Electronic Journal of Probability

Vol. 15 (2010), Paper no. , pages 1743–1771.

Journal URL
http://www.math.washington.edu/~ejpecp/

A class of $\mathbb{F}$-doubly stochastic Markov chains *

Jacek Jakubowski
Institute of Mathematics, University of Warsaw
Banacha 2, 02-097 Warszawa, Poland
and
Faculty of Mathematics and Information Science
Warsaw University of Technology
Plac Politechniki 1, 00-661 Warszawa, Poland
E-mail: jakub@mimuw.edu.pl

Mariusz Niewęgłowski
Faculty of Mathematics and Information Science,
Warsaw University of Technology
Plac Politechniki 1, 00-661 Warszawa, Poland
E-mail: M.Nieweglowski@mini.pw.edu.pl

Abstract

We define a new class of processes, very useful in applications, $\mathbb{F}$-doubly stochastic Markov chains which contains among others Markov chains. This class is fully characterized by some martingale properties, and one of them is new even in the case of Markov chains. Moreover a predictable representation theorem holds and doubly stochastic property is preserved under natural change of measure.

Key words: $\mathbb{F}$-doubly stochastic Markov chain, intensity, Kolmogorov equations, martingale characterization, sojourn time, predictable representation theorem.

AMS 2000 Subject Classification: Primary 60G99; Secondary: 60G55, 60G44, 60G17, 60K99.

Submitted to EJP on March 29, 2010, final version accepted September 28, 2010.

*Research supported in part by Polish KBN Grant P03A 034 29 “Stochastic evolution equations driven by Lévy noise” and by Polish MNiSW grant N N201 547838.

1 Introduction

Our goal is to find a class of processes with good properties which can be used for modeling different kind of phenomena. So, in Section 2 we introduce a class of processes, which we call $\mathbb{F}$-doubly stochastic Markov chains. The reason for the name is that there are two sources of uncertainty in their definition, so in analogy to Cox processes, called doubly stochastic Poisson processes, we chose the name "$\mathbb{F}$-doubly stochastic Markov chains". This class contains Markov chains, compound Poisson processes with jumps in $\mathbb{Z}$, Cox processes, the process of rating migration constructed by Lando [16] and also the process of rating migration obtained by the canonical construction in Bielecki and Rutkowski [3]. We stress that the class of $\mathbb{F}$-doubly stochastic Markov chains contains processes that are not Markov. In the following we use the shorthand "$\mathbb{F}$–DS Markov chain" for the "$\mathbb{F}$-doubly stochastic Markov chain". Note that an $\mathbb{F}$-doubly stochastic Markov chain is a different object than a doubly stochastic Markov chain which is a Markov chain with a doubly stochastic transition matrix. On the end of this section we give examples of $\mathbb{F}$-doubly stochastic Markov chains. Section 3 is devoted to investigation of basic properties of $\mathbb{F}$-doubly stochastic Markov chains. In the first part we prove that an $\mathbb{F}$–DS Markov chain C is a conditional Markov chain and that any $\mathbb{F}$-martingale is a $\mathbb{F} \vee \mathbb{F}^C$-martingale. This means that the immersion property for $(\mathbb{F}, \mathbb{F} \vee \mathbb{F}^C)$, so called hypothesis H, holds. Moreover, the family of transition matrices satisfies the Chapman-Kolmogorov equations. In the second part and until the end of the paper we restricted ourselves to a class of $\mathbb{F}$–DS Markov chains with values in a finite set $\mathscr{K} = \{1, \ldots, K\}$. We introduce the notion of intensity of an $\mathbb{F}$–DS Markov chain and formulate conditions which ensure its existence. In section 4 we prove that an $\mathbb{F}$–DS Markov chain C with intensity is completely characterized by the martingale property of the compensated process describing the position of C as well as by the martingale property of the compensated processes counting the number of jumps of C from one state to another (Theorem 4.1). The equivalence between points iii) and iv) in Theorem 4.1 in a context of Markov chains has not yet been known according to the best of our knowledge. In a view of the above characterizations, the $\mathbb{F}$–DS Markov chains can be described as the class of processes that behave like time inhomogeneous Markov chains conditioned on $\mathscr{F}_\infty$. Moreover these equivalences and the fact that the class of $\mathbb{F}$-DS Markov chain contains the most of $\mathbb{F}$ conditional $\mathbb{F} \vee \mathbb{F}^C$ Markov chains used for modeling in finance, indicate that the class of $\mathbb{F}$-DS Markov chains is a natural, and very useful in applications, subspace of $\mathbb{F}$ conditional $\mathbb{F} \vee \mathbb{F}^C$ Markov chains. Next, an $\mathbb{F}$–DS Markov chain with a given intensity is constructed. Section 5 is devoted to investigation of properties of distribution of C and the distribution of sojourn time in fixed state j under assumption that C does not stay in j forever. We find some conditional ditributions which among others allows to find a conditional probability of transition from one state to another given that tansition occurs at known time. In Section 6 a kind of predictable representation theorem is given. Such theorems are very important for applications, for example in finance and backward stochastic differential equations (see Pardoux and Peng [18] and El Karoui, Peng and Quenez [7]). By the way we prove that $\mathbb{F}$–DS Markov chain with intensity and arbitrary $\mathbb{F}$ adapted process do not have jumps at the same time. Our results allows to describe and investigate a credit risk for a single firm. In such case the predictable representation theorem (Theorem 6.5) generalize the Kusuoka theorem [14]. In the last section we study how replacing the probability measure by an equivalent one affects the properties of an $\mathbb{F}$–DS Markov chain.

Summing up, the class of $\mathbb{F}$–DS Markov chains is a class with very good and desirable properties in modeling. It can be applied to model rating migration in financial markets. More precisely, it can be used for modeling a credit rating migration process and which contains processes usually taken for this purpose. This allows us to include rating migration in the process of valuation of defaultable claims and generalize the case where only two states are considered: default and non-default (for such generalizations see Jakubowski and Niewęgłowski [12]). These processes can also be applied in other fields where system evolves in a way depending on random environment, e.g., in insurance.

2 Definition and examples

In this section we introduce and investigate a new class of processes, which will be called $\mathbb{F}$-doubly stochastic Markov chains. This class contains among others Markov chains and Cox processes. We assume that all processes are defined on a complete probability space $(\Omega, \mathscr{F}, \mathbf{P})$. We also fix a filtration $\mathbb{F}$ satisfying usual conditions, which plays the role of a reference filtration.

Definition 2.1. A càdlàg process C is called an $\mathbb{F}$–doubly stochastic Markov chain with state space $\mathscr{K} \subset \mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\}$ if there exists a family of stochastic matrices $P(s,t) = (p_{i,j}(s,t))_{i,j\in\mathscr{K}}$ for $0 \le s \le t$ such that
1) the matrix $P(s,t)$ is $\mathscr{F}_t$–measurable, and $P(s,\cdot)$ is $\mathbb{F}$ progressively measurable,
2) for any $t \ge s \ge 0$ and every $i, j \in \mathscr{K}$ we have

$$\mathbf{P}(C_t = j \mid \mathscr{F}_\infty \vee \mathscr{F}_s^C)\mathbf{1}_{\{C_s=i\}} = \mathbf{1}_{\{C_s=i\}} p_{i,j}(s,t). \tag{2.1}$$

The process P will be called the conditional transition probability process of C.

The equality (2.1) implies that $P(t,t) = \mathbb{I}$ a.s. for all $t \ge 0$. Definition 2.1 extends the notion of Markov chain with continuous time (when $\mathscr{F}_\infty$ is trivial). A process satisfying 1) and 2) is called a doubly stochastic Markov chain by analogy with Cox processes (doubly stochastic Poisson processes). In both cases there are two sources of uncertainty. As mentioned in the Introduction, we use the shorthand "$\mathbb{F}$–DS Markov chain" for the "$\mathbb{F}$–doubly stochastic Markov chain". Now, we give a few examples of processes which are $\mathbb{F}$–DS Markov chains.

Example 1. (Compound Poisson process) Let X be a compound Poisson process with jumps in $\mathbb{Z}$, i.e., $X_t = \sum_{i=1}^{N_t} Y_i$, where N is a Poisson process with intensity λ, Y_i is a sequence of independent identically distributed random variables with values in $\mathbb{Z}$ and distribution ν, moreover $(Y_i)_{i\ge1}$ and N are independent. By straightforward calculations we see that:
a) X is an $\mathbb{F}$–DS Markov chain with $\mathbb{F} = \mathbb{F}^N$ and

$$p_{i,j}(s,t) = \nu^{\otimes(N_t - N_s)}(j-i).$$

b) X is an $\mathbb{F}$–DS Markov chain with respect to $\mathbb{F}$ being the trivial filtration, and with deterministic transition matrix given by the formula

$$p_{i,j}(s,t) = \sum_{k=0}^{\infty} \nu^{\otimes k}(j-i) \frac{[\lambda(t-s)]^k}{k!} e^{-\lambda(t-s)}.$$

From these examples we have seen that the conditional transition probability matrix depends on the choice of the reference filtration $\mathbb{F}$, and $P(s,t)$ can be either continuous with respect to s,t or discontinuous.

Example 2. (Cox process) The process C with cadlag trajectories such that

$$\mathbf{P}(C_t - C_s = k \mid \mathscr{F}_\infty \vee \mathscr{F}_s^C) = e^{-\int_s^t \lambda_u du} \frac{\left(\int_s^t \lambda_u du\right)^k}{k!} \tag{2.2}$$

for some $\mathbb{F}$–adapted process λ such that $\lambda \geq 0$, $\int_0^t \lambda_s ds < \infty$ for all $t \geq 0$ we call a Cox process. This definition implies that

$$\mathbf{P}(C_t - C_s = k \mid \mathscr{F}_\infty \vee \mathscr{F}_s^C) = \mathbf{P}(C_t - C_s = k \mid \mathscr{F}_\infty),$$

so the increments and the past (i.e. $\mathscr{F}_s^C$) are conditionally independent given $\mathscr{F}_\infty$. Therefore for $j \geq i$,

$$\begin{aligned}\mathbf{P}(C_t = j \mid \mathscr{F}_\infty \vee \mathscr{F}_s^C)\mathbf{1}_{\{C_s = i\}} &= \mathbf{1}_{\{C_s=i\}}\mathbf{P}(C_t - C_s = j - i \mid \mathscr{F}_\infty \vee \mathscr{F}_s^C) \\ &= \mathbf{1}_{\{C_s=i\}} e^{-\int_s^t \lambda_u du} \frac{\left(\int_s^t \lambda_u du\right)^{j-i}}{(j-i)!}.\end{aligned}$$

Thus

$$p_{i,j}(s,t) = \begin{cases} \frac{\left(\int_s^t \lambda_u du\right)^{j-i}}{(j-i)!} e^{-\int_s^t \lambda_u du} & \text{for } j \geq i, \\ 0 & \text{for } j < i, \end{cases}$$

satisfy conditions 1) and 2) of Definition 2.1. A Cox process C is therefore an $\mathbb{F}$–DS Markov chain with $\mathscr{K} = \mathbb{N}$. Usually in a definition of Cox process there is one more assumption on intensity, namely $\int_0^\infty \lambda_s ds = \infty$ a.s. Under this assumption C has properties similar to Poisson process and is called conditional Poisson process (or doubly stochastic Poisson process). So our definition of Cox process is a slight generalization of a classical one.

Example 3. (Time changed discrete Markov chain) Assume that $\bar{C}$ is a discrete time Markov chain with values in $\mathscr{K} = \{1, \ldots, K\}$, N is a Cox process and the processes $(\bar{C}_k)_{k\geq 0}$ and $(N_t)_{t\geq 0}$ are independent and conditionally independent given $\mathscr{F}_\infty$. Then the process $C_t := \bar{C}_{N_t}$ is an $\mathbb{F}$–DS Markov chain (see Jakubowski and Niewęgłowski [11, Theorem 7 and 9]).

Example 4. (Truncated Cox process) Simple calculations give us that the process $C_t := \min\{N_t, K\}$, where N is a Cox process and $K \in \mathbb{N}$, is an $\mathbb{F}$–DS Markov chain with state space $\mathscr{K} = \{0, \ldots, K\}$.

3 Basic properties

3.1 General case

In this subsection we consider the case of an arbitrary countable state space $\mathscr{K}$. We study basic properties of transition matrices and martingale invariance property of $\mathbb{F}$ with respect to $\mathbb{F} \vee \mathbb{F}^C$. Moreover we prove that the class of $\mathbb{F}$-DS Markov chains is a subclass of $\mathbb{F}$ conditional $\mathbb{F} \vee \mathbb{F}^C$ Markov chains.

For the rest of the paper we assume that $C_0 = i_0$ for some $i_0 \in \mathscr{K}$. We start the investigation of $\mathbb{F}$–DS Markov chains from the very useful lemma describing conditional finite–dimensional distributions of C, which is a counterpart of the well known result for Markov chains.

