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The cluster size distribution for a forest-fire process on $\mathbb{Z}$

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Abstract

Consider the following forest-fire model where trees are located on sites of $\mathbb{Z}$. A site can be vacant or be occupied by a tree. Each vacant site becomes occupied at rate 1, independently of the other sites. Each site is hit by lightning with rate λ, which burns down the occupied cluster of that site instantaneously. As $\lambda \downarrow 0$ this process is believed to display self-organised critical behaviour.

This paper is mainly concerned with the cluster size distribution in steady-state. Drossel, Clar and Schwabl (3) claimed that the cluster size distribution has a certain power law behaviour which holds for cluster sizes that are not too large compared to some explicit cluster size s_{max}. The latter can be written in terms of λ approximately as $s_{max}\ln(s_{max}) = 1/\lambda$. However, Van den Berg and Járai (1) showed that this claim is not correct for cluster sizes of order s_{max}, which left the question for which cluster sizes the power law behaviour *does* hold. Our main result is a rigorous proof of the power law behaviour up to cluster sizes of the order $s_{max}^{1/3}$. Further, it proves the existence of a stationary translation invariant distribution, which was always assumed but never shown rigorously in the literature

Key words: forest-fires, self-organised criticality, cluster size distribution

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1 Introduction and background

This paper discusses a forest-fire model in one dimension. In time, trees can grow, or disappear by fire. Let $\Omega = \{0,1\}^{\mathbb{Z}}$ be the state space. A '1' represents a tree and a '0' represents a vacant space. To each site of $\mathbb{Z}$ we assign a Poisson process with rate 1, independently of the other sites. If such a Poisson clock rings, there is a birth attempt. If the site is empty, a tree grows. If a tree is already present, nothing happens. Further, to each site we assign a Poisson process with rate λ, independently of the other sites. If such a Poisson clock rings, there is a lightning attempt. The lightning destroys instantaneously all the trees that are in the cluster of the site that is hit. If the site was vacant, nothing happens. In what follows, a Poisson event is the ring of a Poisson clock (either a birth attempt or a lightning attempt). It is expected that this model displays so-called self-organised critical behaviour as the lightning rate λ goes to zero.

If we start from any configuration with infinitely many vacant sites on both half-lines, then (with probability one), infinitely many of those sites in $\mathbb{Z}$ stay vacant during the time interval $[0,t]$. This 'breaks up' the line in finite pieces, and hence the process is easily constructed (for example by using a graphical representation, see (5), (6)). In higher dimensions this is not evident; recently M. Dürre (2) has made a more abstract construction which does work for higher dimensions. Although the existence of the process on $\mathbb{Z}$ is relatively easy, it is not immediately clear whether there exists a stationary translation-invariant measure. We come back to this issue later. For now, let μ_λ denote any stationary translation-invariant measure.

For any $\omega \in \Omega$, we write $\omega = (\cdots \omega_{-1}\omega_0\omega_1 \cdots)$ where $\omega_i \in \{0,1\}$ denotes the state of the site at position i. In short notation, we write $\mu_\lambda(0) = \mu_\lambda(\omega_0 = 0)$ for the probability that the origin is empty in the steady-state, and $\mu_\lambda(1)$ for the probability that the origin is occupied. It seems likely that the probability that a site is empty goes to zero as the lightning rate goes to zero. This is indeed the case and the speed at which this happens is known: it has been proved in (1) that there exist positive constants A_1 and A_2 such that for $\lambda < 1$ and any measure μ_λ invariant under the dynamics,

$$\frac{A_1}{\ln(1/\lambda)} \leq \mu_\lambda(0) \leq \frac{A_2}{\ln(1/\lambda)}. \tag{1.1}$$

This forest-fire model is closely related to the well-known Drossel-Schwabl forest-fire model (4). In that model the state space is large but finite, the speed of fires is finite and time is discrete. The self-organised critical behaviour is expected when the volume and the speed of fires go to infinity, and the lightning rate to zero in a suitable way. In dimension 1, the behaviour of this model has been studied in (3) but their results are not rigorous and some of them need significant correction (see below).

Van den Berg and Járai (1) have studied our version of the model rigorously. For the overall behaviour, it does not matter much which model we consider. The proof of the main result can easily be adjusted to the Drossel-Schwabl model.

