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Decay Rates of Solutions of Linear Stochastic Volterra Equations

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Abstract

The paper studies the exponential and non–exponential convergence rate to zero of solutions of scalar linear convolution Itô-Volterra equations in which the noise intensity depends linearly on the current state. By exploiting the positivity of the solution, various upper and lower bounds in first mean and almost sure sense are obtained, including Liapunov exponents .

Key words: almost sure exponential asymptotic stability, Liapunov exponent, subexponential distribution, subexponential function, Volterra equations, Itô-Volterra equations.

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1 Introduction

The stability and decay rate of linear deterministic Volterra integrodifferential equations to point equilibria has attracted much attention, in part because of applications to physics, biology and other sciences (cf., e.g., [13, Ch.1]). Since such real–world systems can be modelled to include random effects, it is natural to ask how the presence of such stochastic perturbations can influence the stability and convergence rate of solutions to these equilibria.

Particular applications of Itô-Volterra equations arise in physics and mathematical finance. In material physics, for example, partial integro–differential equations of Itô type are used to model the temperature in heat conductors with memory (cf., e.g., [9]). The behaviour of viscoelastic materials under external stochastic loads has also been analysed using Itô-Volterra equations (cf., e.g, [12]). In financial mathematics, the presence of inefficiency in real markets can be modelled by using stochastic functional differential equations. Anh et al. [1; 2] have recently posited models of the evolution of the asset returns using stochastic Volterra equations with infinite memory.

This paper is a mathematical contribution to the literature on the stability and rate of convergence to equilibrium. The non–exponential convergence reported in this paper is particularly interesting in economics, as it may be used in future work to represent the persistent memory of a shock in a financial system in the distant past.

The asymptotic behaviour of the solution of the deterministic linear convolution integrodifferential equation

$$x'(t) = -ax(t) + \int_0^t k(t-s)x(s)\,ds, \quad t \geq 0, \tag{1}$$

with $x(0) \neq 0$, is often analysed by considering the solutions of the characteristic equation

$$z + a - \hat{k}(z) = 0, \tag{2}$$

where $\hat{k}$ is the Laplace transform of k (cf., e.g., [13]). If all solutions λ of this equation obey $\operatorname{Re}\lambda < 0$, then the solution decays to zero exponentially fast (cf., e.g., [16]). However it can happen that (2) has no solutions, in which case this equation gives no information about the rate of decay of solutions. Nonetheless, in this situation, it is possible for the rate of decay to be determined exactly when k is asymptotic to a positive weight function γ in the class $\mathcal{U}(\mu)$, as introduced in [11]. In this case the solution of (1) is asymptotic as $t \to \infty$ to $\gamma(t)$ times a constant (cf., e.g., [7]). The class $\mathcal{U}(\mu)$ has the property that for $\gamma \in \mathcal{U}(\mu)$, $\gamma(t)$ is asymptotic to the exponential function $e^{\mu t}$ times a slowly decaying function.

It is natural to ask how the asymptotic behaviour of solutions of (1) will change when it is perturbed by a simple state–dependent stochastic perturbation, which depends only on the present value and preserves the zero equilibrium of (1). The stochastic Itô–Volterra equation which results from such a linear perturbation is

$$dX(t) = \left(-aX(t) + \int_0^t k(t-s)X(s)\,ds\right)\,dt + \sigma X(t)\,dB(t), \tag{3}$$

where $\sigma \neq 0$ and $\{B(t)\}_{t\geq 0}$ is a one-dimensional standard Brownian motion. Assumptions are imposed on the data which that ensure solutions are non–negative.

In this paper we examine the asymptotic decay rate of solutions of (3) in both first mean and almost sure senses. By exploiting the scalar nature of (3) and positivity of solutions we obtain both lower and upper estimates on almost sure decay rates, and exact decay rates on the first mean. In fact we establish estimates closely related to those found for the deterministic equation, irrespective of whether (2) has solutions or not.

In particular, the following results are established. In the case when k has a Liapunov exponent i.e., when there exists $\mu \le 0$ such that

$$\lim_{t\to\infty} \frac{1}{t} \log k(t) = \mu,$$

it is shown that

$$\lim_{t\to\infty} \frac{1}{t} \log |X(t)| = \mu, \quad \text{a.s.}$$

in the cases when $\mu = 0$ or when $k \in \mathcal{U}(\mu)$ for $\mu < 0$. In neither of these cases does (2) have a solution. When (2) has a solution, it can be shown that there is a constant $\eta > 0$ which depends only on k such that

$$-\eta \le \liminf_{t\to\infty} \frac{1}{t} \log |X(t)| \le \limsup_{t\to\infty} \frac{1}{t} \log |X(t)| \le \lambda, \quad \text{a.s.}$$

where $\lambda < 0$ is the unique positive solution of (2). A consequence of this result is that it is impossible to reduce arbitrarily the a.s. Liapunov exponent of the solution by increasing the noise intensity σ. This contrasts with the situation for scalar linear stochastic differential equations. Finally, in the case when k has a zero Liapunov exponent, and is in $\mathcal{U}(0)$, we can establish for each $\varepsilon > 0$ the estimates

$$\limsup_{t\to\infty} \frac{|X(t)|}{k(t)} = \infty, \quad \limsup_{t\to\infty} \frac{|X(t)|}{k(t)t^{1+\varepsilon}} = 0, \quad \text{a.s.}$$

on the solution.

Other studies which consider the convergence rates of solutions of stochastic Volterra equations with state–dependent noise intensities include [3; 6; 18; 19]. [18] deals with exponential stability of solutions, while [3] examines the asymptotic stability of solutions, but not the rate of decay to equilibrium. The remaining papers look at non-exponetial decay rates. Further results on non-exponential decay of solutions of stochastic Volterra equations with state–dependent noise, as well as a review of literature, may be found in [4].

2 Mathematical Preliminaries

2.1 Notation

We first fix some standard notation. Denote by $\mathbb{R}^+$ the set $[0,\infty)$. The maximum of the real numbers x, y is denoted by $x \vee y$; their minimum is denoted by $x \wedge y$.

If $F : \mathbb{R}^+ \to \mathbb{R}$ is in $L^1(\mathbb{R}^+)$ and $G : \mathbb{R}^+ \to \mathbb{R}$ has bounded variation, we put

$$(F \star G)(t) = \int_0^t F(t-s)\, dG(s), \quad t \ge 0.$$

Similarly, for $f, g \in L^1(\mathbb{R}^+)$, the convolution of f with g is defined to be

$$(f * g)(t) = \int_0^t f(t-s)g(s)\,ds, \quad t \geq 0.$$

We denote by $\mathbb{C}$ the set of complex numbers, and the real part of $z \in \mathbb{C}$ by $\operatorname{Re} z$. If $f : \mathbb{R}^+ \to \mathbb{R}$ satisfies $\int_0^\infty |f(t)|e^{-\alpha t}\,dt < \infty$ for some $\alpha \in \mathbb{R}$, we can define the Laplace transform of f for all $\operatorname{Re} z \geq -\alpha$ by

$$\hat{f}(z) = \int_0^\infty f(t)e^{-zt}\,dt.$$

We also define $e_\alpha(t) = e^{-\alpha t}$ for all $t \in \mathbb{R}^+$.

