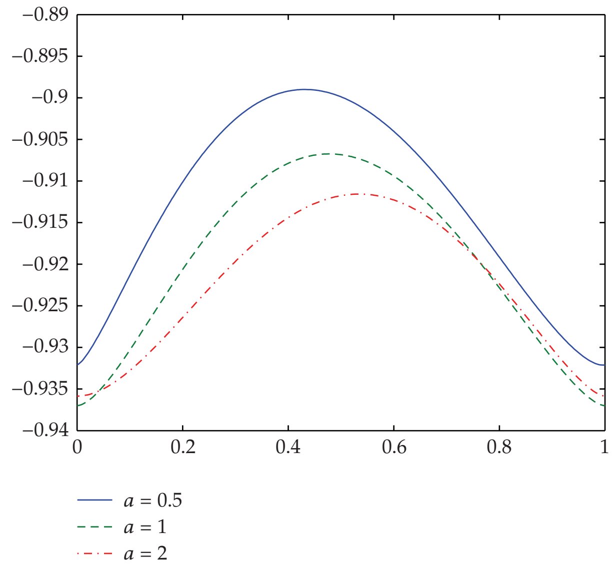


Figure 4: Illustrating Theorem 5.6: solutions of differential equation (5.43), subject to periodic boundary conditions (5.1a). See graph legend for the values of a.

Consequently,

$$-\int_{t_0}^{t_1} \frac{u_n''(t)}{\omega(-u_n'(t))} \mathrm{d}t \geq -\int_{t_0}^{t_1} \left(-u_n'(t) + \psi(t) + \frac{1}{c}(B - A)\right) \mathrm{d}t,$$
$$\int_0^{\rho} \frac{\mathrm{d}s}{\omega(s)} \leq u_n(t_0) - u_n(t_1) + \|\psi\|_1 + \frac{T}{c}(B - A) < r. \tag{5.32}$$

Hence, according to (5.15), we again have $\rho < \rho^*$.

Step 4 (convergence of $\{u_n\}$). Consider the sequence $\{u_n\}$ of solutions of problems (5.21), $n \in \mathbb{N}$, $n > 1/T$. It has been shown in Steps 2 and 3 that (5.22) and (5.27) hold, which means that the sequences $\{u_n\}$ and $\{u_n'\}$ are bounded in $C[0, T]$. Therefore $\{u_n\}$ is equicontinuous on $[0, T]$. According to (5.17), (5.19), and (5.21), we obtain for $t \in [1/n, T]$,

$$u_n'(t) = -\int_t^T \left(f_n(s, u_n(s), u_n'(s)) + \frac{u_n(s)}{n} \right) \mathrm{d}s$$
$$= -\int_t^T \left(\frac{a}{s} u_n'(s) + f(s, u_n(s), u_n'(s)) + \frac{u_n(s) - A}{n} \right) \mathrm{d}s. \tag{5.33}$$

Let us now choose an arbitrary compact subinterval $[a_0, T] \subset (0, T]$. Then there exists $n_0 \in \mathbb{N}$ such that $[1/n, T] \subset [a_0, T]$ for each $n \geq n_0$. By (5.33), the sequence $\{u_n'\}$ is equicontinuous on