The forest-fire process has 'natural scales'. If we consider a string of length n, there is a non-trivial (i.e. bounded away from zero and one) probability that all sites grow a tree in a time interval of length $\ln n$. If the lightning parameter is of order $1/(n \ln n)$, there is also a non-trivial probability that a lightning attempt occurs in this string. This and other considerations ((1), (4)) lead to the following definition of a characteristic length: $s_{max} = s_{max}(\lambda)$ is the integer satisfying

$$s_{max} \ln s_{max} \leq \frac{1}{\lambda} < (s_{max}+1)\ln(s_{max}+1). \tag{1.2}$$

Our result concerns $p_\lambda(s)$, where

$$p_\lambda(s) := \mu_\lambda(\omega_0 = 0, \omega_1 = \cdots = \omega_s = 1, \omega_{s+1} = 0).$$

The probabilities $p_\lambda(s), s \geq 0$ are called the cluster size distribution in (3), although strictly speaking, they are not a distribution but the probabilities of the event that the origin is the left-most site of a cluster of size s. From $p_\lambda(s)$, it is easy to recover the true cluster size distribution. By translation invariance, the probability that a fixed site is in a cluster of size s is $sp_\lambda(s)$.

In (3) it was shown that for fixed s, the probabilities $p_\lambda(s)$ satisfy

$$p_\lambda(s) \simeq \mu_\lambda(0)s^{-2}, \tag{1.3}$$

where the symbol '$\simeq$' means that the quotient of the left and right side is bounded from above and below as $\lambda \downarrow 0$. Further, it is an 'ansatz' in their paper that (1.3) holds for s up to s_{max}. Although this ansatz led to a correct prediction in (3) of the asymptotic behaviour (1.1) of $\mu_\lambda(0)$, it was shown in (1) that (1.3) does not hold for s of the order s_{max}. This raises the question for which $s < s_{max}$ the relation (1.3) does hold. In this paper we partly answer this question by showing that, loosely formulated, (1.3) holds for s up to $s_{max}^{1/3}$ and hence (by (1.1))

$$p_\lambda(s) \simeq s^{-2}/\ln(1/\lambda) \text{ for all } s \text{ up to } s_{max}^{1/3}.$$

A precise formulation is stated in the following section. Section 3 handles some preliminaries and in Section 4 we give the proof of the main result. In Section 5, we address the issue of the existence of a stationary translation-invariant measure for the one-dimensional forest-fire process.

2 Statement of the main result

The main result is as follows.

Theorem 2.1. *Let $\alpha < 1/3$. There exists positive constants B_1 and B_2 such that for all $\lambda < 1$ and any stationary, translation-invariant measure μ_λ of the forest-fire model with parameter λ on $\mathbb{Z}$,*

$$\frac{B_1}{s^2\ln(1/\lambda)} \leq \mu_\lambda(\omega_0 = 0, \omega_1 = \cdots = \omega_s = 1, \omega_{s+1} = 0) \leq \frac{B_2}{s^2\ln(1/\lambda)}, \tag{2.1}$$

for all $s \leq s_{max}^\alpha$.

The theorem above is only useful when there is at least one stationary translation-invariant measure for the one-dimensional forest-fire model. We will show that this is the case in Section 5.

3 Preliminaries

Note that on a finite interval, the total rate of the Poisson processes assigned to the sites in this interval is finite. This implies that the probability of two Poisson events occurring inside this finite space interval in a time interval of length t is $o(t)$, as $t \downarrow 0$. We repeatedly use this fact when proving statements about the stationary measure. However, when we consider $\mathbb{Z}$, infinitely many Poisson events occur in any time interval; far away a tree could be hit by lightning and the fire thus started could travel over a very large distance, creating long-range dependencies. We have no a priori bound on these dependencies. Unfortunately, this complicates the argument. In the rest of this section, we show that in some sense 'there are enough vacant sites at all times', which gives us a bound on the size of clusters, and hence a bound on the size of fires. We make this precise in Proposition 3.3. We need an auxiliary model where destructions are local.

