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Research Article

Almost Periodic Viscosity Solutions of Nonlinear Parabolic Equations

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We generalize the comparison result 2007 on Hamilton-Jacobi equations to nonlinear parabolic equations, then by using Perron's method to study the existence and uniqueness of time almost periodic viscosity solutions of nonlinear parabolic equations under usual hypotheses.

1. Introduction

In this paper we will study the time almost periodic viscosity solutions of nonlinear parabolic equations of the form

$$\begin{gathered}\partial_t u + H\left(x, u, Du, D^2 u\right) = f(t), \quad (x,t) \in \Omega \times \mathbb{R}, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R},\end{gathered} \tag{1.1}$$

where $\Omega \in \mathbb{R}^N$ is a bounded open subset and $\partial\Omega$ is its boundary. Here $H : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N) \to \mathbb{R}$ and $\mathcal{S}(N)$ denotes the set of symmetric $N \times N$ matrices equipped with its usual order (i.e., for $X, Y \in \mathcal{S}(N)$, we say that $X \leq Y$ if and only if $p^t X p \leq p^t Y p$, $(\forall p \in \mathbb{R}^N)$); Du and $D^2 u$ denote the gradient and Hessian matrix, respectively, of the function u w.r.t the argument x. f is almost periodic in t. Most notations and notions of this paper relevant to viscosity solutions are borrowed from the celebrated paper of Crandall et al. [1]. Bostan and Namah [2] have studied the time periodic and almost periodic viscosity solutions of first-order Hamilton-Jacobi equations. Nunziante considered the existence and uniqueness of viscosity solutions of parabolic equations with discontinuous time dependence in [3, 4], but the time almost periodic viscosity solutions of parabolic equations have not been studied yet as far as we know. We are going to use Perron's Method to study the existence of time almost periodic viscosity solutions of (1.1). Perron's Method was introduced by Ishii [5] in the proof of existence of viscosity solutions of first-order Hamilton-Jacobi equations, Crandall et al. had applications of Perron's Method to second-order partial differential equations in [1] except to parabolic case.

To study the existence and uniqueness of viscosity solutions of (1.1), we will use some results on the Cauchy-Dirichlet problem of the form

$$\begin{gathered}
\partial_t u + H\left(x,t,u,Du,D^2u\right) = 0, \quad \text{in } \Omega \times (0,T), \\
u(x,t) = 0, \quad \text{for } x \in \partial\Omega,\ 0 \le t < T, \\
u(x,0) = u_0(x), \quad \text{for } x \in \overline{\Omega},
\end{gathered} \tag{1.2}$$

where $u_0(x) \in C(\overline{\Omega})$ is given. Crandall et al. studied the comparison result of the Cauchy-Dirichlet problem in [1], and it follows the maximum principle of Crandall and Ishii [6].

This paper is structured as follows. In Section 2, we present the definition and some properties of almost periodic functions. In Section 3, first we list some hypotheses and some results that will be used for existence and uniqueness of viscosity solutions, here we give an improvement of comparison result in paper [2] to fit for second-order parabolic equations; then we prove the uniqueness and existence of time almost periodic viscosity solutions. In the end, we concentrate on the asymptotic behavior of time almost periodic solutions for large frequencies.

2. Almost Periodic Functions

In this section we recall the definition and some fundamental properties of almost periodic functions. For more details on the theory of almost periodic functions and its application one can refer to Corduneanu [7] or Fink [8].

Proposition 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. The following conditions are equivalent:*

(i) $\forall \varepsilon > 0, \exists l(\varepsilon) > 0$ *such that* $\forall a \in \mathbb{R}, \exists \tau \in [a, a + l(\varepsilon))$ *satisfying*

$$\left|f(t+\tau) - f(t)\right| < \varepsilon, \quad \forall t \in \mathbb{R}; \tag{2.1}$$

(ii) $\forall \varepsilon > 0$, *there is a trigonometric polynomial* $T_\varepsilon(t) = \Sigma_{k=1}^{n}\{a_k \cdot \cos(\lambda_k t) + b_k \cdot \sin(\lambda_k t)\}$ *where* $a_k, b_k, \lambda_k \in \mathbb{R}, 1 \le k \le n$ *such that* $|f(t) - T_\varepsilon(t)| < \varepsilon,\ \forall t \in \mathbb{R}$;

(iii) *for all real sequence* $(h_n)_n$ *there is a subsequence* $(h_{n_k})_k$ *such that* $(f(\cdot + h_{n_k}))_k$ *converges uniformly on* $\mathbb{R}$.

Definition 2.2. One says that a continuous function f is almost periodicif and only if f satisfies one of the three conditions of Proposition 2.1.

A number τ verifying (2.1) is called ε almost period. By using Proposition 2.1 we get the following property of almost periodic functions.

Proposition 2.3. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is almost periodic. Then f is bounded uniformly continuous function.*

Proposition 2.4. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is almost periodic. Then $(1/T)\int_a^{a+T} f(t)dt$ converges as $T \to +\infty$ uniformly with respect to $a \in R$. Moreover the limit does not depend on a and it is called the average of f:*

$$\exists\langle f\rangle := \lim_{T\to+\infty}\frac{1}{T}\int_a^{a+T} f(t)dt, \quad \textit{uniformly} \;\; \textit{w.r.t.} \; a \in R. \tag{2.2}$$

Proposition 2.5. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is almost periodic and denote by F a primitive of f. Then F is almost periodic if and only if F is bounded.*

For the goal of applications to the differential equations, Yoshizawa [9] extended almost periodic functions to so called uniformly almost periodic functions.

Definition 2.6 ([9]). One says that $u : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is almost periodic in t uniformly with respect to x if u is continuous in t uniformly with respect to x and $\forall\varepsilon > 0$, $\exists l(\varepsilon) > 0$ such that all interval of length $l(\varepsilon)$ contain a number τ which is ε almost periodic for $u(x,\cdot)$, $\forall x \in \overline{\Omega}$

$$|u(x,t+\tau) - u(x,t)| < \varepsilon, \quad \forall(x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{2.3}$$

3. Almost Periodic Viscosity Solutions

In this section we get some results for almost periodic viscosity solutions.

We consider the following two equations to get some results used for the existence and uniqueness of almost periodic viscosity solutions. That is, the Dirichlet problems of the form

$$\begin{gathered} \partial_t u + H\left(x,t,u,Du,D^2u\right) = 0, \quad \text{in } \Omega \times (0,T), \\ u(x,t) = 0, \quad \text{for } x \in \partial\Omega,\ 0 \le t < T, \end{gathered} \tag{3.1}$$

$$\begin{gathered} H\left(x,u,Du,D^2u\right) = 0, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{3.2}$$

in (3.2) Ω is an arbitrary open subset of $\mathbb{R}^N$.

In [1], Crandall et al. proved such a theorem.