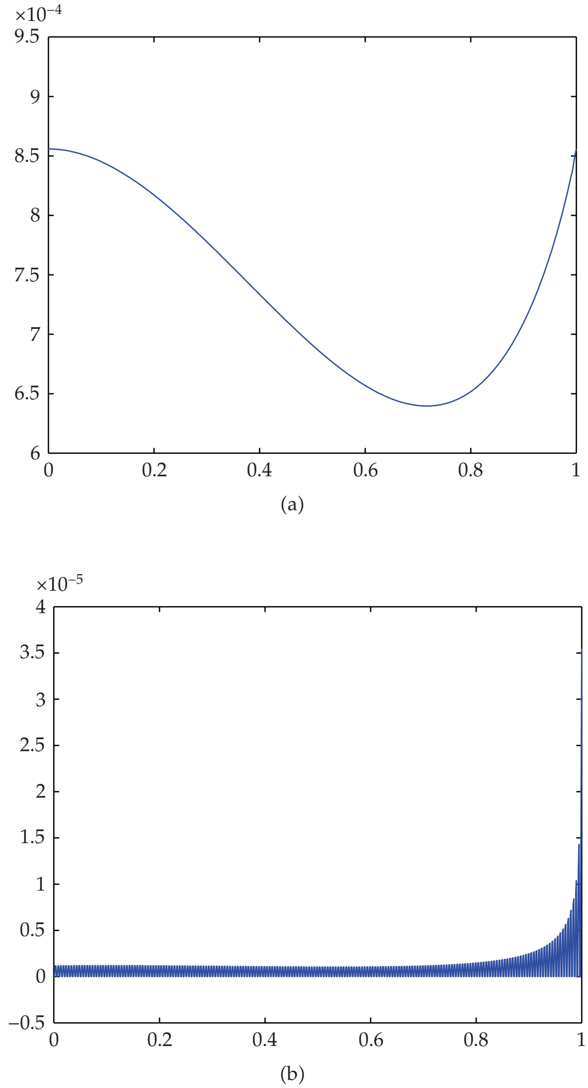


Figure 5: Error estimate (a) and residual (b) for (5.43)-(5.1a), $a = 1$.

$[a_0, T]$. Therefore, we can find a subsequence $\{u_m\}$ such that $\{u_m\}$ converges uniformly on $[0, T]$, and $\{u'_m\}$ converges uniformly on $[a_0, T]$. By the diagonalization theorem; see [11], we can find a subsequence $\{u_\ell\}$ such that there exists $u \in C[0, T] \cap C^1(0, T]$ with

$$\lim_{\ell \to \infty} u_\ell(t) = u(t) \text{ uniformly on } [0, T],$$
$$\lim_{\ell \to \infty} u'_\ell(t) = u'(t) \text{ locally uniformly on } (0, T]. \tag{5.34}$$

Therefore $u(0) = u(T)$ and $u'(T) = 0$. For $\ell \to \infty$ in (5.33), Lebesgue's dominated

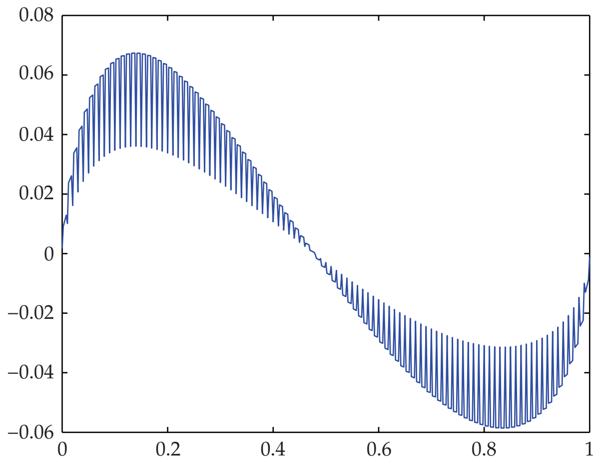


Figure 6: First derivative of the numerical solution to (5.43)-(5.1a) with $a = 1$.

convergence theorem yields

$$u'(t) = -\int_t^T \left(\frac{a}{s}u'(s) + f\left(s, u(s), u'(s)\right)\right) \mathrm{d}s, \quad t \in (0, T]. \tag{5.35}$$

Consequently, $u \in \mathrm{AC}^1_{\mathrm{loc}}(0, T]$ satisfies equation (5.1a) a.e. on $[0, T]$. Moreover, due to (5.22) and (5.27), we have

$$A \le u(t) \le B \quad \text{for } t \in [0, T], \qquad \left|u'(t)\right| \le \rho^* \quad \text{for } t \in (0, T]. \tag{5.36}$$

Hence (4.1) is satisfied. Applying Theorem 4.1, we conclude that $u \in \mathrm{AC}^1[0, T]$ and $u'(0) = 0$. Therefore u satisfies the periodic conditions on $[0, T]$. Thus u is a solution of problem (5.1a) and (5.1b) and $A \le u \le B$ on $[0, T]$.

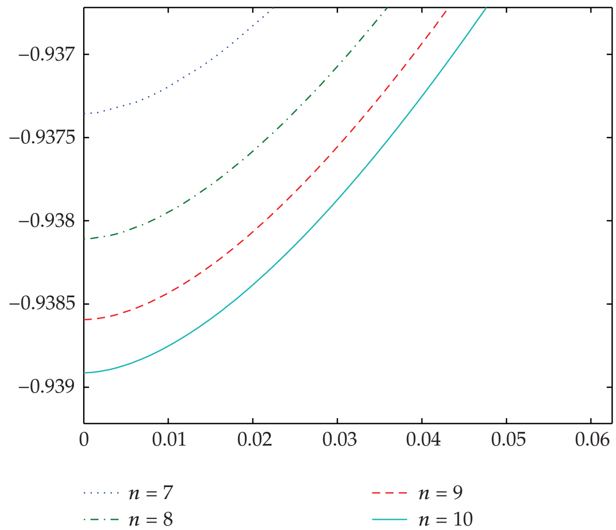
□

Example 5.4. Let $T = 1$, $k \in \mathbb{N}$, $\varepsilon = \pm 1$, $h \in L_p[0, 1]$ for some $p > 1$, and $c_0 \in C(0, 1)$. Moreover, let h be nonnegative, and let c_0 be bounded on $[0, 1]$. Then in Theorem 5.3 the following class of functions f is covered:

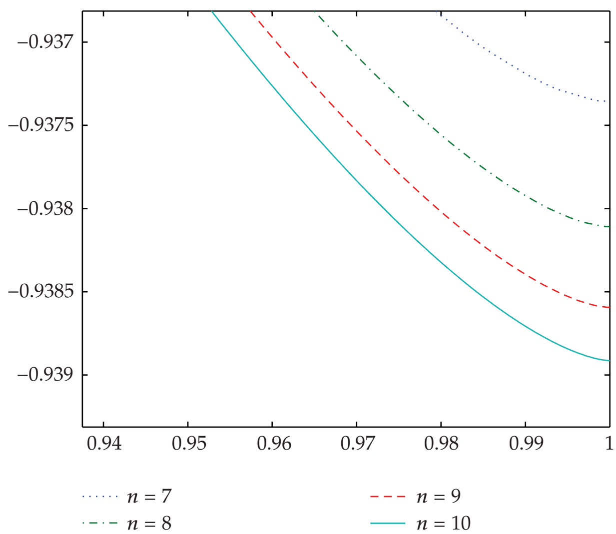
$$f\left(t, x, y\right) = h(t)\left(x^{2k+1} + \varepsilon e^x y^n + c_0(t)\cos\left(\sqrt{|x|}\right)\right) \tag{5.37}$$

for a.e. $t \in [0, 1]$ and all $x, y \in \mathbb{R}$, provided $n = 2m + 1$ if $\varepsilon = 1$ and $n = 1$ if $\varepsilon = -1$. In particular, for $t \in (0, 1]$, $x, y \in \mathbb{R}$

$$f_1\left(t, x, y\right) = \frac{1}{\sqrt{1-t}}\left(x^3 + e^x y^5 + \cos\frac{1}{t}\cos\sqrt{|x|}\right), \tag{5.38}$$



(a)



(b)

Figure 7: Numerical solutions of (5.43)-(5.1a) and $a = 1$ in the vicinity of $t = 0$ (a) and $t = 1$ (b). The step size is decreasing according to $h = 1/2^n$.

or

$$f_2(t, x, y) = \frac{1}{\sqrt{1-t}}\left(x^3 - e^x y + \cos\frac{1}{t}\cos\sqrt{|x|}\right). \tag{5.39}$$

In order to show the existence of solutions to the periodic boundary value problem (5.1a) and (5.1b), the Fredholm-type Existence Theorem is used, see for example, in [20, Theorem 4], [11, Theorem 2.1] or [21, page 25]. For convenience, we provide its simple formulation suitable for our purpose below.

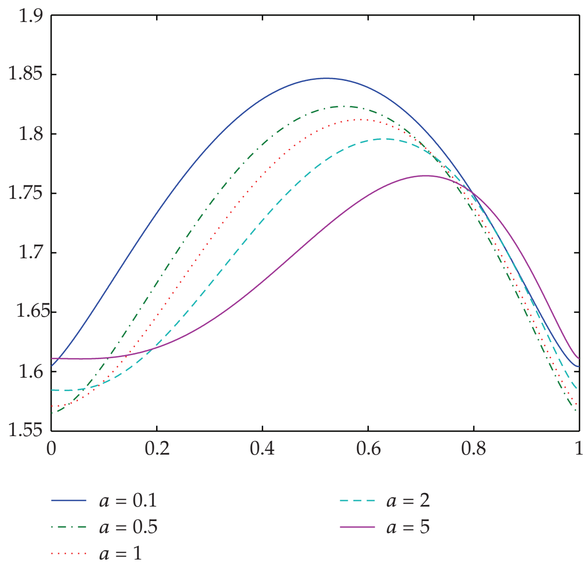


Figure 8: Illustrating Theorem 5.6: solutions of the boundary value problem (5.44)-(5.1a). See graph legend for the values of a.