The model with local destructions is coupled to the original one by using the same Poisson clocks. The state space is $\{0,1\}^{\mathbb{Z}}$ as before. Each time a growth clocks rings, a tree tries to grow. As for the lightning attempts: a lightning attempt destroys a tree (if present) instantaneously. The difference to the original model is that the rest of its cluster remains intact. This implies that each site behaves independently of the other sites. Hence the distribution of the configuration at time t converges to ν_λ, which is the product measure with density $1/(\lambda+1)$, as $t \to \infty$. This happens for any initial configuration. Now suppose that we take a configuration x in $\{0,1\}^{\mathbb{Z}}$ and start both the processes from that configuration. Using the same Poisson events for both models gives us a natural coupling. Let $\eta^x(t)$ denote the configuration at time t when we start in configuration x for the model with local destructions. For the original model we define $\omega^x(t)$ likewise. Then from the definition of the processes it should be clear that for all initial configurations $x \in \{0,1\}^{\mathbb{Z}}$, and all times $t \geq 0$,

$$\omega^x(t) \leq \eta^x(t). \tag{3.1}$$

Consider the interval $I = [i_1, i_m] \subset \mathbb{Z}$. We define for any configuration $x \in \{0,1\}^{\mathbb{Z}}$,

$$\begin{aligned} i_l = i_l[x] &:= \max\{j \leq i_1 - 2 : x_j + x_{j+1} = 0\}, \\ i_r = i_r[x] &:= \min\{j \geq i_m + 2 : x_{j-1} + x_j = 0\}, \\ S_I(x) &:= [i_l, i_r]. \end{aligned} \tag{3.2}$$

In words, $S_I(x)$ is the smallest set of consecutive sites containing the interval $[i_1, i_m]$, 2 consecutive zeros to the left of i_1 and 2 consecutive zeros to the right of i_m. In many of the applications below, I will be the set determining some cylinder event. We prove the following lemma.

Lemma 3.1. *For any $\lambda < 1$, any stationary measure μ_λ and any interval $I = [i_1, i_m]$, there exists a $D = D(\lambda) \in (0,1)$ such that for all $s \geq m+4$,*

$$\mu_\lambda(|S_I| \geq s) \leq 2D^{\lfloor (s-m) \rfloor}.$$

Proof. Let I be as in the statement of the lemma. Let μ_λ be a stationary measure and let $x \in \{0,1\}^{\mathbb{Z}}$. Let $\mathcal{P}_\lambda$ denote the measure governing the Poisson clocks. By (3.1), for any $t \geq 0$,

$$\mathcal{P}_\lambda(|S_I(\omega^x(t))| \geq s) \leq \mathcal{P}_\lambda(|S_I(\eta^x(t))| \geq s) \tag{3.3}$$

This inequality remains valid if we integrate over x with respect to μ_λ. The left side of (3.3) is then simply $\mu_\lambda(|S_I| \geq s)$. Recall that the η-process converges to its stationary measure ν_λ for every initial configuration, when we take the limit $t \to \infty$. Taking this limit, we obtain

$$\mu_\lambda(|S_I| \geq s) \quad \leq \quad \nu_\lambda(|S_I| \geq s). \tag{3.4}$$

Suppose the size of S_I is at least s. Then there are at least $\lfloor (s-m)/2 \rfloor$ sites in S_I directly to the right of i_m (or directly to the left of i_1). These sites can be divided into disjoint pairs, where each pair, apart from the right-most (or left-most respectively), has at least one occupied site, by definition of S_I. This gives us

$$\nu_\lambda(|S_I| \geq s) \leq 2\Big(1 - \big(\frac{\lambda}{1+\lambda}\big)^2\Big)^{\lfloor (s-m)/4 \rfloor - 1}. \tag{3.5}$$

Combining (3.4) and (3.5) proves the lemma. $\square$

Note that in the proof above, we did not need translation invariance for μ_λ.

Remark 3.2. An immediate consequence of Lemma 3.1 is that occupied clusters are a.s. finite.

