Electronic Journal of Probability

Vol. 4 (1999) Paper no. 6, pages 1–29.

Journal URL
http://www.math.washington.edu/~ejpecp/
Paper URL
http://www.math.washington.edu/~ejpecp/EjpVol4/paper6.abs.html

EXTENDING MARTINGALE MEASURE STOCHASTIC INTEGRAL WITH APPLICATIONS TO SPATIALLY HOMOGENEOUS S.P.D.E'S

Robert C. Dalang[1]
Département de Mathématiques
Ecole Polytechnique Fédérale
CH-1015 Lausanne
robert.dalang@epfl.ch

Abstract

We extend the definition of Walsh's martingale measure stochastic integral so as to be able to solve stochastic partial differential equations whose Green's function is not a function but a Schwartz distribution. This is the case for the wave equation in dimensions greater than two. Even when the integrand is a distribution, the value of our stochastic integral process is a real-valued martingale. We use this extended integral to recover necessary and sufficient conditions under which the linear wave equation driven by spatially homogeneous Gaussian noise has a process solution, and this in any spatial dimension. Under this condition, the non-linear three dimensional wave equation has a global solution. The same methods apply to the damped wave equation, to the heat equation and to various parabolic equations.

[1]The research of the author is partially supported by the Swiss National Foundation for Scientific Research.

AMS 1991 subject classifications. Primary 60H15; Secondary 60H05, 35R60, 35D10.
Key words and phrases: stochastic wave equation, stochastic heat equation, Gaussian noise, process solution.

Submitted to EJP on August 4, 1998. Final version accepted on March 24, 1999.

1 Introduction

In his lectures in Saint-Flour [31], J.B. Walsh introduced the notions of martingale measures and of stochastic integrals with respect to such martingale measures. These were used to give rigorous meaning to stochastic partial differential equations (s.p.d.e.'s), primarily parabolic equations driven by space-time white noise, though Walsh also considered the wave equation in one spatial dimension, and various linear equations in higher dimensions. In the latter case, solutions to the equations only exist as random (Schwartz) distributions, and therefore there is no entirely satisfactory formulation of non-linear equations of this type.

On the other hand, there has been considerable interest recently in stochastic equations in higher dimensions, beginning with [2, 7, 19, 23] for the wave equation in $\mathbb{R}^d$ for the case $d \in \{1, 2\}$, and [3, 8, 21, 22, 24] for the heat equation. Albeverio, Haba and Russo [2] introduce an approach to non-linear s.p.d.e.'s driven by white noise via the notion of "Colombeau solution," which is an extension of the theory of distributions akin to nonstandard analysis. In order to create a theory of non-linear s.p.d.e.'s in higher dimensions, a different approach was suggested by Dalang and Frangos [7]: rather than consider equations driven by white noise, these authors proposed to consider noise with a spatial correlation and to find the weakest possible conditions on this correlation that makes it possible to solve linear and non-linear equations in the space of real-valued stochastic processes. This program was carried out in [7] in the case of the wave equation in two spatial dimensions. Following this paper, this approach was also considered for the heat equation and for the wave equation in higher dimensions [13, 25].

The study of the wave equation in dimensions $d \geq 3$ presents an added difficulty, namely the Green's function (or fundamental solution) of the equation is in fact *not* a function but a distribution. This does not occur for the heat equation, whose kernel is very smooth in all dimensions. This difference is one reason why the papers [7, 15, 16, 19] only considered the case $d \in \{1, 2\}$.

Walsh's martingale measure stochastic integral therefore appears ill suited to study equations of the form

$$Lu = \dot{F}, \tag{1}$$

in which L is a partial differential operator and $\dot{F}$ is a Gaussian noise, typically white in time but possibly with some spatial correlation, in the case where the Green's function Γ associated with L is a distribution (the typical example is the wave equation $Lu = \frac{\partial^2 u}{\partial t^2} - \Delta u$ when $d \geq 3$). Indeed, the solution to (1) with vanishing initial conditions should be

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\, \dot{F}(ds, dy). \tag{2}$$

The most natural interpretation of the integral in (2) would be the martingale measure stochastic integral, except that this integral is only defined in the case where $\Gamma(t,x)$ is a function.

The main objective of this paper is to give an extension of Walsh's stochastic integral that gives meaning to the integral in (2) even when Γ is a distribution that is not a function. The extension uses an isometry between a space $\mathcal{E}$ of simple processes and the space of continuous martingales, the key being the appropriate choice of the norm on the space $\mathcal{E}$ and the identification of elements in the completion of $\mathcal{E}$ with respect to this norm (see Theorems 2 and 3). The norm makes use of the Fourier transform of Schwartz distributions. Even though the integrand Γ may be a distribution, the value of the stochastic integral is always an ordinary real-valued random variable, and the stochastic integral process is a square-integrable martingale. Bounds on the L^2 and L^p-norms of this random variable are provided in Theorems 2 and 5, respectively. Results in Section 4 show that the conditions under which our stochastic integral is defined are essentially optimal (see Remark 12). An attractive feature of this integral is that the functional analysis aspects are on the same level as those used to define the classical Itô integral [6].

We apply this extension of the martingale measure stochastic integral in particular to the study of wave equations of the form

$$\frac{\partial^2 u}{\partial t^2}(t,x) - \Delta\, u(t,x) = \alpha(u)\dot{F}(t,x) + \beta(u), \qquad t > 0,\ x \in I\!R^d, \tag{3}$$

where $\dot{F}(t,x)$ is a spatially homogeneous Gaussian noise with covariance of the form

$$E(\dot{F}(t,x)\,\dot{F}(s,y)) = \delta(t-s)\,f(x-y). \tag{4}$$

In this equation, $\delta(\cdot)$ denotes the Dirac delta function. The case $f(x) = \delta(x)$ would correspond to the case of space-time white noise. We are mainly interested in the case where f is continuous on $I\!R^d \setminus \{0\}$ and unbounded at the origin.

The case $d = 2$ and f bounded was considered by Mueller in [19]. For the linear equation ($\alpha \equiv 1$ and $\beta \equiv 0$) and $d = 2$, with $f(x-y)$ replaced by $f(|x-y|)$, Dalang and Frangos [7] showed that equation (3) has a process solution if and only if

$$\int_0^1 f(r)\,r\log\left(\frac{1}{r}\right)\,dr < \infty. \tag{5}$$

This line of investigation has since been pursued by several authors. Using our extension of the martingale measure stochastic integral, we easily recover the long-term existence result of Millet and Sanz [15] for the non-linear wave equation in two spatial dimensions; their result improved a local existence result of [7]. We also recover the necessary and sufficient condition on f for existence of a process solution to the linear wave equation discovered by Karkzeska and Zabczyk [13] when $d \geq 3$: condition (5) should be replaced by

$$\int_0^1 f(r)\,r\,dr < +\infty. \tag{6}$$

Moreover, we recover the very recent (in fact, nearly simultaneous) long-term existence result of Peszat and Zabczyk [25] for the non-linear wave equation in three spatial dimensions. These last two articles used a more abstract approach via stochastic equations in infinite dimensions [12]. In that approach, one introduces non-intrinsic Hilbert spaces of functions that are to contain the solution, while the martingale measure approach avoids this. A very general theory of stochastic integrals of this kind is presented in [17].

Our approach is quite robust: because the hypotheses are made on the Fourier transform of the Green's function of the equation, a typically accessible quantity, we can handle a variety of equations. Indeed, with little effort, we extend the results on the linear wave equation to the linear *damped* wave equation (also referred to as the *telegraph equation:* see Example 7). Our approach also applies to the heat equation, both linear and non-linear, and we recover results of [24, 25] for these equations. We can also handle parabolic equations with time-dependent coefficients (see Example 9): this case appears not to be covered by previous results.

We should mention that equation (3) with $d = 3$ was considered by Mueller in [18]. However, the stochastic integrals that appear in the integral formulation of the equation were nowhere defined in that paper. Our extension of the stochastic integral gives meaning to the integrals and should allow a formal verification of the results announced in [18].

In the proof of existence of a solution to non-linear equations, we need an extension of Gronwall's lemma: consider a sequence (f_n) of non-negative functions of a real variable and a locally integrable function $g \geq 0$ such that for $n \geq 1$,

$$f_n(t) \leq \int_0^t (k + f_{n-1}(s))g(s)\, ds. \tag{7}$$

Gronwall's lemma asserts that $\sum_n f_n(t)$ then converges uniformly over compact sets. More difficult to handle is the case in which $g(s)$ is replaced by $g(t-s)$ on the right-hand side of (7), so the inequality becomes

$$f_n(t) \leq \int_0^t (k + f_{n-1}(s))g(t-s)\, ds.$$

This is the extension of Gronwall's lemma that appears in the theory of s.p.d.e's (see [31, Lemma 3.3]), and has been used in later references (e.g. [5, 7]). We prove that under the above condition on g, $\sum_n f_n(t)$ again converges uniformly (see Lemma 15). Rather surprisingly, our proof is purely probabilistic and uses some results from the theory of large deviations!

The outline of the paper is as follows. In Section 2, we introduce our extension of the martingale measure stochastic integral, both with respect to Gaussian noise and with respect to certain stochastic integrals of the Gaussian noise. The latter is essential to define the Picard iteration sequence that is needed to study non-linear equations. In Theorems 2 and 3, we identify distributions with respect to which the stochastic integral is defined, along with L^p bounds on the stochastic integrals in Theorem 5.

In Section 3, we discuss several examples of distributions such as Green's functions of the wave equation, the damped wave equation, the heat equation and various other parabolic equations. We identify the condition on the covariance function f in (4) under which the stochastic integrals of these Green's functions are defined. It turns out that the condition is the same both for the heat and wave equations, because even though the Green's functions are very different, the integrals over a time interval of their Fourier transforms behave similarly (see Remark 10).

In Section 4, we consider linear stochastic equations such as the wave and heat equations. We determine the necessary and sufficient condition under which such an equation has a real-valued process solution, recovering results of [7, 13]. For the wave and heat equations, this condition is the same as the one under which the stochastic integral of the Green's function is defined, and this implies that our definition of the stochastic integral is in a sense optimal.

Finally, in Section 5, under the conditions that guarantee existence of solutions to the linear wave or heat equation, we use our definition of the stochastic integral to show that the non-linear form of the three dimensional wave equation and the heat equation in any dimension also have a global solution that is L^p-bounded and L^2-continuous (see Theorem 13). This proof considerably simplifies even the case $d = 2$ considered in [15] and recovers the result of [25]. It is for this proof that we establish the extension of Gronwall's lemma mentioned above. The case of parabolic equations with time-dependent coefficients is discussed in Remark 20.

2 Extending the stochastic integral

Let $\mathcal{D}(\mathbb{R}^{d+1})$ be the topological vector space of functions $\varphi \in C_0^\infty(\mathbb{R}^{d+1})$ with the topology that corresponds to the following notion of convergence [1, p.19]: $\varphi_n \to \varphi$ if and only if the following two conditions hold:

1. There is a compact subset K of $\mathbb{R}^{d+1}$ such that $\mathrm{supp}(\varphi_n - \varphi) \subset K$, for all n.
2. $\lim_{n\to\infty} D^a \varphi_n = D^a \varphi$ uniformly on K for each multi-index a.