Theorem 3.1 (see [1]). *Let $\mathcal{O}_i$ be a locally compact subset of $\mathbb{R}^{N_i}$ for $i = 1,\ldots,k$,*

$$\mathcal{O} = \mathcal{O}_1 \times \cdots \times \mathcal{O}_k, \tag{3.3}$$

$u_i \in USC(\mathcal{O}_i)$, and φ be twice continuously differentiable in a neighborhood of $\mathcal{O}$. Set

$$w(x) = u_1(x_1) + \cdots + u_k(x_k) \quad \textit{for } x = (x_1,\ldots,x_k) \in \mathcal{O}, \tag{3.4}$$

and suppose $\hat{x} = (\hat{x}_1, \dots, \hat{x}_k) \in \mathcal{O}$ *is a local maximum of* $w - \varphi$ *relative to* $\mathcal{O}$*. Then for each* $\varepsilon > 0$ *there exists* $X_i \in \mathcal{S}(N_i)$ *such that*

$$\left(D_{x_i}\varphi(\hat{x}), X_i\right) \in \overline{J}_{\mathcal{O}_i}^{2,+} u_i(\hat{x}_i) \quad \textit{for } i = 1, \dots, k, \tag{3.5}$$

and the block diagonal matrix with entries X_i *satisfies*

$$-\left(\frac{1}{\varepsilon} + \|A\|\right) I \le \begin{pmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_k \end{pmatrix} \le A + A^2, \tag{3.6}$$

where $A = D^2\varphi(\hat{x}) \in \mathcal{S}(N)$, $N = N_1 + \cdots + N_k$.

Put $k = 2$, $\mathcal{O}_1 = \mathcal{O}_2 = \Omega$, $u_1 = u$, $u_2 = -v$, $\varphi(x, y) = (\alpha/2)|x - y|^2$, where $\alpha > 0$, recall that $\overline{J}_{\Omega}^{2,-} v = -\overline{J}_{\Omega}^{2,+}(-v)$, then, from Theorem 3.1, at a local maximum $(\hat{x}, \hat{y})$ of $u(x) - v(y) - \varphi(x, y)$, we have

$$\begin{gathered} D_x\varphi(\hat{x}, \hat{y}) = -D_y\varphi(\hat{x}, \hat{y}) = \alpha(\hat{x} - \hat{y}), \\ A = \alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \quad A^2 = 2\alpha A, \quad \|A\| = 2\alpha. \end{gathered} \tag{3.7}$$

We conclude that for each $\varepsilon > 0$, there exists $X, Y \in \mathcal{S}(N)$ such that

$$\begin{gathered} \left(\alpha(\hat{x} - \hat{y}), X\right) \in \overline{J}_{\Omega}^{2,+} u(\hat{x}), \qquad \left(\alpha(\hat{x} - \hat{y}), Y\right) \in \overline{J}_{\Omega}^{2,-} v(\hat{y}), \\ -\left(\frac{1}{\varepsilon} + 2\alpha\right)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \alpha(1 + 2\varepsilon\alpha)\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \end{gathered} \tag{3.8}$$

Choosing $\varepsilon = 1/\alpha$ one can get

$$-3\alpha\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3\alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{3.9}$$

To prove the existence and uniqueness of viscosity solutions, let us see the following main hypotheses first.

As in Crandall et al. [1], we present a fundamental monotonicity condition of H, that is,

$$H(x, r, p, X) \le H(x, s, p, Y) \quad \text{whenever } r \le s, \ Y \le X, \tag{3.10}$$

where $r, s \in \mathbb{R}$, $x \in \Omega$, $p \in \mathbb{R}^N$, $X, Y \in \mathcal{S}(N)$. Then we will say that H is *proper*.

Assume there exists $\gamma > 0$ such that

$$\gamma(r-s) \le H(x,r,p,X) - H(x,s,p,X), \quad \text{for } r \ge s,\ (x,p,X) \in \overline{\Omega} \times \mathbb{R}^N \times \mathcal{S}(N), \tag{3.11}$$

and there is a function $\omega : [0,\infty] \to [0,\infty]$ that satisfies $\omega(0+) = 0$ such that

$$\begin{gathered} H(y,r,\alpha(x-y),Y) - H(x,r,\alpha(x-y),X) \le \omega\left(\alpha|x-y|^2 + |x-y|\right) \\ \text{whenever } x,y \in \Omega,\ r \in \mathbb{R},\ X,Y \in \mathcal{S}(N),\ \text{and (3.9) holds.} \end{gathered} \tag{3.12}$$

Now we can easily prove the following result. There is a similar result for first-order Hamilton-Jacobi equations in the book of Barles [10].

Lemma 3.2. *Assume that $H \in C(\overline{\Omega} \times (0,T] \times \mathbb{R} \times \mathbb{R}^N \times S(N))$ and $u \in C(\overline{\Omega} \times (0,T])$ is a viscosity subsolution (resp., supersolution) of $\partial_t u + H(x,t,u,Du,D^2u) = 0, (x,t) \in \Omega \times (0,T)$. Then u is a viscosity subsolution (resp., supersolution) of $\partial_t u + H(x,t,u,Du,D^2u) = 0, (x,t) \in \Omega \times (0,T]$.*

Proof. Since $u \in C(\overline{\Omega} \times (0,T])$ is a viscosity subsolution of $\partial_t u + H(x,t,u,Du,D^2u) = 0, (x,t) \in \Omega \times (0,T)$, if $\forall \varphi \in C^2(\Omega \times (0,T])$ and local maximum $(\hat{x},\hat{t}) \in (\Omega \times (0,T))$ of $u - \varphi$, we have

$$\partial_t \varphi\left(\hat{x},\hat{t}\right) + H\left(\hat{x},\hat{t},u\left(\hat{x},\hat{t}\right),D\varphi\left(\hat{x},\hat{t}\right),D^2\varphi\left(\hat{x},\hat{t}\right)\right) \le 0. \tag{3.13}$$

Now we prove that if (x_0,T) is a local maximum of $u - \varphi$ in $\Omega \times (0,T]$, then

$$\partial_t \varphi(x_0,T) + H\left(x_0,T,u(x_0,T),D\varphi(x_0,T),D^2\varphi(x_0,T)\right) \le 0. \tag{3.14}$$

Suppose that (x_0,T) is a strict local maximum of $u - \varphi$ in $\Omega \times (0,T]$, we consider the function

$$\varphi_\varepsilon(x,t) = u(x,t) - \varphi(x,t) - \varepsilon(T-t)^{-1} \tag{3.15}$$

for small $\varepsilon > 0$. Then we know that the function $\varphi_\varepsilon(x,t)$ has a local maximum point $(x_\varepsilon,t_\varepsilon)$ such that $t_\varepsilon < T$ and $(x_\varepsilon,t_\varepsilon) \to (x_0,T)$ when $\varepsilon \to 0$. So at the point $(x_\varepsilon,t_\varepsilon)$ we deduce that

$$\partial_t \varphi(x_\varepsilon,t_\varepsilon) + \frac{\varepsilon}{(T-t_\varepsilon)^2} + H\left(x_\varepsilon,t_\varepsilon,u(x_\varepsilon,t_\varepsilon),D\varphi(x_\varepsilon,t_\varepsilon),D^2\varphi(x_\varepsilon,t_\varepsilon)\right) \le 0. \tag{3.16}$$