2.2 Linear Itô-Volterra Equations

In this paper, we concentrate upon the asymptotic behaviour of the solution of the scalar linear convolution Itô-Volterra equation

$$dX(t) = \left(-aX(t) + \int_0^t k(t-s)X(s)\,ds\right)dt + \sigma X(t)\,dB(t), \quad t \geq 0. \tag{4}$$

We have as a standing hypothesis

$$k \in C(\mathbb{R}^+;\mathbb{R}^+) \cap L^1(\mathbb{R}^+). \tag{5}$$

$(B(t))_{t\geq 0}$ is an standard one-dimensional Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}^B(t))_{t\geq 0}, \mathbb{P})$, where the filtration is the natural one, namely $\mathcal{F}^B(t) = \sigma\{B(s) : 0 \leq s \leq t\}$. We introduce the initial condition

$$X(0) = X_0, \tag{6}$$

where X_0 is a random variable, independent of $(B(t))_{t\geq 0}$, with $\mathbb{E}[X_0^2] < \infty$. Then there is a unique almost surely continuous solution to (4) (cf., e. g., Theorem 2E of [8]), satisfying (4) and (6). More precisely the solution is the unique adapted process satisfying (4) and (6) on the augmented filtered space $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t\geq 0}, \mathbb{P})$, where $(\mathcal{F}(t))_{t\geq 0}$ is the augmented filtration such that B is a $(\mathcal{F}(t))_{t\geq 0}$–standard Brownian motion, and X_0 is $\mathcal{F}(0)$–measurable. We denote a realisation of the solution of (4) and (6) by $t \mapsto X(t, \omega; X_0)$. Usually this is abbreviated to $t \mapsto X(t, \omega)$, and the value of the process at time t by $X(t)$.

We assume without loss of generality that

$$\sigma > 0. \tag{7}$$

The case that $\sigma = 0$ is the deterministic problem. For $\sigma < 0$ we may consider the Brownian motion $B^-(t) = -B(t)$ on $(\Omega, \mathcal{F}, (\mathcal{F}^B(t))_{t\geq 0}, \mathbb{P})$: for then, we can rewrite (4) as

$$dX(t) = \left(-aX(t) + \int_0^t k(t-s)X(s)\,ds\right)dt + \sigma^- X(t)\,dB^-(t),$$

with $\sigma^- = -\sigma > 0$, which is in exactly the form of (4). The augmentation of the filtration required to accommodate a random initial condition can then be achieved as indicated above.

We will impose additional hypotheses on the decay rate of k to obtain different decay rates on the solution X.

If $X(0) = 0$, then $X(t) = 0$ for all $t \geq 0$, almost surely. This is called the *zero solution* of (4).

Remark 1. The solution of (4) satisfying (6) satisfies $X(t,\omega;X_0)=X(t,\omega;1)X_0$. For this reason, we often take $X(0)=1$ in proofs without loss of generality.

2.3 Stochastic Analysis Preliminaries

We collect in this subsection a few results concerning solutions of Itô-Volterra equations.

We say that the zero solution of (4) is *almost surely globally asymptotically stable* if, for each solution of (4) and (6),

$$\mathbb{P}\left[\{\omega\in\Omega:\lim_{t\to\infty}X(t,\omega)=0\}\right]=1.$$

We say that the zero solution is *almost surely exponentially asymptotically stable* if there exists a postive random variable β_0 such that

$$\mathbb{P}\left[\left\{\omega\in\Omega:\limsup_{t\to\infty}\frac{1}{t}\log|X(t,\omega)|\le-\beta_0(\omega)\right\}\right]=1.$$

We say that a process $X=\{X(t);t\ge 0\}$ with $|X(t)|>0$ for all $t\ge 0$ a.s., has an *a.s. lower Liapunov exponent* μ if there is a deterministic μ such that

$$\liminf_{t\to\infty}\frac{1}{t}\log|X(t)|\ge\mu,\quad\text{a.s.}\tag{8}$$

A scalar process $X=\{X(t);t\ge 0\}$ has an *a.s. upper Liapunov exponent* ν if there is a deterministic ν such that

$$\limsup_{t\to\infty}\frac{1}{t}\log|X(t)|\le\nu,\quad\text{a.s.}\tag{9}$$

The positive process X has *a.s. Liapunov exponent* β if there is a deterministic β which is both a lower and upper Liapunov exponent.

The following result follows directly by using Problem 5.3.15 in [15].

Lemma 2.1. *If* $(X(t))_{t\ge 0}$ *is a solution of* (4) *and* (6)*, then for each* $0\le p\le 2$*, the mapping* $t\mapsto\mathbb{E}[|X(t)|^p]$ *is continuous on* $\mathbb{R}^+$*.*

3 Estimates on Lower Liapunov Exponents

In this section, we recall results which show that it is not destabilising to add the noise perturbation $\sigma X(t)dB(t)$ to (1), yielding (4). We also establish lower bounds on the decay rate of solutions. In each case, the results are established without any additional hypotheses being imposed on the kernel k. We will appeal to the following result frequently in the sequel.

Lemma 3.1. *Suppose that* X *is the solution of* (4) *and* (6)*. If* k *is continuous, then* $\mathbb{E}[|X(t)|]=r(t)\mathbb{E}[|X_0|]$ *for* $t\ge 0$*, where* r *is the differential resolvent of* k *defined by*

$$r'(t)=-ar(t)+\int_0^t k(t-s)r(s)\,ds,\quad t>0;\qquad r(0)=1.\tag{10}$$

A consequence of this is the following, which is Theorem 6 of [3].

Theorem 3.2. *Suppose that k satisfies* (5)*. If the solution r of* (10) *is in $L^1(\mathbb{R}^+)$, then the solution of* (4) *and* (6) *satisfies*

$$\lim_{t\to\infty} X(t) = 0, \quad a.s. \tag{11}$$

$$X \in L^1(\mathbb{R}^+), \quad a.s. \tag{12}$$

To establish lower bounds on the decay rate of solutions of (4), we observe a result essentially contained in the proof of [6, Theorem 1].

Lemma 3.3. *Suppose that k satisfies* (5)*. If X is the solution of* (4) *and* (6) *with $X_0 \neq 0$ a.s., then*

$$|X(t)| > 0 \quad \text{for all } t \geq 0, \text{ a.s.}$$

The next result gives a deterministic lower estimate on the a.s. rate of decay of $|X(t)|$ as $t \to \infty$.

Theorem 3.4. *Let $a + \sigma^2/2 > 0$ and k satisfies* (5)*. If X is the solution of* (4) *and* (6) *with $X_0 \neq 0$ a.s., then*

$$\liminf_{t\to\infty} \frac{1}{t} \log |X(t)| \geq \liminf_{t\to\infty} \frac{1}{t} \log(k * e_{a+\sigma^2/2})(t), \quad a.s. \tag{13}$$

Proof. As remarked earlier, we take $X(0) = 1$ without loss of generality. Define the process $\varphi = \{\varphi(t); \mathcal{F}^B(t); 0 \leq t < \infty\}$ by

$$\varphi(t) = 1 + \int_0^t -a\varphi(s)\,ds + \int_0^t \sigma\varphi(s)\,dB(s), \quad t \geq 0. \tag{14}$$

Setting $\lambda = a + \sigma^2/2 > 0$, φ is given by $\varphi(t) = \exp(\lambda t + \sigma B(t))$, $t \geq 0$. Since $\varphi(t) > 0$ for all $t \geq 0$, a.s., we may define the process y according to $y(t) = X(t)/\varphi(t)$, $t \geq 0$. Then, by Lemma 3.3, it follows that $y(t) > 0$ for all $t \geq 0$ a.s. Moreover, by (stochastic) integration by parts, we can show that y obeys

$$y(t) = 1 + \int_0^t \frac{1}{\varphi(s)} \int_0^s k(s-u)X(u)\,du\,ds, \quad t \geq 0.$$