Lemma 3.1. *If C is an $\mathbb{F}$–DS Markov chain, then*

$$\begin{aligned}&\mathbf{P}(C_{u_1}=i_1,\dots C_{u_n}=i_n \mid \mathscr{F}_\infty \vee \mathscr{F}^C_{u_0})\mathbf{1}_{\{C_{u_0}=i_0\}} \\ &= \mathbf{1}_{\{C_{u_0}=i_0\}} p_{i_0,i_1}(u_0,u_1)\prod_{k=1}^{n-1} p_{i_k,i_{k+1}}(u_k,u_{k+1})\end{aligned} \tag{3.1}$$

for arbitrary $0 \le u_0 \le \dots \le u_n$ *and* $(i_0,\dots,i_n)\in\mathscr{K}^{n+1}$.

Proof. The proof is by induction on n. For $n=1$ the above formula obviously holds. Assume that it holds for n, arbitrary $0\le u_0\le\dots\le u_n$ and $(i_0,\dots,i_n)\in\mathscr{K}^{n+1}$. We will prove it for $n+1$ and arbitrary $0\le u_0\le\dots\le u_n\le u_{n+1}$, $(i_0,\dots,i_n,i_{n+1})\in\mathscr{K}^{n+2}$. Because

$$\begin{aligned}&\mathbf{E}(\mathbf{1}_{\{C_{u_1}=i_1,\dots C_{u_{n+1}}=i_{n+1}\}} \mid \mathscr{F}_\infty\vee\mathscr{F}^C_{u_0})\mathbf{1}_{\{C_{u_0}=i_0\}} \\ &= \mathbf{E}\left(\mathbf{E}\left(\mathbf{1}_{\{C_{u_2}=i_2,\dots,C_{u_{n+1}}=i_{n+1}\}} \mid \mathscr{F}_\infty\vee\mathscr{F}^C_{u_1}\right)\mathbf{1}_{\{C_{u_1}=i_1\}} \mid \mathscr{F}_\infty\vee\mathscr{F}^C_{u_0}\right)\mathbf{1}_{\{C_{u_0}=i_0\}},\end{aligned}$$

by the induction assumption applied to $u_1\le\dots\le u_{n+1}$ and $(i_1,\dots,i_{n+1})\in\mathscr{K}^{n+1}$ we know that the left hand side of (3.1) is equal to

$$\mathbf{E}\left(\mathbf{1}_{\{C_{u_1}=i_1\}} p_{i_1,i_2}(u_1,u_2)\prod_{k=2}^{n} p_{i_k,i_{k+1}}(u_k,u_{k+1}) \mid \mathscr{F}_\infty\vee\mathscr{F}^C_{u_0}\right)\mathbf{1}_{\{C_{u_0}=i_0\}} = I.$$

Using $\mathscr{F}_\infty$–measurability of family of transition probabilities $(P(s,t))_{0\le s\le t<\infty}$, and the definition of $\mathbb{F}$–DS Markov chain, we obtain

$$\begin{aligned}I &= \mathbf{E}\left(\mathbf{1}_{\{C_{u_1}=i_1\}} \mid \mathscr{F}_\infty\vee\mathscr{F}^C_{u_0}\right)\mathbf{1}_{\{C_{u_0}=i_0\}}\left(p_{i_1,i_2}(u_1,u_2)\prod_{k=2}^{n} p_{i_k,i_{k+1}}(u_k,u_{k+1})\right) \\ &= \mathbf{1}_{\{C_{u_0}=i_0\}} p_{i_0,i_1}(u_0,u_1)\left(p_{i_1,i_2}(u_1,u_2)\prod_{k=2}^{n} p_{i_k,i_{k+1}}(u_k,u_{k+1})\right) \\ &= \mathbf{1}_{\{C_{u_0}=i_0\}} p_{i_0,i_1}(u_0,u_1)\prod_{k=1}^{n} p_{i_k,i_{k+1}}(u_k,u_{k+1})\end{aligned}$$

and this completes the proof. □

Remark 3.2. *Of course, if* (3.1) *holds, then condition 2) of Definition 2.1 of* $\mathbb{F}$*–DS Markov chain is satisfied. Therefore* (3.1) *can be viewed as an alternative to equality* (2.1).

As a consequence of our assumption that $C_0=i_0$ and Lemma 3.1 we obtain

Proposition 3.3. *If C is an $\mathbb{F}$–DS Markov chain, then for arbitrary* $0\le u_1\le\dots\le u_n\le t$ *and* $(i_1,\dots,i_n)\in\mathscr{K}^n$ *we have*

$$\mathbf{P}(C_{u_1}=i_1,\dots C_{u_n}=i_n \mid \mathscr{F}_\infty) = p_{i_0,i_1}(0,u_1)\prod_{k=0}^{n-1} p_{i_k,i_{k+1}}(u_k,u_{k+1}), \tag{3.2}$$

and

$$\mathbf{P}(C_{u_1}=i_1,\dots C_{u_n}=i_n \mid \mathscr{F}_\infty) = \mathbf{P}(C_{u_1}=i_1,\dots C_{u_n}=i_n \mid \mathscr{F}_t). \tag{3.3}$$

The following hypothesis is standard for many applications e.g., in credit risk theory.

HYPOTHESIS H: For every bounded $\mathcal{F}_\infty$–measurable random variable Y and for each $t \geq 0$ we have

$$\mathbf{E}(Y \mid \mathcal{F}_t \vee \mathcal{F}_t^C) = \mathbf{E}(Y \mid \mathcal{F}_t).$$

It is well known that hypothesis H for the filtrations $\mathbb{F}$ and $\mathbb{F}^C$ is equivalent to the martingale invariance property of the filtration $\mathbb{F}$ with respect to $\mathbb{F} \vee \mathbb{F}^C$ (see Brémaud and Yor [4] or Bielecki and Rutkowski [3, Lemma 6.1.1, page 167]) i.e. any $\mathbb{F}$ martingale is an $\mathbb{F} \vee \mathbb{F}^C$ martingale. From results in [4] one can deduce that hypothesis H implies (3.3). So, by Proposition 3.3, we can expect that for a class of $\mathbb{F}$-DS Markov chains hypothesis H is satisfied. Indeed,

Proposition 3.4. *If C is an $\mathbb{F}$–DS Markov chain then hypothesis H holds.*

Proof. According to [11, Lemma 2] we know that hypothesis H is equivalent to (3.3). This and Proposition 3.3 complete the proof. □

Now, we will show that each $\mathbb{F}$–DS Markov chain is a conditional Markov chain (see [3, page 340] for a precise definition). For an example of a process which is an $\mathbb{F}$ conditional $\mathbb{F} \vee \mathbb{F}^C$ Markov chain and is not an $\mathbb{F}$–DS Markov chain we refer to Section 3 of Becherer and Schweizer [2].

Proposition 3.5. *Assume that C is an $\mathbb{F}$–DS Markov chain. Then C is an $\mathbb{F}$ conditional $\mathbb{F} \vee \mathbb{F}^C$ Markov chain.*

Proof. We have to check that for $s \leq t$,

$$\mathbf{P}(C_t = i \mid \mathcal{F}_s \vee \mathcal{F}_s^C) = \mathbf{P}(C_t = i \mid \mathcal{F}_s \vee \sigma(C_s)).$$

By the definition of an $\mathbb{F}$–DS Markov chain,

$$\mathbf{P}(C_t = i \mid \mathcal{F}_s \vee \mathcal{F}_s^C) = \mathbf{E}(\mathbf{E}(\mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_\infty \vee \mathcal{F}_s^C) \mid \mathcal{F}_s \vee \mathcal{F}_s^C)$$

$$= \mathbf{E}\Big(\sum_{j \in \mathcal{K}} \mathbf{1}_{\{C_s=j\}} p_{j,i}(s,t) | \mathcal{F}_s \vee \mathcal{F}_s^C \Big) = \sum_{j \in \mathcal{K}} \mathbf{1}_{\{C_s=j\}} \mathbf{E}\left(p_{j,i}(s,t) | \mathcal{F}_s \vee \mathcal{F}_s^C \right) = I.$$

But

$$I = \sum_{j \in \mathcal{K}} \mathbf{1}_{\{C_s=j\}} \mathbf{E}\left(p_{j,i}(s,t) \mid \mathcal{F}_s \right)$$

since hypothesis H holds (Proposition 3.4), and this ends the proof. □

Now we define processes H^i, which play a crucial role in our characterization of the class of $\mathbb{F}$–DS Markov chains:

$$H_t^i := \mathbf{1}_{\{C_t=i\}} \tag{3.4}$$

for $i \in \mathcal{K}$. The process H_t^i tells us whether at time t the process C is in state i or not. Let $H_t := (H_t^\alpha)^\top_{\alpha \in \mathcal{K}}$, where $\top$ denotes transposition.

We can express condition (2.1) in the definition of an $\mathbb{F}$–DS Markov chain, for $t \leq u$, in the form

$$H_t^i \mathbf{E}(H_u^j \mid \mathcal{F}_\infty \vee \mathcal{F}_t^C) = H_t^i p_{i,j}(t,u),$$

or equivalently

$$\mathbf{E}(H_u^j \mid \mathscr{F}_\infty \vee \mathscr{F}_t^C) = \sum_{i\in\mathscr{K}} H_t^i p_{i,j}(t,u)$$

and so (2.1) is equivalent to

$$\mathbf{E}\left(H_u \mid \mathscr{F}_\infty \vee \mathscr{F}_t^C\right) = P(t,u)^\top H_t. \tag{3.5}$$

The next theorem states that the family of matrices $P(s,t) = [p_{i,j}(s,t)]_{i,j\in\mathscr{K}}$ satisfies the Chapman-Kolmogorov equations.

Theorem 3.6. *Let C be an $\mathbb{F}$–DS Markov chain with transition matrices $P(s,t)$. Then for any $u \geq t \geq s$ we have*

$$P(s,u) = P(s,t)P(t,u) \ \ a.s., \tag{3.6}$$

so on the set $\{C_s = i\}$ we have

$$p_{i,j}(s,u) = \sum_{k\in\mathscr{K}} p_{i,k}(s,t)p_{k,j}(t,u).$$

Proof. It is enough to prove that (3.6) holds on each set $\{C_s = i\}$, $i \in \mathscr{K}$. So we have to prove that

$$H_s^\top P(s,u) = H_s^\top P(s,t)P(t,u).$$

By the chain rule for conditional expectation, equality (3.5) and the fact that $P(t,u)$ is $\mathscr{F}_\infty$–measurable it follows that for $s \leq t \leq u$,

$$\begin{aligned} P(s,u)^\top H_s &= \mathbf{E}\left(H_u \mid \mathscr{F}_\infty \vee \mathscr{F}_s^C\right) = \mathbf{E}\left(\mathbf{E}\left(H_u \mid \mathscr{F}_\infty \vee \mathscr{F}_t^C\right) \mid \mathscr{F}_\infty \vee \mathscr{F}_s^C\right) \\ &= \mathbf{E}\left(P(t,u)^\top H_t \mid \mathscr{F}_\infty \vee \mathscr{F}_s^C\right) = P(t,u)^\top \mathbf{E}\left(H_t \mid \mathscr{F}_\infty \vee \mathscr{F}_s^C\right) \\ &= P(t,u)^\top P(s,t)^\top H_s = (P(s,t)P(t,u))^\top H_s, \end{aligned}$$

and this completes the proof. □

3.2 The case of a finite state space

From this subsection, until the end of the paper we restrict ourselves to a finite set $\mathscr{K}$, i.e. $\mathscr{K} = \{1,\ldots,K\}$, with $K < \infty$. It is enough for the most of applications, e.g., in finance to model markets with rating migrations we use processes with values in a finite set. In this case $H_t := (H_t^1 \ldots, H_t^K)^\top$. We recall the standing assumption that $C_0 = i_0$ for some $i_0 \in \mathscr{K}$.

The crucial concept in this subsection and in study of properties of $\mathbb{F}$–DS Markov chains is the concept of intensity, analogous to that for continuous time Markov chains.

Definition 3.7. We say that an $\mathbb{F}$–DS Markov chain C has an intensity if there exists an $\mathbb{F}$–adapted matrix-valued process $\Lambda = (\Lambda(s))_{s\geq 0} = (\lambda_{i,j}(s))_{s\geq 0}$ such that:
1) Λ is locally integrable, i.e. for any $T > 0$

$$\int_{]0,T]} \sum_{i\in\mathscr{K}} \left|\lambda_{ii}(s)\right| ds < \infty. \tag{3.7}$$

2) Λ satisfies the conditions:

$$\lambda_{i,j}(s) \geq 0 \quad \forall i,j \in \mathcal{K}, i \neq j, \quad \lambda_{i,i}(s) = -\sum_{j \neq i} \lambda_{i,j}(s) \quad \forall i \in \mathcal{K}, \tag{3.8}$$

the Kolmogorov backward equation: for all $v \leq t$,

$$P(v,t) - \mathbb{I} = \int_v^t \Lambda(u) P(u,t) du, \tag{3.9}$$

the Kolmogorov forward equation: for all $v \leq t$,

$$P(v,t) - \mathbb{I} = \int_v^t P(v,u) \Lambda(u) du. \tag{3.10}$$

A process Λ satisfying the above conditions is called an intensity of the $\mathbb{F}$–DS Markov chain C.