Let $F = (F(\varphi),\ \varphi \in \mathcal{D}(\mathbb{R}^{d+1}))$ be an $L^2(\Omega, F, P)$-valued mean zero Gaussian process with covariance functional of the form $(\varphi, \psi) \mapsto E(F(\varphi)F(\psi)) = J(\varphi, \psi)$, where

$$J(\varphi, \psi) = \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\ \varphi(t, x) f(x - y) \psi(t, y), \tag{8}$$

and $f : \mathbb{R}^d \to \mathbb{R}_+$ is continuous on $\mathbb{R}^d \setminus \{0\}$.

The fact that (8) is to be a covariance functional imposes certain requirements on f. Indeed, in order that there exist a Gaussian process with the covariance functional in (8), it is necessary and sufficient that the functional $J(\cdot, \cdot)$ be non-negative definite ([20, Prop.3.4]). According to [27, Chap.VII, Théorème XVII], this implies that f is symmetric ($f(x) = f(-x)$, for all $x \in \mathbb{R}^d$), and is equivalent to the existence of a non-negative tempered measure μ on $\mathbb{R}^d$ whose Fourier transform is f. More precisely, let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwarz space of rapidly decreasing C^∞ test-functions, and for $\varphi \in \mathcal{S}(\mathbb{R}^d)$, let $\mathcal{F}\varphi$ denote the Fourier transform of φ :

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} \exp(-2i\pi\ \xi \cdot x) \varphi(x)\, dx.$$

The relationship between μ and f is, by definition of the Fourier transform on the space $\mathcal{S}'(I\!R^d)$ of tempered distributions, that for all $\varphi \in \mathcal{S}(I\!R^d)$,

$$\int_{I\!R^d} f(x)\varphi(x)\,dx = \int_{I\!R^d} \mathcal{F}\varphi(\xi)\,\mu(d\xi).$$

Furthermore, according to [27, Chap.VII, Théorème VII], there is an integer $\ell \geq 1$ such that

$$\int_{I\!R^d} (1+|\xi|^2)^{-\ell}\,\mu(d\xi) < \infty \tag{9}$$

(here and throughout this paper, $|\xi|$ denotes the Euclidean norm of $\xi \in I\!R^d$).

Elementary properties of convolution and Fourier transform show that for all $\varphi, \psi \in \mathcal{S}(I\!R^d)$,

$$\begin{aligned}\int_{I\!R^d} dx \int_{I\!R^d} dy\ \varphi(x) f(x-y)\psi(y) &= \int_{I\!R^d} dx\ f(x)\,(\varphi * \tilde{\psi})(x) \\ &= \int_{I\!R^d} \mu(d\xi)\,\mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}.\end{aligned} \tag{10}$$

In this formula, $\tilde{\psi}$ is the function defined by $\tilde{\psi}(x) = \psi(-x)$, and $\overline{z}$ is the complex conjugate of z. We note in passing that f, as the Fourier transform of a tempered measure, also defines a tempered measure: f is locally integrable and satisfies a growth condition analogous to (9).

Example 1 For $x \in I\!R^d$ and $0 < \alpha < d$, let $f_\alpha(x) = |x|^{-\alpha}$. Then $f_\alpha = c_\alpha \mathcal{F} f_{d-\alpha}$ (see [28, Chap.V,§1, Lemma 2(a)]), and therefore the $f_\alpha(\cdot)$ are typical examples of functions that can be used in (8).

Extending F to a worthy martingale measure

In order to define stochastic integrals with respect to F, we first extend F to a worthy martingale measure [31, p.289-290]. For this, we proceed as in [7]: the function $\varphi \mapsto F(\varphi)$ is first extended to a σ-finite L^2-valued measure $A \mapsto F(A)$ defined for bounded Borel sets $A \subset I\!R_+ \times I\!R^d$, then one sets

$$M_t(B) = F([0,t] \times B), \qquad B \in \mathcal{B}_b(I\!R^d), \tag{11}$$

and defines a filtration

$$\mathcal{F}_t^0 = \sigma(M_s(B),\ s \leq t,\ B \in \mathcal{B}_b(I\!R^d)), \qquad \mathcal{F}_t = \mathcal{F}_t^0 \vee \mathcal{N},$$

where $\mathcal{B}_b(I\!R^d)$ denotes the bounded Borel subsets of $I\!R^d$ and $\mathcal{N}$ is the σ-field generated by P-null sets. The martingale measure

$$(M_t(B),\ \mathcal{F}_t,\ t \geq 0,\ B \in \mathcal{B}_b(I\!R^d))$$

is then a worthy martingale measure, with covariation measure defined by

$$Q([0,t] \times A \times B) = \langle M(A), M(B)\rangle_t = t \int_{I\!R^d} dx \int_{I\!R^d} dy\ 1_A(x) f(x-y) 1_B(y)$$

and dominating measure $K \equiv Q$. By construction, $t \mapsto M_t(B)$ is a continuous martingale and

$$F(\varphi) = \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \varphi(t,x)\, M(dt,dx).$$

Stochastic integrals

Like the classical Itô stochastic integral, the stochastic integral $\int X\, dM$ of a process $X = (X(t,x))$ with respect to the worthy martingale measure M is defined when X is in the completion of a suitable space of elementary functions. In order to explain our extension of the stochastic integral, we first recall Walsh's construction [31]. A function $(s,x;\omega) \mapsto g(s,x;\omega)$ is *elementary* if it is of the form

$$g(s,x;\omega) = 1_{]a,b]}(s) 1_A(x) X(\omega),$$

where $0 \leq a < b$, $A \in \mathcal{B}_b(\mathbb{R}^d)$ and X is a bounded and $\mathcal{F}_a$-measurable random variable. For such g, the stochastic integral $g \cdot M$ is a martingale measure defined in the obvious way:

$$(g \cdot M)_t(B) = \int_0^t \int_B g(s,x;\cdot)\, M(ds,dx) = (M_{t\wedge b}(A \cap B) - M_{t\wedge a}(A \cap B))X(\cdot).$$

This definition is extended by linearity to the set $\mathcal{E}$ of all finite linear combinations of elementary functions. The σ-field on $\mathbb{R}_+ \times \mathbb{R}^d \times \Omega$ generated by elements of $\mathcal{E}$ is termed the *predictable* σ-field.

For $T > 0$, given a predictable function g, define

$$\|g\|_+ = E\left(\int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\ |g(s,x;\cdot)|\ f(x-y)\ |g(s,y;\cdot)|\right) \tag{12}$$

and let $\mathcal{P}_+$ be the set of all predictable g for which $\|g\|_+ < +\infty$. Then $\mathcal{P}_+$ is complete for $\|\cdot\|_+$, as $\|g\|_+ < \infty$ implies that

$$(s,x,y;\omega) \mapsto g(s,x;\omega) g(s,y;\omega) \tag{13}$$

is integrable with respect to the non-negative measure $f(x-y) dx\, dy\, ds\, dP(\omega)$, and L^1-limits of functions of the product form (13) are of the same product form.

One easily checks that for $g \in \mathcal{E}$,

$$E\left(((g \cdot M)_T(B))^2\right) = E\left(\int_0^T ds \int_B dx \int_B dy\ g(s,x;\cdot)\ f(x-y)\ g(s,y;\cdot)\right) \tag{14}$$

$$\leq \|g\|_+, \tag{15}$$

and the bound (15) is used in [31] to define the stochastic integral $g \cdot M$ for all $g \in \mathcal{P}_+$.

Extension of the stochastic integral

Our extension of the stochastic integral is based on the small difference between (12) and (14): there are no absolute values in (14). Consider the inner-product on $\mathcal{E}$ defined by

$$\langle g, h\rangle = E\left(\int_0^T ds \int_{I\!R^d} dx \int_{I\!R^d} dy\ g(s,x;\cdot) f(x-y) h(s,y;\cdot)\right) \tag{16}$$

and the associated norm

$$\|g\|_0 = \langle g, g\rangle^{1/2}. \tag{17}$$

Since $\langle g, h\rangle$ defined by (16) is bilinear, symmetric and $\langle g, g\rangle \geq 0$, formula (16) does indeed define an inner product provided we identify pairs of functions (g, h) such that $\|g-h\|_0 = 0$. With this identification, $(\mathcal{E}, \|\cdot\|_0)$ is a pre-Hilbert space.

In [31], the norm $\|\cdot\|_+$ is used in order to guarantee that $g \cdot M$ is again a worthy martingale measure, with covariation measure

$$\begin{aligned} Q_g([0,t] \times A \times B) &= \langle (g \cdot M)(A), (g \cdot M)(B)\rangle_t \\ &= E\left(\int_0^t ds \int_A dx \int_B dy\ g(s,x;\cdot) f(x-y) g(s,y;\cdot)\right) \end{aligned} \tag{18}$$

and dominating measure

$$K_g([0,t] \times A \times B) = E\left(\int_0^t ds \int_A dx \int_B dy\ |g(s,x;\cdot)|\ f(x-y)\ |g(x,y;\cdot)|\right). \tag{19}$$

This is useful if one wants to integrate with respect to the new martingale measure $g \cdot M$.

However, if one is merely looking to define a martingale

$$t \mapsto \int_0^t \int_{I\!R^d} g(s,x;\cdot)\, M(ds, dx),$$

then it turns out that one can define this stochastic integral for all elements of the completion $\mathcal{P}_0$ of $(\mathcal{E}, \|\cdot\|_0)$. Indeed, because of (14), the map $g \mapsto g \cdot M$, where $g \cdot M$ denotes the martingale $t \mapsto (g \cdot M)_t(I\!R^d)$, is an isometry between $\mathcal{P}_0$ and the Hilbert space $\mathcal{M}$ of continuous square-integrable $(\mathcal{F}_t)$-martingales $X = (X_t,\ 0 \leq t \leq T)$ equipped with the norm $\|X\| = (E(X_T^2))^{1/2}$.

Because $\|\cdot\|_0 \leq \|\cdot\|_+$, a Cauchy sequence in $\|\cdot\|_+$ is also a Cauchy sequence in $\|\cdot\|_0$, and therefore $\mathcal{P}_+ \subset \mathcal{P}_0$. However, the key point is that $\mathcal{P}_0$ can be much larger than $\mathcal{P}_+$ (see Theorem 2).

Identifying elements of $\mathcal{P}_0$

The general theory of Hilbert spaces tells us that $\mathcal{P}_0$ can be formally identified with the bidual of $\mathcal{E}$ ([4, Chap.V, §2]). However, for our purposes, it is more useful to embed $\mathcal{E}$ topologically into a well-known space $\overline{\mathcal{P}}$, and then any element of the closure of $\mathcal{E}$ in $\overline{\mathcal{P}}$ can be taken as an element in the completion of $\mathcal{E}$ (cf. [4, Chap.V, §2, Ex.6]).