Lemma 3.1 gives us the tools to bound influences from far away. When the initial configuration is drawn from μ_λ, we define $\mathcal{P}^{\mu_\lambda}$ to be the measure governing this initial configuration and the Poisson clocks, hence determining the forest-fire process. Recall that $\omega(t)$ denotes the state of the model at time t. To make the statement at the beginning of this section precise we prove the following:

Proposition 3.3. *Let I be an interval $[i_1, i_m] \subset \mathbb{Z}$ and let $t > 0$. Recall the definition of $S_I(\omega)$ and let $M(I, \omega(0), t)$ denote the number of Poisson events occurring in the set $S_I(\omega(0))$ in the time-interval $[0, t]$. For any stationary measure μ_λ and interval I,*

$$\mathcal{P}^{\mu_\lambda}(M(I, \omega(0), t) > 1) = o(t),$$

as $t \downarrow 0$.

Proof. For fixed sets S_I it is immediately clear that the probability of two or more Poisson events in S_I is $o(t)$ as $t \downarrow 0$. But now S_I is random; at this point we use that the size distribution of S_I decays exponentially. Further, we use that for a Poisson process $X(t)$ with $EX(t) = \alpha t$ we have $\mathcal{P}(X(t) > 1) \leq (\alpha t)^2$.

$$\begin{aligned}
&\mathcal{P}^{\mu_\lambda}(M(I, \omega(0), t) > 1) \\
&\quad \leq \sum_{j \geq m+4} \mathcal{P}^{\mu_\lambda}(M(I, \omega(0), t) > 1 \mid |S_I(\omega(0))| = j)\mu_\lambda(|S_I(\omega(0))| = j) \\
&\quad \leq \sum_{j \geq m+4} j^2(\lambda+1)^2 t^2 \mu_\lambda(|S_I(\omega(0))| = j).
\end{aligned}$$

The sum over j is finite by Lemma 3.1, which proves the proposition. $\square$

4 Proof of the main result

The proof of our main result is based on ideas from (7). We introduce some notation and state some lemmas first. From now, let $\lambda > 0$ be fixed and suppose that μ_λ is a stationary translation-invariant measure. In what follows, when we write $o(t)$, we implicitly take the limit $t \downarrow 0$.

In the proof of the main theorem, we need to bound the probability of the event that a tree on the edge of a cluster is burnt. The next lemma shows that although large clusters may arise, the probability that a boundary tree is on fire is not that large.

Define for $i \in \mathbb{Z}$ and $t > 0$,

$$\begin{aligned} B_i^+(t) &:= \{\omega_i(0) = 0, \omega_{i+1}(0) = 1, \omega_{i+1}(t) = 0\}, \\ B_i^-(t) &:= \{\omega_i(0) = 0, \omega_{i-1}(0) = 1, \omega_{i-1}(t) = 0\}. \end{aligned}$$

Note that by translation invariance, the probability of $B_i^+(t)$ does not depend on i. The same holds for B_i^-.

Lemma 4.1. *Let $B_i^+(t), B_i^-(t)$ be defined as above. Then for all i and all $t > 0$,*

$$\begin{aligned} \mathcal{P}^{\mu_\lambda}(B_i^+(t)) &\leq \lambda t + o(t), \\ \mathcal{P}^{\mu_\lambda}(B_i^-(t)) &\leq \lambda t + o(t). \end{aligned}$$

Proof. We prove the lemma only for $B_i^+(t)$. The proof for $B_i^-(t)$ is completely similar. Note first that we can bound the possibility that more than one Poisson event influences the event $B_i^+(t)$. This will give us for example, that the event that the site $i+1$ is occupied, becomes vacant, again becomes occupied and again vacant (which is in principle possible by the definition of $B_i^+(t)$) has probability $o(t)$. We take $I = \{i+1\}$ and apply Proposition 3.3:

$$\mathcal{P}^{\mu_\lambda}(B_i^+(t)) \leq \mathcal{P}^{\mu_\lambda}(B_i^+(t) \cap M(\{i+1\}, \omega(0), t) \leq 1) + o(t). \tag{4.1}$$

On $\{M(\{i+1\}, \omega(0), t) \leq 1\}$, only Poisson events inside $S_{\{i+1\}}$ can cause $B_i^+(t)$ to occur: the two zeros on the left and right boundary of $S_{\{i+1\}}$ (and the fact that at most one of these turns into a one) prevents fires from the outside to reach $i+1$. For the same reason,

$$\mathcal{P}^{\mu_\lambda}(B_i^+(t) \cap M(\{i+1\}, \omega(0), t) = 0) = 0. \tag{4.2}$$