It is not obvious that if we have a solution to the Kolmogorov backward equation then it also solves the Kolmogorov forward equation. This fact follows from the theory of differential equations, namely we have

Theorem 3.8. *Assume that Λ is locally integrable. Then the random ODE's*

$$dX(t) = -\Lambda(t) X(t) dt, \quad X(0) = \mathbb{I}, \tag{3.11}$$

$$dY(t) = Y(t) \Lambda(t) dt, \quad Y(0) = \mathbb{I}, \tag{3.12}$$

have unique solutions, and in addition $X(t) = Y^{-1}(t)$. Moreover, $Z(s,t) := X(s)Y(t)$ is a unique solution to the Kolmogorov forward equation (3.10) *and to the Kolmogorov backward equation* (3.9).

Proof. The existence and uniqueness of solutions of the ODE's (3.11) and (3.12) follows by standard arguments. To deduce that $X(t) = Y^{-1}(t)$ we apply integration by parts to the product $X(t)Y(t)$ of finite variation continuous processes and get

$$\begin{aligned} d(Y(t)X(t)) &= Y(t) dX(t) + (dY(t)) X(t) \\ &= Y(t)(-\Lambda(t) X(t) dt) + Y(t) \Lambda(t) X(t) dt = 0. \end{aligned}$$

From $Y(0) = X(0) = \mathbb{I}$ we have $Y(t)X(t) = \mathbb{I}$, which means that $X(t)$ is a right inverse matrix of $Y(t)$. It is also the left inverse, since we are dealing with square matrices.

Now we check that $Z(s,t)$ are solutions to the Kolmogorov backward equation and also the Kolmogorov forward equation. Indeed,

$$d_s Z(s,t) = (dX(s)) Y(t) = -\Lambda(s) X(s) Y(t) ds = -\Lambda(s) Z(s,t) ds,$$

and

$$d_t Z(s,t) = X(s) dY(t) = X(s) Y(t) \Lambda(t) dt = Z(s,t) \Lambda(t) dt.$$

This ends the proof since $X(t) = Y^{-1}(t)$ implies that $Z(t,t) = \mathbb{I}$ for every $t \geq 0$. □

Corollary 3.9. *If an $\mathbb{F}$–DS–Markov chain C has intensity, then the conditional transition probability process $P(s,t)$ is jointly continuous at (s,t) for $s \leq t$.*

Proof. This follows immediately from Theorem 3.8, since

$$P(s,t) = X(s)Y(t)$$

and both factors are continuous in s and t, respectively. □

Theorem 3.8 gives us existence and uniqueness of solutions to (3.9) and (3.10). Next proposition provides the form of these solutions.

Proposition 3.10. *Let a matrix process* $(\Lambda(s))_{s\geq 0}$ *satisfies conditions* (3.7) *and* (3.8)*. Then the solution to* (3.9) *is given by the formula*

$$P(v,t) = \mathbb{I} + \sum_{n=1}^{\infty} \int_{v}^{t} \int_{v_1}^{t} \dots \int_{v_{n-1}}^{t} \Lambda(v_1)\dots\Lambda(v_n)dv_n \dots dv_1,$$

and the solution to (3.10) *is given by*

$$P(v,t) = \mathbb{I} + \sum_{n=1}^{\infty} \int_{v}^{t} \int_{v}^{v_1} \dots \int_{v}^{v_{n-1}} \Lambda(v_n)\dots\Lambda(v_1)dv_n \dots dv_1.$$

Proof. It is a special case of Theorem 5 in Gill and Johansen [8], see also Rolski et al. [20, § 8.4.1]. □

Proposition 3.11. *Let* $P = (P(s,t))$, $0 \leq s \leq t$, *be a family of stochastic matrices such that the matrix* $P(s,t)$ *is* $\mathscr{F}_t$*–measurable, and* $P(s,\cdot)$ *is* $\mathbb{F}$*–progressively measurable. Let* $\Lambda = (\Lambda(s))_{s\geq 0}$ *be an* $\mathbb{F}$*–adapted matrix-valued locally integrable process such that the Kolmogorov backward equation* (3.9) *and Kolmogorov forward equation* (3.10) *hold. Then*
i) For each $s \in [0,t]$ *there exists an inverse matrix of* $P(s,t)$ *denoted by* $Q(s,t)$*.*
ii) There exists a version of $Q(\cdot,t)$ *such that the process* $Q(\cdot,t)$ *is a unique solution to the integral (backward) equation*

$$dQ(s,t) = Q(s,t)\Lambda(s)ds, \quad Q(t,t) = \mathbb{I}. \tag{3.13}$$

This unique solution is given by the following series:

$$Q(s,t) = \mathbb{I} + \sum_{k=1}^{\infty} (-1)^k \int_{s}^{t} \int_{u_1}^{t} \dots \int_{u_{k-1}}^{t} \Lambda(u_k)\dots\Lambda(u_1)du_k \dots du_1. \tag{3.14}$$

iii) There exists a version of $Q(s,\cdot)$ *such that the process* $Q(s,\cdot)$ *is a unique solution to the integral (forward) equation*

$$dQ(s,t) = -\Lambda(t)Q(s,t)dt, \quad Q(s,s) = \mathbb{I}. \tag{3.15}$$

This unique solution is given by the following series:

$$Q(s,t) = \mathbb{I} + \sum_{k=1}^{\infty} (-1)^k \int_{s}^{t} \int_{s}^{u_1} \dots \int_{s}^{u_{k-1}} \Lambda(u_1)\dots\Lambda(u_k)du_k \dots du_1. \tag{3.16}$$

Proof. i) From Theorem 3.8 it follows that $P(s,t) = X(s)Y(t)$, where X, Y are solutions to the random ODE's (3.11), (3.12) and moreover $Y = X^{-1}$. Therefore the matrix $P(s,t)$ is invertible and its inverse $Q(s,t)$ is given by $Q(s,t) = X(t)Y(s)$.
ii) We differentiate $Q(s,t)$ with respect to the first argument and obtain

$$d_s Q(s,t) = X(t)dY(s) = X(t)Y(s)\Lambda(s)ds = Q(s,t)\Lambda(s)ds.$$

Moreover $Q(t,t) = X(t)Y(t) = \mathbb{I}$. So $Q(\cdot,t)$ solves (3.13). Uniqueness of solutions to (3.13) follows by standard arguments based on Gronwall's lemma. Formula (3.14) is derived analogously to a similar formula for $P(s,t)$ in § 8.4.1, page 348 of Rolski et al. [20].
iii) The proof of iii) is analogous to that of ii). □

In the next theorem we prove that under some conditions imposed on the conditional transition probability process P, an $\mathbb{F}$–DS Markov chain C has intensity. Using this intensities we can construct martingale intensities for different counting processes building in natural way from $\mathbb{F}$–DS Markov chains (see Theorem 4.1). Therefore, Theorem 3.12 is in a spirit of approaches of Dellacherie (Meyer's Laplacian see Delacherie [6], Guo and Zeng [9]) and of Aven [1]. Theorem 3.12 generalizes for $\mathbb{F}$–DS Markov chains results from [1].

Theorem 3.12 (Existence of Intensity). *Let C be an $\mathbb{F}$–DS–Markov chain with conditional transition probability process P. Assume that*
1) P as a matrix–valued mapping is measurable, i.e.

$$P : (\mathbb{R}_+ \times \mathbb{R}_+ \times \Omega, \mathscr{B}(\mathbb{R}_+ \times \mathbb{R}_+) \otimes \mathscr{F}) \to (\mathbb{R}^{K\times K}, \mathscr{B}(\mathbb{R}^{K\times K})).$$

2) There exists a version of P which is continuous in s and in t.
3) For every $t \geq 0$ the following limit exists almost surely

$$\Lambda(t) := \lim_{h\downarrow 0} \frac{P(t,t+h) - \mathbb{I}}{h}, \tag{3.17}$$

and is locally integrable.
Then Λ is the intensity of C.

Proof. By assumption 3 the process Λ is well defined and by 1) it is $(\mathbb{R}_+ \times \Omega, \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F})$ measurable. By assumption 3, $\Lambda(t)$ is $\mathscr{F}_{t+}$–measurable, but $\mathbb{F}$ satisfies the usual conditions, so $\Lambda(t)$ is $\mathscr{F}_t$–measurable. It is easy to see that (3.8) holds.
It remains to prove that equations (3.9) and (3.10) are satisfied. Fix t. From the assumptions and the Chapman–Kolmogorov equations it follows that for $v \leq v + h \leq t$,

$$\begin{aligned} P(v+h,t) - P(v,t) &= P(v+h,t) - P(v,v+h)P(v+h,t) \\ &= -(P(v,v+h) - \mathbb{I})P(v+h,t), \end{aligned}$$

so

$$\frac{P(v+h,t) - P(v,t)}{h} = -\frac{(P(v,v+h) - \mathbb{I})}{h} P(v+h,t).$$

Therefore $\frac{\partial^+}{\partial v} P(v,t)$ exists for a.e. v and is $(\mathbb{R}_+ \times \mathbb{R}_+ \times \Omega, \mathscr{B}(\mathbb{R}_+ \times \mathbb{R}_+) \otimes \mathscr{F})$ measurable. Using assumptions 2 and 3 we finally have

$$\frac{\partial^+}{\partial v} P(v,t) = -\Lambda(v)P(v,t), \quad P(t,t) = \mathbb{I}. \tag{3.18}$$

Since elements of $P(u,t)$ are bounded by 1, and Λ is integrable over $[v,t]$ (by assumption 3), we see that $\frac{\partial^+}{\partial u}P(u,t)$ is Lebesgue integrable on $[v,t]$, so (see Walker [21])

$$\mathbb{I} - P(v,t) = \int_v^t \frac{\partial^+}{\partial u} P(u,t)du.$$

Hence, by (3.18), we have

$$P(v,t) - \mathbb{I} = \int_v^t \Lambda(u)P(u,t)du,$$

and this is exactly the Kolmogorov backward equation (3.9).

Similar arguments apply to the case of right derivatives of $P(v,t)$ with respect to the second variable. Since for $v \le t \le t+h$,

$$P(v,t+h) - P(v,t) = P(v,t)(P(t,t+h) - \mathbb{I}),$$

we obtain

$$\frac{\partial^+}{\partial t} P(v,t) = P(v,t)\Lambda(t), \quad P(v,v) = \mathbb{I},$$

which gives (3.10),

$$P(v,t) - \mathbb{I} = \int_v^t P(v,u)\Lambda(u)du.$$

□

Now, we find the intensity for the processes described in Examples 3 and 4.

Example 5. If $C_t = \min\{N_t, K\}$, where N is a Cox process with càdlàg intensity process $\tilde{\lambda}$, then C has the intensity process of the form

$$\lambda_{i,j}(t) = \begin{cases} -\tilde{\lambda}(t) & \text{for } i = j \in \{0, \dots K-1\}; \\ \tilde{\lambda}(t) & \text{for } j = i+1 \ \text{ with } i \in \{0, \dots K-1\}; \\ 0 & \text{otherwise.} \end{cases}$$

Example 6. If an $\mathbb{F}$–DS Markov chain C is defined as in Example 3 with a discrete time Markov chain $\bar{C}$ with a transition matrix P and a Cox process N with càdlàg intensity process $\tilde{\lambda}$, then

$$P(s,t) = e^{(P-I)\int_s^t \tilde{\lambda}(u)du}$$

(see Theorem 9 in [11]), so the intensity of C is given by

$$\lambda_{i,j}(t) = (P-I)_{i,j}\tilde{\lambda}(t).$$

4 Martingale properties

We prove that under natural assumptions, belonging of a process X to the class of $\mathbb{F}$-DS Markov chains is fully characterized by the martingale property of some processes strictly connected with X. These nice martingale characterizations allow to check by using martingales whenever process is an $\mathbb{F}$-DS Markov chain. To do this, we introduce a filtration $\hat{\mathbb{G}} = (\hat{\mathscr{G}}_t)_{t\geq 0}$, where

$$\hat{\mathscr{G}}_t := \mathscr{F}_\infty \vee \mathscr{F}_t^C. \tag{4.1}$$

Theorem 4.1. *Let $(C_t)_{t\geq 0}$ be a $\mathscr{K}$–valued stochastic process and $(\Lambda(t))_{t\geq 0}$ be a matrix valued process satisfying* (3.7) *and* (3.8). *The following conditions are equivalent:*
i) The process C is an $\mathbb{F}$–DS Markov chain with intensity process Λ.
ii) The processes

$$M_t^i := H_t^i - \int_{]0,t]} \lambda_{C_u,i}(u)du, \quad i \in \mathscr{K}, \tag{4.2}$$

are $\hat{\mathbb{G}}$ local martingales.
iii) The processes $M^{i,j}$ defined by

$$M_t^{i,j} := H_t^{i,j} - \int_{]0,t]} H_s^i \lambda_{i,j}(s)ds, \quad i,j \in \mathscr{K},\ i \neq j, \tag{4.3}$$

where

$$H_t^{i,j} := \int_{]0,t]} H_{u-}^i dH_u^j, \tag{4.4}$$

are $\hat{\mathbb{G}}$ local martingales.
iv) The process L defined by

$$L_t := Q(0,t)^\top H_t, \tag{4.5}$$

where $Q(0,t)$ is a unique solution to the random integral equation

$$dQ(0,t) = -\Lambda(t)Q(0,t)dt, \quad Q(0,0) = \mathbb{I}, \tag{4.6}$$

is a $\hat{\mathbb{G}}$ local martingale.