As the term $\varepsilon/(T-t_\varepsilon)^2$ is positive, so we obtain

$$\partial_t \varphi(x_\varepsilon,t_\varepsilon) + H\left(x_\varepsilon,t_\varepsilon,u(x_\varepsilon,t_\varepsilon),D\varphi(x_\varepsilon,t_\varepsilon),D^2\varphi(x_\varepsilon,t_\varepsilon)\right) \le 0. \tag{3.17}$$

The results following upon letting $\varepsilon \to 0$. This process can be easily applied to the viscosity supersolution case. $\square$

By time periodicity one gets the following.

Proposition 3.3. *Assume that $H \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times S(N))$ and $u \in C(\overline{\Omega} \times \mathbb{R})$ are T periodic such that u is a viscosity subsolution (resp., supersolution) of $\partial_t u + H(x,t,u,Du,D^2u) = 0, (x,t) \in \Omega \times (0,T)$. Then u is a viscosity subsolution (resp., supersolution) of $\partial_t u + H(x,t,u,Du,D^2u) = 0, (x,t) \in \Omega \times \mathbb{R}$.*

Crandall et al. have proved the following two comparison results.

Theorem 3.4 (see [6]). *Let Ω be a bounded open subset of $\mathbb{R}^N$, $F \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N))$ be proper and satisfy* (3.11), (3.12). *Let $u \in USC(\overline{\Omega})$ (resp., $v \in LSC(\overline{\Omega})$) be a subsolution (resp., supersolution) of $F = 0$ in Ω and $u \le v$ on $\partial\Omega$. Then $u \le v$ in $\overline{\Omega}$.*

Theorem 3.5 (see [1]). *Let $\Omega \in \mathbb{R}^N$ be open and bounded. Let $H \in C(\overline{\Omega} \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N))$ be continuous, proper, and satisfy* (3.12) *for each fixed $t \in [0,T)$, with the same function ω. If u is a subsolution of* (1.2) *and v is a supersolution of* (1.2)*, then $u \le v$ on $[0,T) \times \Omega$.*

We generalize the comparison result in article [2] for first-order Hamilton-Jacobi equations, and get two theorems for second-order parabolic equations. Let us see a proposition we will need in the proof of the comparison result (see [1]).

Proposition 3.6 (see [1]). *Let $\mathcal{O}$ be a subset of $\mathbb{R}^M$, $\Phi \in USC(\mathcal{O})$, $\Psi \in LSC(\mathcal{O})$, $\Psi \ge 0$, and*

$$M_\alpha = \sup_{\mathcal{O}} (\Phi(x) - \alpha\Psi(x)) \tag{3.18}$$

for $\alpha > 0$. Let $-\infty < \lim_{\alpha \to \infty} M_\alpha < \infty$ and $x_\alpha \in \mathcal{O}$ be chosen so that

$$\lim_{\alpha \to \infty} (M_\alpha - (\Phi(x_\alpha) - \alpha\Psi(x_\alpha))) = 0. \tag{3.19}$$

Then the following holds:

$$\begin{aligned} &(i) \quad \lim_{\alpha \to \infty} \alpha\Psi(x_\alpha) = 0, \\ &(ii) \quad \Psi(\hat{x}) = 0, \qquad \lim_{\alpha \to \infty} M_\alpha = \Phi(\hat{x}) = \sup_{\{\Psi(x)=0\}} \Phi(x) \\ &\qquad \text{whenever } \hat{x} \in \mathcal{O} \text{ is a limit point of } x_\alpha \text{ as } \alpha \longrightarrow \infty. \end{aligned} \tag{3.20}$$

Remark 3.7. In Proposition 3.6, when M, $\mathcal{O}$, x, $\Phi(x)$, $\Psi(x)$ are replaced by $2N$, $\mathcal{O} \times \mathcal{O}$, (x,y), $u(x) - v(y)$, $(1/2)|x-y|^2$, respectively, we can get the following results:

$$\begin{aligned} &\text{(i)} \quad \lim_{\alpha \to \infty} \alpha \left| x_\alpha - y_\alpha \right|^2 = 0, \\ &\text{(ii)} \quad \Psi(\hat{x}) = 0, \qquad \lim_{\alpha \to \infty} M_\alpha = u(\hat{x}) - v(\hat{x}) = \sup_{\mathcal{O}} (u(x) - v(x)) \\ &\qquad \text{whenever } \hat{x} \in \mathcal{O} \text{ is a limit point of } x_\alpha \text{ as } \alpha \longrightarrow \infty. \end{aligned} \tag{3.21}$$

Now we have the following.

Theorem 3.8. *Let $\Omega \in \mathbb{R}^N$ be open and bounded. Assume $H \in C(\overline{\Omega} \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N))$ be continuous, proper, and satisfy* (3.11), (3.12) *for each fixed $t \in [0,T)$. Let u, v be bounded u.s.c. subsolution of $\partial_t u + H(x,t,u,Du,D^2u) = f(x,t)$ in $\Omega \times (0,T)$, $u(x,t) = 0$ for $x \in \partial\Omega$ and $0 \le t < T$, respectively, l.s.c. supersolution of $\partial_t v + H(x,t,v,Dv,D^2v) = g(x,t)$ in $\Omega \times (0,T)$, $v(x,t) = 0$ for $x \in \partial\Omega$ and $0 \le t < T$ where $f, g \in BUC(\overline{\Omega} \times [0,T])$.*

$$\lim_{t\searrow 0}(u(x,t) - u(x,0))_+ = \lim_{t\searrow 0}(v(x,t) - v(x,0))_- = 0, \quad \text{uniformly for } x \in \overline{\Omega},$$
$$u(\cdot,0) \in BUC\left(\overline{\Omega}\right) \quad \text{or} \quad v(\cdot,0) \in BUC\left(\overline{\Omega}\right). \tag{3.22}$$

Then one has for all $t \in [0,T]$

$$e^{\gamma t}\|(u(\cdot,t) - v(\cdot,t))\|_{L^\infty(\overline{\Omega})} \le \|(u(\cdot,0) - v(\cdot,0))_+\|_{L^\infty(\overline{\Omega})} + \int_0^t e^{\gamma s}\|(f(\cdot,s) - g(\cdot,s))\|_{L^\infty(\overline{\Omega})}ds, \tag{3.23}$$

where $\gamma = \gamma_{R_0}$, $R_0 = \max(\|u\|_{L^\infty(\overline{\Omega}\times(0,T))}, \|v\|_{L^\infty(\overline{\Omega}\times(0,T))})$.