Now, using that the cluster of site i is finite a.s. we consider all possibilities that cause $B_i^+(t)$ to occur. Let L_j denote the event that site j is hit by lightning in the time interval $[0, t]$. We get

$$\begin{aligned} &\mathcal{P}^{\mu_\lambda}(B_i^+(t) \cap M(\{i+1\}, \omega(0), t) = 1) \\ &\leq \sum_{j=0}^{\infty} \mathcal{P}^{\mu_\lambda}(\omega_i(0) = 0, \omega_{i+1}(0) = \cdots = \omega_{i+j}(0) = 1 \text{ and } L_j) \\ &= \sum_{j=0}^{\infty} \mu_\lambda(\omega_{-j} = 0, \omega_{-j+1} = \cdots = \omega_0 = 1) \mathcal{P}^{\mu_\lambda}(L_j) \\ &\leq \mu_\lambda(1)\lambda t + o(t) \leq \lambda t + o(t). \end{aligned} \tag{4.3}$$

Once more, we have used translation invariance in the equality above. Combining (4.1), (4.2) and (4.3) proves the lemma. □

We now concentrate on what happens in a finite string. Define

$$\Omega_n[k] \quad := \quad \{\omega \in \Omega : \sum_{i=0}^{n-1} \omega_i = k\},$$

We consider the event that in a string of n consecutive sites there are exactly k occupied sites, and the ends of the string are empty. We define for $k \leq n-1$,

$$A_n^k = \{\omega \in \Omega_n[k] : \omega_0 = \omega_{n-1} = 0\}. \tag{4.4}$$

Note that we are particularly interested in A_{n+2}^n. The proof of the main theorem is based heavily on the following relation for the A_n^k's.

Lemma 4.2.

$$\left|\mu_\lambda(A_n^0) \quad - \quad \frac{\mu_\lambda(0)}{n}\right| \leq \lambda n \tag{4.5}$$

$$\left|\mu_\lambda(A_n^k) \quad - \quad \frac{n-k-1}{n-k}\mu_\lambda(A_n^{k-1})\right| \leq \frac{4\lambda n}{n-k}. \tag{4.6}$$

Note that the event A_n^0 contains only configurations that are equal zero to on $[0, n-1]$.

Proof. For any measurable $A \subset \{0,1\}^{\mathbb{Z}}$ and any $t > 0$, we have

$$\mathcal{P}^{\mu_\lambda}(\omega(0) \in A) = \mathcal{P}^{\mu_\lambda}(\omega(t) \in A),$$

and hence

$$\mathcal{P}^{\mu_\lambda}(\omega(0) \notin A, \omega(t) \in A) = \mathcal{P}^{\mu_\lambda}(\omega(0) \in A, \omega(t) \notin A). \tag{4.7}$$

The equation (4.7) is called the steady-state equation. We refer to the l.h.s. of (4.7) as 'going in' side and to the r.h.s. as 'going out' side, for obvious reasons.

We first show (4.5). To this end, we apply (4.7) to $A = \{\omega \in \Omega : \omega_0 = \omega_1 = \cdots = \omega_{n-2} = 0, \omega_{n-1} = 1\}$. Now on the 'going in' side of the steady-state equation for A we get the following contributions.

- At time 0 we see the configuration $\{\omega_0(0) = \cdots = \omega_{n-1}(0) = 0\}$ and a tree grows at site $n-1$ in the time interval $[0, t]$.
- We see a configuration at time 0 where there is a cluster of trees with rightmost site between 0 and $n-3$ and this cluster is burnt during the time interval $[0, t]$. This has probability at most $\lambda(n-2)t + o(t)$ by Lemma 4.1.

All other possibilities have probability $o(t)$ by Proposition 3.3. This gives us a contribution of at least

$$\begin{aligned}&\mu_\lambda(\omega_0 = \cdots = \omega_{n-1} = 0)t + o(t)\\ &\text{and at most}\\ &\mu_\lambda(\omega_0 = \cdots = \omega_{n-1} = 0)t + (n-2)\lambda t + o(t).\end{aligned} \tag{4.8}$$

On the 'going out' side of the steady-state equation we get contributions from growing trees and fires as well.