Proof. Denoting by M the vector valued process with coordinates M^i, we can write M as follows

$$M_t := H_t - \int_{]0,t]} \Lambda^\top(u) H_u du. \tag{4.7}$$

"$i) \Rightarrow ii)$" Assume that C is an $\mathbb{F}$–DS Markov chain with intensity process $(\Lambda(t))_{t\geq 0}$. Fix $t \geq 0$ and set

$$N_s := P(s,t)^\top H_s \quad \text{for } 0 \leq s \leq t. \tag{4.8}$$

The process C satisfies (3.5), which is equivalent to N being a $\hat{\mathbb{G}}$ martingale for $0 \leq s \leq t$. Using integration by parts and the Kolmogorov backward equation (3.9) we find that

$$\begin{aligned} dN_s &= (dP(s,t))^\top H_s + P^\top(s,t)dH_s = \\ &- P^\top(s,t)\Lambda^\top(s)H_s ds + P^\top(s,t)dH_s = P^\top(s,t)dM_s. \end{aligned} \tag{4.9}$$

Hence, using $Q(s,t)$ (the inverse of $P(s,t)$; we know that it exists, see Proposition 3.11iii)), we conclude that

$$M_s - M_0 = \int_{]0,s]} Q^\top(u,t) P^\top(u,t) dM_u = \int_{]0,s]} Q^\top(u,t) dN_u.$$

Therefore, by the $\widehat{\mathbb{G}}$ martingale property of N, we see that M is a $\widehat{\mathbb{G}}$ local martingale.

"$ii) \Rightarrow i)$" Assume that the process M associated with C and Λ is a $\widehat{\mathbb{G}}$ martingale. Fix $t \geq 0$. To prove that C is an $\mathbb{F}$–DS Markov chain it is enough to show that for some process $(P(s,t))_{0\leq s\leq t}$ the process N defined by (4.8) is a $\hat{\mathbb{G}}$ martingale on $[0,t]$. Let $P(s,t) := X(s)Y(t)$ with X, Y being solutions to the random ODE's (3.11) and (3.12). We know that $P(\cdot,t)$ satisfies the integral equation (3.9) (see Theorem 3.8). We also know that $P(s,t)$ is $\mathscr{F}_t$–measurable (Remark 3.10) and continuous in t, hence $\mathbb{F}$ progressively measurable. Using the same arguments as before, we find that (4.9) holds. So, using the martingale property of M we see that N is a local martingale. The definition of N implies that N is bounded (since H and P are bounded, see Last and Brandt [17, §7.4]). Therefore N has an integrable supremum, so it is a $\hat{\mathbb{G}}$ martingale, which implies that C is an $\mathbb{F}$–DS Markov chain with transition matrix P. From Theorem 3.8 it follows that Λ is the intensity matrix process of C.

"$ii) \Leftrightarrow iii)$" and "$iii) \Leftrightarrow iv)$" These equivalences follows from Lemmas 4.3 and 4.4 below, respectively, with $\mathbb{A} = \widehat{\mathbb{G}}$ given by (4.1).
The proof is complete. □

Remark 4.2. *The equivalence between i) and ii) in the above proposition corresponds to a well known martingale characterization of a Markov chain with a finite state space. In a view of this characterization of $\mathbb{F}$–DS Markov chains with intensities, they can be described as the class of processes that conditioned on $\mathscr{F}_\infty$ behave as time inhomogeneous Markov chains with intensities. Hence, we can also see that the name $\mathbb{F}$–doubly stochastic Markov chain well describe this class of processes. The equivalence between iii) and iv) in a context of Markov chains has not yet been known according to the best of our knowledge.*

The equivalence between points ii), iii) and iv) in Theorem 4.1 is a simple consequence of slightly more general results, which we formulate in two separate lemmas. It is worth to note that equivalences in lemmas below follow from general stochastic integration theory and in the proofs we do not use the doubly stochastic property.

Lemma 4.3. *Let $\mathbb{A}$ be a filtration such that Λ and H are adapted to $\mathbb{A}$. Assume that Λ satisfies* (3.7) *and* (3.8)*. The processes M^i, defined by* (4.2) *for $i \in \mathscr{K}$, are $\mathbb{A}$ local martingales if and only if for all $i,j \in \mathscr{K}$, $i \neq j$, the processes $M^{i,j}$ defined by* (4.3) *are $\mathbb{A}$ local martingales.*

Proof. $\Rightarrow$ Fix $i \neq j$, $i,j \in \mathscr{K}$. Using the definition of $M^{i,j}_t$ and M^i_t we have

$$\begin{aligned} M^{i,j}_t &= \int_{]0,t]} H^i_{u-} dH^j_u - \int_{]0,t]} H^i_u \lambda_{i,j}(u) du = \int_{]0,t]} H^i_{u-} dH^j_u - \int_{]0,t]} H^i_u \lambda_{C_u,j}(u) du \\ &= \int_{]0,t]} H^i_{u-} dH^j_u - \int_{]0,t]} H^i_{u-} \lambda_{C_u,j}(u) du = \int_{]0,t]} H^i_{u-} dM^j_u. \end{aligned}$$

Hence $M^{i,j}$ is an $\mathbb{A}$ local martingale, since M^j is one and H^i_{u-} is a bounded process.

$\Leftarrow$ Assume that the $M_t^{i,j}$ are $\mathbb{A}$ local martingales for all $i \neq j,\ i,j \in \mathscr{K}$. First notice that H^i can be obtained from $H^{j,i}$ by the formula

$$H_t^i = H_0^i + \sum_{j\neq i}\left(H_t^{j,i} - H_t^{i,j}\right).$$

Here and in what follows we use the notation $\sum_{j\neq i} = \sum_{j\in\mathscr{K}\setminus\{i\}}$. Indeed, from (4.4) it follows that

$$\begin{aligned}\sum_{j\neq i}(H_t^{j,i} - H_t^{i,j}) &= \int_{]0,t]}\left(\sum_{j\neq i}H_{u-}^j\right)dH_u^i + \int_{]0,t]}H_{u-}^i d\left(-\sum_{j\neq i}H_u^j\right)\\ &= \int_{]0,t]}(1-H_{u-}^i)dH_u^i + \int_{]0,t]}H_{u-}^i dH_u^i = H_t^i - H_0^i.\end{aligned}$$

Next, by (4.3),

$$\begin{aligned}H_t^i &= H_0^i + \sum_{j\neq i}(M_t^{j,i} - M_t^{i,j}) + \sum_{j\neq i}\left(\int_{]0,t]}H_s^j\lambda_{j,i}(s) - H_s^i\lambda_{i,j}(s)ds\right)\\ &= H_0^i + \sum_{j\neq i}\left(M_t^{j,i} - M_t^{i,j}\right) + \int_{]0,t]}\sum_{j=1}^K H_s^j\lambda_{j,i}(s)ds\\ &= H_0^i + \sum_{j\neq i}\left(M_t^{j,i} - M_t^{i,j}\right) + \int_{]0,t]}\lambda_{C_s,i}(s)ds,\end{aligned}$$

which implies that

$$M_t^i = H_t^i - \int_{]0,t]}\lambda_{C_s,i}(s)ds = H_0^i + \sum_{j\neq i}\left(M_t^{j,i} - M_t^{i,j}\right),$$

and therefore M^i is an $\mathbb{A}$ local martingale for each $i \in \mathscr{K}$ as a finite sum of $\mathbb{A}$ local martingales. □

The process $H^{i,j}$ defined by (4.4) counts the number of jumps from state i to j over the time interval $(0,t]$. Indeed, it is easy to see that

$$H_t^{i,j} = \sum_{0<u\leq t}H_{u-}^i H_u^j.$$

Lemma 4.4. *Let $\mathbb{A}$ be a filtration such that Λ and H are adapted to $\mathbb{A}$. Assume that Λ satisfies* (3.7) *and* (3.8). *The processes M^i, $i \in \mathscr{K}$, are $\mathbb{A}$ local martingales if and only if the process L defined by* (4.5) *is an $\mathbb{A}$ local martingale.*

Proof. Integration by parts formula gives

$$dL_t = Q(0,t)^\top dM_t, \tag{4.10}$$

indeed

$$\begin{aligned}dL_t &= Q(0,t)^\top dH_t + d(Q(0,t)^\top)H_t\\ &= Q(0,t)^\top dH_t - Q(0,t)^\top\Lambda(t)^\top H_t dt = Q(0,t)^\top dM_t.\end{aligned}$$

We know that Q is $\mathbb{A}$ predictable and locally bounded. So, if M is an $\mathbb{A}$ local martingale, then L is also an $\mathbb{A}$ local martingale.
On the other hands, taking $P(0,t)$ as an unique solution to the integral equation

$$dP(0,t) = P(0,t)\Lambda(t)dt, \quad P(0,0) = \mathbb{I}$$

and noting that $P(0,t)$ is the inverse of $Q(0,t)$ (see Proposition 3.11) we have

$$P(0,t)^\top dL_t = P(0,t)^\top Q(0,t)^\top dM_t = dM_t.$$

Therefore, if L is an $\mathbb{A}$ local martingale, then M is also an $\mathbb{A}$ local martingale. □

Corollary 4.5. *If C is an $\mathbb{F}$–DS Markov chain, then M^i are $\mathbb{G}$ local martingales with $\mathscr{G}_t = \mathscr{F}_t \vee \mathscr{F}_t^C$.*

Proof. This follows from the fact that the M^i are adapted to $\mathbb{G}$, and $\mathbb{G}$ is a subfiltration of $\tilde{\mathbb{G}}$. □

Remark 4.6. *The process C obtained by the canonical construction in [3] is an $\mathbb{F}$–DS Markov chain. This is a consequence of Theorem 4.1, because Λ in the canonical construction is bounded and calculations analogous to those in [3, Lemma 11.3.2 page 347] show that M^i are $\tilde{\mathbb{G}}$ martingales. In a similar way one can check that if C is a process of rating migration given by Lando [16], then M^i are $\tilde{\mathbb{G}}$ local martingales, so C is an $\mathbb{F}$–DS Markov chain.*

The martingale M is orthogonal to all square integrable $\mathbb{F}$ martingales.

Proposition 4.7. *Let C be an $\mathbb{F}$–DS Markov chain. The martingale M given by* (4.7) *is strongly orthogonal to all square integrable $\mathbb{F}$ martingales.*

Proof. Denote by N an arbitrary $\mathbb{R}^d$–valued, square integrable $\mathbb{F}$ martingale. Since C is an $\mathbb{F}$–DS Markov chain we have, by Proposition 3.5, that H hypothesis holds and therefore N is also $\mathbb{F} \vee \mathbb{F}^C$ martingale. Since M is an $\mathbb{F} \vee \mathbb{F}^C$ martingale, we need to show that the process $N^j M^i$ is an $\mathbb{F} \vee \mathbb{F}^C$ martingale for every $i \in \mathscr{K}$ and $j \in \{1,\dots,d\}$. Fix arbitrary $t \geq 0$ and any $s \leq t$, then

$$\begin{aligned}\mathbf{E}\big(N_t^j M_t^i \,\big|\, \mathscr{F}_s \vee \mathscr{F}_s^C\big) &= \mathbf{E}\left(N_t^j \mathbf{E}\left(M_t^i \,\middle|\, \mathscr{F}_\infty \vee \mathscr{F}_s^C\right) \,\middle|\, \mathscr{F}_s \vee \mathscr{F}_s^C\right) \\ &= \mathbf{E}\left(N_t^j M_s^i \,\middle|\, \mathscr{F}_s \vee \mathscr{F}_s^C\right) = M_s^i \mathbf{E}\left(N_t^j \,\middle|\, \mathscr{F}_s \vee \mathscr{F}_s^C\right) = N_s^j M_s^i\end{aligned}$$

and hence the result follows. □

Now we construct an $\mathbb{F}$–DS Markov chain with intensity given by an arbitrary $\mathbb{F}$ adapted matrix-valued locally bounded stochastic process which satisfies condition (3.8).