We take as space $\overline{\mathcal{P}}$ the set of all predictable functions $t \mapsto S(t)$ from $[0,T] \times \Omega$ into $\mathcal{S}'(\mathbb{R}^d)$, with the property that $\mathcal{F}S(t)$ is *a.s.* a function and $\|S\|_0 < \infty$, where

$$\|S\|_0^2 = E\left(\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)\ |\mathcal{F}S(t)(\xi)|^2\right). \tag{20}$$

Let $\mathcal{E}_0$ be the subset of $\mathcal{P}_+$ that consists of functions $g(s,x;\omega)$ such that $x \mapsto g(s,x;\omega)$ belongs to $\mathcal{S}(\mathbb{R}^d)$, for all s and ω. Clearly, $\mathcal{E}_0 \subset \overline{\mathcal{P}}$, and by (10), the two definitions (20) and (17) of $\|\cdot\|_0$ agree on $\mathcal{E}_0$. Therefore, any element S of $\overline{\mathcal{P}}$ for which we can find a sequence (g_n) of elements of $\mathcal{E}_0$ such that $\lim_{n\to\infty} \|S - g_n\|_0 = 0$ will correspond to an element of $\mathcal{P}_0$, and the stochastic integral $S \cdot M$ will be defined for such S.

In order to avoid repetition, we shall directly consider a more general class of martingale measures than M. This class is needed in the study of non-linear equations in Section 5.

The martingale measures M^Z

Let $(Z(t,x),\ 0 \leq t \leq T,\ x \in \mathbb{R}^d)$ be a predictable process such that

$$\sup_{0\leq t\leq T} \sup_{x\in\mathbb{R}^d} E(Z(t,x)^2) < +\infty, \tag{21}$$

and let M^Z be the worthy martingale measure $Z \cdot M$, more precisely,

$$M_t^Z(A) = \int_0^t \int_A Z(s,y)\ M(ds,dy).$$

The covariation and dominating measure of M^Z are Q_Z and K_Z respectively, as in (18) and (19) with g replaced by Z.

Consider the norms $\|\cdot\|_{0,Z}$ and $\|\cdot\|_{+,Z}$ defined by

$$\|g\|_{0,Z}^2 = E\left(\int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\ g(s,x;\cdot)Z(s,x)f(x-y)Z(s,y)g(s,y;\cdot)\right), \tag{22}$$

$$\|g\|_{+,Z}^2 = E\left(\int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\ |g(s,x;\cdot)\, Z(s,x)\, f(x-y)\, Z(s,y)\, g(s,y;\cdot)|\right), \tag{23}$$

and let $\mathcal{P}_{0,Z}$ (resp. $\mathcal{P}_{+,Z}$) be the completion of $(\mathcal{E}_0, \|\cdot\|_{0,Z})$ (resp. $(\mathcal{E}_0, \|\cdot\|_{+,Z})$). According to [31, Ex.2.5], $\mathcal{P}_{+,Z}$ is the set of predictable g for which $\|g\|_{+,Z} < +\infty$, but because $\|\cdot\|_{0,Z} \leq \|\cdot\|_{+,Z}$, $\mathcal{P}_{0,Z}$ will in general be much larger. The stochastic integral of $g \in \mathcal{P}_{0,Z}$ with respect to M^Z is defined through the isometry between $(\mathcal{P}_{0,Z}, \|\cdot\|_{0,Z})$ and $\mathcal{M}$, and is denoted $g \cdot M^Z$; we also use the notations

$$\begin{aligned}(g \cdot M^Z)_t &= \int_0^t \int_{\mathbb{R}^d} g(s,x;\cdot)\, M^Z(ds,dx), \\ &= \int_0^t \int_{\mathbb{R}^d} g(s,x;\cdot)Z(s,x;\cdot)\, M(ds,dx). \end{aligned} \tag{24}$$

Since we are interested in spatially homogeneous situations, we make the following assumption.

Hypothesis A. For all $x, y \in I\!\!R^d$, $E(Z(s,x)Z(s,y)) = E(Z(s,0)Z(s,x-y))$.

Under this hypothesis, the function $g_s : I\!\!R^d \to I\!\!R$ defined by

$$g_s(z) = E(Z(s,x)Z(s,x+z)) \tag{25}$$

does not depend on the choice of x and, as a covariance function, is non-negative definite. We may write $g(s,z)$ instead of $g_s(z)$. For fixed s, the product $f^Z(s,x) = f(x)g(s,x)$ is again a non-negative definite function of x [27, Chap.VII, Théorème XIX], and so there is a non-negative measure μ_s^Z on $I\!\!R^d$ such that

$$f^Z(s,\cdot) = \mathcal{F}\mu_s^Z.$$

Notice that according to (22) and (25), for any deterministic function φ,

$$\|\varphi\|_{0,Z}^2 = \int_0^T ds \int_{I\!\!R^d} dx \int_{I\!\!R^d} dy\ \varphi(s,x)f(x-y)g(s,x-y)\varphi(s,y),$$

so if $\varphi(s,\cdot) \in \mathcal{S}(I\!\!R^d)$ for each s, as in (10), we get

$$\|\varphi\|_{0,Z}^2 = \int_0^T ds \int_{I\!\!R^d} \mu_s^Z(d\xi)\ |\mathcal{F}\varphi(s,\cdot)(\xi)|^2.$$

Let $\mathcal{E}_{0,\mathrm{d}}$ be the deterministic elements of $\mathcal{E}_0$. We are going to identify elements of the completion of $(\mathcal{E}_{0,\mathrm{d}}, \|\cdot\|_{0,Z})$, which will clearly also belong to $\mathcal{P}_{0,Z}$ (in a more classical setting, this type of question is considered in [14, Theorem 6.1, p.355]). Let $\overline{\mathcal{P}}^Z$ be the set of deterministic functions $t \mapsto S(t)$ from $[0,T]$ into $\mathcal{S}'(I\!\!R^d)$ with the property that $\mathcal{F}S(t)$ is a function and $\|S\|_{0,Z} < \infty$, where

$$\|S\|_{0,Z}^2 = \int_0^T dt \int_{I\!\!R^d} \mu_t^Z(d\xi)\ |\mathcal{F}S(t)(\xi)|^2.$$

We note that when $S(t) \in \mathcal{S}(I\!\!R^d)$ for each t, this definition of $\|\cdot\|_{0,Z}$ agrees with that of (22). Therefore, any element $S \in \overline{\mathcal{P}}^Z$ for which we can find a sequence (g_n) of elements of $\mathcal{E}_{0,\mathrm{d}}$ such that $\lim_{n\to\infty} \|S - g_n\|_{0,Z} = 0$ will correspond to an element of $\mathcal{P}_{0,Z}$, and the stochastic integral $S \cdot M^Z$ will be defined for such S and will satisfy

$$E((S \cdot M^Z)_T^2) = \|S\|_{0,Z}^2.$$

Recall that a distribution S is *non-negative* if $\langle S, \varphi \rangle \geq 0$ for all $\varphi \geq 0$. An important subset of $\mathcal{S}'(I\!\!R^d)$ is the set of *distributions with rapid decrease* [27, Chap.VII, §5]. Key properties of such distributions are that their Fourier transform is a C^∞-function [27, Chap.VII, Théorème XV] and the convolution with any other distribution is well-defined [27, Chap.VII, Théorème XI]. Typically, fundamental solutions of p.d.e.'s (as in Section 3) are distributions with rapid decrease in the space variable.

Theorem 2 *Let* $(Z(t,x),\ 0 \le t \le T,\ x \in \mathbb{R}^d)$ *be a process for which* (21) *and Hypothesis* A *are satisfied. Let* $t \mapsto S(t)$ *be a deterministic function with values in the space of non-negative distributions with rapid decrease, such that*

$$\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)\ |\mathcal{F}S(t)(\xi)|^2 < +\infty. \tag{26}$$

Then S *belongs to* $\mathcal{P}_{0,Z}$ *and*

$$E((S \cdot M^Z)_t^2) = \int_0^t ds \int_{\mathbb{R}^d} \mu_s^Z(d\xi)\ |\mathcal{F}S(s)(\xi)|^2 \tag{27}$$

$$\le \int_0^t ds\ \left(\sup_{x\in\mathbb{R}^d} E(Z(s,x)^2)\right) \int_{\mathbb{R}^d} \mu(d\xi)\ |\mathcal{F}S(s)(\xi)|^2. \tag{28}$$

PROOF. Fix $\psi \in C_0^\infty(\mathbb{R}^d)$ such that $\psi \ge 0$, the support of ψ is contained in the unit ball of $\mathbb{R}^d$ and $\int_{\mathbb{R}^d} \psi(x)\, dx = 1$. For $n \ge 1$, set

$$\psi_n(x) = n^d\, \psi(nx).$$

Then $\psi_n \to \delta_0$ in $\mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{F}\psi_n(\xi) = \mathcal{F}\psi(\xi/n)$, therefore $|\mathcal{F}\psi_n(\cdot)|$ is bounded by 1.

Let $S_n(t) = \psi_n * S(t)$, where $*$ denotes convolution in the x-variable. Then for each t, $S_n(t) \in \mathcal{S}(\mathbb{R}^d)$ (see [27, Chap.VII, §5, p.245]) and

$$\|S_n\|_{+,Z}^2 = E\left(\int_0^T dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\ |S_n(t,x)Z(t,x)|\ f(x-y)\ |Z(t,y)S_n(t,y)|\right). \tag{29}$$

Because the hypotheses imply that $S_n(t,x) \ge 0$, we remove the absolute values around $S_n(t,x)$ and $S_n(t,y)$ and conclude that

$$\|S_n\|_{+,Z}^2 = \int_0^T dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\ S_n(t,x) f(x-y) S_n(t,y) E(|Z(t,x)Z(t,y)|).$$

Because the integrands are non-negative, we use the Cauchy-Schwartz inequality to bound the expectation by $\sup_x E(Z(t,x)^2)$, and then apply (10) to get

$$\|S_n\|_{+,Z}^2 \le \int_0^T dt\ \left(\sup_{x\in\mathbb{R}^d} E(Z(t,x)^2)\right) \int_{\mathbb{R}^d} \mu(d\xi)\ |\mathcal{F}S_n(t)(\xi)|^2. \tag{30}$$

Because

$$|\mathcal{F}S_n(t)(\xi)| = |\mathcal{F}\psi_n(\xi)|\ |\mathcal{F}S(t)(\xi)| \le |\mathcal{F}S(t)(\xi)|, \tag{31}$$

we conclude from (21) and (26) that

$$\sup_n \|S_n\|_{+,Z} < \infty. \tag{32}$$

It follows that $S_n \in \mathcal{P}_{+,Z} \subset \mathcal{P}_{0,Z}$ and $E((S_n \cdot M^Z)_t^2)$ is bounded by the right-hand side of (28).