- A tree grows on one of the vacant sites, which gives us a factor $(n-1)\mu_\lambda(\omega_0 = \cdots = \omega_{n-2} = 0, \omega_{n-1} = 1)t + o(t)$,
- The tree on site $n-1$ is burnt. This gives us a contribution of exactly $\mathcal{P}^{\mu_\lambda}(B^+_{n-2}(t))$, which is at most $\lambda t + o(t)$ by Lemma 4.1.

All other possibilities have probability $o(t)$ by Proposition 3.3. This gives us a contribution for the 'going out' side of at least

$$\begin{aligned}&(n-1)\mu_\lambda(\omega_0 = \cdots = \omega_{n-2} = 0, \omega_{n-1} = 1)t + o(t)\\ &\text{and at most}\\ &(n-1)\mu_\lambda(\omega_0 = \cdots = \omega_{n-2} = 0, \omega_{n-1} = 1)t + \lambda t + o(t).\end{aligned} \tag{4.9}$$

Combining (4.7), (4.8) and (4.9) we obtain

$$\Big|\mu_\lambda(\omega_0 = \cdots = \omega_{n-1} = 0)t - (n-1)\mu_\lambda(\omega_0 = \cdots = \omega_{n-2} = 0, \omega_{n-1} = 1)t\Big| < \lambda n t + o(t).$$

We divide by t and subsequently let $t \downarrow 0$:

$$\Big|\mu_\lambda(\omega_0 = \cdots = \omega_{n-1} = 0) - (n-1)\mu_\lambda(\omega_0 = \cdots = \omega_{n-2} = 0, \omega_{n-1} = 1)\Big| < \lambda n. \tag{4.10}$$

Note that

$$\begin{aligned}\mu_\lambda(\omega_0 = \cdots = \omega_{n-2} = 0, \omega_{n-1} = 1) \quad &+ \quad \mu_\lambda(\omega_0 = \cdots = \omega_{n-1} = 0)\\ &= \quad \mu_\lambda(\omega_0 = \cdots = \omega_{n-2} = 0).\end{aligned} \tag{4.11}$$

Now, combining (4.11) with (4.10) we obtain

$$\Big|\mu_\lambda(A^0_n) - \frac{n-1}{n}\mu_\lambda(A^0_{n-1})\Big| < \lambda. \tag{4.12}$$

Iterating (4.12) we get (4.5).

Now take $n-1 \geq k > 0$. As before, we consider the steady-state equation (4.7), but now for the event A^k_n to prove (4.6). Again we ignore multiple Poisson events using Proposition 3.3.

On the 'going in' side of the equation we obtain the following contributions.

- A tree grows in a configuration with $k-1$ trees. There are $n-2-(k-1)$ possible locations; recall that there are no trees allowed at site 0 or $n-1$ in a configuration in A^k_n.

- The possibility to get into a configuration in A_n^k by a fire is contained in $\cup_{i=0}^{n-2}(B_i^+(t) \cup B_{i+1}^-(t))$ so applying Lemma 4.1 gives a contribution of at most $2(n-1)\lambda t + o(t)$.

The total contribution on the 'going in' side is at least

$$\begin{aligned}&(n-k-1)\mu_\lambda(A_n^{k-1})t + o(t)\\ &\text{and at most}\\ &(n-k-1)\mu_\lambda(A_n^{k-1})t + 2(n-1)\lambda t + o(t).\end{aligned} \tag{4.13}$$

On the 'going out' side of the equation we get the following contributions.

- In a configuration in A_n^k a tree grows. There are $n-k$ possible locations.
- The possibility to leave a configuration in A_n^k by a fire is contained in $\cup_{i=0}^{n-3}(B_i^+(t) \cup B_{i+2}^-(t))$. As before, Lemma 4.1 bounds this probability by $2(n-2)\lambda t + o(t)$.

The total contribution on the 'going out' side is at least

$$\begin{aligned}&(n-k)\mu_\lambda(A_n^k)t + o(t)\\ &\text{and at most}\\ &(n-k)\mu_\lambda(A_n^k)t + 2(n-2)\lambda t + o(t).\end{aligned} \tag{4.14}$$

Combining (4.7), (4.13) and (4.14) and subsequently dividing by t (and $n-k$) we obtain as $t \downarrow 0$,

$$\left|\mu_\lambda(A_n^k) - \frac{n-k-1}{n-k}\mu_\lambda(A_n^{k-1})\right| \leq \frac{4(n-1)\lambda}{n-k}.$$

This proves (4.6). □

Now we are ready for the proof of Theorem 2.1.