Therefore, as φ, k and X possess a.s. continuous sample paths, $y \in C^1(0,\infty)$ and so we have

$$y'(t) = \frac{1}{\varphi(t)} \int_0^t k(t-s)\varphi(s)y(s)\,ds, \quad t > 0.$$

Since k is non–negative, and φ and y are positive a.s., it follows that $y'(t) \geq 0$ for all $t > 0$. Thus $y(t) \geq 1$ and so

$$y'(t) \geq \frac{1}{\varphi(t)} \int_0^t k(t-s)\varphi(s)\,ds = \int_0^t k(t-s)e^{\lambda(t-s)}e^{-\sigma(B(t)-B(s))}\,ds. \tag{15}$$

Now, by the Law of the Iterated Logarithm, it follows that there exists an a.s. event Ω^* such that for each $\omega \in \Omega^*$ there exists a $T(\omega) > 0$ such that $|\sigma B(t)| \leq t^{2/3}$ for all $t \geq T(\omega)$. Then for $t \geq s \geq T(\omega)$ we have

$$e^{-\sigma(B(t)-B(s))} \geq e^{-t^{2/3}-s^{2/3}} \geq e^{-2t^{2/3}}.$$

Therefore, applying this estimate to (15), for $t \geq T(\omega)$, we get

$$\begin{aligned} y'(t) &\geq \int_T^t k(t-s)e^{\lambda(t-s)}e^{-\sigma(B(t)-B(s))}\,ds \geq \int_T^t k(t-s)e^{\lambda(t-s)}e^{-2t^{2/3}}\,ds \\ &= e^{-2t^{2/3}} \int_0^{t-T} k(u)e^{\lambda u}\,du. \end{aligned}$$

Define $\alpha := \liminf_{t\to\infty} \frac{1}{t} \log \int_0^t k(u)e^{\lambda u}\,du$. Then $\alpha \geq 0$ (we do not exclude a priori the possibility that $\alpha = \infty$). Since

$$\frac{1}{t}\log y'(t) \geq -2\frac{1}{t^{1/3}} + \frac{t-T}{t} \cdot \frac{1}{t-T} \log \int_0^{t-T} k(u)e^{\lambda u}\,du, \quad t \geq T,$$

we see that

$$\liminf_{t\to\infty} \frac{1}{t} \log y'(t) \geq \alpha, \quad \text{a.s.} \tag{16}$$

If $\alpha = 0$, since y is increasing, we have that $\liminf_{t\to\infty} \frac{1}{t} \log y(t) \geq 0$. If $\alpha > 0$ is finite, (16) implies that

$$\liminf_{t\to\infty} \frac{1}{t} \log y(t) \geq \alpha, \quad \text{a.s.} \tag{17}$$

In the case $\alpha = \infty$, (16) implies (17). Hence (17) holds whatever the value of $\alpha \geq 0$. The Strong Law of Large numbers implies that $\lim_{t\to\infty} \frac{1}{t} \log \varphi(t) = -\lambda$, a.s., so we have

$$\liminf_{t\to\infty} \frac{1}{t} \log |X(t)| \geq \alpha - \lambda, \quad \text{a.s.}$$

Finally, (13) holds because

$$\begin{aligned} \alpha - \lambda &= \lim_{t\to\infty} \frac{1}{t} \log e^{-\lambda t} + \liminf_{t\to\infty} \frac{1}{t} \log \int_0^t k(u)e^{\lambda u}\,du \\ &= \liminf_{t\to\infty} \frac{1}{t} \log \int_0^t k(u)e^{-\lambda(t-u)}\,du \\ &= \liminf_{t\to\infty} \frac{1}{t} \log \int_0^t k(u)e^{-(a+\sigma^2/2)(t-u)}\,du, \end{aligned}$$

which establishes (13). □

The lower estimate furnished by Theorem 3.4 can be simplified in the case when

$$\liminf_{t\to\infty} \frac{1}{t} \log k(t) \geq \mu. \tag{18}$$

In order to see this, it is first necessary to prove the following preliminary lemma.

Lemma 3.5. *Let $k \in C([0,\infty);(0,\infty))$ and suppose that there exists a $\mu \leq 0$ such that (18) holds. Suppose that $a + \sigma^2/2 + \mu > 0$. Then*

$$\liminf_{t\to\infty} \frac{1}{t} \log \int_0^t k(s)e_{a+\sigma^2/2}(t-s)\,ds \geq \mu. \tag{19}$$

Proof. For every $\varepsilon \in (0, a+\sigma^2/2+\mu)$ there exists $T = T(\varepsilon) > 0$ such that $k(t) > e^{(\mu-\varepsilon)t}$ for all $t > T(\varepsilon)$. Then, for $t > T$, we have

$$\begin{aligned} e^{(-\mu+\varepsilon)t}\int_0^t k(s)e^{-(a+\sigma^2/2)(t-s)}\,ds &\geq e^{(-\mu+\varepsilon)t}\int_T^t k(s)e^{-(a+\sigma^2/2)(t-s)}\,ds \\ &\geq \int_T^t e^{-(a+\sigma^2/2+\mu-\varepsilon)(t-s)}\,ds = \int_0^{t-T} e^{-(a+\sigma^2/2+\mu-\varepsilon)u}\,du. \end{aligned}$$

Thus

$$\liminf_{t\to\infty} e^{(-\mu+\varepsilon)t}\int_0^t k(s)e^{-(a+\sigma^2/2)(t-s)}\,ds \geq \frac{1}{a+\frac{\sigma^2}{2}+\mu-\varepsilon},$$

which implies

$$\liminf_{t\to\infty} t\left(-\mu+\varepsilon+\frac{1}{t}\log\int_0^t k(s)e^{-(a+\sigma^2/2)(t-s)}\,ds\right) \geq \log\frac{1}{a+\frac{\sigma^2}{2}+\mu-\varepsilon},$$

Hence, for every $\varepsilon \in (0, a+\sigma^2/2+\mu)$ we have

$$\liminf_{t\to\infty}\frac{1}{t}\log(k * e_{a+\sigma^2/2})(t) = \liminf_{t\to\infty}\frac{1}{t}\log\int_0^t k(s)e^{-(a+\sigma^2/2)(t-s)}\,ds \geq \mu-\varepsilon,$$

so, letting $\varepsilon \to 0$, we get (19). □

We may now combine Lemma 3.5 and Theorem 3.4 to obtain explicit and simple estimates on the Liapunov exponent and lower Liapunov exponent of $|X|$.

Corollary 3.6. *Suppose that k satisfies* (5) *and*

$$\lim_{t\to\infty}\frac{1}{t}\log k(t) = 0. \tag{20}$$

Let $X_0 \neq 0$ a.s. If the solution of (4) *qand* (6) *obeys $X(t) \to 0$ as $t \to \infty$ a.s., then*

$$\lim_{t\to\infty}\frac{1}{t}\log|X(t)| = 0, \quad \textit{a.s.} \tag{21}$$

Proof. Once more we let $X(0) = 1$ without loss of generality. Suppose as in the proof of Theorem 3.4 that φ obeys (14) and $y = X/\varphi$. In this case, the proof of Theorem 3.4 implies that $y(t) \geq 1$ and so $X(t) \geq \varphi(t)$. Therefore $X(t) \to 0$ a.s. implies that $\varphi(t) \to 0$ as $t \to \infty$ a.s. This implies that $a + \sigma^2/2 > 0$. Theorem 3.4 now gives

$$\liminf_{t\to\infty}\frac{1}{t}\log X(t) \geq \liminf_{t\to\infty}\frac{1}{t}\log\int_0^t k(s)e^{-(a+\sigma^2/2)(t-s)}\,ds, \quad \text{a.s.}$$

Now by Lemma 3.5 with $\mu = 0$ we get

$$\liminf_{t\to\infty}\frac{1}{t}\log\int_0^t k(s)e^{-(a+\sigma^2/2)(t-s)}\,ds \geq 0.$$