In order to show that $S \in \mathcal{P}_{0,Z}$ we show that $\|S_n - S\|_{0,Z} \to 0$. Using the definition of $\|\cdot\|_{0,Z}$ and (31), we see that

$$\begin{aligned}
\|S_n - S\|_{0,Z}^2 &= \int_0^T dt \int_{\mathbb{R}^d} \mu_t^Z(d\xi)\ |\mathcal{F}(S_n(t) - S(t))(\xi)|^2 \\
&= \int_0^T dt \int_{\mathbb{R}^d} \mu_t^Z(d\xi)\ |\mathcal{F}\psi_n(\xi) - 1|^2\ |\mathcal{F}S(t)(\xi)|^2.
\end{aligned}$$

The integrand converges pointwise to 0. In order to conclude that the same is true for the integral, we shall apply the Dominated Convergence Theorem. Because $|\mathcal{F}\psi_n(\xi) - 1|^2 \leq 4$, it suffices to check that

$$\|S\|_{0,Z}^2 = \int_0^T dt \int_{\mathbb{R}^d} \mu_t^Z(d\xi)\ |\mathcal{F}S(t)(\xi)|^2 < +\infty.$$

By (31),

$$\|S_n\|_{0,Z}^2 = \int_0^T dt \int_{\mathbb{R}^d} \mu_t^Z(d\xi)\ |\mathcal{F}\psi_n(\xi)|^2\ |\mathcal{F}S(t)(\xi)|^2.$$

The integrand is non-negative and converges to $|\mathcal{F}S(t)(\xi)|^2$, so we can apply Fatou's Lemma to get

$$\|S\|_{0,Z}^2 \leq \liminf_{n\to\infty} \|S_n\|_{0,Z}^2 \leq \liminf_{n\to\infty} \|S_n\|_{+,Z}^2 < +\infty$$

by (32). These inequalities and (30) imply (28) with $t = T$, and (27) with $t = T$ results from the definition of $S \cdot M$. The validity of (27) and (28) for any $t \in [0,T]$ results from what has just been proved and the fact that T could be replaced by t in the definition of $\|\cdot\|_{0,Z}$. This completes the proof. $\square$

The non-negativity assumption on $S(t)$ was used in a strong way to establish (30) and (32), and (32) was also needed in the last lines of the proof. In the case where $Z(t,x) \equiv 1$, we can remove this assumption on $S(t)$, provided $\mathcal{F}S(t)$ satisfies a slightly stronger condition.

Theorem 3 *Let S be a deterministic space-time distribution such that $t \mapsto S(t)$ is a function with values in the space of distributions with rapid decrease, such that* (26) *holds and*

$$\lim_{h\downarrow 0} \int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi) \sup_{|r-t|<h} |\mathcal{F}S(r)(\xi) - \mathcal{F}S(t)(\xi)|^2 = 0. \tag{33}$$

Then $S \in \mathcal{P}_0$ and

$$E((S\cdot M)_t^2) = \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\ |\mathcal{F}S(s)(\xi)|^2.$$

Remark 4 Condition (33) is clearly implied by the following condition: for all $\xi \in \mathbb{R}^d$, $t \mapsto \mathcal{F}S(t)(\xi)$ is continuous and there is a function $k : \mathbb{R}^d \to \mathbb{R}_+$ such that $|\mathcal{F}S(t)(\xi)| \leq k(\xi)$, for all $t \in [0,T]$ and $\xi \in \mathbb{R}^d$, and

$$\int_{\mathbb{R}^d} \mu(d\xi)\, k(\xi)^2 < +\infty.$$

Proof of Theorem 3. Define ψ_n and S_n as in the proof of Theorem 2. As in that proof, but more simply (because $\|S\|_0 < \infty$ by hypothesis (26)), we conclude that $\|S - S_n\|_0 \to 0$. Therefore, we only need to show that $S_n \in \mathcal{P}_0$ for each n. For this, we notice using (31) that the hypotheses of the theorem are satisfied with S replaced by S_n, and so it suffices to prove the theorem under the additional assumption that $S(t) \in \mathcal{S}(\mathbb{R}^d)$ for each t. We shall write $\varphi(t,x)$ instead of $S(t,x)$. Set

$$\varphi_n(t,x) = \sum_{k=0}^{2^n-1} \varphi(t_n^k, x) 1_{[t_n^k, t_n^{k+1}[}(t),$$

where $t_n^k = kT2^{-n}$. Then $\varphi_n(t,\cdot) \in \mathcal{S}(\mathbb{R}^d)$ for each t and

$$\begin{aligned} \|\varphi_n\|_+^2 &= \int_0^T dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\ |\varphi_n(t,x)|\ f(x-y)\ |\varphi_n(t,y)| \\ &= \sum_{k=0}^{2^n-1} T2^{-n} \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\ |\varphi(t_n^k,x)|\ f(x-y)\ |\varphi(t_n^k,y)|. \end{aligned}$$

The k^{th} term of this sum is equal to

$$\int_{\mathbb{R}^d} dz\ f(z) \int_{\mathbb{R}^d} dx\ |\varphi(t_n^k,x)|\ |\varphi(t_n^k, x-z)| = \int_{\mathbb{R}^d} dz\ f(z)\ (|\varphi(t_n^k,\cdot)| * |\tilde{\varphi}(t_n^k,\cdot)|)(z),$$

where $\tilde{\varphi}(t_n^k, y) = \varphi(t_n^k, -y)$. According to Leibnitz' formula ([29, Ex.26.4 p.283]), $z \mapsto (|\varphi(t_n^k,\cdot)| * |\tilde{\varphi}(t_n^k,\cdot)|)(z)$ decreases more rapidly than any polynomial of $|z|$, and therefore the integral above is finite because f, as the Fourier transform of a tempered measure, is a tempered function that satisfies a condition analogous to (9). We conclude that $\|\varphi_n\|_+ < \infty$, and therefore $\varphi_n \in \mathcal{P}_+ \subset \mathcal{P}_0$. Furthermore, by (18),

$$\begin{aligned} E((\varphi_n \cdot M)_t^2) &= \int_0^t ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\ \varphi_n(s,x) f(x-y) \varphi_n(s,y) \\ &= \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\ |\mathcal{F}\varphi_n(s,\cdot)(\xi)|^2. \end{aligned} \tag{34}$$

In order to conclude that $S \in \mathcal{P}_0$, it remains to show that $\|\varphi - \varphi_n\|_0 \to 0$. Indeed,

$$\|\varphi - \varphi_n\|_0^2 = \int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)\ |\mathcal{F}(\varphi(t,\cdot) - \varphi_n(t,\cdot))(\xi)|^2.$$

For $t \in [t_n^k, t_n^{k+1}[$, $\mathcal{F}\varphi_n(t,\cdot)(\xi) = \mathcal{F}\varphi(t_n^k,\cdot)(\xi)$ for all ξ, so the right-hand side converges to 0 by (33).

Finally, using the isometry, we get $E((\varphi_n \cdot M)_t^2) \to E((\varphi \cdot M)_t^2)$, and by (34),

$$\begin{aligned} E((\varphi_n \cdot M)_t^2) &= \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\ |\mathcal{F}\varphi_n(s,\cdot)(\xi)|^2 \\ &\to \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\ |\mathcal{F}\varphi(s,\cdot)(\xi)|^2 \end{aligned}$$

by (33). This completes the proof. □

We shall also need a bound on the L^p-norm of $S \cdot M$. Under the assumptions of Theorem 3, $S \cdot M$ is a Gaussian random variable, so the L^p-norm is essentially the $p/2$-power of the L^2-norm. We therefore provide a bound under the assumptions of Theorem 2.

Theorem 5 *Under the assumptions of Theorem* 2, *suppose in addition that for some* $p \geq 2$,

$$\sup_{0\leq t\leq T}\sup_{x\in I\!\!R^d} E(|Z(t,x)|^p) < +\infty. \tag{35}$$

Then

$$E(|(S\cdot M^Z)_t|^p) \leq c_p\,(\nu_t)^{\frac{p}{2}-1}\int_0^t ds \left(\sup_{x\in I\!\!R^d} E(|Z(t,x)|^p)\right)\int_{I\!\!R^d}\mu(d\xi)\ |\mathcal{F}S(s)(\xi)|^2, \tag{36}$$

where

$$\nu_t = \int_0^t ds \int_{I\!\!R^d}\mu(d\xi)\ |\mathcal{F}S(s)(\xi)|^2.$$

Proof. We first prove the inequality under the additional assumption that $S(t) \in \mathcal{S}(I\!\!R^d)$, for all t. In this case, (23), (35) and (26) imply that $S \in \mathcal{P}_{+,Z}$ and we can apply [31, Theorem 2.5] to conclude that

$$\langle S\cdot M^Z\rangle_t = \int_0^t ds\int_{I\!\!R^d} dx \int_{I\!\!R^d} dy\ S(s,x)Z(s,x)f(x-y)Z(s,y)S(s,y). \tag{37}$$

Apply Burkholder's inequality [26, Chap.IV §4] to the continuous martingale $S\cdot M^Z$:

$$E(|(S\cdot M^Z)_t|^p) \leq c_p E(\langle S\cdot M^Z\rangle_t^{p/2}).$$

We replace $\langle S\cdot M^Z\rangle_t$ by the expression in (37), then apply Hölder's inequality in the form

$$E(|Y_1Y_2|)^q \leq E(|Y_1|^q\,|Y_2|)E(|Y_2|)^{q-1} \qquad (q\geq 1) \tag{38}$$

to the case $Y_1 = Z(s,x)Z(s,y)$, $Y_2 = S(s,x)f(x-y)S(s,y)$, $q = p/2$, to see that $E(|(S\cdot M^Z)_t|^p)$ is not greater than

$$\begin{aligned} c_pE&\left(\int_0^t ds\int_{I\!\!R^d}dx\int_{I\!\!R^d}dy\ |Z(s,x)Z(s,y)|^{p/2}S(s,x)f(x-y)S(s,y)\right)\\ &\times\left(\int_0^t ds\int_{I\!\!R^d}dx\int_{I\!\!R^d}dy\ S(s,x)f(x-y)S(s,y)\right)^{\frac{p}{2}-1}. \end{aligned} \tag{39}$$

Because of the non-negativity of $S(s,x)$ and $f(x-y)$, we apply the Cauchy-Schwarz inequality to bound $E(|Z(s,x)Z(s,y)|^{p/2})$ by $\sup_x E(|Z(s,x)|^p)$. Together with (10), this proves the theorem in the case where $S(t)\in\mathcal{S}(I\!\!R^d)$, for all t.

We now assume only that S satisfies the assumptions of the theorem. Let ψ_n and S_n be as in the proof of Theorem 2. By the special case just established, the conclusion of the theorem holds for S_n. Because $\|S_n - S\|_{0,Z}\to 0$, $(S_n\cdot M^Z)_t$ converges in $L^2(\Omega, \mathcal{F}, P)$ to $(S \cdot M^Z)_t$, and therefore a subsequence converges a.s. By Fatou's lemma,

$$E(|(S \cdot M^Z)_t|^p) \leq \liminf_{n\to\infty} E(|(S^n \cdot M^Z)|^p).$$

Now $E(|(S^n \cdot M^Z)_t|^p)$ is bounded by the expression in (39) with S replaced by S^n. By (31) and the fact that $\mathcal{F}\psi_n(\xi) \to 1$, we can apply the Dominated Convergence Theorem to conclude that this bound converges to the right-hand side of (36). The proof is complete. □

3 Examples

The basic examples of distributions in $\mathcal{P}_0$ and $\mathcal{P}_{0,Z}$ are fundamental solutions of various partial differential equations.