Theorem 4.8. *Let $(\Lambda(t))_{t\geq 0}$ be an arbitrary $\mathbb{F}$ adapted matrix-valued stochastic process which satisfies conditions* (3.7) *and* (3.8). *Then there exists an $\mathbb{F}$–DS Markov chain with intensity $(\Lambda(t))_{t\geq 0}$.*

Proof. We assume that on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$ with a filtration $\mathbb{F}$ we have a family of Cox processes $N^{i,j}$ for $i,j \in \mathscr{K}$ with intensities $(\lambda_{i,j}(t))$ such that the $N^{i,j}$ are conditionally independent given $\mathscr{F}_\infty$ (otherwise we enlarge the probability space). We construct on $(\Omega, \mathscr{F}, \mathbf{P})$ an $\mathbb{F}$–DS Markov chain C with intensity $(\Lambda(t))_{t\geq 0}$ and given initial state i_0. It is a pathwise construction inspired by

Lando [16]. First, we define a sequence $(\tau_n)_{n\geq 1}$ of jump times of C and a sequence $(\bar{C}_n)_{n\geq 0}$ which describes the states of rating after change. We define these sequences by induction. We put

$$\bar{C}_0 = i_0, \qquad \tau_1 := \min_{j\in\mathscr{K}\setminus\bar{C}_0} \inf\left\{t > 0 : \Delta N_t^{\bar{C}_0,j} > 0\right\}$$

If $\tau_1 = \infty$, then C never jumps so $\bar{C}_1 := i_0$. If $\tau_1 < \infty$, then we put $\bar{C}_1 := j$, where j is the element of $\mathscr{K} \setminus \bar{C}_0$ for which the above minimum is attained. By conditional independence of $N^{i,j}$ given $\mathscr{F}_\infty$, the processes $N^{i,j}$ have no common jumps, so $\bar{C}_1$ is uniquely determined. We now assume that $\tau_1, \dots, \tau_k$, $\bar{C}_1, \dots, \bar{C}_k$ are defined, $\tau_k < \infty$ and we construct τ_{k+1} as the first jump time of the Cox processes after τ_k, i.e.

$$\tau_{k+1} := \min_{j\in\mathscr{K}\setminus\bar{C}_k} \inf\left\{t > \tau_k : \Delta N_t^{\bar{C}_k,j} > 0\right\}.$$

If $\tau_{k+1} = \infty$ then $\bar{C}_{k+1} = \bar{C}_k$, and if $\tau_{k+1} < \infty$ we put $\bar{C}_{k+1} := j$, where j is the element of $\mathscr{K} \setminus \bar{C}_k$ for which the above minimum is attained. Arguing as before, we see that τ_{k+1} and $\bar{C}_{k+1}$ are well defined.

Having the sequences $(\tau_n)_{n\geq 0}$ and $(\bar{C}_n)_{n\geq 0}$ we define a process C by the formula

$$C_t := \sum_{k=0}^{\infty} 1_{[\tau_k,\tau_{k+1})}(t)\,\bar{C}_k. \tag{4.11}$$

This process C is càdlàg and adapted to the filtration $\mathbb{A} = (\mathscr{A}_t)_{t\geq 0}$, where $\mathscr{A}_t := \mathscr{F}_t \vee \left(\bigvee_{i\neq j} \mathscr{F}_t^{N^{i,j}}\right)$, and hence it is also adapted to the larger filtration $\tilde{\mathbb{A}} = (\tilde{\mathscr{A}}_t)_{t\geq 0}$, $\tilde{\mathscr{A}}_t := \mathscr{F}_\infty \vee \left(\bigvee_{i\neq j} \mathscr{F}_t^{N^{i,j}}\right)$. Notice that $H^{i,j}$ defined by (4.4) is equal to

$$H_t^{i,j} = \int_{]0,t]} \mathbf{1}_{\{i\}}(C_{s-})dN_s^{i,j} \qquad a.s.$$

The processes $N_t^{i,j} - \int_{]0,t]} \lambda_{i,j}(s)ds$ are $\tilde{\mathbb{A}}$ local martingales (since they are compensated Cox's processes, see e.g. [3]). Likewise, each $M^{i,j}$ defined by (4.3) is an $\tilde{\mathbb{A}}$ local martingale, since

$$\begin{aligned} M_t^{i,j} &= H_t^{i,j} - \int_{]0,t]} H_s^i \lambda_{i,j}(s)ds = H_t^{i,j} - \int_{]0,t]} H_{s-}^i \lambda_{i,j}(s)ds \\ &= \int_{]0,t]} \mathbf{1}_{\{i\}}(C_{s-})d(N_s^{i,j} - \lambda_{i,j}(s)ds). \end{aligned}$$

Recall that $\hat{\mathscr{G}}_t = \mathscr{F}_\infty \vee \mathscr{F}_t^C$, so $\hat{\mathbb{G}} \subseteq \tilde{\mathbb{A}}$. Therefore each $M^{i,j}$ is also a $\hat{\mathbb{G}}$ local martingale, since $M^{i,j}$ is $\hat{\mathbb{G}}$ adapted. Hence, using Theorem 4.1, we see that C is an $\mathbb{F}$–DS Markov chain. □

Remark 4.9. *Suppose that in the above construction $\tau_k < \infty$ a.s. and $C_{\tau_k} = i$. Then*

a) $\tau_{k+1} < \infty$ *a.s. provided*

$$\exists j \in \mathscr{K} \setminus \{i\} \qquad \int_0^\infty \lambda_{i,j}(s)ds = \infty \qquad a.s.,$$

b) $\tau_{k+1} = \infty$ *a.s. provided*

$$\forall j \in \mathscr{K} \setminus \{i\} \qquad \int_{\tau_k}^\infty \lambda_{i,j}(s)ds = 0 \qquad a.s.$$

5 Distribution of C and sojourn times

In this section we investigate properties of distribution of C and the distribution of sojourn time in a fixed state j under assumption that C does not stay in j forever. These properties give, among others, some interpretation to

$$\lambda_j(t) := \sum_{l \in \mathcal{K}\setminus\{j\}} \lambda_{j,l}(t) = -\lambda_{j,j}(t) \tag{5.1}$$

and to the ratios $\frac{\lambda_{i,j}}{\lambda_{i,j}}$. The first result says that the sojourn time of a given state has an exponential distribution in an appropriate time scale.

Proposition 5.1. *Let C be an $\mathbb{F}$–DS Markov chain with intensity Λ, $\tau_0 = 0$, and*

$$\tau_k = \inf\{t > \tau_{k-1} : C_t \neq C_{\tau_{k-1}}, C_{t-} = C_{\tau_{k-1}}\}. \tag{5.2}$$

If $\tau_k < \infty$ a.s., then the random variable

$$E_k := \int_{\tau_{k-1}}^{\tau_k} \lambda_{C_{\tau_{k-1}}}(u)du \tag{5.3}$$

is independent of $\widehat{\mathcal{G}}_{\tau_{k-1}}$ and E_k is exponentially distributed with parameter equal to 1. Moreover, $(E_i)_{1\le i\le k}$ is a sequence of independent random variables.

Proof. For arbitrary $j \in \mathcal{K}$ define the process that counts number of jumps from j

$$N_t^j := \sum_{l\in\mathcal{K}\setminus\{j\}} H_t^{j,l}.$$

Let

$$\widetilde{Y}_t^j := N_t^j - \int_0^t H_{u-}^j \lambda_j(u)du.$$

$\widetilde{Y}^j$ is a $\hat{\mathbb{G}}$ local martingale by Theorem 4.1. Moreover, a sequence of stopping times defined by

$$\sigma_n := \inf\left\{ t \ge 0 : N_t^j \ge n \vee \int_0^t H_{u-}^j \lambda_j(u)du \ge n \right\}$$

reduces $\widetilde{Y}^j$. In what follows we denote

$$Y_t^n := \widetilde{Y}_{t\wedge\sigma_n}^j.$$

The definition of σ_n implies that Y^n is a bounded $\hat{\mathbb{G}}$ martingale for every n. Using the Itô lemma on the interval $]\tau_{k-1}, \tau_k]$ we get

$$\begin{aligned} e^{iuY^n_{\tau_k}} &= e^{iuY^n_{\tau_{k-1}}} + \int_{]\tau_{k-1},\tau_k]} iue^{iuY^n_{s-}} dY_s^n \\ &+ \sum_{\tau_{k-1}<s\le\tau_k} \left(e^{iuY_s^n} - e^{iuY^n_{s-}} - iue^{iuY^n_{s-}} \Delta Y_s^n \right) \\ &= e^{iuY^n_{\tau_{k-1}}} + \int_{]\tau_{k-1},\tau_k]} iue^{iuY^n_{s-}} dY_s^n + \sum_{\tau_{k-1}<s\le\tau_k} e^{iuY^n_{s-}} \left(e^{iu} - 1 - iu \right) \Delta N_s^j. \end{aligned}$$

For every n the boundedness of (Y^n) and (e^{iuY^n}) imply that the process

$$\left(\int_{]0,t]} iue^{iuY_s^n} dY_s^n\right)_{t\geq 0}$$

is a uniformly integrable $\widehat{\mathbb{G}}$ martingale. Therefore, by the Doob optional sampling theorem we have

$$\mathbf{E}\left(e^{iu\left(Y_{\tau_k}^n - Y_{\tau_{k-1}}^n\right)} - \sum_{\tau_{k-1}<s\leq\tau_k} e^{iu\left(Y_{s-}^n - Y_{\tau_{k-1}}^n\right)}\left(e^{iu}-1-iu\right)\Delta N_{s\wedge\sigma_n}^j \,\middle|\, \widehat{\mathcal{G}}_{\tau_{k-1}}\right) = 1. \tag{5.4}$$

On the set $\left\{C_{\tau_{k-1}} = j\right\}$ we have

$$Y_{\tau_k}^n - Y_{\tau_{k-1}}^n = 1\!\!1_{\{\tau_k\leq\sigma_n\}}\left(1-\int_{\tau_{k-1}}^{\tau_k}\lambda_j(s)ds\right) - 1\!\!1_{\{\tau_{k-1}\leq\sigma_n<\tau_k\}}\left(\int_{\tau_{k-1}}^{\sigma_n}\lambda_j(s)ds\right)$$

and

$$Y_{\tau_k-}^n - Y_{\tau_{k-1}}^n = -\int_{\tau_{k-1}\wedge\sigma_n}^{\tau_k\wedge\sigma_n}\lambda_j(u)du.$$

Hence using (5.4) and facts that $\Delta N_{\tau_k\wedge\sigma_n}^j = 1\!\!1_{\{\tau_k\leq\sigma_n\}}$ and $\Delta N_{s\wedge\sigma_n}^j = 0$ for $\tau_{k-1} < s < \tau_k$ we infer that on the set $\left\{C_{\tau_{k-1}} = j\right\}$:

$$\begin{aligned}\mathbf{E}\Big(1\!\!1_{\{\tau_k\leq\sigma_n\}}e^{iu\left(1-\int_{\tau_{k-1}}^{\tau_k}\lambda_j(s)ds\right)} + 1\!\!1_{\{\sigma_n<\tau_k\}}e^{-iu\int_{\tau_{k-1}\wedge\sigma_n}^{\sigma_n}\lambda_j(s)ds}\\ - 1\!\!1_{\{\tau_k\leq\sigma_n\}}e^{-iu\int_{\tau_{k-1}}^{\tau_k}\lambda_j(s)ds}\left(e^{iu}-1-iu\right)\,\Big|\,\widehat{\mathcal{G}}_{\tau_{k-1}}\Big) = 1,\end{aligned}$$

and therefore

$$\begin{aligned}\mathbf{E}\Big(1\!\!1_{\{\tau_k\leq\sigma_n\}}e^{-iu\int_{\tau_{k-1}}^{\tau_k}\lambda_j(s)ds}(1+iu)+\\ 1\!\!1_{\{\sigma_n<\tau_k\}}e^{-iu\int_{\tau_{k-1}\wedge\sigma_n}^{\sigma_n}\lambda_j(s)ds}\,\Big|\,\widehat{\mathcal{G}}_{\tau_{k-1}}\Big) = 1.\end{aligned}$$

Applying Lebesgue's dominated convergence theorem and the fact that $\sigma_n \uparrow +\infty$ yields

$$\mathbf{E}\left(e^{-iu\int_{\tau_{k-1}}^{\tau_k}\lambda_j(u)du}\,\middle|\,\widehat{\mathcal{G}}_{\tau_{k-1}}\right)1\!\!1_{\left\{C_{\tau_{k-1}}=j\right\}} = \frac{1}{1+iu}.$$

Which implies the first statement of theorem. The second statement follows immediately from the above considerations. □

As a consequence of the Proposition 5.1 we have the following result which states that the exit times are in some sense exponentially distributed. This is a generalization of the well known property of Markov chains.