Therefore, by Theorem 3.4 we get

$$\liminf_{t\to\infty}\frac{1}{t}\log X(t) \geq 0 \quad \text{a.s.} \tag{22}$$

Finally, $X(t) \to 0$ as $t \to \infty$ a.s. implies that $\limsup_{t\to\infty}\frac{1}{t}\log X(t) \leq 0$ a.s. Combining this with (22) gives the result. □

Corollary 3.7. *Suppose that k satisfies* (5) *and there exists $\mu < 0$ such that* (18). *Assume that $a + \sigma^2/2 + \mu > 0$. Let $X_0 \neq 0$ a.s. If the solution X of* (4) *and* (6) *obeys $X(t) \to 0$ as $t \to \infty$ a.s., then*

$$\liminf_{t\to\infty} \frac{1}{t} \log |X(t)| \geq \mu \quad a.s. \tag{23}$$

Proof. Theorem 3.4 gives

$$\liminf_{t\to\infty} \frac{1}{t} \log X(t) \geq \liminf_{t\to\infty} \frac{1}{t} \log \int_0^t k(s) e^{-(a+\sigma^2/2)(t-s)}\, ds, \quad \text{a.s.}$$

Now, as k satisfies all the hypotheses of Lemma 3.5, we have

$$\liminf_{t\to\infty} \frac{1}{t} \log \int_0^t k(s) e^{-(a+\sigma^2/2)(t-s)}\, ds \geq \mu.$$

Therefore, we must have (23), as required. □

4 Decay Rate when Characteristic Equation has no Roots

Corollary 3.6 shows that rate of decay of $X(t)$ to zero as $t \to \infty$, cannot be exponential if the kernel obeys (20). However it does not precisely quantify the relationship between the decay rate of $X(t)$ and that of the kernel $k(t)$. In this section, we investigate this under additional assumptions on the decay of k. Moreover the corresponding problem for kernels obeying (18) can be reduced to one in which it obeys (20), by considering the equation for $X_\mu(t) = e^{-\mu t} X(t)$, which is

$$dX_\mu(t) = \left(-(\mu + a) X_\mu(t) + \int_0^t k_\mu(t-s) X_\mu(s)\, ds \right) dt + \sigma X_\mu(t) dB(t). \tag{24}$$

Recall that $k_\mu(t) = e^{-\mu t} k(t)$, and that in our case $\mu \leq 0$.

4.1 k is in $\mathcal{U}(\mu)$

We recall the definition of a class of decaying functions employed in [7]: it was based on the hypotheses of Theorem 3 of [11].

Definition 4.1. Let $\mu \in \mathbb{R}$. A function $\gamma : [0, \infty) \to \mathbb{R}$ is in $\mathcal{U}(\mu)$ if it is continuous with $\gamma(t) > 0$ for all $t \geq 0$, and

$$\hat{\gamma}(\mu) = \int_0^\infty \gamma(t) e^{-\mu t}\, dt < \infty, \tag{25}$$

$$\lim_{t\to\infty} \frac{(\gamma * \gamma)(t)}{\gamma(t)} = 2\hat{\gamma}(\mu), \tag{26}$$

$$\lim_{t\to\infty} \frac{\gamma(t-s)}{\gamma(t)} = e^{-\mu s} \quad \text{uniformly for } 0 \leq s \leq S, \text{ for all } S > 0. \tag{27}$$

The functions in $\mathcal{U}(0)$ are called subexponential. The reason for this nomenclature is that, if k is subexponential, then $k(t)e^{\epsilon t} \to \infty$ as $t \to \infty$, for each $\epsilon > 0$: consequently k obeys (20). The connection between $\mathcal{U}(\mu)$ and $\mathcal{U}(0)$ is that, if k is in $\mathcal{U}(\mu)$, then k_μ is subexponential.

Many slowly decaying functions belong to the class $\mathcal{U}(0)$. For instance a positive function which is regularly varying at infinity with index $\alpha < -1$ (such as $k(t) = (t+1)^{-\alpha}$ for $\alpha < -1$) is a member of $\mathcal{U}(0)$. Functions such as $k(t) = e^{-t^\alpha}$, for $\alpha \in (0,1)$, which decay faster than any polynomial, but more slowly than any negative exponential, are also in $\mathcal{U}(0)$. We refer the reader to [5] for a further discussion regarding the properties of subexponential functions.

For $k \in \mathcal{U}(\mu)$, applying [7, Theorem 15] to the scalar equation (10), we see that the solution r of (10) inherits precisely the decay rate of k.

Lemma 4.2. *Suppose that k satisfies* (5)*, is in $\mathcal{U}(\mu)$ and*

$$\mu + a - \hat{k}(\mu) > 0. \tag{28}$$

Then the solution of (10) *obeys*

$$\lim_{t\to\infty} \frac{r(t)}{k(t)} = \frac{1}{(\mu + a - \hat{k}(\mu))^2}. \tag{29}$$

We also state a corollary of Theorem 4.2 in [5], which is used later.

Lemma 4.3. *Let f be in $C(\mathbb{R}^+;\mathbb{R}) \cap L^1(\mathbb{R}^+)$ and $k \in \mathcal{U}(0)$. If $f(t)/k(t) \to \alpha < \infty$ as $t \to \infty$, then*

$$\lim_{t\to\infty} \frac{(f * k)(t)}{k(t)} = \alpha \int_0^\infty k(s)\,ds + \int_0^\infty f(s)\,ds. \tag{30}$$

We start by determining the exact asymptotic decay rate of solutions in first mean. Its proof is a corollary of Lemmata 3.1 and 4.2.

Theorem 4.4. *Suppose that k satisfies* (5)*, is in $\mathcal{U}(\mu)$ and* (28) *holds. Then the solution of* (4) *and* (6) *obeys*

$$\lim_{t\to\infty} \frac{\mathbb{E}[|X(t)|]}{k(t)} = \frac{\mathbb{E}[|X_0|]}{(\mu + a - \hat{k}(\mu))^2}. \tag{31}$$

The result of Theorem 4.4 is interesting in itself, and may be used to obtain an upper bound on the almost sure asymptotic behaviour of solutions of (4).

Theorem 4.5. *Suppose that k satisfies* (5)*, is in $\mathcal{U}(\mu)$ and* (28) *holds. Then the solution of* (4) *and* (6) *obeys*

$$\lim_{t\to\infty} \frac{|X(t)|}{k(t)t^\gamma} = 0, \quad \textit{a.s., for every } \gamma > 1. \tag{32}$$

It suffices to suppose that k is in $\mathcal{U}(0)$, since we may consider (24) instead of (4). The idea of the proof of Theorem 4.5 is similar to that used to prove the main result in [3], which in turn uses the strategy employed to prove Theorem 4.3.1 in [17]. We use a carefully chosen partition $\mathcal{P} = \cup_{n=1}^\infty [a_n, a_{n+1})$ of the positive real half-line, where $a_n \nearrow \infty$, as $n \to \infty$. By the appropriate choice of the partition, and using the properties of subexponential functions, together with Theorem 4.4 above, we can show that for sufficiently large n there is a finite $\tilde{B} > 0$ such that

$$\mathbb{E}\left[\sup_{a_n \le t \le a_{n+1}} |X(t)|\right] \le \tilde{B} k(a_n).$$

A Borel-Cantelli argument along the lines of [17, Theorem 4.3.1] is then used to obtain an upper bound on the almost sure decay rate.