Example 6 *The wave equation.* Let Γ_1 be the fundamental solution of the wave equation $\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$. Explicit formulas for $\Gamma_1(t)$ are well-known (see [10, Chap.5]): if σ_t denotes uniform surface measure on the d-dimensional sphere of radius t, then

$$\Gamma_1(t) = c_d \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(d-3)/2} \frac{\sigma_t^d}{t}, \qquad \text{if } d \geq 3 \text{ and } d \text{ odd},$$

$$\Gamma_1(t,x) = c_d \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(d-2)/2} (t^2 - |x|^2)_+^{-1/2}, \qquad \text{if } d \geq 2 \text{ and } d \text{ even},$$

and $\Gamma_1(t,x) = \frac{1}{2}1_{\{|x|<t\}}$ if $d = 1$. In particular, for each t, $\Gamma_1(t)$ has compact support. Furthermore, for all dimensions d (see [30, §7]),

$$\mathcal{F}\Gamma_1(t)(\xi) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}, \qquad \xi \in \mathbb{R}^d.$$

Elementary estimates show that there are positive constants c_1 and c_2 depending on T such that

$$\frac{c_1}{1+|\xi|^2} \leq \int_0^T ds\ \frac{\sin^2(2\pi s|\xi|)}{4\pi^2|\xi|^2} \leq \frac{c_2}{1+|\xi|^2}.$$

Therefore $S = \Gamma_1$ satisfies (26) if and only if

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < +\infty. \tag{40}$$

If this condition is fulfilled, then the hypotheses of Theorem 2 are satisfied for $S = \Gamma_1$ when $d = 1, 2$ or 3, because in these dimensions, Γ_1 is non-negative. The hypotheses of Theorem 3 are satisfied in all dimensions (take $k(\xi) = c/(1+|\xi|^2)$ and use Remark 4). It is not difficult to express condition (40) as a condition on f: see Remark 10.

Example 7 *The damped wave equation.* Let Γ_2 be the fundamental solution of the equation

$$\frac{\partial^2 u}{\partial t^2} + 2c\frac{\partial u}{\partial t} - \Delta u = 0.$$

The case $c > 0$ corresponds to "damping", the case $c < 0$ to "excitation". The Fourier transform $v(\xi)(t) = \mathcal{F}\Gamma_2(t)(\xi)$ is, for fixed $\xi \in I\!\!R^d$, solution of the ordinary differential equation

$$\ddot{v} + 2c\,\dot{v} + 4\pi^2|\xi|^2 v = 0, \quad v(\xi)(0) = 0,\ \dot{v}(\xi)(0) = 1.$$

Thus,

$$v(t,\xi) = (c^2 - 4\pi^2|\xi|^2)^{-\frac{1}{2}} e^{-ct} \sinh\left(t\sqrt{c^2 - 4\pi^2|\xi|^2}\right).$$

Observe that for $2\pi|\xi| < c$ and $0 \leq t \leq T$, $|v(t,\xi)|$ is bounded, and for $2\pi|\xi| > c$,

$$v(t,\xi) = (4\pi^2|\xi|^2 - c^2)^{-\frac{1}{2}} e^{-ct} \sin\left(t\sqrt{4\pi^2|\xi|^2 - c^2}\right).$$

As in Example 6, we conclude that for all $d \geq 1$, (26) holds for $S = \Gamma_2$ if and only if (40) holds. In this case, the hypotheses of Theorem 3 are satisfied with $S = \Gamma_2$.

Example 8 *The heat equation.* Let Γ_3 be the fundamental solution of the heat equation $\frac{\partial u}{\partial t} - \frac{1}{2}\Delta u = 0$. Then

$$\Gamma_3(t,x) = (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right) \quad \text{and} \quad \mathcal{F}\Gamma_3(t)(\xi) = \exp(-4\pi^2 t|\xi|^2).$$

Because

$$\int_0^t ds \exp(-4\pi^2 s|\xi|^2) = \frac{1}{4\pi^2|\xi|^2}(1 - \exp(-4\pi^2 t|\xi|^2)),$$

we conclude that the hypotheses of Theorem 2 (and 3) hold for $S = \Gamma_3$ if and only if (40) holds.

Example 9 *Parabolic equations with time-dependent coefficients.* Let

$$L_4 u(t,x) = \frac{\partial u}{\partial t} - \left(\frac{1}{2}\sum_{i,j=1}^n a_{i,j}(t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(t)\frac{\partial u}{\partial x_i} + c(t)u\right),$$

where $a_{i,j}$, b_i and c are bounded continuous functions on $[0,T]$. Assume further that the following coercivity condition holds:

(A_1) there is $\varepsilon > 0$ such that

$$\sum_{i,j=1}^n a_{i,j}(t)\xi_i\xi_j \geq \varepsilon|\xi|^2,$$

for all $t \in [0,t]$, $\xi = (\xi_1,\dots,\xi_d) \in I\!\!R^d$.

According to [11, Chap.6, Theorems 4.5 and 5.4], there are then positive constants c and C such that the fundamental solution Γ_4 of the equation $\frac{\partial}{\partial t} - Lu = 0$ satisfies

$$0 \leq \Gamma_4(t,s;x-y) \leq C(t-s)^{-d/2} \exp\left(-c\,\frac{|x-y|^2}{t-s}\right). \tag{41}$$

We conclude that if (40) holds, then $\Gamma_4(t,\cdot\,;x-\cdot) \in \mathcal{P}_{+,Z}$ for any $(Z(s,y))$ satisfying the hypotheses of Theorem 2, because by (29), $\|\Gamma_4(t,\cdot\,;x-\cdot)\|_{+,Z}$ is equal to

$$E\left(\int_0^t ds \int_{I\!R^d} dy \int_{I\!R^d} dz \ |\Gamma_4(t,s;x-y)Z(s,y)| \ f(y-z) \ |Z(s,z)\Gamma_4(t,s;x-z)|\right)$$
$$\leq K \int_0^t ds \int_{I\!R^d} dy \int_{I\!R^d} dz \ \varphi(t-s,x-y) f(y-z) \varphi(t-s,x-z),$$

where $\varphi(t-s,x-y)$ is the right-hand side of (41). By (10), we conclude that $\|\Gamma_4(t,\cdot;x-\cdot)\|_{+,Z}$ is finite provided

$$\int_0^T dt \int_{I\!R^d} \mu(d\xi) \ |\mathcal{F}\varphi(t,\cdot)(\xi)|^2 < +\infty.$$

Since $\varphi(t,x)$ is essentially a Gaussian density, the considerations in Example 8 show that this integral is finite provided (40) holds.

Remark 10 (a) From Examples 6 and 8 above, it is apparent that condition (40) is essential for solving the linear stochastic wave and heat equations, and this will be confirmed in the next section. Though the fundamental solutions Γ_1 and Γ_3 are very different, the function

$$\xi \mapsto \int_0^T ds \ |\mathcal{F}\Gamma_i(s)(\xi)|^2$$

has similar behavior for $i=1$ and $i=3$. Under condition (40), it is also natural to study non-linear forms of these equations, which is what we shall turn to in Section 5.

(b) Condition (40) can be expressed in terms of the covariance function f in (8) as follows. Let G_d be the fundamental solution of $u + \frac{1}{4\pi^2}\Delta u = 0$ in $I\!R^d$. Taking Fourier transforms, we find that $\mathcal{F}G_d + |\xi|^2 \mathcal{F}G_d = 1$, or equivalently,

$$\mathcal{F}G_d(\xi) = \frac{1}{1+|\xi|^2}.$$

The left-hand side of (40) is equal to

$$\langle \mu, \mathcal{F}G_d \rangle = \langle \mathcal{F}\mu, G_d \rangle = \langle f, G_d \rangle = \int_{I\!R^d} G_d(x) f(x)\, dx.$$

Because $\xi \mapsto \mathcal{F}G_d(\xi)$ is not in $\mathcal{S}(I\!R^d)$, the first equality requires some justification: for this, we refer to [13, Theorem 4], in which standard results concerns the behavior of G_d at 0 and $+\infty$ are used to show that (40) always holds when $d=1$, and for $d \geq 2$, (40) holds if and only if

$$\int_{|x|\leq 1} f(x) \log \frac{1}{|x|} \, dx < +\infty \quad \text{and} \quad d=2$$

or

$$\int_{|x|\leq 1} f(x) \, \frac{1}{|x|^{d-2}} \, dx < +\infty \quad \text{and} \quad d \geq 3.$$

In the case where $f(x)$ only depends on $|x|$, this condition is precisely that in (5) when $d=2$, and is equivalent to condition (6) for $d \geq 3$.

4 Linear spatially homogeneous s.p.d.e.'s

We consider equations of the form

$$Lu = \dot{F} \tag{42}$$

with vanishing initial conditions, where L is typically a second-order partial differential operator with constant coefficients, or at least coefficients that do not depend on the x-variable. The basic examples that we have in mind are the four examples of Section 3.

Let $\Gamma(t,x)$ be the fundamental solution of the equation $Lu = 0$. If $\dot{F}$ were a smooth function, then the solution of (42) would be

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) \dot{F}(s,y)\, dsdy.$$

Therefore, a natural candidate solution of (42) is

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\, M(ds, dy), \tag{43}$$

where M is the martingale measure defined in (11). However, the stochastic integral in (43) is well-defined only if $\Gamma(t - \cdot, x - \cdot) \in \mathcal{P}_0$. On the other hand, if we consider $\dot{F}$ as a random variable with values in the space $\mathcal{D}'(\mathbb{R}^{d+1})$, then (42) always has a (random) distribution-valued solution. Formally, one multiplies (43) by a test-function $\varphi \in \mathcal{D}(\mathbb{R}^{d+1})$, integrates both sides and applies Fubini's Theorem to the right-hand side:

$$\begin{aligned}
&\int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \varphi(t,x) u(t,x)\, dtdx \\
&\qquad = \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^d} dx\ \varphi(t,x) \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\, M(ds, dy) \\
&\qquad = \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \Big(\int_s^{+\infty} dt \int_{\mathbb{R}^d} dx\ \varphi(t,x) \Gamma(t-s, x-y) \Big)\, M(ds, dy) \\
&\qquad = \int_{\mathbb{R}^+} \int_{\mathbb{R}^d} (\Gamma \underset{(t,x)}{*} \tilde{\varphi})(-s, -y)\, M(ds, dy),
\end{aligned}$$

where $\tilde{\varphi}(r,z) = \varphi(-r, -z)$ (" $\underset{(t,x)}{*}$ " denotes convolution in both the time and space variables). In fact, it is not difficult to check that the formula

$$\langle u, \varphi \rangle = \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} (\Gamma \underset{(t,x)}{*} \tilde{\varphi})(-s, -y)\, M(ds, dy) \quad (= \langle \dot{F}, (\Gamma \underset{(t,x)}{*} \tilde{\varphi})^{\sim} \rangle) \tag{44}$$

does define the distribution-valued solution of (42). Indeed, formula (44) is just another way of writing (cf. [27, Chap.VI, §2]) that $u = \Gamma \underset{(t,x)}{*} \dot{F}$ is the classical distribution-valued solution of (42).

A natural question is whether or not the solution (44) corresponds to a solution in the space of real-valued stochastic processes. We address this question in the next theorem.