Corollary 5.2. *Let C be an $\mathbb{F}$–DS Markov chain with intensity Λ and let $\tau_k < \infty$ a.s., where τ_k is defined by* (5.2)*. Then*

$$\mathbf{P}(\tau_k - \tau_{k-1} > t \,|\, \widehat{\mathscr{G}}_{\tau_{k-1}}) = e^{-\int_{\tau_{k-1}}^{\tau_{k-1}+t} \lambda_{C_{\tau_{k-1}}}(u)du}. \tag{5.5}$$

Moreover, on the set $\left\{C_{\tau_{k-1}} = i\right\}$ we have

$$\int_{\tau_{k-1}}^{\infty} \lambda_i(u)du = \infty. \tag{5.6}$$

Proof. Because

$$\tau_k - \tau_{k-1} = \inf\left\{ t \geq 0 : \int_{\tau_{k-1}}^{\tau_{k-1}+t} \lambda_{C_{\tau_{k-1}}}(u)du \geq E_k \right\},$$

we obtain (5.5). Indeed

$$\begin{aligned}&\mathbf{P}(\tau_k - \tau_{k-1} \geq t \,|\, \widehat{\mathscr{G}}_{\tau_{k-1}}) \\ &= \mathbf{P}\left(\int_{\tau_{k-1}}^{\tau_{k-1}+t} \lambda_{C_{\tau_{k-1}}}(u)du \leq E_k \,\middle|\, \widehat{\mathscr{G}}_{\tau_{k-1}} \right) = e^{-\int_{\tau_{k-1}}^{\tau_{k-1}+t} \lambda_{C_{\tau_{k-1}}}(u)du}.\end{aligned}$$

(5.5) yields (5.6) by letting $t \to \infty$. □

The next proposition describes the conditional distribution of the vector $(C_{\tau_k}, \tau_k - \tau_{k-1})$ given $\widehat{\mathscr{G}}_{\tau_{k-1}}$.

Proposition 5.3. *Let C be an $\mathbb{F}$-DS Markov chain with an intensity Λ and τ_k be given by* (5.2)*. If $\tau_k < \infty$ a.s., then we have*

$$\begin{aligned}&\mathbf{P}(C_{\tau_k} = j, \tau_k - \tau_{k-1} \leq t \,\big|\, \widehat{\mathscr{G}}_{\tau_{k-1}}) \\ &= \int_0^t e^{-\int_0^u \lambda_i(v+\tau_{k-1})dv} \lambda_{i,j}(\tau_{k-1} + u)du\end{aligned} \tag{5.7}$$

on the set $\left\{C_{\tau_{k-1}} = i\right\}$.

Proof. First we note that RHS of (5.7) is well defined. Fix arbitrary $j \in \mathscr{K}$. Let N^j be the process that counts number of jumps of C to the state j, i.e.

$$N_t^j := \sum_{i \in \mathscr{K} \setminus \{j\}} H_t^{i,j},$$

and

$$Y_t^j := N_t^j - \int_0^t \sum_{i \in \mathscr{K} \setminus \{j\}} H_{u-}^i \lambda_{i,j}(u)du.$$

Then Y^j is a $\widehat{\mathbb{G}}$ local martingale by Theorem 4.1. Therefore there exist a sequence of $\widehat{\mathbb{G}}$ stopping times $\sigma_n \uparrow +\infty$ such that $(Y^j_{t\wedge\sigma_n})_{t\geq 0}$ is an uniformly integrable martingale. By Doob's optional sampling theorem we have on the set $\left\{C_{\tau_{k-1}} = i\right\}$

$$\begin{aligned}
&\mathbf{E}(N^j_{(t+\tau_{k-1})\wedge\tau_k\wedge\sigma_n} - N^j_{\tau_{k-1}\wedge\sigma_n} \,|\, \widehat{\mathscr{G}}_{\tau_{k-1}}) \\
&= \mathbf{E}\left(\int_{\tau_{k-1}\wedge\sigma_n}^{(t+\tau_{k-1})\wedge\tau_k\wedge\sigma_n} \lambda_{i,j}(u)du \,\middle|\, \widehat{\mathscr{G}}_{\tau_{k-1}}\right).
\end{aligned} \tag{5.8}$$

Since on the set $\left\{C_{\tau_{k-1}} = i\right\}$

$$N^j_{(t+\tau_{k-1})\wedge\tau_k\wedge\sigma_n} - N^j_{\tau_{k-1}\wedge\sigma_n} = \begin{cases} 1 & \text{if } X_{\tau_k} = j, \tau_k - \tau_{k-1} \leq t, \tau_k \leq \sigma_n, \\ 0 & \text{otherwise ,} \end{cases}$$

we can pass to the limit on the LHS of (5.8) and obtain

$$\lim_{n\to\infty} \mathbf{E}(N^j_{(t+\tau_{k-1})\wedge\tau_k\wedge\sigma_n} - N^j_{\tau_{k-1}\wedge\sigma_n} \,|\, \widehat{\mathscr{G}}_{\tau_{k-1}}) = \mathbf{P}(X_{\tau_k} = j, \tau_k - \tau_{k-1} \leq t \,|\, \widehat{\mathscr{G}}_{\tau_{k-1}}).$$

The RHS of (5.8) we can write as a sum

$$\mathbf{E}\left(\int_{\tau_{k-1}\wedge\sigma_n}^{(t+\tau_{k-1})\wedge\tau_k\wedge\sigma_n} \lambda_{i,j}(u)du \,\middle|\, \widehat{\mathscr{G}}_{\tau_{k-1}}\right) = I_1(n) + I_2(n),$$

where

$$\begin{aligned}
I_1(n) &:= \mathbf{E}\left(1\!\!1_{\{\tau_k \leq \sigma_n\}} \int_0^{(\tau_k - \tau_{k-1})\wedge t} \lambda_{i,j}(s + \tau_{k-1})ds \,\middle|\, \widehat{\mathscr{G}}_{\tau_{k-1}}\right), \\
I_2(n) &:= \mathbf{E}\left(1\!\!1_{\{\tau_{k-1} \leq \sigma_n < \tau_k\}} \int_0^{(\sigma_n - \tau_{k-1})\wedge t} \lambda_{i,j}(s + \tau_{k-1})ds \,\middle|\, \widehat{\mathscr{G}}_{\tau_{k-1}}\right).
\end{aligned}$$

By a monotone convergence theorem and fact that $\sigma_n \uparrow +\infty$ we obtain

$$\lim_{n\to\infty} I_1(n) = \mathbf{E}\left(\int_0^{(\tau_k - \tau_{k-1})\wedge t} \lambda_{i,j}(s + \tau_{k-1})ds \,\middle|\, \widehat{\mathscr{G}}_{\tau_{k-1}}\right).$$

On the other hand

$$\lim_{n\to\infty} I_2(n) = 0, \ a.s.$$

Summing up, we have proved the following equality holds on the set $\left\{C_{\tau_{k-1}} = i\right\}$

$$\begin{aligned}
&\mathbf{P}(X_{\tau_k} = j, \tau_k - \tau_{k-1} \leq t \,|\, \widehat{\mathscr{G}}_{\tau_{k-1}}) \\
&= \mathbf{E}\left(\int_0^{(\tau_k - \tau_{k-1})\wedge t} \lambda_{i,j}(s + \tau_{k-1})ds | \widehat{\mathscr{G}}_{\tau_{k-1}}\right).
\end{aligned} \tag{5.9}$$

To finish the proof it is enough to transform the RHS of (5.9). Corollary 5.2 and Fubbini theorem yield

$$\mathbf{E}\left(\int_0^{(\tau_k - \tau_{k-1})\wedge t} \lambda_{i,j}(s + \tau_{k-1})ds \,\middle|\, \widehat{\mathscr{G}}_{\tau_{k-1}}\right)$$

$$
\begin{aligned}
&= \int_0^\infty \left(\int_0^{r\wedge t} \lambda_{i,j}(s+\tau_{k-1})ds \right) e^{-\int_0^r \lambda_i(v+\tau_{k-1})dv} \lambda_i(r+\tau_{k-1})dr \\
&= \int_0^t \left(\int_0^{r} \lambda_{i,j}(s+\tau_{k-1})ds \right) e^{-\int_0^r \lambda_i(v+\tau_{k-1})dv} \lambda_i(r+\tau_{k-1})dr \\
&\quad + \int_t^\infty \left(\int_0^{t} \lambda_{i,j}(s+\tau_{k-1})ds \right) e^{-\int_0^r \lambda_i(v+\tau_{k-1})dv} \lambda_i(r+\tau_{k-1})dr \\
&= \int_0^t \left(\int_s^{t} \lambda_{i,j}(s+\tau_{k-1}) e^{-\int_0^r \lambda_i(v+\tau_{k-1})dv} \lambda_i(r+\tau_{k-1})dr \right) ds \\
&\quad + \int_0^t \left(\int_t^{\infty} \lambda_{i,j}(s+\tau_{k-1}) e^{-\int_0^r \lambda_i(v+\tau_{k-1})dv} \lambda_i(r+\tau_{k-1})dr \right) ds \\
&= \int_0^t \lambda_{i,j}(s+\tau_{k-1}) \left(\int_s^{\infty} e^{-\int_0^r \lambda_i(v+\tau_{k-1})dv} \lambda_i(r+\tau_{k-1})dr \right) ds \\
&= \int_0^t \lambda_{i,j}(s+\tau_{k-1}) e^{-\int_0^s \lambda_i(v+\tau_{k-1})dv} ds.
\end{aligned}
$$

In the last equality we use (5.6). The proof is now complete. □

Remark 5.4. *Using Corollary 5.2 in the same way as in the last part of the proof of Proposition 5.3 we obtain on the set* $\left\{C_{\tau_{k-1}} = i\right\}$

$$
\begin{aligned}
&\mathbf{E}\left(\frac{\lambda_{i,j}(\tau_k)}{\lambda_i(\tau_k)} \mathbb{1}_{\{0\le \tau_k - \tau_{k-1} \le t\}} \,\middle|\, \widehat{\mathscr{G}}_{\tau_{k-1}} \right) \\
&\quad = \int_0^t \frac{\lambda_{i,j}(s+\tau_{k-1})}{\lambda_i(s+\tau_{k-1})} e^{-\int_0^s \lambda_i(v+\tau_{k-1})dv} \lambda_i(s+\tau_{k-1})ds,
\end{aligned}
$$

which, by (5.7)*, is equal to* $\mathbf{P}(C_{\tau_k} = j, \tau_k - \tau_{k-1} \le t \,|\, \widehat{\mathscr{G}}_{\tau_{k-1}})$*. This together with*

$$
\sum_{j\in\mathscr{K}\setminus\{i\}} \frac{\lambda_{i,j}(\tau_k)}{\lambda_i(\tau_k)} = 1
$$

suggest that the ratio $\frac{\lambda_{i,j}(\tau_k)}{\lambda_i(\tau_k)}$ *can be treated as a conditional probability of transition from* i *to* j *given that transition occurs at the time* τ_k*. The next corollary confirms this suggestion.*

Corollary 5.5. *On the set* $\left\{C_{\tau_{k-1}} = i\right\}$*, under assumptions of Proposition 5.3, we have*

$$
\mathbf{P}(C_{\tau_k} = j \,|\, \widehat{\mathscr{G}}_{\tau_{k-1}} \vee \{\tau_k - \tau_{k-1} = t\}) = \frac{\lambda_{i,j}(t+\tau_{k-1})}{\lambda_i(t+\tau_{k-1})}. \tag{5.10}
$$

Proof. In a view of Proposition 5.3 and Corollary 5.2 we have on the set $\left\{C_{\tau_{k-1}} = i\right\}$:

$$
\begin{aligned}
&\mathbf{P}(C_{\tau_k} \in dx, \tau_k - \tau_{k-1} \in dt \,\big|\, \widehat{\mathscr{G}}_{\tau_{k-1}}) \\
&= \sum_{j \in \mathscr{K} \setminus \{i\}} \delta_{\{j\}}(dx) e^{-\int_0^t \lambda_i(v+\tau_{k-1})dv} \lambda_{i,j}(t+\tau_{k-1})dt \\
&= \sum_{j \in \mathscr{K} \setminus \{i\}} \delta_{\{j\}}(dx) \frac{\lambda_{i,j}(t+\tau_{k-1})}{\lambda_i(t+\tau_{k-1})} e^{-\int_{\tau_0}^t \lambda_i(v+\tau_{k-1})dv} \lambda_i(t+\tau_{k-1})dt \\
&= \sum_{j \in \mathscr{K} \setminus \{i\}} \delta_{\{j\}}(dx) \frac{\lambda_{i,j}(t+\tau_{k-1})}{\lambda_i(t+\tau_{k-1})} \mathbf{P}(\tau_k - \tau_{k-1} \in dt \,\big|\, \widehat{\mathscr{G}}_{\tau_{k-1}}),
\end{aligned}
$$

which implies (5.10). □

6 Predictable representation theorem

In this section we prove a predictable representation theorem in the form useful for applications. At first we study a hypothesis which often appears in literature as hypothesis that a given ρ avoids $\mathbb{F}$ stopping times. This hypothesis says that for every $\mathbb{F}$ stopping time σ it holds $\mathbf{P}(\rho = \sigma) = 0$.