Proof. [Thm 2.1] Note that $\{\omega \in \Omega : \omega_0 = 0, \omega_1 = \cdots = \omega_n = 1, \omega_{n+1} = 0\} = A_{n+2}^n$. We apply (4.5) and (4.6) repeatedly to obtain

$$\left|\mu_\lambda(A_{n+2}^n) \quad - \quad \frac{1}{2}\frac{2}{3}\cdot\dots\cdot\frac{n}{n+1}\frac{\mu_\lambda(0)}{n+2}\right| \leq \frac{n\lambda}{n+1} + \sum_{j=1}^{n+1}\frac{4n\lambda}{j},$$

which tells us

$$\left|\mu_\lambda(A_{n+2}^n) \quad - \quad \frac{\mu_\lambda(0)}{(n+1)(n+2)}\right| \leq (4n[log(n+1)+1]+1)\lambda.$$

The question is now for which cluster sizes the error can be bounded uniformly in λ. Using (1.1), the relative error is

$$\frac{(n+1)(n+2)(4n[\ln(n+1)+1]+1)\lambda}{\mu_\lambda(0)} \simeq n^3\ln(n)\lambda\ln(1/\lambda).$$

Now choose $\alpha < 1/3$ and suppose that $n \leq s_{max}^\alpha$. The right hand side is dominated by $\alpha s_{max}^{3\alpha-1}\ln(1/\lambda)$. By definition of s_{max} (equation (1.2)) this dominating factor goes to zero as $\lambda \downarrow 0$ and this proves Theorem 2.1. □

5 Existence of a stationary translation-invariant measure

Proposition 5.1. *For the forest-fire model with parameter λ on $\mathbb{Z}$, there exists at least one stationary, translation-invariant measure.*

Since the forest-fire process is not a Feller process, the proposition does not follow from the standard theory, see for example (6).

Proof. Fix $\lambda > 0$. Let $\omega(\cdot)$ denote the forest-fire process on $\mathbb{Z}$. Let $k \in \mathbb{N}$ and let $\omega^{(k)}(\cdot)$ be an auxiliary forest-fire process where on $[-k, k]$ we have the dynamics of the ordinary forest-fire process, but with the understanding that we consider sites $-k$ and k to be neighbours i.e., we consider the forest-fire process on a one-dimensional torus embedded in $\mathbb{Z}$. For instance if $-k$ and k are both occupied we consider them to be in the same occupied cluster for this process. Outside the interval $[-k, k]$ nothing happens, i.e. all sites are kept vacant.

Essentially, $\omega^{(k)}$ is just the forest-fire process on a circle with $2k + 1$ sites. This is a Markov chain with a unique stationary distribution which we denote by $\mu^{(k)}$. Note that by the above description,

$$\mu^{(k)}(\omega_i^{(k)} = 0) = 1 \text{ for all } |i| > k.$$

By standard arguments the sequence $\mu^{(k)}, k = 1, 2, \dots$ has a weakly convergent subsequence $\mu^{(k_i)}, i = 1, 2 \dots$. We denote its limit by μ. By the above mentioned correspondence to circles, where we have rotation invariance for $\mu^{(k)}, k > 0$, it follows immediately that μ is translation invariant. We will show that μ is a stationary distribution for the process $\omega(\cdot)$.

First some notation. Let ν be a distribution on $\{0, 1\}^{\mathbb{Z}}$, assigning infinitely many zeros to both half-lines (so that the process starting in a configuration drawn from this distribution exists). Let $\mathcal{P}^{\nu}$ denote the law governing all Poisson processes and the initial configuration. In particular, $\mathcal{P}^{\nu}(\omega(t) \in \cdot)$ is the distribution of the configuration at time t for the forest-fire process starting in a configuration drawn from ν. Similarly, when $\nu(k)$ is a distribution on $\{0, 1\}^{[-k,k]}$ we write $\mathcal{P}^{\nu(k)}(\omega^{(k)}(t) \in \cdot)$ for the auxiliary process $\omega^{(k)}$, starting in a configuration drawn from $\nu(k)$. For a distribution ν on $\{0, 1\}^{\mathbb{Z}}$ and a subset J of $\mathbb{Z}$ we denote by ν_J the restriction of ν to J, i.e. its projection on J.