Theorem 11 *Suppose that the fundamental solution* Γ *of* $Lu = 0$ *is such that* $(s, \xi) \mapsto \mathcal{F}\Gamma(s, \cdot)(\xi)$ *is a jointly measurable function and for each* ξ, $s \mapsto \mathcal{F}\Gamma(s, \cdot)(\xi)$ *is locally Lebesgue-integrable. Let* u *be the distribution-valued solution to the linear s.p.d.e.* (42) *given by formula* (44). *If there exists a jointly measurable locally mean-square bounded process* $X : (t, x, \omega) \mapsto X(t, x, \omega)$ *such that a.s., for all* $\varphi \in \mathcal{D}(\mathbb{R}^{d+1})$,

$$\langle u, \varphi \rangle = \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} X(t, x)\varphi(t, x)\, dtdx, \tag{45}$$

then for all $T > 0$,

$$\int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi)\ |\mathcal{F}\Gamma(s, \cdot)(\xi)|^2 < +\infty. \tag{46}$$

Remark 12 (a) The first hypothesis of Theorem 11 is weaker than that of Theorem 3, and is satisfied by the examples $\Gamma_1, \ldots, \Gamma_4$ of Section 3. For these examples, (46) is equivalent to (40) (see also Remark 10).

(b) Condition (46) happens to be necessary for the stochastic integral in (43) to be well-defined. Therefore, if (42) has a process solution, then one can check that it is given by formula (43). This shows that condition (46) (which is also (26)) is essentially the optimal condition under which an extension of the martingale measure stochastic integral of a distribution Γ can be defined.

Proof of Theorem 11. We assume existence of the process X and compute $E(\langle u, \varphi \rangle^2)$ in two different ways. From (45), we get

$$E(\langle u, \varphi \rangle^2) = \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} dy\ \varphi(t, x)\varphi(s, y)E(X(t, x)X(s, y)). \tag{47}$$

Because X is locally mean-square bounded, the function

$$g(t, x, s, y) = E(X(t, x)X(s, y))$$

is locally integrable. If we replace $\varphi(t, x)$ by $\varphi_n(t, x) = \lambda_n(t - t_0)\psi_n(x - x_0)$, where ψ_n is as in the proof of Theorem 2 and λ_n is defined in the same way as ψ_n but for $d = 1$, then as $n \to \infty$, (47) converges to $g(t_0, x_0, t_0, x_0) = E(X(t_0, x_0)^2)$ for a.a. (t_0, x_0) by the Lebesgue Differentiation Theorem [32, Chap. 7, Exercise 2]. On the other hand, we can compute $E(\langle u, \varphi \rangle^2)$ from (44):

$$E(\langle u, \varphi \rangle^2) = \|(\Gamma \underset{(t,x)}{*} \tilde{\varphi})^{\sim}\|_0^2. \tag{48}$$

If $\varphi(t, x) = \lambda(t)\psi(x)$, then

$$\begin{aligned} \Gamma \underset{(t,x)}{*} \tilde{\varphi}(-s, -y) &= \int_{\mathbb{R}_+} dr\ \lambda(s + r) \int_{\mathbb{R}^d} dz\ \Gamma(r, z)\psi(y + z) \\ &= \int_{\mathbb{R}_+} dr\ \lambda(s + r)\ \Gamma(r, \cdot) * \tilde{\psi}(-y). \end{aligned}$$

Therefore, by the isometry property and (20),

$$\begin{aligned} E(\langle u,\varphi\rangle^2) &= \int_{I\!R_+} ds \int_{I\!R^d} \mu(d\xi)\ |\mathcal{F}(\Gamma \underset{(t,x)}{*} \tilde\varphi(-s,-\cdot))(\xi)|^2 \\ &= \int_{I\!R_+} ds \int_{I\!R^d} \mu(d\xi) \left|\int_{I\!R_+} dr\ \lambda(s+r)\mathcal{F}(\Gamma(r,\cdot) * \tilde\psi)(\xi)\right|^2 \\ &= \int_{I\!R_+} ds \int_{I\!R^d} \mu(d\xi) \left|\mathcal{F}\tilde\psi(\xi) \int_{I\!R_+} dr\ \lambda(s+r)\mathcal{F}\Gamma(r,\cdot)(\xi)\right|^2. \end{aligned}$$

If we replace $\psi(\cdot)$ by $\psi_n(\cdot - x_0)$ and $\lambda(\cdot)$ by $\lambda_n(\cdot - t_0)$, then as $n \to \infty$ the quantity inside the modulus converges by the hypothesis on $r \mapsto \mathcal{F}\Gamma(r,\cdot)(\xi)$ (by the same Lebesgue Differentiation Theorem as above) to $\mathcal{F}\Gamma(t_0 - s,\cdot)(\xi)1_{\{s<t_0\}}$ for almost all s. Equating (48) with (47) and applying Fatou's Lemma, we conclude that

$$\int_0^{t_0} ds \int_{I\!R^d} \mu(d\xi)\ |\mathcal{F}\Gamma(t_0 - s,\cdot)(\xi)|^2 \le E(X(t_0,x_0)^2) < +\infty.$$

This proves the theorem. □

5 Non-linear spatially homogeneous s.p.d.e.'s

We are interested in solutions of equations of the form

$$\begin{aligned} Lu &= \alpha(u)\dot F(t,x) + \beta(u), \qquad (49) \\ u(0,x) &\equiv 0, \\ \frac{\partial u}{\partial t}(0,x) &\equiv 0, \end{aligned}$$

under standard assumptions on $\alpha(\cdot)$ and $\beta(\cdot)$, where L is a second order partial differential operator, typically as in one of the examples of Section 3.

Hypothesis B. The fundamental solution Γ of $Lu = 0$ is a non-negative measure of the form $\Gamma(t,dy)dt$ such that $\Gamma(t, I\!R^d) \le C_T < +\infty$ for $0 \le t \le T$ and all $T > 0$, and the hypotheses of Theorems 2 and 3 are satisfied with $S(t) = \Gamma(t,\cdot)$.

By solution to (49), we mean a jointly measurable adapted process $(u(t,x),\ (t,x) \in I\!R_+ \times I\!R^d)$ such that

$$\begin{aligned} u(t,x) &= \int_0^t \int_{I\!R^d} \Gamma(t-s,x-y)\alpha(u(s,y))\, M(ds,dy) \\ &\qquad + \int_0^t ds \int_{I\!R^d} \beta(u(t-s,x-y))\, \Gamma(s,dy). \qquad (50) \end{aligned}$$

The stochastic integral above is defined as explained in (24).

Theorem 13 *If Hypothesis* B *is satisfied and $\alpha(\cdot)$ and $\beta(\cdot)$ are Lipschitz functions, then* (49) *has a unique solution $(u(t,x))$. Moreover, this solution is L^2-continuous and for any $T > 0$ and $p \ge 1$,*

$$\sup_{0\le t\le T}\ \sup_{x\in I\!R^d} E(|u(t,x)|^p) < \infty.$$

Remark 14 (a) As mentioned earlier, for Γ_1 of Example 6 (wave equation), the hypotheses of this theorem are satisfied if and only if (40) holds and $d \in \{1, 2, 3\}$, while for Example 8 (heat equation), they are satisfied in all dimensions if and only if (40) holds (see also Remark 10). Therefore, by Theorems 11 and 13, (40) is the necessary and sufficient condition for existence of a process solution to (42) when L is the heat or wave operator. For Example 9, see Remark 20.

(b) It is not difficult to check that if a process $(u(t,x))$ satisfies (50) and if $\alpha(\cdot) \geq \epsilon > 0$, then (46) must hold: see [15, Remark 1.3]. Therefore, (46) is also a necessary condition for the existence of a process that satisfies (50).

Proof of Theorem 13. We will follow a standard Picard iteration scheme. If $\alpha(\cdot)$ and $\beta(\cdot)$ have Lipschitz constant K, then

$$|\alpha(u)| \leq K(1 + |u|) \qquad \text{and} \qquad |\beta(u)| \leq K(1 + |u|). \tag{51}$$

Define $u_0(t,x) \equiv 0$, and, for $n \geq 0$ and assuming that u_n has been defined, set

$$Z_n(s,y) = \alpha(u_n(s,y))$$

and

$$\begin{aligned} u_{n+1}(t,x) &= \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) Z_n(s,y)\, M(ds,dy) \\ &\qquad + \int_0^t ds \int_{\mathbb{R}^d} \beta(u_n(t-s, x-y))\, \Gamma(s,dy). \end{aligned} \tag{52}$$

Assume by induction that for any $T > 0$,

$$\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^d} E(u_n(t,x)^2) < +\infty, \tag{53}$$

that $u_n(t,x)$ is $\mathcal{F}_t$-measurable for all x and t and that $(t,x) \mapsto u_n(t,x)$ is L^2-continuous. To see that the stochastic integral in (52) is well defined, observe by Lemma 19 below that $(t,x;\omega) \mapsto u_n(t,x;\omega)$ has a jointly measurable version and that conditions of Proposition 2 of [7] are satisfied. Furthermore, Hypothesis A holds for (Z_n) by Lemma 18 below. Therefore, the martingale measure M^{Z_n} is well defined, and by Theorem 2, $\Gamma(t - \cdot, x - \cdot)$ belongs to $\mathcal{P}_{0,Z_n}$. It follows that $u_{n+1}(t,x)$ is well defined and by Theorem 5, (53), (51) and (26), for any $T > 0$,

$$\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^d} E(u_{n+1}(t,x)^2) < +\infty.$$

This proves that the sequence (u_n) is well defined.

Similar to the argument in [15, Theorem 1], we first prove that for $T > 0$ and $p \geq 2$,

$$\sup_{n \geq 0} \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^d} E(|u_n(t,x)|^p) < +\infty. \tag{54}$$

For $n \geq 0$, note that

$$E(|u_{n+1}(t,x)|^p) \leq C_p \left(E(|A_n(t,x)|^p) + E(|B_n(t,x)|^p) \right),$$

where $A_n(t,x)$ is the first term in (52) and $B_n(t,x)$ is the second term in the same equation. By Theorem 5,

$$E(|A_n(t,x)|^p) \leq c_p\,(\nu_t)^{\frac{p}{2}-1} \int_0^t ds\ \left(\sup_{x\in I\!R^d} E\left(|\alpha(u_n(s,x))|^p\right)\right) J(t-s),$$

where

$$J(s) = \int_{I\!R^d} \mu(d\xi)\ |e^{i\,\xi\cdot x} \mathcal{F}\Gamma(s,\cdot)(\xi)|^2.$$

Hölder's inequality (38) implies that

$$E(|B_n(t,x)|^p) \leq E\left(\int_0^t ds \int_{I\!R^d} \Gamma(s,dy)\ |\beta(u_n(t-s,x-y))|^p \right) (\Gamma([0,T]\times I\!R^d))^{p-1}.$$

Because $\Gamma(s, I\!R^d)$ is bounded, we now conclude from (51) that

$$E(|u_{n+1}(t,x)|^p) \leq C_p \int_0^t ds\ \left(1 + \sup_{x\in I\!R^d} E(|u_n(s,x)|^p)\right) (J(t-s)+1).$$

We now conclude that (54) holds by Lemma 15 below.