Proposition 6.1. *Every τ_k defined by* (5.2) *such that $\tau_k < \infty$ a.s. avoids $\mathbb{F}$ stopping times.*

Proof. Fix k and let σ be an arbitrary $\mathbb{F}$ stopping time. Using the random variable E_{k+1} defined by (5.3) we have

$$
\begin{aligned}
\mathbf{P}(\tau_{k+1} = \sigma) &= \mathbf{P}\left(\int_{\tau_k}^{\tau_{k+1}} \lambda_{C_{\tau_k}}(u)du = \int_{\tau_k}^{\sigma} \lambda_{C_{\tau_k}}(u)du\right) \\
&= \mathbf{E}\left(\mathbf{E}\left(1_{\left\{E_{k+1} = \int_{\tau_k}^{\sigma} \lambda_{C_{\tau_k}}(u)du\right\}} \,\Big|\, \mathscr{F}_\infty \vee \mathscr{F}^C_{\tau_k}\right)\right) \\
&= \mathbf{E}\left(\mathbf{E}\left(1_{\{E_{k+1} = x\}}\right)\Big|_{x = \int_{\tau_k}^{\sigma} \lambda_{C_{\tau_k}}(u)du}\right) = 0.
\end{aligned}
$$

The third equality follows from measurability of $\int_{\tau_k}^{\sigma} \lambda_{C_{\tau_k}}(u)du$ with respect to $\mathscr{F}_\infty \vee \mathscr{F}^C_{\tau_k}$ and independence of E_{k+1} and $\mathscr{F}_\infty \vee \mathscr{F}^C_{\tau_k}$. Moreover $\mathbf{P}(E_{k+1} = x) = 0$ since random variable E_{k+1} has an exponential distribution (see Proposition 5.1), so the fourth equality holds, and the assertion follows. □

This proposition immediately implies

Corollary 6.2. *An $\mathbb{F}$–DS Markov chain with intensity and an arbitrary $\mathbb{F}$ adapted process do not have jumps at the same time.*

Remark 6.3. *The assertion of Proposition 4.7 follows immediately from Corollary 6.2, since $[M,N] = 0$ for any square integrable $\mathbb{F}$ martingale N.*

Now we precise the notion of predictable representation property for square integrable martingales (although this definition has sense for local martingales, see Klebaner [13, Chapter 8§12], Cont and Tankov [5, Chapter 9§2]).

Definition 6.4. Let N be a square integrable martingale. We say that *the predictable representation property* (PRP) holds for N if for every $T > 0$ and every $\mathscr{F}_T^N$ measurable square integrable random variable X there exists an $\mathbb{F}^N$ predictable stochastic process Z such that

$$X = \mathbf{E}X + \int_0^T Z_t^\top dN_t. \tag{6.1}$$

Assume that PRP holds for N and let C be an $\mathbb{F}^N$-DS Markov chain. It turns out that the PRP holds for an (N, M) where M is given by (4.7), so M is a martingale which characterizes C. Indeed, the following form of a predictable representation theorem holds:

Theorem 6.5. *Assume that the predictable representation property holds for N and let C be an $\mathbb{F}^N$–DS Markov chain. Then the predictable representation property holds for (N, M), where M is a martingale given by*

$$M_t := H_t - \int_{]0,t]} \Lambda^\top(u) H_u du.$$

Proof. Proposition 3.4 implies that N is an $\mathbb{F}^N \vee \mathbb{F}^C$ martingale and, by Theorem 4.1, the process M is $\mathbb{F}^N \vee \mathbb{F}^C$ martingale. For every square integrable $\mathscr{F}_T^N \vee \mathscr{F}_T^C$ measurable random variable X and $T > 0$ we have to find an $\mathbb{F}^N \vee \mathbb{F}^C$ predictable stochastic process such that (6.1) holds with (N, M) instead of N. Fix arbitrary $T > 0$. In the proof we will use a monotone class theorem. Let

$$\mathscr{N} = \Big\{ X \in L^2(\mathscr{F}_T^N \vee \mathscr{F}_T^C) : X = \mathbf{E}X + \int_{]0,T]} R_s^\top dN_s + \int_{]0,T]} S_s^\top dM_s, \\ \text{where } R, S \text{ are } \mathbb{F}^N \vee \mathbb{F}^C \text{ predictable stochastic processes} \Big\}.$$

We claim that $\mathscr{N}$ is a monotone vector space. Obviously, it is a vector space over $\mathbf{R}$ and $1 \in \mathscr{N}$. From closedness of the space of integrals follows that the bounded limit of monotone sequence of elements from $\mathscr{N}$ belongs to $\mathscr{N}$. Let

$$\mathscr{M} := \Big\{ X : \; X = Y \prod_{k=1}^n 1_{\{C_{t_k} = i_k\}} \text{ where } Y \in L^\infty(\mathscr{F}_T^N), \\ (t_1, \dots, t_n) \in [0, T]^n, (i_1, \dots, i_n) \in \mathscr{K}^n : n \in \mathbb{N} \Big\}. \tag{6.2}$$

It is easy to see that $\mathscr{M}$ is a multiplicative class. To finish the proof it is enough to prove that $\mathscr{M} \subset \mathscr{N}$, because by a monotone class theorem, $\mathscr{N}$ contains all bounded functions that are measurable with respect to the σ-algebra generated by $\mathscr{M}$ i.e. $L^\infty(\mathscr{F}_T^N \vee \mathscr{F}_T^C) \subset \mathscr{N}$. Hence, by standard arguments we obtain $L^2(\mathscr{F}_T^N \vee \mathscr{F}_T^C) \subset \mathscr{N}$, so $\mathscr{N} = L^2(\mathscr{F}_T^N \vee \mathscr{F}_T^C)$.

Therefore, we need to derive predictable representation for an arbitrary random variable in $\mathscr{M}$. Fix $X \in \mathscr{M}$. Without loss of generality we can assume that $t_1 \leq \ldots \leq t_n$. Let Z be a martingale defined by $Z_t := \mathbf{E}(X \mid \mathscr{F}_t^N \vee \mathscr{F}_t^C)$, so it has the form

$$Z_t = \mathbf{E}\left(Y \prod_{k=1}^{n} 1_{\left\{C_{t_k}=i_k\right\}} \,\middle|\, \mathscr{F}_t^N \vee \mathscr{F}_t^C \right).$$

Fix $m \in \{1, \ldots, n\}$. For K dimensional vectors e_{i_k} with 1 on i_k-th component and zero otherwise, and $t \in (t_{m-1}, t_m]$ we have

$$\begin{aligned} Z_t &= \left(\prod_{k=1}^{m-1} 1_{\left\{C_{t_k}=i_k\right\}} \right) \mathbf{E}\left(Y \prod_{k=m}^{n} 1_{\left\{C_{t_k}=i_k\right\}} \,\middle|\, \mathscr{F}_t^N \vee \mathscr{F}_t^C \right) \\ &= \left(\prod_{k=1}^{m-1} 1_{\left\{C_{t_k}=i_k\right\}} \right) \sum_{i \in \mathscr{K}} H_t^i \mathbf{E}\left(Y P_{i,i_m}(t, t_m) \prod_{k=m+1}^{n} P_{i_{k-1},i_k}(t_{k-1}, t_k) \,\middle|\, \mathscr{F}_t^N \right) \\ &= \left(\prod_{k=1}^{m-1} 1_{\left\{C_{t_k}=i_k\right\}} \right) H_t^\top \mathbf{E}\left(P(t, t_m) e_{i_m} Y \prod_{k=m+1}^{n} P_{i_{k-1},i_k}(t_{k-1}, t_k) \,\middle|\, \mathscr{F}_t^N \right) \\ &= \left(\prod_{k=1}^{m-1} 1_{\left\{C_{t_k}=i_k\right\}} \right) H_t^\top Q(0,t) \mathbf{E}\left(P(0, t_m) e_{i_m} Y \prod_{k=m+1}^{n} P_{i_{k-1},i_k}(t_{k-1}, t_k) \,\middle|\, \mathscr{F}_t^N \right), \end{aligned}$$

where we have used conditional Chapman-Kolmogorov equation, i.e., $P(0, t_m) = P(0, t)P(t, t_m)$, together with invertibility of $P(0, t)$. Define a bounded martingale N^m by

$$N_t^m = \mathbf{E}\left(P(0, t_m) e_{i_m} Y \prod_{k=m+1}^{n} P_{i_{k-1},i_k}(t_{k-1}, t_k) \,\middle|\, \mathscr{F}_t^N \right).$$

Then for $t \in (t_{m-1}, t_m]$, using L defined by (4.5), we have

$$Z_t = \left(\prod_{k=1}^{m-1} 1_{\left\{C_{t_k}=i_k\right\}} \right) (N_t^m)^\top L_t. \tag{6.3}$$

By assumption, N^m admits the following predictable decomposition:

$$N_T^m = N_0^m + \int_{]0,T]} (R_t^m)^\top dN_t$$

for some $\mathbb{F}$-predictable stochastic process R^m. Hence, using (4.10) and Proposition 6.1, we have

$$\begin{aligned} d((N_t^m)^\top L_t) &= (N_{t-}^m)^\top dL_t + d(N_t^m)^\top L_{t-} = (N_{t-}^m)^\top dL_t + (L_{t-}^\top dN_t^m)^\top \\ &= (N_{t-}^m)^\top Q(0,t)^\top dM_t + H_{t-}^\top Q(0,t) dN_t^m \\ &= (Q(0,t) N_{t-}^m)^\top dM_t + (H_{t-}^\top Q(0,t)) dN_t^m \\ &= (Q(0,t) N_{t-}^m)^\top dM_t + (H_{t-}^\top Q(0,t)) \left(R_t^m\right)^\top dN_t. \end{aligned}$$

Therefore,

$$(N_{t_m}^m)^\top L_{t_m} = (N_{t_{m-1}}^m)^\top L_{t_{m-1}} + \int_{]t_{m-1},t_m]} (Q(0,u) N_{u-}^m)^\top dM_u \tag{6.4}$$

$$+\int_{]t_{m-1},t_m]} \left(R^m_u Q(0,u)^\top H_{u-}\right)^\top dN_u.$$

Moreover,

$$\begin{aligned}(N^m_{t_m})^\top L_{t_m} &= (N^m_{t_m})^\top Q(0,t_m)^\top H_{t_m} = (Q(0,t_m)N^m_{t_m})^\top H_{t_m} \\ &= e^\top_{i_m} H_{t_m} \mathbf{E}\left(Y \prod_{k=m+1}^{n} P_{i_{k-1},i_k}(t_{k-1},t_k) \,\middle|\, \mathcal{F}^N_{t_m}\right)\end{aligned}$$

and

$$\begin{aligned}&(N^{m+1}_{t_m})^\top L_{t_m} = (N^{m+1}_{t_m})^\top Q(0,t_m)^\top H_{t_m} = (Q(0,t_m)N^{m+1}_{t_m})^\top H_{t_m} \\ &= \left(\mathbf{E}\left(Q(0,t_m)P(0,t_{m+1})e_{i_{m+1}} Y \prod_{k=m+2}^{n} P_{i_{k-1},i_k}(t_{k-1},t_k) \,\middle|\, \mathcal{F}^N_{t_m}\right)\right)^\top H_{t_m} \\ &= \left(\mathbf{E}\left(P(t_m,t_{m+1})e_{i_{m+1}} Y \prod_{k=m+2}^{n} P_{i_{k-1},i_k}(t_{k-1},t_k) \,\middle|\, \mathcal{F}^N_{t_m}\right)\right)^\top H_{t_m}.\end{aligned}$$

These imply that

$$Z_{t_m} = \left(\prod_{k=1}^{m} 1_{\left\{C_{t_k}=i_k\right\}}\right)(N^{m+1}_{t_m})^\top L_{t_m}, \tag{6.5}$$

By (6.4), (6.3), (6.5) for $m-1$ and properties of stochastic integral we have

$$\begin{aligned}Z_{t_m} = Z_{t_{m-1}} &+ \int_{]t_{m-1},t_m]} \left(\prod_{k=1}^{m-1} 1_{\left\{C_{t_k}=i_k\right\}}\right)(Q(0,u)N^m_{u-})^\top dM_u \\ &+ \int_{]t_{m-1},t_m]} \left(\prod_{k=1}^{m-1} 1_{\left\{C_{t_k}=i_k\right\}}\right)\left(R^m_u Q(0,u)^\top H_{u-}\right)^\top dN_u.\end{aligned}$$

Thus defining processes R, S by

$$R_t := \sum_{m=1}^{n} 1_{(t_{m-1},t_m]}(t)\left(\prod_{k=1}^{m-1} 1_{\left\{C_{t_k}=i_k\right\}}\right) R^m_u Q(0,u)^\top H_{u-}$$

$$S_t := \sum_{m=1}^{n} 1_{(t_{m-1},t_m]}(t)\left(\prod_{k=1}^{m-1} 1_{\left\{C_{t_k}=i_k\right\}}\right) Q(0,t)N^m_{t-}$$

we obtain the desired predictable representation. □

7 Change of probability and doubly stochastic property

Now, we investigate how changing the probability measure to an equivalent one affects the properties of an $\mathbb{F}$–DS Markov chain. We start from a lemma

Lemma 7.1. *Let* **Q**, **P** *be equivalent probability measures with density factorizing as*

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathscr{F}_\infty \vee \mathscr{F}^C_{T^*}} := \eta_1\eta_2, \tag{7.1}$$

where η_1 *is an* $\mathscr{F}_\infty$*–measurable strictly positive random variable and* η_2 *is an* $\mathscr{F}_\infty \vee \mathscr{F}^C_{T^*}$*–measurable strictly positive random variable integrable under* **P**. *Let* $(\eta_2(t))_{t\in[0,T^*]}$ *be defined by the formula*

$$\eta_2(t) := \mathbf{E_P}(\eta_2 \mid \mathscr{F}_\infty \vee \mathscr{F}^C_t), \quad \eta_2(0) = 1. \tag{7.2}$$

Then $(N(t))_{t\in[0,T^*]}$ *is a* $\hat{\mathbb{G}}$ *martingale (resp. local martingale) under* **Q** *if and only if* $(N(t)\eta_2(t))_{t\in[0,T^*]}$ *is a* $\hat{\mathbb{G}}$ *martingale (resp. local martingale) under* **P**.