Lemma 5.5 (Fredholm-type existence theorem). *Let f satisfy* (2.2)*, let matrices $B_0, B_1 \in \mathbb{R}^{2\times 2}$, vector $\beta \in \mathbb{R}^2$ be given, and let $c_1, c_2 \in L_1[0,T]$. Let us denote by $U(t) := (u(t), u'(t))^T$, and assume that the linear homogeneous boundary value problem*

$$u'' + c_1(t)u' + c_2(t)u = 0, \qquad B_0U(0) + B_1U(T) = 0 \tag{5.40}$$

has only the trivial solution. Moreover, let us assume that there exists a function $m \in L_p[0,T]$ such that

$$|f(t,x,y)| \leq m(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } x, y \in \mathbb{R}. \tag{5.41}$$

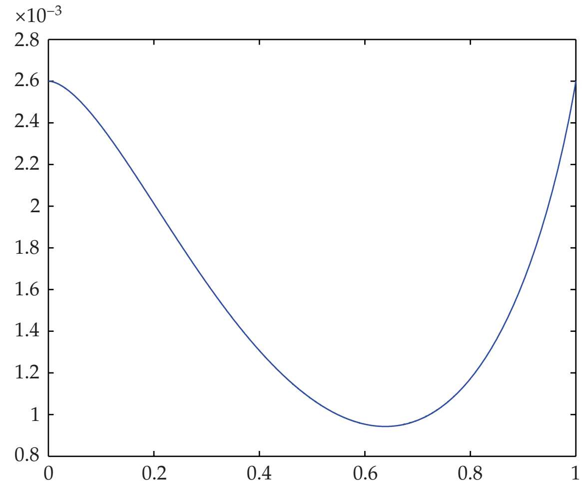
Then the problem

$$u'' + c_1(t)u' + c_2(t)u = f(t,u,u'), \qquad B_0U(0) + B_1U(T) = \beta \tag{5.42}$$

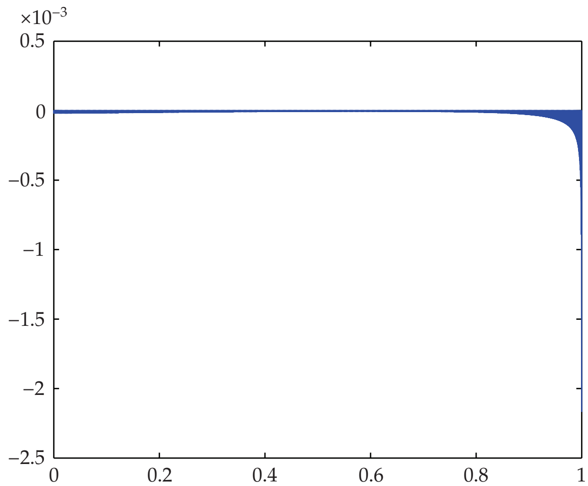
has a solution $u \in \mathrm{AC}^1[0,T]$.

If we combine Theorems 5.2 and 5.3, we obtain conditions sufficient for the solution of (5.1a) and (5.1b) to be unique.

Theorem 5.6 (existence and uniqueness). *Let all assumptions of Theorems 5.2 and 5.3 hold. Then problem* (5.1a) *and* (5.1b) *has a unique solution u. Moreover u satisfies* (5.14).



(a)



(b)

Figure 9: Error estimate (a) and residual (b) for (5.44)-(5.1a), $a = 1$.

Example 5.7. Functions satisfying assumptions of Theorem 5.6 can have the form

$$f(t,x,y) = \frac{a}{\sqrt{1-t}}\left(x^3 + e^x y^5 + t\right), \tag{5.43}$$

$$f(t,x,y) = \frac{a}{\sqrt{1-t}}\left(x^3 - e^{-x} y\right) - 16\sqrt{t}, \tag{5.44}$$

for $t \in (0,1]$, $x, y \in \mathbb{R}$.

We now illustrate the above theoretical findings by means of numerical simulations. Figure 4 shows graphs of solutions of problem (5.43), (5.1a). In Figure 5 we display the error estimate for the global error of the numerical solution and the so-called residual (defect)

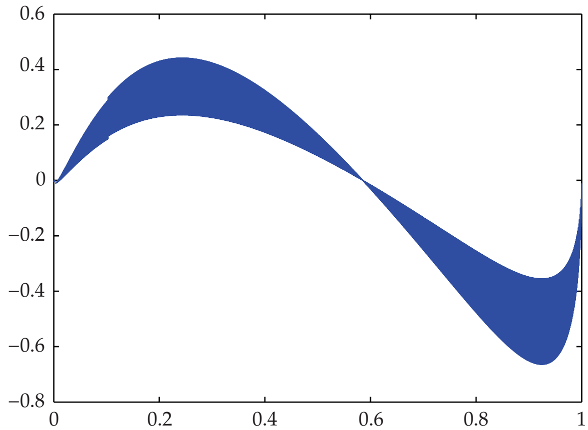


Figure 10: First derivative of the numerical solution to (5.44)-(5.1a) with $a = 1$.