Proof. Let us consider the function given by

$$w_\alpha(x,y,t) = u(x,t) - v(y,t) - \varphi(x,y,t), \tag{3.24}$$

where $\varphi(x,y,t) = (\alpha/2)(|x-y|^2 + \phi(t))$, and $\phi(t) \in C^1([0,T])$. As we know that u and v are bounded semicontinuous in $\overline{\Omega} \times [0,T]$ and $\Omega \in \mathbb{R}^N$ is open and bounded, we can find $(\hat{x}(t_\alpha), \hat{y}(t_\alpha)) \in \overline{\Omega} \times \overline{\Omega}$, for $t_\alpha \in [0,T]$ such that $M_\alpha(t_\alpha) := \sup_{\overline{\Omega}\times\overline{\Omega}}(u(x,t_\alpha) - v(y,t_\alpha) - \varphi(x,y,t_\alpha)) = u(\hat{x}(t_\alpha),t_\alpha) - v(\hat{y}(t_\alpha),t_\alpha) - \varphi(\hat{x}(t_\alpha),\hat{y}(t_\alpha),t_\alpha)$, here without loss of generality, we can assume that $M_\alpha(t_\alpha) = 0$. Since $\overline{\Omega} \times \overline{\Omega} \times [0,T]$ is compact, these maxima $(\hat{x}(t_\alpha), \hat{y}(t_\alpha), t_\alpha)$ converge to a point of the form $(z(t), z(t), t)$ from Remark 3.7. From Theorem 3.1 and its following discussion, there exists $X_\alpha, Y_\alpha \in \mathcal{S}(N)$ such that

$$(\alpha(\hat{x}(t_\alpha) - \hat{y}(t_\alpha)), X_\alpha) \in \overline{J}_\Omega^{2,+}u(\hat{x}(t_\alpha),t_\alpha), \qquad (\alpha(\hat{x}(t_\alpha) - \hat{y}(t_\alpha)), Y_\alpha) \in \overline{J}_\Omega^{2,-}v(\hat{y}(t_\alpha),t_\alpha),$$
$$-3\alpha\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X_\alpha & 0 \\ 0 & -Y_\alpha \end{pmatrix} \le 3\alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{3.25}$$

which implies $X_\alpha \le Y_\alpha$. At the maximum point, from the definition of u being a subsolution and v being a supersolution we arrive at the following:

$$\partial_{t_\alpha}\varphi(\hat{x}(t_\alpha),\hat{y}(t_\alpha),t_\alpha) + H(\hat{x}(t_\alpha),t_\alpha,u(\hat{x}(t_\alpha),t_\alpha),\alpha(\hat{x}(t_\alpha) - \hat{y}(t_\alpha)),X_\alpha)$$
$$- H(\hat{y}(t_\alpha),t_\alpha,v(\hat{y}(t_\alpha),t_\alpha),\alpha(\hat{x}(t_\alpha) - \hat{y}(t_\alpha)),Y_\alpha) \le f(\hat{x}(t_\alpha),t_\alpha) - g(\hat{y}(t_\alpha),t_\alpha), \tag{3.26}$$

by the *proper* condition of H, we have

$$\begin{aligned} &H\left(\widehat{y}(t_\alpha), t_\alpha, v\left(\widehat{y}(t_\alpha), t_\alpha\right), \alpha\left(\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)\right), Y_\alpha\right) \\ &\quad \leq H\left(\widehat{y}(t_\alpha), t_\alpha, v\left(\widehat{y}(t_\alpha), t_\alpha\right), \alpha\left(\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)\right), X_\alpha\right), \end{aligned} \tag{3.27}$$

as we know that H satisfying (3.12) then we deduce that

$$\begin{aligned} &H\left(\widehat{x}(t_\alpha), t_\alpha, u(\widehat{x}(t_\alpha), t_\alpha), \alpha\left(\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)\right), X_\alpha\right) \\ &\qquad - H\left(\widehat{y}(t_\alpha), t_\alpha, v\left(\widehat{y}(t_\alpha), t_\alpha\right), \alpha\left(\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)\right), X_\alpha\right) \\ &\quad = H\left(\widehat{x}(t_\alpha), t_\alpha, u(\widehat{x}(t_\alpha), t_\alpha), \alpha\left(\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)\right), X_\alpha\right) \\ &\qquad - H\left(\widehat{y}(t_\alpha), t_\alpha, u(\widehat{x}(t_\alpha), t_\alpha), \alpha\left(\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)\right), X_\alpha\right) \\ &\qquad + H\left(\widehat{y}(t_\alpha), t_\alpha, u(\widehat{x}(t_\alpha), t_\alpha), \alpha\left(\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)\right), X_\alpha\right) \\ &\qquad - H\left(\widehat{y}(t_\alpha), t_\alpha, v\left(\widehat{y}(t_\alpha), t_\alpha\right), \alpha\left(\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)\right), X_\alpha\right) \\ &\quad \geq H\left(\widehat{y}(t_\alpha), t_\alpha, u(\widehat{x}(t_\alpha), t_\alpha), \alpha\left(\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)\right), X_\alpha\right) \\ &\qquad - H\left(\widehat{y}(t_\alpha), t_\alpha, v\left(\widehat{y}(t_\alpha), t_\alpha\right), \alpha\left(\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)\right), X_\alpha\right) \\ &\qquad - \omega\left(\alpha\left|\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)\right|^2 + \left|\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)\right|\right), \end{aligned} \tag{3.28}$$

hence we get

$$\begin{aligned} &\partial_{t_\alpha}\varphi\left(\widehat{x}(t_\alpha), \widehat{y}(t_\alpha), t_\alpha\right) + H\left(\widehat{y}(t_\alpha), t_\alpha, u(\widehat{x}(t_\alpha), t_\alpha), \alpha\left(\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)\right), X_\alpha\right) \\ &\qquad - H\left(\widehat{y}(t_\alpha), t_\alpha, v\left(\widehat{y}(t_\alpha), t_\alpha\right), \alpha\left(\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)\right), X_\alpha\right) \\ &\qquad - \omega\left(\alpha\left|\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)\right|^2 + \left|\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)\right|\right) \\ &\quad \leq h(t_\alpha), \end{aligned} \tag{3.29}$$

where $h(t_\alpha) = f(\widehat{x}(t_\alpha), t_\alpha) - g(\widehat{y}(t_\alpha), t_\alpha),\ \forall t_\alpha \in [0, T]$. For any $t_\alpha \in [0, T]$ consider