Before we start the proof of Theorem 4.5, we recall that a Burkholder-Davis-Gundy inequality (see, for example, [15], p.166) implies that for every a.s. locally square integrable scalar process $\{Y(t), \mathcal{F}^B(t); 0 \le t < \infty\}$, and every $0 \le t_1 \le t_2$, there exists a universal (i.e., Y-, t_1-, t_2-independent) positive constant c_2 such that

$$\mathbb{E}\left[\left|\sup_{t_1 \le t \le t_2} \int_{t_1}^{t} Y(s)\, dB(s)\right|\right] \le c_2 \mathbb{E}\left[\left(\int_{t_1}^{t_2} Y(s)^2\, ds\right)^{1/2}\right]. \tag{33}$$

Now, consider an increasing sequence $\{a_n\}_{n=0}^{\infty}$ with $a_0 = 0$, such that $a_n = n^{\eta}$ for some $\eta \in (0,1)$. Then

$$a_{n+1} - a_n < 1, \quad \text{for all } n \ge 1, \tag{34}$$

and for all $n > n_0(\eta)$, we have

$$c_2|\sigma|(a_{n+1} - a_n)^{1/2} < \frac{1}{2}. \tag{35}$$

Proof of Theorem 4.5. Suppose that k is in $\mathcal{U}(0)$ and $X(0) = 1$. It follows that$\mathbb{E}[|X(t)|] = r(t)$, where r obeys (10). Fix $\varepsilon > 0$. Let $\eta \in (0,1)$ be given by $\eta = 1/(1+\varepsilon/2)$, and $\varepsilon' = 1 - \eta \in (0,1)$. Define $\lambda = -(1+\varepsilon)$. Then

$$\lambda = -\frac{1+\varepsilon'}{\eta}. \tag{36}$$

Let $\{a_n\}_{n=0}^{\infty}$ be as defined in the preamble to this proof. For every $t \in \mathbb{R}^+$, there exists $n = n(t) \in \mathbb{N}$ such that $a_n \le t < a_{n+1}$ so

$$X(t) = X(a_n) + \int_{a_n}^{t} \left(-aX(s) + \int_0^s k(s-u)X(u)\, du\right) ds \\ + \int_{a_n}^{t} \sigma X(s)\, dB(s).$$

Using the triangle inequality, taking suprema over $[a_n, a_{n+1})$, and then expectations, and availing of the continuity of X, we arrive at

$$\mathbb{E}\left[\sup_{a_n \le t \le a_{n+1}} |X(t)|\right] \le r(a_n) + \int_{a_n}^{a_{n+1}} (ar(s) + (k*r)(s)\, ds \\ + \mathbb{E}\left[\sup_{a_n \le t \le a_{n+1}} \left|\int_{a_n}^{t} \sigma X(s)\, dB(s)\right|\right]. \tag{37}$$

Using (35), (33), we can bound the third term on the right hand side of (37) for $n > n_0$ as follows:

$$\mathbb{E}\left[\sup_{a_n \le t \le a_{n+1}} \left|\int_{a_n}^t \sigma X(s)\, dB(s)\right|\right] \le c_2 \mathbb{E}\left[\left(\int_{a_n}^{a_{n+1}} \sigma^2 X(s)^2\, ds\right)^{1/2}\right]$$
$$\le c_2 (a_{n+1} - a_n)^{1/2} |\sigma| \mathbb{E}\left[\sup_{a_n \le s \le a_{n+1}} |X(s)|\right]$$
$$< \frac{1}{2} \mathbb{E}\left[\sup_{a_n \le s \le a_{n+1}} |X(s)|\right]. \tag{38}$$

Combining (37), (38) for $n > n_0$, we obtain

$$\mathbb{E}\left[\sup_{a_n \le s \le a_{n+1}} |X(s)|\right] \le 2r(a_n) + 2\int_{a_n}^{a_{n+1}} ar(s)\, ds + 2\int_{a_n}^{a_{n+1}} (k * r)(s)\, ds. \tag{39}$$

We now obtain asymptotic estimates on the behaviour of the second and third terms on the right hand side of (39). Considering the second term first, remembering that $k(t) > 0$ and that (34) holds, we have

$$\int_{a_n}^{a_{n+1}} r(s)\, ds \le \sup_{a_n \le s \le a_{n+1}} \frac{r(s)}{k(s)} \frac{k(s)}{k(a_n)} k(a_n)$$
$$\le k(a_n) \sup_{a_n \le s \le a_{n+1}} \frac{r(s)}{k(s)} \sup_{a_n \le s \le a_{n+1}} \frac{k(s)}{k(a_n)}. \tag{40}$$

Since $k \in \mathcal{U}(0)$, we can use (27) to give

$$\lim_{n\to\infty} \sup_{a_n \le s \le a_{n+1}} \frac{k(s)}{k(a_n)} = 1,$$

while Theorem 4.4 yields

$$\lim_{n\to\infty} \sup_{a_n \le s \le a_{n+1}} \frac{r(s)}{k(s)} = A_1 := \frac{1}{(a - \int_0^\infty k(s)\, ds)^2}.$$

Inserting these estimates into (40) implies that for $n > n_0'$ there exists $B_1 > 0$ such that

$$\int_{a_n}^{a_{n+1}} ar(s)\, ds \le B_1 k(a_n).$$

We proceed similarly for the third term on the right hand side of (39). First, we obtain

$$\int_{a_n}^{a_{n+1}} (k * r)(s)\, ds \le k(a_n) \sup_{a_n \le s \le a_{n+1}} \frac{(k * r)(s)}{k(s)} \sup_{a_n \le s \le a_{n+1}} \frac{k(s)}{k(a_n)}. \tag{41}$$

Since r is integrable, by using Lemma 4.3, we have

$$\lim_{n\to\infty} \sup_{a_n \le s \le a_{n+1}} \frac{(k * r)(s)}{k(s)} = A_1 \int_0^\infty k(s)\, ds + \int_0^\infty r(s)\, ds.$$

Using (27) as before, and the above limit, we can infer from (41) the existence of $n_0'' \in \mathbb{N}$, and $B_2 > 0$ such that $n > n_0''$ implies

$$\int_{a_n}^{a_{n+1}} (k * r)(s)\, ds \le B_2 k(a_n).$$

By Theorem 4.4, there exists $n_0''' \in \mathbb{N}$ and $B_3 > 0$ such that

$$\frac{r(a_n)}{k(a_n)} \le B_3, \quad \text{for all } n > n_0'''.$$

Choosing $n_1 = n_0 \vee n_0' \vee n_0'' \vee n_0''' \vee 1$, for $n > n_1$, from (39) and the above estimates, we have

$$\mathbb{E}\left[\sup_{a_n \le t \le a_{n+1}} |X(t)|\right] \le \tilde{B} k(a_n). \tag{42}$$

where $\tilde{B} = 2(B_1 + B_2 + B_3)$. Hence, by Markov's inequality, for every $\gamma' > 0$ and $n > n_1$, we have

$$\mathbb{P}\left[\sup_{n^\eta \le t \le (n+1)^\eta} |X(t)| \frac{1}{k(n^\eta)} \frac{1}{n^{1+\varepsilon'}} > \gamma'\right] \le \frac{\tilde{B}}{\gamma'} \frac{1}{n^{1+\varepsilon'}}.$$

The summability of this sequence of probabilities enables us to conclude, by the first Borel-Cantelli lemma and (36), that

$$\lim_{n\to\infty} \sup_{a_n \le t \le a_{n+1}} \frac{|X(t)|}{k(a_n)} n^{\eta\lambda} = 0 \quad \text{a.s.} \tag{43}$$

Therefore, using (27) and (43), we get

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{a_n \le t \le a_{n+1}} \frac{|X(t)|}{k(t)} n^{\eta\lambda} \\ &\qquad \le \limsup_{n\to\infty} \sup_{a_n \le t \le a_{n+1}} \frac{|X(t)|}{k(a_n)} n^{\eta\lambda} \cdot \limsup_{n\to\infty} \sup_{a_n \le t \le a_{n+1}} \frac{k(a_n)}{k(t)} = 0 \quad \text{a.s.} \end{aligned} \tag{44}$$