Table 1: Estimated convergence order for the periodic boundary value problem (5.43)-(5.1a) and $a = 1$.

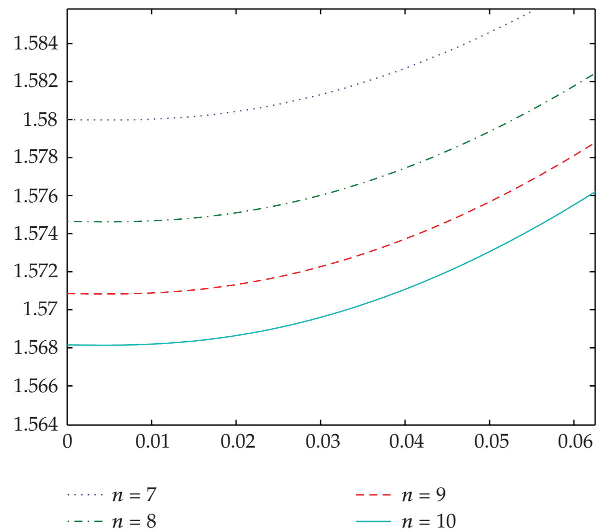
i	Error estimate	Conv. order
1	5.042446e–003	—
2	2.850171e–003	0.823075
3	1.681410e–003	0.761377
4	1.029876e–003	0.707200
5	6.514046e–004	0.660845
6	4.231359e–004	0.622433
7	2.807926e–004	0.591616
8	1.894611e–004	0.567604
9	1.294654e–004	0.549335
10	8.930836e–005	0.535699

obtained from the substitution of the numerical solution into the differential equation. Both quantities are rather small and indicate that we have found a solution to the analytical problem (5.43)-(5.1a).

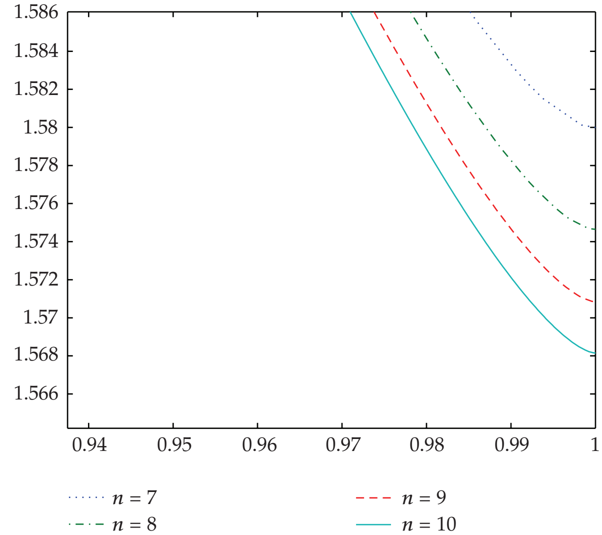
We now pose that question about the values of the first derivative at the end points of the interval of integration, $t = 0$ and $t = 1$. According to the theory, it holds that $u'(0) = u'(1) = 0$. Therefore, we approximate the values of the first derivative of the numerical solution and show these values in Figure 6. One can see that indeed $u'(0) \approx 0$, $u'(1) \approx 0$. Also, to support this observation, we plotted in Figure 7 the numerical solutions obtained for the step size h tending to zero, or equivalently, grids becoming finer.

We finally observe experimentally the order of convergence of the numerical method (collocation). Clearly, we do not expect very hight order to hold, since the analytical solution has nonsmooth higher derivatives. However, the method is convergent and, according to Table 1, we observe that its order tends to $1/2$.

The results of the numerical simulation for the boundary value problem (5.44)-(5.1a), can be found in Figures 8, 9, 10, and 11.



(a)



(b)

Figure 11: Numerical solutions of (5.44)-(5.1a) and $a = 1$ in the vicinity of $t = 0$ (a) and $t = 1$ (b). The step size is decreasing according to $h = 1/2^n$.

Acknowledgments

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