Proof. $\Rightarrow$ By the abstract Bayes rule and the fact that η_1 is $\mathscr{F}_\infty$ measurable and hence also $\mathscr{G}_u$ measurable for all $u \geq 0$, we obtain, for $s < t$,

$$\begin{aligned} N(s) &= \mathbf{E_Q}\left(N(t) \mid \hat{\mathscr{G}}_s\right) = \frac{\mathbf{E_P}\left(N(t)\eta_1\eta_2 \mid \hat{\mathscr{G}}_s\right)}{\mathbf{E_P}\left(\eta_1\eta_2 \mid \hat{\mathscr{G}}_s\right)} \\ &= \mathbf{E_P}\left(N(t)\frac{\mathbf{E_P}\left(\eta_1\eta_2 \mid \hat{\mathscr{G}}_t\right)}{\mathbf{E_P}\left(\eta_1\eta_2 \mid \hat{\mathscr{G}}_s\right)} \,\middle|\, \hat{\mathscr{G}}_s\right) = \mathbf{E_P}\left(N(t)\frac{\mathbf{E_P}\left(\eta_2 \mid \hat{\mathscr{G}}_t\right)}{\mathbf{E_P}\left(\eta_2 \mid \hat{\mathscr{G}}_s\right)} \,\middle|\, \hat{\mathscr{G}}_s\right) \\ &= \mathbf{E_P}\left(N(t)\frac{\eta_2(t)}{\eta_2(s)} \,\middle|\, \hat{\mathscr{G}}_s\right) = \frac{\mathbf{E_P}\left(N(t)\eta_2(t) \mid \hat{\mathscr{G}}_s\right)}{\eta_2(s)}. \end{aligned}$$

$\Leftarrow$ The proof is similar. ☐

The next lemma is a standard one, describing the change of compensator of a pure jump martingale under a change of probability measure. Although it can be proved using Girsanov-Meyer theorem (see e.g. [10], [15], [19]), we give a short self-contained proof.

Lemma 7.2. *Let* C *be an* $\mathbb{F}$*–DS Markov chain under* **P** *with intensity* $(\lambda_{i,j})$ *and suppose that* η_2 *defined by* (7.2) *satisfies*

$$d\eta_2(t) = \eta_2(t-)\left(\sum_{k,l\in\mathscr{K}:k\neq l} \kappa_{k,l}(u)dM^{k,l}_u\right) \tag{7.3}$$

with some $\mathbb{G}$ *predictable stochastic processes* $\kappa_{i,j}$, $i, j \in \mathscr{K}$, *such that* $\kappa_{i,j} > -1$. *Then*

$$\tilde{M}^{i,j}_t = H^{i,j}_t - \int_{]0,t]} H^i_u(1 + \kappa_{i,j}(u))\lambda_{i,j}(u)du, \tag{7.4}$$

$i, j \in \mathscr{K}$, *is a* $\hat{\mathbb{G}}$ *local martingale under* **Q** *defined by* (7.1).

Proof. By Lemma 7.1 it is enough to prove that $\tilde{M}^{i,j}\eta_2$ is a $\hat{\mathbb{G}}$ local martingale under **P**. Integration by parts yields

$$d(\tilde{M}^{i,j}_t\eta_2(t)) = \tilde{M}^{i,j}_{t-}d\eta_2(t) + \eta_2(t-)d\tilde{M}^{i,j}_t + \Delta\tilde{M}^{i,j}_t\Delta\eta_2(t) =: I.$$

Since

$$\tilde{M}^{i,j}_t = M^{i,j}_t - \int_{]0,t]} H^i_u\kappa_{i,j}(u)\lambda_{i,j}(u)du,$$

we have

$$d\tilde{M}_t^{i,j} = dM_t^{i,j} - H_t^i \kappa_{i,j}(t)\lambda_{i,j}(t)dt$$

and

$$\begin{aligned}\Delta\tilde{M}_t^{i,j}\Delta\eta_2(t) &= \Delta M_t^{i,j}\eta_2(t-)\left(\sum_{k,l\in\mathcal{K}:k\neq l}\kappa_{k,l}(t)\Delta M_t^{k,l}\right)\\ &= \eta_2(t-)\kappa_{i,j}(t)\left(\Delta M_t^{i,j}\right)^2 = \eta_2(t-)\kappa_{i,j}(t)\left(\Delta H_t^{i,j}\right)^2\\ &= \eta_2(t-)\kappa_{i,j}(t)\Delta H_t^{i,j}.\end{aligned}$$

Hence

$$\begin{aligned}I &= \tilde{M}_{t-}^{i,j}\eta_2(t-)\left(\sum_{k,l\in\mathcal{K}:k\neq l}\kappa_{k,l}(t)dM_t^{k,l}\right) + \eta_2(t-)dM_t^{i,j}\\ &\quad + \eta_2(t-)\kappa_{i,j}(t)(\Delta H_t^{i,j} - H_t^i\lambda_{i,j}(t)dt)\\ &= \tilde{M}_{t-}^{i,j}\eta_2(t-)\left(\sum_{k,l\in\mathcal{K}:k\neq l}\kappa_{k,l}(t)dM_t^{k,l}\right) + \eta_2(t-)(1+\kappa_{i,j}(t))dM_t^{i,j},\end{aligned}$$

which completes the proof. □

Hence and from Theorem 4.1 we deduce that the doubly stochastic property is preserved by a wide class of equivalent changes of probability measures.

Theorem 7.3. *Let C be an $\mathbb{F}$–DS Markov chain under* **P** *with intensity* $(\lambda_{i,j})$, *and* **Q** *be an equivalent probability measure with density given by* (7.1) *and* η_2 *satisfying* (7.3) *with an $\mathbb{F}$ predictable matrix-valued process κ. Then C is an $\mathbb{F}$–DS Markov chain under* **Q** *with intensity* $((1+\kappa_{i,j})\lambda_{i,j})$.

Now, we exhibit a broad class of equivalent probability measures such that the factorization (7.1) in Lemma 7.1 holds.

Example 7. Let $\mathbb{F} = \mathbb{F}^W$ be the filtration generated by some Brownian motion W under **P**, and let C be an $\mathbb{F}$–DS Markov chain with intensity matrix process Λ. Let **Q** be a probability measure equivalent to **P** with Radon-Nikodym density process given as a solution to the SDE

$$d\eta_t = \eta_{t-}\left(\gamma_t dW_t + \sum_{k,l\in\mathcal{K}:k\neq l}\kappa_{k,l}(u)dM_u^{k,l}\right), \qquad \eta_0 = 1,$$

with $\mathbb{F}$ predictable stochastic processes γ and κ. It is easy to see that this density can be written as a product of the following two Doleans-Dade exponentials:

$$d\eta_1(t) = \eta_1(t-)\gamma_t dW_t, \qquad \eta_1(0) = 1;$$

and

$$d\eta_2(t) = \eta_2(t-)\left(\sum_{k,l\in\mathcal{K}:k\neq l}\kappa_{k,l}(u)dM_u^{k,l}\right), \qquad \eta_2(0) = 1.$$

Therefore a factorization

$$\eta(t) = \eta_1(t)\eta_2(t)$$

as in Lemma 7.1 holds, since η_1 is $\mathscr{F}_\infty$ measurable. As an immediate consequence we find that C is an $\mathbb{F}$–DS Markov chain under $\mathbf{Q}$ with intensity $[\lambda^{\mathbf{Q}}]_{i,j} = ((1+\kappa_{i,j})\lambda_{i,j})$ and moreover the process defined by $W_t^* := W_t - \int_0^t \gamma_u du$ is a Brownian motion under $\mathbf{Q}$.

Acknowledgments The authors thank to anonymous referee for her/his bibliographical comments.

References

[1] T. Aven. A Theorem for Determining the Compensator of a Counting Process. *Scand. J. Statist.* **12**(1985), No. 1, 69–72. MR0804226

[2] D. Becherer, M. Schweizer. Classical solutions to reaction-diffusion systems for hedging problems with interacting Itô and point processes. *Ann. Appl. Probab.* **15**(2005), No. 2, 1111–1144. MR2134099

[3] T. Bielecki and M. Rutkowski. *Credit Risk: Modeling, Valuation and Hedging*. Springer Finance. Springer-Verlag Berlin Heidelberg, New York, 2001.

[4] P. Brémaud and M. Yor. Changes of Filtrations and of Probability Measures. *Z. Wahrsch. Verw. Gebiete* **45**(1978), no. 4, 269–295. MR0511775

[5] R. Cont and P. Tankov *Financial modelling with jump processes* . Chapman & Hall, 2004. MR2042661

[6] C. Dellacherie. *Capacités et processus stochastiques*. Springer, Berlin. (1972).

[7] N. El Karoui, S. Peng, and M.C. Quenez. Backward Stochastic Differential Equations in Finance. *Math. Finance* **7**(1997), No. 1, 1–71. MR1434407

[8] R.D. Gill and S. Johansen A survey of product-integration with a view toward application in survival analysis. *Ann. Statist.* **18**(1990), No. 4, 1501–1555. MR1074422

[9] X. Guo and Y. Zeng Intensity Process And Compensator: A New Filtration Expansion Approach And The Jeulin-Yor Theorem. *Ann. Appl. Probab.* **18**(2008), No. 1, 120–142. MR2380894

[10] Jacod, Jean and Shiryaev, Albert N. *Limit theorems for stochastic processes.*, Grundlehren der Mathematischen Wissenschaften, 288. Springer-Verlag, Berlin, 1987. MR0959133

[11] J. Jakubowski and M. Niewęgłowski. Pricing bonds and CDS in the model with rating migration induced by Cox process. *Advances in Mathematics of Finance, ed. L. Stettner, Banach Centre Publication*. **83**(2008), 159–182. MR2509232

[12] J. Jakubowski and M. Niewęgłowski. Hedging strategies of payment streams and application to defaultable rating-sensitive claims. *To appear in AMaMeF volume*

[13] F.C. Klebaner. *Introduction to stochastic calculus with applications*. Second edition. Imperial College Press, London, 2005. MR2160228

[14] S. Kusuoka. A remark on default risk models. *Adv. Math. Econ.* **1**(1999), 69–82. MR1722700

[15] H. Kunita. Itô's stochastic calculus: Its surprising power for applications *Stoch. Proc. Appl.* **120**(2010), No. 5, 622–652. MR2603057

[16] D. Lando. On Cox processes and credit risky securities. *Rev. Deriv. Res.* **2**(1998), 99–120.

[17] G. Last and A. Brandt. *Marked Point Processes on the Real Line: The Dynamic Approach*. Springer-Verlag Berlin Heidelberg, New York, 1995. MR1353912

[18] E. Pardoux, and S. Peng. Adapted solution of a backward stochastic differential equation. *Systems Control Lett.* **14** (1990), No. 1, 55–61. MR1037747

[19] P. Protter. *Stochastic Integration and Differential Equations*. Springer-Verlag Berlin Heidelberg, New York, 2004. MR2020294

[20] T. Rolski, H. Schmidli, V. Schmidt and J. Teugels. *Stochastic Processes for Insurance and Finance*. Wiley Series in Probability and Statistics, 1999. MR1680267

[21] P. L. Walker. On Lebesgue Integrable Derivatives. *Am. Math. Mon.* **84**(1977), No. 4, 287–288. MR0432835