$$\begin{aligned} r(t_\alpha) = {} & \frac{1}{u(\widehat{x}(t_\alpha), t_\alpha) - v\left(\widehat{y}(t_\alpha), t_\alpha\right)}\left(H\left(\widehat{y}(t_\alpha), t_\alpha, u(\widehat{x}(t_\alpha), t_\alpha), \alpha\left(\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)\right), X_\alpha\right)\right. \\ &- \gamma u(\widehat{x}(t_\alpha), t_\alpha) - H\left(\widehat{y}(t_\alpha), t_\alpha, v\left(\widehat{y}(t_\alpha), t_\alpha\right), \alpha\left(\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)\right), X_\alpha\right) \\ &\left. + \gamma v\left(\widehat{y}(t_\alpha), t_\alpha\right)\right), \end{aligned} \tag{3.30}$$

if $u(\widehat{x}(t_\alpha), t_\alpha) \neq v(\widehat{y}(t_\alpha), t_\alpha)$, and $r(t_\alpha) = 0$ otherwise. From hypothesis (3.11) we deduce that $H(x, t, z, p, X) - \gamma \cdot z$ is nondecreasing with respect to z, then we have $r(t_\alpha) \geq 0$ for all $t_\alpha \in [0, T]$. Hence we have

$$\begin{aligned} &H(\widehat{y}(t_\alpha), t_\alpha, u(\widehat{x}(t_\alpha), t_\alpha), \alpha(\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)), X_\alpha) \\ &\quad - H(\widehat{y}(t_\alpha), t_\alpha, v(\widehat{y}(t_\alpha), t_\alpha), \alpha(\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)), X_\alpha) \\ &= (\gamma + r(t_\alpha))(u(\widehat{x}(t_\alpha), t_\alpha) - v(\widehat{y}(t_\alpha), t_\alpha)), \quad \forall t_\alpha \in [0, T]. \end{aligned} \tag{3.31}$$

Notice that $u(\widehat{x}(t_\alpha), t_\alpha) - v(\widehat{y}(t_\alpha), t_\alpha) = \varphi(\widehat{x}(t_\alpha), \widehat{y}(t_\alpha), t_\alpha)$, we get

$$\begin{aligned} &\partial_{t_\alpha}\varphi(\widehat{x}(t_\alpha), \widehat{y}(t_\alpha), t_\alpha) + (\gamma + r(t_\alpha))\varphi(\widehat{x}(t_\alpha), \widehat{y}(t_\alpha), t_\alpha) \\ &\quad - \omega\left(\alpha\left|\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)\right|^2 + \left|\widehat{x}(t_\alpha) - \widehat{y}(t_\alpha)\right|\right) \\ &\leq h(t_\alpha). \end{aligned} \tag{3.32}$$

Replacing $u(\widehat{x}(t_\alpha), t_\alpha) - v(\widehat{y}(t_\alpha), t_\alpha)$ by $\varphi(\widehat{x}(t_\alpha), \widehat{y}(t_\alpha), t_\alpha)$ in the expression of $r(t_\alpha)$ we know that $r(\cdot)$ is integrable and denote by $A(t_\alpha)$ the function $A(t_\alpha) = \int_0^{t_\alpha}\{\gamma + r(\sigma)\}d\sigma$, $t_\alpha \in [0, T]$. After integration one gets

$$\varphi(t_\alpha) \leq e^{-A(t_\alpha)}\left(\varphi(0) + \int_0^{t_\alpha} e^{A(s_\alpha)} \cdot \left(h(s_\alpha) + \omega\left(\alpha\left|\widehat{x}(s_\alpha) - \widehat{y}(s_\alpha)\right|^2 + \left|\widehat{x}(s_\alpha) - \widehat{y}(s_\alpha)\right|\right)\right) ds_\alpha\right), \tag{3.33}$$

$t_\alpha \in [0, T]$. Now taking $u(\widehat{x}(t_\alpha), t_\alpha) - v(\widehat{y}(t_\alpha), t_\alpha)$ instead of $\varphi(\widehat{x}(t_\alpha), \widehat{y}(t_\alpha), t_\alpha)$ for any $t_\alpha \in [0, T]$ and letting $\alpha \to \infty$ we can get

$$u(z(t), t) - v(z(t), t) \leq e^{-A(t)}\left(u(z(0), 0) - v(z(0), 0) + \int_0^t e^{A(s)} \cdot h(s) ds\right), \quad t \in [0, T]. \tag{3.34}$$

Finally we deduce that for all $t \in [0, T]$

$$\begin{aligned} e^{\gamma t}\|(u(\cdot, t) - v(\cdot, t))\|_{L^\infty(\overline{\Omega})} &\leq \|(u(\cdot, 0) - v(\cdot, 0))_+\|_{L^\infty(\overline{\Omega})} \\ &\quad + \int_0^t e^{\gamma s}\|(f(\cdot, s) - g(\cdot, s))\|_{L^\infty(\overline{\Omega})} ds. \end{aligned} \tag{3.35}$$

□

Theorem 3.9. *Let $\Omega \in \mathbb{R}^N$ be open and bounded. Assume $H \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N))$ be continuous, proper, T periodic, and satisfy* (3.11), (3.12). *Let u be a bounded time periodic viscosity u.s.c. subsolution of $\partial_t u + H(x, t, u, Du, D^2u) = f(x, t)$ in $\Omega \times \mathbb{R}$, $u(x, t) = 0$ for $(x, t) \in \partial\Omega \times \mathbb{R}$*

and v a bounded time periodic viscosity l.s.c. supersolution of $\partial_t v + H(x,t,v,Dv,D^2v) = g(x,t)$ in $\Omega \times \mathbb{R}, v(x,t) = 0$ for $(x,t) \in \partial\Omega \times \mathbb{R}$, where $f, g \in BUC(\overline{\Omega} \times \mathbb{R})$. Then one has

$$\sup_{x\in\overline{\Omega}}(u(x,t) - v(x,t)) \leq \sup_{s\leq t}\int_s^t \sup_{x\in\overline{\Omega}}\bigl(f(x,\sigma) - g(x,\sigma)\bigr)d\sigma. \tag{3.36}$$

Proof. As the proof of Theorem 3.8, we get equation (3.34)

$$u(z(t),t) - v(z(t),t) \leq e^{-A(t)}\left(u(z(0),0) - v(z(0),0) + \int_0^t e^{A(s)} \cdot h(s)ds\right), \quad t \in [0,T]. \tag{3.37}$$

We introduce that $F(s) = -\int_s^t h(\sigma)d\sigma,\ s,t \in [0,T]$. By integration by parts we have

$$\begin{aligned}\int_0^t e^{A(s)}h(s)ds &= \int_0^t e^{A(s)}F'(s)ds \\ &= \int_0^t h(\sigma)d\sigma + \int_0^t e^{A(s)}A'(s)\int_s^t h(\sigma)d\sigma\, ds \\ &\leq \int_0^t h(\sigma)d\sigma + \left(e^{A(t)} - 1\right)\sup_{0\leq s\leq t}\int_s^t h(\sigma)d\sigma.\end{aligned} \tag{3.38}$$