In order to conclude that the sequence $(u_n(t,x),\ n\geq 0)$ converges in L^p, let

$$M_n(t) = \sup_{0\leq s\leq t}\ \sup_{x\in I\!R^d} E(|u_{n+1}(s,x) - u_n(s,x)|^p).$$

Using the Lipschitz properties of $\alpha(\cdot)$ and $\beta(\cdot)$, we conclude as above that

$$M_n(t) \leq K_p \int_0^t ds\ M_{n-1}(s)(J(t-s)+1).$$

By Theorem 5 and Hypothesis B, $\sup_{0\leq s\leq T} M_0(s) < \infty$, so we conclude by Lemma 15 below that $(u_n(t,x),\ n\geq 0)$ converges uniformly in $L^p(\Omega,\mathcal{F},P)$ to a limit $u(t,x)$. Because each u_n is L^2-continuous (see Lemma 19), the same is true of $u(t,x)$. Therefore $(u(t,x),\ t\geq 0,\ x\in I\!R^d)$ has a jointly measurable version which is easily seen to satisfy (50). Uniqueness of the solution to (49) is checked by a standard argument.

□

The following lemma is a variation on Gronwall's classical lemma that improves a result established in [31, Lemma 3.3].

Lemma 15 *(Extension of Gronwall's Lemma.) Let $g : [0,T] \to I\!R_+$ be a non-negative function such that*

$$\int_0^T g(s)\,ds < +\infty.$$

Then there is a sequence $(a_n, n\in I\!N)$ of non-negative real numbers such that $\Sigma_{n=1}^\infty a_n < \infty$ with the following property. Let $(f_n, n\in I\!N)$ be a sequence of non-negative functions on $[0,T]$ and k_1, k_2 be non-negative numbers such that for $0\leq t\leq T$,

$$f_n(t) \leq k_1 + \int_0^t (k_2 + f_{n-1}(s))g(t-s)\,ds. \tag{55}$$

If $\sup_{0\leq s\leq T} f_0(s) = M$, *then for* $n \geq 1$,

$$f_n(t) \leq k_1 + (k_1+k_2)\sum_{i=1}^{n-1} a_i + (k_2+M)a_n. \tag{56}$$

In particular, $\sup_{n\geq 0}\sup_{0\leq t\leq T} f_n(t) < \infty$, *and if* $k_1 = k_2 = 0$, *then* $\Sigma_{n\geq 0} f_n(t)$ *converges uniformly on* $[0,T]$.

Proof. Set $G(t) = \int_0^t g(s)\,ds$. In order to avoid the trivial case $g \equiv 0$, we assume that $G(T) > 0$. Let $(X_n,\ n \in I\!N)$ be an i.i.d. sequence of random variables with values in $[0,T]$ and density $g(s)/G(T)$. Set $S_n = X_1 + \cdots + X_n$. Condition (55) can be written

$$f_n(t) \leq k_1 + G(T)\, E\left(1_{\{X_1\leq t\}}(k_2 + f_{n-1}(t-X_1))\right).$$

Therefore $E(1_{\{X_1\leq t\}} f_{n-1}(t-X_1))$ is bounded above by

$$\int dP(\omega_1)\ 1_{\{X_1(\omega_1)\leq t\}}(k_1 + G(T)\int dP(\omega_2)\ 1_{\{X_2(\omega_2)\leq t-X_1(\omega_1)\}} \times(k_2 + f_{n-2}(t-X_1(\omega_1)-X_2(\omega_2)))),$$

and so

$$\begin{aligned} f_n(t) &\leq k_1 + (k_1+k_2)G(T)P\{X_1\leq t\} \\ &\quad + G(T)^2E(1_{\{X_1+X_2\leq t\}}(k_2 + f_{n-2}(t-X_1-X_2))). \end{aligned}$$

Proceeding by induction, we conclude that

$$\begin{aligned} f_n(t) &\leq k_1 + (k_1+k_2)\sum_{i=1}^{m-1} G(T)^i P\{S_i \leq t\} \\ &\quad + G(T)^m E(1_{\{S_m\leq t\}}(k_2 + f_{n-m}(t-S_m))). \end{aligned} \tag{57}$$

Letting $m = n$ and $a_n = G(T)^n P\{S_n \leq t\}$, we see immediately that (56) holds. Finally, $\Sigma_{n=1}^{\infty} a_n < \infty$ by Lemma 17 below. □

Remark 16 In the classical lemma of Gronwall, (55) is replaced by

$$f_n(t) \leq k_1 + \int_0^t (k_2 + f_{n-1}(s))g(s)\,ds.$$

In this case, proceeding as above, (57) becomes

$$\begin{aligned} f_n(t) &\leq k_1 + (k_1+k_2)\sum_{i=1}^{m-1} G(T)^i P\{X_i \leq \cdots \leq X_1 \leq t\} \\ &\quad + G(T)^m E(1_{\{X_m\leq\cdots\leq X_1\leq t\}}(k_2 + f_{n-m}(X_m))). \end{aligned}$$

Because the order statistics for an arbitrary i.i.d. sequence of continuous random variables is the same as for an i.i.d. sequence of uniform random variables on $[0,1]$, $P\{X_m \leq \cdots \leq X_1 \leq t\} \leq 1/m!$ and (56) can be replaced by

$$f_n(t) \leq k_1 + (k_1+k_2)e^{G(T)} + (k_2+M)\frac{G(T)^n}{n!}.$$

Lemma 17 *Let F be the common distribution function of an i.i.d. sequence $(X_n,\ n \in \mathbb{N})$ of non-negative random variables. Suppose that $F(0) = 0$ and set $S_n = X_1 + \cdots + X_n$. Then for any $a \geq 1$ (and trivially, for $0 \leq a < 1$) and $t > 0$,*

$$\sum_{n=1}^{\infty} a^n P\{S_n \leq t\} < +\infty. \tag{58}$$

Proof. Fix $a \geq 1$. For $\varepsilon > 0$, set $p_\varepsilon = 1 - F(\varepsilon)$. Because $F(0) = 0$, we can fix $\varepsilon > 0$ so that $F(\varepsilon) \leq 1/a$, or equivalently,

$$\log\left(\frac{1}{1 - p_\varepsilon}\right) > \log a. \tag{59}$$

Define

$$\Lambda_\varepsilon^*(\lambda) = \lambda \log\left(\frac{\lambda}{p_\varepsilon}\right) + (1 - \lambda)\log\left(\frac{1 - \lambda}{1 - p_\varepsilon}\right),$$

and note that

$$\lim_{\lambda \downarrow 0} \Lambda_\varepsilon^*(\lambda) = \log\left(\frac{1}{1 - p_\varepsilon}\right).$$

By (59), we can therefore choose $\lambda > 0$ sufficiently small so that $\Lambda_\varepsilon^*(\lambda) > \log a$. Finally, choose $\eta > 0$ sufficiently small so that

$$\Lambda_\varepsilon^*(\lambda) - \eta > \log a. \tag{60}$$

Now set $Y_{\varepsilon,n} = 1_{\{X_n \geq \varepsilon\}}$ and $S_{\varepsilon,n} = Y_{\varepsilon,1} + \cdots + Y_{\varepsilon,n}$. Then $S_n \geq \varepsilon S_{\varepsilon,n}$ and so

$$P\{S_n \leq t\} \leq P\{S_{\varepsilon,n} \leq t\varepsilon^{-1}\} = P\left\{\frac{S_{\varepsilon,n}}{n} \leq \frac{t\varepsilon^{-1}}{n}\right\}. \tag{61}$$

For n large enough so that $t\varepsilon^{-1}/n < \lambda$, this probability is bounded by $P\{S_{\varepsilon,n}/n \leq \lambda\}$. According to Cramer's large deviation theorem applied to the i.i.d. sequence $(Y_{\varepsilon,n},\ n \in \mathbb{N})$ of Bernoulli random variables with parameter p_ε (see [9, Theorem 2.2.3 and Ex. 2.2.23(b)]), for sufficiently large n,

$$P\left\{\frac{S_{\varepsilon,n}}{n} \leq \lambda\right\} \leq \exp(-(\Lambda_\varepsilon^*(\lambda) - \eta)n),$$

and by (61) and (60), (58) holds. □

Definition 5.1 For $z \in \mathbb{R}^d$, let $z + B = \{z + y : y \in B\}$ and define a martingale measure $(M_s^{(z)}(B))$ by $M_s^{(z)}(B) = M_s(z + B)$. Also, given a process $(Z(s,x),\ (s,x) \in \mathbb{R}_+ \times \mathbb{R}^d)$, set $Z^{(z)}(s,x) = Z(s, z + x)$. We say that $(Z(s,x))$ has *property* (S) if for all $z \in \mathbb{R}^d$, the finite-dimensional distributions of

$$((Z^{(z)}(s,x),\ (s,x) \in \mathbb{R}_+ \times \mathbb{R}^d), (M_s^{(z)}(B),\ s \in \mathbb{R}_+,\ B \in \mathcal{B}_b(\mathbb{R}^d)))$$

do not depend on z.

Lemma 18 *For $n \geq 0$, if $(u_n(s,x))$ has property* (S)*, then $(u_{n+1}(s,x))$ defined by* (52) *does too.*

PROOF. From (52), it is easy to check that

$$\begin{aligned} u_{n+1}(t,x) &= \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,-y)\alpha(u_n^{(x)}(s,y))\, M^{(x)}(ds,dy) \\ &\quad + \int_0^t ds \int_{\mathbb{R}^d} \beta(u_n^{(x)}(t-s,-y))\Gamma(s,dy), \end{aligned}$$

and therefore, $u_{n+1}(t,x)$ is an (abstract) function Φ of $u_n^{(x)}$ and $M^{(x)}$: $u_{n+1}(t,x) = \Phi(u_n^{(x)}, M^{(x)})$, and similarly, $u_{n+1}^{(z)}(t,x) = \Phi(u_n^{(z+x)}, M^{(z+x)})$. Therefore, for any $z \in \mathbb{R}^d$, $(s_1,\dots,s_k)$, $(t_1,\dots,t_k) \in \mathbb{R}_+^k$, $(x_1,\dots,x_k) \in (\mathbb{R}^d)^k$ and for all bounded Borel sets $B_1,\dots,B_k$ of $\mathbb{R}^d$, the joint distribution of

$$(u_{n+1}^{(z)}(s_1,x_1),\dots,u_{n+1}^{(z)}(s_k,x_k), M_{t_1}^{(z)}(B_1),\dots,M_{t_k}^{(z)}(B_k)),$$

is a function of the joint distribution of

$$u_n^{(z+x_1)}(\cdot,\cdot),\dots,u_n^{(z+x_n)}(\cdot,\cdot), M_\cdot^{(z+x_1)}(\cdot),\dots,M_\cdot^{(z+x_k)}(\cdot), M_{t_1}^{(z)}(B_1),\dots,M_{t_k}^{(z)}(B_k).$$

By property (S) for u_n, this joint distribution does not depend on z. Therefore, property (S) holds for u_{n+1}. □

Lemma 19 *Under the assumptions of Theorem* 13*, each of the processes $(u_n(t,x), 0 \leq t \leq T,\ x \in \mathbb{R}^d)$ defined in the proof of that theorem is L^2-continuous.*