To show that μ is a stationary distribution for the process $\omega(\cdot)$, it is sufficient to show that for all cylinder events A and all $t < 1$

$$\mathcal{P}^{\mu}(\omega(t) \in A) = \mu(A). \tag{5.1}$$

So let A be a cylinder event and $t < 1$. Let $I = [i_1, i_m]$ be an interval such that A is determined by the configuration on I. Take positive integers L and N. We define $J = J(L) = [i_1 - L, i_m + L]$ and let $k > \max\{|i_1 - L|, |i_m + L|\}$. If the ordinary forest-fire process $\omega(\cdot)$ and the auxiliary process $\omega^{(k)}(\cdot)$ start with initial configurations which agree on J and we use the same Poisson clocks, then the only way to have disagreement between $\omega(t)$ and $\omega^{(k)}(t)$ on I is by influences (fires) from outside J. These fires can only reach I if all sites in the interval $[i_1 - L, i_1)$ that were vacant at time 0, become occupied before time t, or if all sites in the interval $(i_m, i_m + L]$ that were vacant at time 0 become occupied before time t. If in both intervals, the number of zeros is at least N, the above event clearly has probability at most $2t^N$.

Now we couple the process $\omega(\cdot)$ with initial distribution μ and the auxiliary process $\omega^{(k)}(\cdot)$ with initial configuration $\mu^{(k)}$ by using the same Poisson clocks and by optimally coupling μ_J and $\mu_J^{(k)}$. Using such coupling and the argument from the previous paragraph, gives

$$\begin{aligned} \left|\mathcal{P}^{\mu}(\omega(t) \in A) - \mu^{(k)}(A)\right| &= \left|\mathcal{P}^{\mu}(\omega(t) \in A) - \mathcal{P}^{\mu^{(k)}}(\omega^{(k)}(t) \in A)\right| \\ &\leq d_V(\mu_J, \mu_J^{(k)}) + 2t^N + \mu(B_I(L,N)), \end{aligned} \tag{5.2}$$

where d_V denotes variational distance and where

$$B_I(L,N) = \{< N \text{ vacant sites in } [i_1 - L, i_1) \text{ or in } (i_m, i_m + L]\}.$$

Now let $k \to \infty$ along the subsection mentioned in the beginning of this section. Since $\mu^{(k)}$ converges weakly to μ along that subsequence, (5.2) then becomes

$$\left|\mathcal{P}^{\mu}(\omega(t) \in A) - \mu(A)\right| \leq 2t^N + \mu(B_I(L,N)). \tag{5.3}$$

Now we first let $L \to \infty$. By exactly the same reasons as in the beginning of Section 3, μ is dominated by a product measure with a positive density of zero. Hence, the last term in (5.3) goes to zero as $L \to \infty$. Finally, we let $N \to \infty$ to finish the proof.

$\square$

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References

[1] van den Berg, J. and Járai, A.A. (2005). On the asypmtotic density in a one-dimensional self-organized critical forest-fire model, *Comm. Math. Phys.* **253**(3) 633-644.

[2] Dürre, M. (2005). Existence of multi-dimensional infinite volume self-organized critical forest-fire models, Preprint.

[3] Drossel, B., Clar, S. and Schwabl, F. (1993). Exact results for the one-dimensional self-organized critical forest-fire model, *Phys. Rev. Lett.* **71**(23) 3739-3742.

[4] Drossel, B. and Schwabl, F. (1992). Self-organized critical forest-fire model, *Phys. Rev. Lett.* **69** 1629-1632.

[5] Durrett, R. (1988). *Lecture notes on particle systems and percolation,* Wadsworth & Brooks/Cole, Pacific Grove, California.

[6] Liggett, T.M. (1985). *Interacting Particle Systems,* Grundlehren der mathematischen Wissenschaften. Springer-Verlag, New York.

[7] Pennanen, J. (2000). Self-organized criticality of the forest fire model. Master's thesis, University of Helsinki.