For each $t \in \mathbb{R}^+$, we define $N(t)$ to be the largest integer such that $N(t) \le t^{1/\eta}$. Hence $N(t)^\eta \le t < (N(t)+1)^\eta$, and so $\lim_{t\to\infty} t/N(t)^\eta = 1$. Using this and (44), we have

$$\begin{aligned} \limsup_{t\to\infty} \frac{|X(t)|}{k(t)} t^\lambda &\le \limsup_{t\to\infty} \frac{|X(t)|}{k(t)} N(t)^{\eta\lambda} \cdot \limsup_{t\to\infty} \left(\frac{t}{N(t)^\eta}\right)^\lambda \\ &\le \limsup_{t\to\infty} \sup_{N(t)^\eta \le s \le (N(t)+1)^\eta} \frac{|X(s)|}{k(s)} N(t)^{\eta\lambda} \\ &= \limsup_{n\to\infty} \sup_{n^\eta \le s \le (n+1)^\eta} \frac{|X(s)|}{k(s)} n^{\eta\lambda} = 0 \quad \text{a.s.} \end{aligned}$$

Putting $\gamma = -\lambda = 1 + \varepsilon$, we have the result. □

The following result can be inferred from Theorem 4.5 and [6, Corollary 2] by considering the equation (24) for X_μ.

Theorem 4.6. *Suppose that k satisfies* (5)*, is in $\mathcal{U}(\mu)$ and* (28) *holds. Then the solution of* (4) *and* (6) *with $X_0 \neq 0$, obeys*

$$\limsup_{t\to\infty} \frac{|X(t)|}{k(t)} = \infty, \quad a.s.,$$
$$\limsup_{t\to\infty} \frac{|X(t)|}{k(t)t^{1+\varepsilon}} = 0 \quad a.s., \text{ for each } \varepsilon > 0$$

When $k \in \mathcal{U}(\mu)$ for $\mu < 0$, by combining the results of Theorem 4.5 and Corollary 3.7, we may show that $|X|$ has a.s. Liapunov exponent μ.

Theorem 4.7. *Suppose that k is $\mathcal{U}(\mu)$ for some $\mu < 0$, and obeys* (28)*. If X is the solution of* (4) *and* (6) *with $X_0 \neq 0$ a.s.,then*

$$\lim_{t\to\infty} \frac{1}{t} \log |X(t)| = \mu, \quad a.s. \tag{45}$$

Proof. For $k \in \mathcal{U}(\mu)$, $\mu < 0$, we have

$$\lim_{t\to\infty} \frac{1}{t} \log k(t) = \mu.$$

By Theorem 4.5, it follows that

$$\limsup_{t\to\infty} \frac{1}{t} \log |X(t)| \leq \mu, \quad \text{a.s.}$$

On the other hand, by Corollary 3.7 it follows that

$$\liminf_{t\to\infty} \frac{1}{t} \log |X(t)| \geq \mu, \quad \text{a.s.}$$

□

We close this section with a discussion on almost sure asymptotic behaviour for $k \in \mathcal{U}(0)$. Just as in the deterministic case, the decay rate of solutions of (4) is closely determined by the decay rate of the kernel. If the kernel decays polynomially in the sense that

$$\lim_{t\to\infty} \frac{\log k(t)}{\log t} = -\alpha,$$

for some $\alpha > 1$, then the solution inherits this polynomial decay, in the sense that

$$-\alpha \leq \limsup_{t\to\infty} \frac{\log |X(t)|}{\log t} \leq -(\alpha - 1), \quad \text{a.s.}$$

For certain kernels whose decay rate is between exponential and polynomial, we can prove a very precise result about the asymptotic behaviour of solutions of (4). We will call these kernels superpolynomial kernels; more precisely, we give the following definition.

Definition 4.8. $k \in \mathcal{U}(0)$ is a *superpolynomial function* if there exists a continuous nonnegative function λ such that

$$\lim_{t\to\infty} \frac{\lambda(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{\lambda(t)}{\log t} = \infty,$$

and

$$\lim_{t\to\infty} \frac{\log k(t)}{\lambda(t)} = -1.$$

We call the function λ a *negative exponent* of k (or say that k has negative exponent λ).

Note that $k(t) = e^{-t^\alpha}$, for $\alpha \in (0,1)$ is an example of a superpolynomial function, with negative exponent $\lambda(t) = t^\alpha$; another is $k(t) = e^{-t/\log(t+2)}$ with $\lambda(t) = t/\log t$.

If k is a superpolynomial function, the solution of (4) inherits the superpolynomial character of the kernel to a significant degree. Indeed, by Theorem 4.6 we have

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\lambda(t)} = -1, \quad \text{a.s.},$$

and therefore

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\log k(t)} = 1, \quad \text{a.s.} \tag{46}$$

It is instructive to compare this result with that which is obtained for (10) when k is superpolynomial, namely (29), which implies

$$\lim_{t\to\infty} \frac{\log|r(t)|}{\log k(t)} = 1. \tag{47}$$

This serves to show the sharpness of (46).

When k is in $\mathcal{U}(\mu)$, we observe an interesting result: although solutions of the stochastic Volterra equation (4) are asymptotically stable whenever those of the deterministic problem (1) are also stable, *the decay rate of solutions of* (4) *is slower than that for solutions of* (1) . Therefore, although the noise seems to be stabilising solutions of the original Volterra equation, it is doing so at the expense of the convergence rate (although that convergence rate cannot be worse by more than a factor of $t^{-(1+\varepsilon)}$, for any $\varepsilon > 0$).

4.2 K is in $\mathcal{S}(\mu)$

We consider another class of kernels for which the characteristic equation (2) does not have a solution. but which is larger than $\mathcal{U}(\mu)$. This enables us to study the decay rate of $\int_t^\infty e^{-\mu t}|X(s)|\,ds$ under weaker assumptions on k than required for $\mathcal{U}(\mu)$. We start by recalling the definition of a *subexponential distribution* introduced by Chistyakov [10].

Definition 4.9. Let F be a distribution function on $\mathbb{R}$. Then F is *subexponential* if $F(0+) = 0$ and

$$\lim_{t\to\infty} \frac{1 - (F \star F)(t)}{1 - F(t)} = 2.$$

The class of subexponential distribution functions is denoted by $\mathcal{S}$.

Chistyakov [10] showed that if F is a subexponential distribution function, then

$$\lim_{t\to\infty} \frac{1-F(t-s)}{1-F(t)} = 1 \tag{48}$$

uniformly for s in compact intervals of $\mathbb{R}^+$. Chistyakov [10] demonstrated that a consequence of (48) is that subexponential distribution functions have "heavy tails" which decay more slowly than any exponential functions. Indeed, he showed that for $F \in \mathcal{S}$ that

$$\lim_{t\to\infty} (1-F(t))e^{\gamma t} = \infty, \quad \text{for all } \gamma > 0. \tag{49}$$

We now extend this definition to include the possibility of exponential decay.

Definition 4.10. Let F be a distribution function on $\mathbb{R}$. Then F is in $\mathcal{S}(\mu)$ if $F(0+) = 0$, there exists $\mu \le 0$ such that $\hat{F}(\mu) := \int_0^\infty e^{-\mu s}\, dF(s)$ is finite, and the distribution G on $\mathbb{R}$, defined by $G(0+) = 0$ and

$$G(t) = \frac{\int_0^t e^{-\mu s}\, dF(s)}{\int_0^\infty e^{-\mu s}\, dF(s)}, \quad t \ge 0,$$

is in $\mathcal{S}$.