We deduce that for all $t \in [0,T]$ we have

$$\begin{aligned}\sup_{x\in\overline{\Omega}}(u(x,t) - v(x,t)) &\leq e^{-\gamma t}\sup_{x\in\overline{\Omega}}(u(x,0) - v(x,0))_+ \\ &\quad + \sup_{0\leq s\leq t}\int_s^t \sup_{x\in\overline{\Omega}}\bigl(f(x,\sigma) - g(x,\sigma)\bigr)d\sigma.\end{aligned} \tag{3.39}$$

Similar to the proof of Corollary 2.2 in paper [2], we can reach the conclusion. □

In order to prove the existence of viscosity solution, we recall the the Perron's method as follows (see [1, 5]). To discuss the method, we assume if $u : \mathcal{O} \to [-\infty,\infty]$ where $\mathcal{O} \subset \mathbb{R}^N$, then

$$\begin{aligned}u^*(x) &= \lim_{r\downarrow 0}\sup\{u(y) : y \in \mathcal{O} \text{ and } |y - x| \leq r\}, \\ u_*(x) &= \lim_{r\downarrow 0}\inf\{u(y) : y \in \mathcal{O} \text{ and } |y - x| \leq r\}.\end{aligned} \tag{3.40}$$

Theorem 3.10 (Perron's method). *Let comparison hold for* (3.2); *that is, if w is a subsolution of* (3.2) *and v is a supersolution of* (3.2), *then $w \leq v$. Suppose also that there is a subsolution $\underline{u}$ and* *a supersolution* $\overline{u}$ *of* (3.2) *that satisfies the boundary condition* $\underline{u}_*(x) = \overline{u}^*(x) = 0$ *for* $x \in \partial\Omega$*. Then*

$$W(x) = \sup\{w(x) : \underline{u} \leq w \leq \overline{u} \ \text{ and } w \text{ is a subsolution of } (3.2)\} \tag{3.41}$$

is a solution of (3.2).

From paper [1], we have the following remarks as a supplement to Theorem 3.10.

Remarks 3.11. Notice that the subset Ω in (3.2) in some part of the proof in Theorem 3.10 was just open in $\mathbb{R}^N$. In order to generalize this and formulate the version of Theorem 3.10 we will need later, we now make some remarks. Suppose $\mathcal{O}$ is locally compact, G^+, G_- are defined on $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N)$ and have the following properties: G^+ is upper semicontinuous, G_- is lower semicontinuous, and classical solutions (twice continuously differentiable solutions in the pointwise sense) of $G^+ \leq 0$ on relatively open subset of $\mathcal{O}$ are solutions of $G_- \leq 0$. Suppose, moreover, that whenever u is a solution of $G_- \leq 0$ on $\mathcal{O}$ and v is a solution of $G^+ \geq 0$ on $\mathcal{O}$ we have $u \leq v$ on $\mathcal{O}$. Then we conclude that the existence of such a subsolution and supersolution guarantees that there is a unique function u, obtained by the Perron's construction, that is a solution of both $G^+ \geq 0$ and $G_- \leq 0$ on $\mathcal{O}$.

Now we will prove the uniqueness and existence of almost periodic viscosity solutions. For the uniqueness we have the following result.

Theorem 3.12. *Let* $\Omega \in \mathbb{R}^N$ *be open and bounded. Assume* $H \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N))$ *be continuous, proper, and satisfy* (3.11), (3.12) *for* $t \in \mathbb{R}$*. Let* u *be a bounded u.s.c. viscosity subsolution of* $\partial_t u + H(x,t,u,Du,D^2u) = f(x,t)$ *in* $\Omega \times \mathbb{R}$, $u(x,t) = 0$ *for* $(x,t) \in \partial\Omega \times \mathbb{R}$*, and* v *a bounded l.s.c. viscosity supersolution of* $\partial_t v + H(x,t,v,Dv,D^2v) = g(x,t)$*, in* $\Omega \times \mathbb{R}$, $v(x,t) = 0$ *for* $(x,t) \in \partial\Omega \times \mathbb{R}$ *where* $f, g \in BUC\ (\overline{\Omega} \times \mathbb{R})$*. Then one has for all* $t \in \mathbb{R}$

$$\sup_{x\in\overline{\Omega}}(u(x,t) - v(x,t))_+ \leq e^{-\gamma t}\int_{-\infty}^{t} e^{\gamma\sigma}\sup_{x\in\overline{\Omega}}\left(f(x,\sigma) - g(x,\sigma)\right)_+ d\sigma. \tag{3.42}$$

Proof. Take $t_0, t \in \mathbb{R}$, $t_0 \leq t$ and by using Theorem 3.8 write for all $x \in \overline{\Omega}$

$$u(x,t) - v(x,t) \leq e^{-\gamma(t-t_0)} \cdot (\|u\|_\infty + \|v\|_\infty) + e^{-\gamma t}\int_{t_0}^{t} e^{\gamma\sigma}\sup_{y\in\overline{\Omega}}\left(f(y,\sigma) - g(y,\sigma)\right)_+ d\sigma, \tag{3.43}$$

where $\gamma = \gamma_{R_0}$, $R_0 = \max(\|u\|_\infty, \|v\|_\infty)$. Then the conclusion follows by passing $t_0 \to -\infty$. $\square$

Now we concentrate on the existence part.

Theorem 3.13. *Let* Ω *be a bounded open subset in* $\mathbb{R}^N$*. Assume* $H \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N))$ *be continuous, proper, and satisfy* (3.11), (3.12)*. Assume that* $f : \mathbb{R} \to \mathbb{R}$ *is almost periodic and* $H(x,-M,0,0) \leq f(t) \leq H(x,M,0,0)$, $\forall(x,t) \in \Omega \times \mathbb{R}$*. Then there is a time almost periodic viscosity solution in* $BUC(\overline{\Omega} \times \mathbb{R})$ *of* (1.1)*, where* $M > 0$ *is a constant.*