We notice that $\mathcal{S}(0) = \mathcal{S}$. There is a fundamental connection between $\mathcal{S}$ and $\mathcal{U}(0)$, and between $\mathcal{S}(\mu)$ and $\mathcal{U}(\mu)$ in general: if $k \in \mathcal{U}(\mu)$, then K defined by $K(0+) = 0$ and

$$K(t) = \frac{\int_0^t k(s)\, ds}{\int_0^\infty k(s)\, ds}, \quad t \ge 0, \tag{50}$$

is in $\mathcal{S}(\mu)$.

By considering the Volterra integrodifferential equation for $r_\mu(t) = e^{-\mu t} r(t)$, and applying Theorem 6.5 of [5], we can infer the following result.

Lemma 4.11. *Suppose that k satisfies* (5)*, that K defined by* (50) *is in $\mathcal{S}(\mu)$ and* (28) *holds. Then the solution of* (10) *obeys*

$$\lim_{t\to\infty} \frac{\int_t^\infty e^{-\mu s} r(s)\, ds}{\int_t^\infty e^{-\mu s} k(s)\, ds} = \frac{1}{(a+\mu-\hat{k}(\mu))\, ds)^2}. \tag{51}$$

We determine the decay rates of solutions of equations whose kernels satisfy the weaker hypothesis that K be in $\mathcal{S}(\mu)$, where K is defined in terms of k by (50). The next result is an immediate corollary of Lemmata 3.1 and 4.11.

Theorem 4.12. *Suppose that k satisfies* (5)*, that K defined by* (50) *is in $\mathcal{S}(\mu)$ and* (28) *holds. Then the solution of* (4) *and* (6) *with $X_0 \neq 0$ a.s., obeys*

$$\lim_{t\to\infty} \frac{\int_t^\infty e^{-\mu s}\mathbb{E}|X(s)|\, ds}{\int_t^\infty e^{-\mu s} k(s)\, ds} = \frac{\mathbb{E}[|X_0|]}{(a+\mu-\hat{k}(\mu))^2}. \tag{52}$$

A similar–though less intricate–argument to that of Theorem 4.5 now enables us to obtain an upper bound on the a.s. decay rate of $\int_t^\infty |X(s)|\,ds$ to zero, as $t \to \infty$. Note that this is meaningful because a conclusion of Theorem 3.2 applied to the solution X_μ of (24) is that X_μ is in $L^1(\mathbb{R}^+)$ a.s.

Theorem 4.13. *Suppose that k satisfies* (5)*, that K is in $\mathcal{S}(\mu)$ and* (28) *holds. Then the solution of* (4) *and* (6) *with $X_0 \neq 0$ obeys*

$$\limsup_{t\to\infty} \frac{\int_t^\infty e^{-\mu s}|X(s)|\,ds}{t^\gamma \int_t^\infty e^{-\mu s}k(s)\,ds} = 0 \quad \textit{a.s., for each } \gamma > 1. \tag{53}$$

Proof. Theorem 4.12 implies the existence of a number $C > 0$ such that

$$\mathbb{E}\int_t^\infty e^{-\mu s}|X(s)|\,ds \leq C\int_t^\infty e^{-\mu s}k(s)\,ds. \tag{54}$$

For any fixed $\gamma > 1$, define the sequence of random variables

$$Y_n = \frac{\int_n^\infty e^{-\mu s}|X(s)|\,ds}{n^\gamma \int_n^\infty e^{-\mu s}k(s)\,ds}.$$

By (54) and Markov's inequality, we have

$$\mathbb{P}\left[|Y_n| \geq \varepsilon\right] \leq \frac{C}{\varepsilon n^\gamma} \quad \text{for every } \varepsilon > 0.$$

The first Borel-Cantelli lemma now gives

$$\lim_{n\to\infty} Y_n = 0, \quad \text{a.s.} \tag{55}$$

For every $t \in \mathbb{R}^+$, let $n(t)$ be the largest integer less than or equal to t. By defining

$$G(t) = \frac{\int_0^t e^{-\mu s}k(s)\,ds}{\int_0^\infty e^{-\mu s}k(s)\,ds},$$

we see that $K \in \mathcal{S}(\mu)$ implies $G \in \mathcal{S}$, so (48) yields

$$\lim_{t\to\infty} \frac{\int_{n(t)}^\infty e^{-\mu s}k(s)\,ds}{\int_t^\infty e^{-\mu s}k(s)\,ds} = \lim_{t\to\infty} \frac{1 - G(n(t))}{1 - G(t)} = 1.$$

Using this in conjunction with (55) and

$$\begin{aligned}\frac{\int_t^\infty e^{-\mu s}|X(s)|\,ds}{t^\gamma \int_t^\infty e^{-\mu s}k(s)\,ds} &\leq \frac{\int_{n(t)}^\infty e^{-\mu s}|X(s)|\,ds}{n(t)^\gamma \int_{n(t)}^\infty e^{-\mu s}k(s)\,ds}\left(\frac{n(t)}{t}\right)^\gamma \frac{\int_{n(t)}^\infty e^{-\mu s}k(s)\,ds}{\int_t^\infty e^{-\mu s}k(s)\,ds}\\ &= Y_{n(t)}\left(\frac{n(t)}{t}\right)^\gamma \frac{\int_{n(t)}^\infty e^{-\mu s}k(s)\,ds}{\int_t^\infty e^{-\mu s}k(s)\,ds},\end{aligned}$$

gives the result, on taking the limit superior as $t \to \infty$. ∎

5 Decay Rates when Characteristic Roots exist

In the previous section, we saw that solutions of (4) decay more slowly that those of its deterministic counterpart (1), for k in a subclass of functions for which the characteristic equation (2) has no solutions. We now consider the asymptotic behaviour of solutions of (4) when (2) has solutions and the differential resolvent r satisfying (10) is in $L^1(\mathbb{R}^+)$. By Grossman and Miller [14], this precludes solutions of (2) being in $\{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$. Furthermore, there exists a $\gamma > 0$ such that

$$\int_0^\infty k(s)e^{\gamma s}\, ds < +\infty. \tag{56}$$

In fact, because (2) has solutions, it follows that there is a $\lambda > 0$ such that

$$-\lambda = \sup\{\operatorname{Re} z \geq -\gamma : z + a - \hat{k}(z) = 0\}. \tag{57}$$

We show in this case that the decay rate of solutions of (4) is at least as fast as that of solutions of (1). This is established by obtaining an upper bound on the a.s. Liapunov exponent of X. Lower bounds on the decay rate are also given. For exponentially integrable kernels satisfying some additional mild conditions, these show that the a.s. Liapunov exponent cannot be less than $-\gamma$.

5.1 Upper Bounds

The following is essentially proven in [16].

Lemma 5.1. *Suppose that k satisfies (5), $a > \int_0^\infty k(s)\, ds$, and suppose the characteristic equation (2) has a solution. Then the solution of (10) satisfies*

$$\lim_{t\to\infty} \frac{r(t)}{e^{-\lambda t}} = \frac{1}{1 + \int_0^\infty sk(s)e^{\lambda s}\, ds},$$

where $\gamma > 0$ is given by (56) and λ in $(0, a \wedge \gamma)$ by (57).

As at the beginning of the last section, we start with an estimate of the decay rate of $\mathbb{E}[|X|]$. It is a consequence of Lemmata 3.1 and 5.1.

Theorem 5.2. *Suppose that k satisfies (5), $a > \int_0^\infty k(s)\, ds$, and suppose the characteristic equation (2) has a solution. Then the solution of (4) and (6) with $X_0 \neq 0$ a.s., satisfies*

$$\lim_{t\to\infty} \frac{\mathbb{E}[|X(t)|]}{e^{-\lambda t}} = \frac{\mathbb{E}[|X_0|]}{1 + \int_0^\infty sk(s)e^{\lambda s}\, ds}, \tag{58}$$

where $\lambda > 0$ is given by (57).

We remark that Lemma 5.1 implies

$$\lim_{t\to\infty} \frac{1}{t} \log|r(t)| = -\lambda. \tag{59}$$

Also, observe that Theorem 5.2 and Lemma 2.1 imply that there exists $C > 0$ such that

$$\mathbb{E}[|X(t)|] \leq Ce^{-\lambda t}, \quad t \geq 0. \tag{60}$$

We now show that X has an a.s. upper Liapunov exponent $-\lambda$, using the estimate on $\mathbb{E}[|X(t)|]$ given by Theorem 5.2. The proof has the same form as that of Theorem 4.5.

Theorem 5.3. *Suppose that k satisfies (5), $a > \int_0^\infty k(s)\,ds$, and suppose the characteristic equation (2) has a solution. Then the solution of (4) and (6) with $X_0 \neq 0$ a.s., satisfies*

$$\limsup_{t\to\infty} \frac{1}{t} \log |X(t)| \leq -\lambda, \quad a.s., \tag{61}$$

where $\lambda > 0$ is given by (57).

Proof. Consider the sequence $a_n = \tau n$ where $\tau > 0$ satisfies

$$c_2 \tau^{1/2} |\sigma| = \frac{1}{2}, \tag{62}$$

and $c_2 > 0$ is the constant in (33). As in Theorem 4.5, we have (37). Using (62), and the same argument used to obtain (38), we get

$$\mathbb{E}\left[\sup_{a_n \leq t \leq a_{n+1}} \left|\int_{a_n}^t \sigma X(s)\,dB(s)\right|\right] \leq \frac{1}{2}\mathbb{E}\left[\sup_{a_n \leq a_{n+1}} |X(s)|\right]. \tag{63}$$

Combining (37), (63) and (60) yields

$$\mathbb{E}\left[\sup_{a_n \leq t \leq a_{n+1}} |X(t)|\right] \leq 2C\Big\{e^{-\lambda a_n} + a\int_{a_n}^{a_{n+1}} e^{-\lambda s}\,ds + \int_{a_n}^{a_{n+1}} \int_0^s k(s-u)e^{-\lambda u}\,du\,ds\Big\}.$$

Noting that $0 < \lambda < \gamma$, we get

$$\mathbb{E}\left[\sup_{a_n \leq t \leq a_{n+1}} |X(t)|\right] \leq 2C\Big\{e^{-\lambda a_n} + \left(a + \int_0^\infty k(s)e^{\lambda s}\,ds\right)\int_{a_n}^{a_{n+1}} e^{-\lambda s}\,ds\Big\}.$$

Hence there exists a number $C_0 > 0$ such that

$$\mathbb{E}\left[\sup_{a_n \leq t \leq a_{n+1}} |X(t)|\right] \leq C_0 e^{-\lambda a_n}.$$

Therefore, for every $\varepsilon \in (0, \lambda)$,

$$\mathbb{P}\left[\sup_{a_n \leq t \leq a_{n+1}} |X(t)|e^{(\lambda-\varepsilon)a_n} \geq 1\right] \leq C_0 e^{-\varepsilon \tau n}.$$

By the first Borel-Cantelli lemma, this implies that for almost all $\omega \in \Omega$

$$\sup_{\tau n \leq t \leq \tau(n+1)} |X(t)| \leq e^{-(\lambda-\varepsilon)\tau n}$$

holds for all but finitely many n. Using the conclusion of the proof of [17, Theorem 4.3.1] now yields (61). ◻

5.2 Lower Bounds

The final result of the paper involves obtaining lower bounds on the top a.s. Liapunov exponent in the case when (2) has a solution. For this problem, with $\gamma > 0$ defined by (56), it transpires that

$$\lim_{t\to\infty} \frac{1}{t} \log |r(t)| > -\gamma.$$

This indicates for the deterministic equation (1) that increasing a, the intensity of the negative instantaneous feedback, will not arbitrarily increase the rate of decay of the solution. This contrasts to the situation for the non-delay problem

$$r_0'(t) = -ar_0(t), \quad r_0(0) = 1.$$

Moreover stabilisation occurs if an instantaneous multiplicative noise term is added to this deterministic non-delay equation, so that the dynamics of the resulting non-trivial process Z is governed by $Z(0) \neq 0$ and

$$dZ(t) = -aZ(t)\,dt + \sigma Z(t)\,dB(t). \tag{64}$$

Then the a.s. Liapunov exponent of Z is defined by

$$\Lambda_Z(a,\sigma) = \lim_{t\to\infty} \frac{1}{t} \log |Z(t)|, \quad \text{a.s.}$$

and is given by the formula

$$\Lambda_Z(a,\sigma) = -a - \frac{1}{2}\sigma^2.$$

For fixed a, this is a decreasing function of $|\sigma|$. Furthermore,

$$\lim_{|\sigma|\to\infty} \Lambda_Z(a,\sigma) = -\infty,$$

so that the decay rate can be made arbitrarily fast by increasing $|\sigma|$. Our final result shows that neither increasing the intensity of the negative instantaneous feedback a, nor that of the noise σ by an arbitrarily large amount, will decrease the a.s. top Liapunov exponent corresponding to (4) arbitrarily.

Theorem 5.4. *Suppose that k satisfies* (5), $a > \int_0^\infty k(s)\,ds$ *and* (2) *has a solution. If there exists $\mu > 0$ such that*

$$\liminf_{t\to\infty} \frac{1}{t} \log k(t) \geq -\mu,$$

then the solution X of (4) *and* (6) *with $X_0 \neq 0$, satisfies*

$$-\{\mu \wedge (a + \sigma^2/2)\} \leq \liminf_{t\to\infty} \frac{1}{t} \log |X(t)| \leq \limsup_{t\to\infty} \frac{1}{t} \log |X(t)| \leq -\lambda, \quad \textit{a.s.} \tag{65}$$

where λ is given by (57).

Proof. As before, we assume without loss of generality that $X(0) = 1$. From Theorem 5.3, it follows that

$$\limsup_{t\to\infty} \frac{1}{t} \log |X(t)| \leq -\lambda, \quad \text{a.s.},$$

proving one half of the inequality (65). We also know that $\lim_{t\to\infty} X(t) = 0$, and $X \in L^1(\mathbb{R}^+)$ a.s., by Theorem 3.2. This forces $a + \sigma^2/2 > 0$.

In the case when $\mu \geq a + \sigma^2/2 > 0$, we have $\mu \wedge (a + \sigma^2/2) = a + \sigma^2/2$. If φ solves (14), we have that y defined by $y = X/\varphi$ obeys $y(t) \geq 1$ for all $t \geq 0$. Hence $X(t) \geq \varphi(t)$, so

$$\liminf_{t\to\infty} \frac{1}{t} \log |X(t)| \geq \liminf_{t\to\infty} \frac{1}{t} \log \varphi(t) = -(a + \sigma^2/2),$$

proving the result in this case. If, on the other hand, $\mu < a + \sigma^2/2$, we have $\mu \wedge (a + \sigma^2/2) = \mu$. All the conditions of Corollary 3.7 apply, so we have

$$\liminf_{t\to\infty} \frac{1}{t} \log |X(t)| \geq -\mu = -\{\mu \wedge (a + \sigma^2/2)\}, \quad \text{a.s.}$$

proving the left half of the inequality (65) in this case also. $\square$

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