Proof. Here we consider the problem

$$\begin{gathered}\partial_t u_n + H\left(x, u_n, Du_n, D^2 u_n\right) = f(t), \quad (x,t) \in \Omega \times (-n, +\infty), \\ u_n(x,t) = 0, \quad (x,t) \in \partial\Omega \times [-n, +\infty), \\ u_n(x, -n) = 0, \quad x \in \overline{\Omega}\end{gathered} \tag{3.44}$$

for all $n \geq 1$. As we know that $H(x, -M, 0, 0) \leq f(t) \leq H(x, M, 0, 0)$, $\forall (x,t) \in \Omega \times \mathbb{R}$, there exists a viscosity solution $u_n(x,t)$ of (3.44) from Theorem 3.5 and Remark 3.11. Then we will prove that for all $t \in \mathbb{R}$, $(u_n(t))_{n \geq -t}$ converges to a almost periodic viscosity solution of (1.1). As we already know that $H(x, -M, 0, 0) \leq f(t) \leq H(x, M, 0, 0), \forall (x,t) \in \Omega \times \mathbb{R}$, we can deduce by Theorem 3.5 that $-M \leq u_n(x,t) \leq M, \forall (x,t) \in \overline{\Omega} \times [-n, +\infty)$. Similar to the proof of Proposition 6.6 in paper [2], using Theorem 3.8, we get for $t = \tilde{t}$ and n large enough

$$\left|u_n\left(x, \tilde{t}\right) - u_n\left(x, \tilde{t} + \tau\right)\right| \leq 2M \cdot e^{-\gamma(\tilde{t} - t_n)} + e^{-\gamma \tilde{t}} \int_{t_n}^{\tilde{t}} e^{\gamma\sigma} \gamma\, \varepsilon \, d\sigma \leq 2M \cdot e^{-\gamma(\tilde{t} - t_n)} + \varepsilon. \tag{3.45}$$

By passing $n \to +\infty$ we have $t_n \to -\infty$ and therefore

$$\left|u\left(x, \tilde{t}\right) - u\left(x, \tilde{t} + \tau\right)\right| \leq \varepsilon, \quad \left(x, \tilde{t}\right) \in \Omega \times \mathbb{R}. \tag{3.46}$$

Since we already know that $u \in \mathrm{BUC}(\overline{\Omega} \times [a,b])$, $\forall a, b \in \mathbb{R}, a \leq b$, by time almost periodicity we deduce also that $u \in \mathrm{BUC}(\overline{\Omega} \times \mathbb{R})$. □

When H does not satisfy the hypothesis (3.11), we study the time almost periodic viscosity solutions of

$$\begin{gathered}\partial_t u + H\left(x, u, Du, D^2 u\right) = f(t), \quad (x,t) \in \Omega \times \mathbb{R}, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}.\end{gathered} \tag{3.47}$$

We introduce also the stationary equation

$$\begin{gathered}H\left(x, u, Du, D^2 u\right) = \langle f \rangle, \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega.\end{gathered} \tag{3.48}$$

Then we can prove our main theorem as follows.

Theorem 3.14. *Let $\Omega \in \mathbb{R}^N$ be open and bounded. Assume $H \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N))$ be continuous, proper, and satisfy* (3.12) *for $t \in \mathbb{R}$. Assume that $f : \mathbb{R} \to \mathbb{R}$ is almost periodic function such that $F(t) = \int_0^t \{f(\sigma) - \langle f \rangle\} d\sigma$ is bounded on $\mathbb{R}$. Then there is a bounded time almost periodic viscosity solution of* (3.47) *and if only if there is a bounded viscosity solution of* (3.48).

Proof. Let $\sup\{|H(x,0,0,0)| : x \in \Omega\} = C$, then $C < +\infty$. Assume that (3.48) has a bounded viscosity solution V, we take $M_\alpha = \|V\|_{L^\infty(\overline{\Omega})} + (1/\alpha)(C + \|f\|_{L^\infty(\overline{\Omega})})$ for $\alpha > 0$, and observe that

$$\alpha(-M_\alpha - V(x)) + H(x, -M_\alpha, 0, 0) \le f(t) \le \alpha(M_\alpha - V(x)) + H(x, M_\alpha, 0, 0), \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{3.49}$$

Then by using Perron's Method from Theorem 3.10 and Remark 3.11 we can construct the family of solutions V_α for

$$\begin{gathered} \alpha(V_\alpha - V(x)) + H\left(x, V_\alpha, DV_\alpha, D^2 V_\alpha\right) = \langle f \rangle, \quad x \in \Omega, \\ V_\alpha(x) = 0, \quad x \in \partial\Omega, \end{gathered} \tag{3.50}$$

and the family of time almost periodic solutions v_α for

$$\begin{gathered} \alpha(v_\alpha - V(x)) + \partial_t v_\alpha + H\left(x, v_\alpha, Dv_\alpha, D^2 v_\alpha\right) = f(t), (x,t) \in \Omega \times \mathbb{R}, \\ v_\alpha(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}. \end{gathered} \tag{3.51}$$

In fact we have $V_\alpha = V$ for any $\alpha > 0$ and by using Theorem 3.9 we have

$$v_\alpha(x,t) - V(x) = v_\alpha(x,t) - V_\alpha(x) \le \sup_{s \le t} \int_s^t \{f(\sigma) - \langle f \rangle\} d\sigma = \sup_{s \le t} \{F(t) - F(s)\} \le 2\|F\|_\infty, \tag{3.52}$$

similarly we can get $V(x) - v_\alpha(x,t) = V_\alpha(x) - v_\alpha(x,t) \le 2\|F\|_\infty$. From the above two inequalities we know that the family $(v_\alpha)_\alpha$ is bounded, thus we know $v_\alpha \in \mathrm{BUC}(\overline{\Omega} \times [a,b])$, $\forall a, b \in \mathbb{R}, a \le b$. Therefore we can extract a subsequence which converges uniformly on compact sets of $\overline{\Omega} \times \mathbb{R}$ to a bounded uniformly continuous function v of (3.47). Next we will check that v is almost periodic. By the hypotheses and Proposition 2.5 we deduce that F is almost periodic and thus, for all $\varepsilon > 0$ there is $l(\varepsilon/2)$ such that any interval of length $l(\varepsilon/2)$ contains an $\varepsilon/2$ almost period of F. Take an interval of length $l(\varepsilon/2)$ and τ an $\varepsilon/2$ almost period of F in this interval. We have for all $\alpha > 0, (x,t) \in \overline{\Omega} \times \mathbb{R}$

$$\begin{aligned} |v_\alpha(x, t+\tau) - v_\alpha(x,t)| &\le \left| \sup_{s \le t} \int_s^t \{f(\sigma + \tau) - f(\sigma)\} d\sigma \right| \\ &= \left| \sup_{s \le t} \left\{ \int_{s+\tau}^{t+\tau} (f(\sigma) - \langle f \rangle) d\sigma - \int_s^t (f(\sigma) - \langle f \rangle) d\sigma \right\} \right| \\ &= \left| \sup_{s \le t} \{(F(t+\tau) - F(t)) - (F(s+\tau) - F(s))\} \right| \\ &\le \varepsilon. \end{aligned} \tag{3.53}$$

After passing to the limit for $\alpha \searrow 0$ one gets $|v(x,t+\tau)-v(x,t)| \leq \varepsilon,\ \forall(x,t)\in\overline{\Omega}\times\mathbb{R}$. Hence we prove the almost periodic of v.

The converse is similar to Theorem 4.1 in paper [2], it can be easily proved from Theorems 3.8, 3.9, and Remark 3.11. □

Now we discuss asymptotic behavior of time almost periodic viscosity solutions for large frequencies, and there is a similar description for Hamilton-Jacobi equations in paper [2]. Let us see the following equation:

$$\begin{gathered}\partial_t u_n + H\left(x,u_n,Du_n,D^2u_n\right) = f_n(t), \quad (x,t)\in\Omega\times\mathbb{R},\\ u_n(x,t)=0, \quad (x,t)\in\partial\Omega\times\mathbb{R},\end{gathered} \tag{3.54}$$

where $f:\mathbb{R}\to\mathbb{R}$ is almost a periodic function. For all $n\geq 1$ notice that $f_n(t)=f(nt),\ \forall t\in\mathbb{R}$ is almost periodic and has the same average as f. Now suppose that such a hypothesis exists

$$\exists M>0 \text{ such that } H(x,-M,0,0)\leq f(t),\forall(x,t)\in\Omega\times\mathbb{R}. \tag{3.55}$$

Theorem 3.15. *Let $\Omega\in\mathbb{R}^N$ be open and bounded. Assume $H\in C(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N\times\mathcal{S}(N))$ be continuous, proper, and satisfy* (3.12) *for $t\in\mathbb{R}$ and* (3.55) *where f is almost periodic function. Suppose also that there is a bounded l.s.c viscosity supersolution $\widetilde{V}\geq -M$ of* (3.48)*, that $t\to F(t)=\int_0^t\{f(s)-\langle f\rangle\}ds$ is bounded and denote by V,v_n the minimal stationary, respectively, time almost periodic l.s.c. viscosity supersolution of* (3.48)*, respectively,* (3.54)*. Then the sequence $(v_n)_n$ converges uniformly on $\overline{\Omega}\times\mathbb{R}$ towards V and $\|v_n-V\|_{L^\infty(\overline{\Omega}\times\mathbb{R})}\leq(2/n)\|F\|_{L^\infty(\mathbb{R})},\ \forall n\geq 1$.*

Proof. As $v_n=\sup_{\alpha>0}v_{n,\alpha}$ is almost periodic, we introduce $w_{n,\alpha}(x,t)=v_{n,\alpha}(x,t/n),\ (x,t)\in\overline{\Omega}\times\mathbb{R}$, which is also almost periodic. As $v_{n,\alpha}$ satisfies in the viscosity sense $\alpha(v_{n,\alpha}+M)+\partial_t v_{n,\alpha}+H(x,v_{n,\alpha},Dv_{n,\alpha},D^2v_{n,\alpha})=f_n(t),\ (x,t)\in\Omega\times\mathbb{R}$, we deduce that $w_{n,\alpha}$ satisfies in the viscosity sense

$$\alpha(w_{n,\alpha}+M)+n\partial_t w_{n,\alpha}+H\left(x,w_{n,\alpha},Dw_{n,\alpha},D^2w_{n,\alpha}\right)=f(t), \quad (x,t)\in\Omega\times\mathbb{R}, \tag{3.56}$$

which can be rewrote as

$$\partial_t w_{n,\alpha}+\frac{1}{n}\left(\alpha w_{n,\alpha}+H\left(x,w_{n,\alpha},Dw_{n,\alpha},D^2w_{n,\alpha}\right)\right)=\frac{1}{n}(f(t)-\alpha M), \quad (x,t)\in\Omega\times\mathbb{R}. \tag{3.57}$$

Recall also that we have in the viscosity sense

$$\frac{1}{n}\left(\alpha V_\alpha+H\left(x,V_\alpha,DV_\alpha,D^2V_\alpha\right)\right)=\frac{1}{n}(\langle f\rangle-\alpha M), \quad x\in\Omega. \tag{3.58}$$

By using Theorem 3.9 we deduce that

$$w_{n,\alpha}(x,t)-V_\alpha(x)\leq\sup_{s\leq t}\frac{1}{n}\int_s^t(f(\sigma)-\langle f\rangle)d\sigma\leq\frac{1}{n}\|F\|_{L^\infty(\mathbb{R})}, \tag{3.59}$$

and similarly $V_\alpha(x) - w_{n,\alpha}(x,t) \le (2/n)\|F\|_{L^\infty(\mathbb{R})}$, $\forall n \ge 1$. We have for all $n \le 1$

$$\left| v_{n,\alpha}\left(x, \frac{t}{n}\right) - V_\alpha(x) \right| \le \frac{2}{n}\|F\|_{L^\infty(\mathbb{R})}, \tag{3.60}$$

and after passing to the limit for $\alpha \searrow 0$ one gets for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$

$$\left| v_{n,\alpha}\left(x, \frac{t}{n}\right) - V(x) \right| \le \frac{2}{n}\|F\|_{L^\infty(\mathbb{R})}. \tag{3.61}$$

Finally we deduce that $\|v_n - V\|_{L^\infty(\overline{\Omega}\times\mathbb{R})} \le (2/n)\|F\|_{L^\infty(\mathbb{R})}$ for all $n \ge 1$. □

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References

[1] M. G. Crandall, H. Ishii, and P.-L. Lions, "User's guide to viscosity solutions of second order partial differential equations," *Bulletin of the American Mathematical Society*, vol. 27, no. 1, pp. 1–67, 1992.
[2] M. Bostan and G. Namah, "Time periodic viscosity solutions of Hamilton-Jacobi equations," *Communications on Pure and Applied Analysis*, vol. 6, no. 2, pp. 389–410, 2007.
[3] D. Nunziante, "Uniqueness of viscosity solutions of fully nonlinear second order parabolic equations with discontinuous time-dependence," *Differential and Integral Equations*, vol. 3, no. 1, pp. 77–91, 1990.
[4] D. Nunziante, "Existence and uniqueness of unbounded viscosity solutions of parabolic equations with discontinuous time-dependence," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 18, no. 11, pp. 1033–1062, 1992.
[5] H. Ishii, "Perron's method for Hamilton-Jacobi equations," *Duke Mathematical Journal*, vol. 55, no. 2, pp. 369–384, 1987.
[6] M. G. Crandall and H. Ishii, "The maximum principle for semicontinuous functions," *Differential and Integral Equations*, vol. 3, no. 6, pp. 1001–1014, 1990.
[7] C. Corduneanu, *Almost Periodic Functions*, Chelsea, New York, NY, USA, 1989.
[8] A. M. Fink, *Almost Periodic Differential Equations*, vol. 377 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1974.
[9] T. Yoshizawa, "Stability properties in almost periodic systems of functional differential equations," in *Functional Differential Equations and Bifurcation*, vol. 799 of *Lecture Notes in Mathmatics*, pp. 385–409, Springer, Berlin, Germany, 1980.
[10] G. Barles, *Solutions de Viscosité des Équations de Hamilton-Jacobi*, vol. 17 of *Mathématiques & Applications*, Springer, Paris, France, 1994.