PROOF. Fix $n \geq 0$, assume by induction that u_n is L^2-continuous, and let $Z_n(t,x) = \alpha(u_n(t,x))$. We begin with time increments. For $t \in [0,T]$, $x \in \mathbb{R}^d$ and $h > 0$, observe from (52) that

$$E((u_{n+1}(t,x) - u_{n+1}(t+h,x))^2) \leq 2(E_1 + E_2),$$

where

$$\begin{aligned} E_1 &= \|\Gamma(t-\cdot,x-\cdot) - \Gamma(t+h-\cdot,x-\cdot)\|_{0,Z_n}^2, \qquad (62) \\ E_2 &= E((\int_0^t ds \int_{\mathbb{R}^d} \beta(u_n(t-s,x-y))\,\Gamma(s,dy) \\ &\quad - \int_0^{t+h} ds \int_{\mathbb{R}^d} \beta(u_n(t+h-s,x-y))\,\Gamma(s,dy))^2). \end{aligned}$$

From (27),

$$\begin{aligned} E_1 &\leq 2\int_0^t ds \int_{\mathbb{R}^d} \mu_s^{Z_n}(d\xi)\, |\mathcal{F}(\Gamma(t-s,x-\cdot) - \Gamma(t+h-s,x-\cdot))(\xi)|^2 \\ &\quad +2\int_t^{t+h} ds \int_{\mathbb{R}^d} \mu_s^{Z_n}(d\xi)\, |\mathcal{F}\Gamma(t+h-s,x-\cdot)(\xi)|^2. \end{aligned}$$

By (28), this is bounded by

$$2 \sup_{0\leq s\leq T} \sup_{x\in I\!R^d} E(Z_n(s,x)^2) \left(\int_0^t ds \int_{I\!R^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t-s,\cdot)(\xi) - \mathcal{F}\Gamma(t+h-s,\cdot)(\xi)|^2 \right.$$
$$\left. + \int_t^{t+h} ds \int_{I\!R^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t+h-s,\cdot)(\xi)|^2 \right).$$

The first integral converges to 0 by hypothesis (33) in Theorem 3, and the second integral does too by (26).

Concerning E_2, observe that $E_2 \leq 2(E_{2,1}+E_{2,2})$, where

$$E_{2,1} = E\left(\left(\int_0^t ds \int_{I\!R^d} (\beta(u_n(t-s,x-y)) - \beta(u_n(t+h-s,x-y)))\, \Gamma(s,dy) \right)^2 \right),$$
$$E_{2,2} = E\left(\left(\int_t^{t+h} ds \int_{I\!R^d} \beta(u_n(t+h-s,x-y))\, \Gamma(s,dy) \right)^2 \right).$$

We only consider $E_{2,1}$, since $E_{2,2}$ is handled in a similar way. From the Cauchy-Schwarz inequality and our hypotheses on Γ and $\beta(\cdot)$,

$$E_{2,1} \leq K \int_0^t ds \int_{I\!R^d} E(|u_n(t-s,x-y) - u_n(t+h-s,x-y)|^2)\, \Gamma(s,dy).$$

By the induction hypothesis and (53), we can apply the Dominated Convergence Theorem to conclude that $E_{2,1}$ is small for small h.

We now consider spatial increments. Observe from (52) that

$$E((u_{n+1}(t,x) - u_{n+1}(t,y))^2) \leq 2(F_1 + F_2),$$

where

$$F_1 = \|\Gamma(t-\cdot, x-\cdot) - \Gamma(t-\cdot, y-\cdot)\|^2_{0,Z_n},$$
$$F_2 = E\left(\left(\int_0^t ds \int_{I\!R^d} (\beta(u_n(t-s,x-z)) - \beta(u_n(t-s,y-z)))\, \Gamma(s,dz) \right)^2 \right).$$

Note from (27) that

$$\begin{aligned} F_1 &= \int_0^t ds \int_{I\!R^d} \mu_s^{Z_n}(d\xi)\, |\mathcal{F}(\Gamma(t-s,x-\cdot) - \Gamma(t-s,y-\cdot))(\xi)|^2 \\ &= \int_0^t ds \int_{I\!R^d} \mu_s^{Z_n}(d\xi)\, |1 - e^{i\,\xi\cdot(x-y)}|^2\, |\mathcal{F}\Gamma(t-s,\cdot)(\xi)|^2. \end{aligned}$$

By (28), this is bounded by

$$\sup_{0\leq s\leq T} \sup_{x\in I\!R^d} E(Z_n(s,x)^2) \int_0^t ds \int_{I\!R^d} \mu(d\xi)\, |1 - e^{i\,\xi\cdot(x-y)}|^2\, |\mathcal{F}\Gamma(s,\cdot)(\xi)|^2.$$

By the Dominated Convergence Theorem, we conclude that the integral converges to 0 as $\|x-y\| \to 0$.

Concerning F_2, observe that by the Cauchy-Schwartz inequality and our hypotheses on Γ and $\beta(\cdot)$,

$$F_2 \leq K \int_0^t ds \int_{\mathbb{R}^d} E(|u_n(t-s,x-z) - u_n(t-s,y-z)|^2)\, \Gamma(s,dz).$$

By the induction hypothesis and (53), we can apply the Dominated Convergence Theorem to conclude that the integral converges to 0 as $\|x-y\| \to 0$. Therefore $x \mapsto u_{n+1}(t,x)$ is uniformly continuous in L^2, and so $(t,x) \mapsto u_{n+1}(t,x)$ is L^2-continuous.

□

Remark 20 *Parabolic equations with time dependent coefficients.* Because we have written the fundamental solution of (49) as $\Gamma(t-s, x-y)$ rather than $\Gamma(t,s;x-y)$, the parabolic equation with time-dependent coefficients might appear not to be covered by Theorem 13. However, if we define a solution of (49) to be a process $(u(t,x))$ such that

$$\begin{aligned} u(t,x) \quad = \quad & \int_0^t \int_{\mathbb{R}^d} \Gamma(t,s;x-y)\alpha(u(s,y))\, M(ds,dy) \\ & \qquad + \int_0^t ds \int_{\mathbb{R}^d} dy\ \Gamma(t,s;x-y)\beta(u(s,y)), \end{aligned}$$

then if (40) holds, the bound (41) shows that the integrals are well-defined provided the hypothesis (26) of Theorem 2 becomes

$$\int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\ |\mathcal{F}\Gamma(t,s;\cdot)(\xi)|^2 < +\infty,$$

and the hypothesis (33) of Theorem 3 becomes

$$\lim_{h\downarrow 0} \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) \sup_{|r-s|<h} |\mathcal{F}\Gamma(t,s;\cdot)(\xi) - \mathcal{F}\Gamma(t,r;\cdot)(\xi)|^2 = 0.$$

In the proof of L^2-continuity of time increments of Lemma 19 (see (62)), we also need to assume that

$$\lim_{h\downarrow 0} \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\ |\mathcal{F}\Gamma(t,s;\cdot)(\xi) - \mathcal{F}\Gamma(t+h,s;\cdot)(\xi)|^2 = 0.$$

The bound (41) shows that all three of these conditions are satisfied if (40) holds (see also Remark 10), and therefore the methods of Theorem 13 prove existence of a solution to (49) when L is the operator L_4 of Example 9 and (40) holds.

Acknowledgement. The author thanks D. Nualart, M. Sanz-Solé, S. Peszat and J. Zabczyk for stimulating discussions, and an anonymous referee for a careful reading of the manuscript.

References

[1] Adams, R.A. *Sobolev Spaces.* Academic Press, New York (1975).

[2] Albeverio, S., Haba, Z. & Russo, F. Trivial solutions for a non-linear two space-dimensional wave equation perturbed by a space-time white noise. *Stochastics and Stochastics Rep.* 56 (1996), 127-160.

[3] Albeverio, S., Haba, Z. & Russo, F. A two space-dimensional semilinear heat equation perturbed by (Gaussian) white noise. *Probab. Th. Rel. Fields* (1999, to appear).

[4] Berberian, S.K. *Introduction to Hilbert Space.* 2nd ed. Chelsea Publ. Co., New York (1976).

[5] Bie, E.S.-L. Etude d'une EDPS conduite par un bruit poissonien. *Probab. Th. Rel. Fields* 111-2 (1998), 287-321.

[6] Chung, K.L. & Williams, R.J. *Introduction to Stochastic Integration.* 2nd ed. Birkhäuser Boston (1990).

[7] Dalang, R.C. & Frangos, N.E. The stochastic wave equation in two spatial dimensions. *Ann. Probab.* 26-1 (1998), 187-212.

[8] Dawson, D.A. & Salehi, H. Spatially homogeneous random evolutions. *J. Mult. Anal.* 10 (1980), 141-180.

[9] Dembo, A. & Zeitouni, O. *Large Deviations Techniques and Applications.* Jones and Bartlett (1993).

[10] Folland, G.B. *Introduction to Partial Differential Equations.* Princeton Univ. Press (1976).

[11] Friedman, A. *Stochastic Differential Equations and Applications* Vol. 1. Academic Press, New York (1975).

[12] Itô, K. *Foundations of stochastic differential equations in infinite dimensional spaces.* CBMS Notes, SIAM, Baton Rouge (1983).

[13] Karkzewska, A. & Zabczyk, J. Stochastic PDE's with function-valued solutions (preprint).

[14] Landkof, N.S. *Foundations of Modern Potential Theory.* Berlin Heidelberg New York: Springer 1972.

[15] Millet, A. & Sanz-Solé, M. A stochastic wave equation in two space dimension: Smoothness of the law. *Annals of Probab.* (1999, to appear).

[16] Millet, A. & Sanz-Solé, M. Approximation and support theorem for a two space-dimensional wave equation (preprint).

[17] Mikulevicius, R. & Rozovkii, B.L. Normalized stochastic integrals in topological vector spaces. Sém. de Proba. XXXII, Lect. N. in Math. 1686, Springer Verlag (1998), 137-165.

[18] Mueller, C. A modulus for the 3-dimensional wave equation with noise: Dealing with a singular kernel. *Canad. J. Math.* 45-6 (1993), 1263-1275.

[19] Mueller, C. Long time existence for the wave equation with a noise term. *Ann. Probab.* 25-1 (1997), 133-151.

[20] Neveu, J. *Processus aléatoires gaussiens.* Presses de L'Univ. de Montréal. Montréal, Canada (1968).

[21] Nobel, J. Evolution equation with Gaussian potential. *Nonlinear Analysis: Theory, Methods and Appl.* 28 (1997), 103-135.

[22] Oberguggenberger, M. & Russo, F. White noise driven stochastic partial differential equations: triviality and non-triviality. In: *Nonlinear Theory of Generalized Functions* (M. Grosser, G. Hörmann, M. Kunzinger & M. Oberguggenberger, eds), Chapman & Hall/CRC Research Notes in Mathematics Series, CRC Press (1999), 315-333.

[23] Oberguggenberger, M. & Russo, F. The non-linear stochastic wave equation. *Integral transforms and special functions* 6 (1997), 58-70.

[24] Peszat, S. & Zabczyk, J. Stochastic evolution equations with a spatially homogeneous Wiener process. *Stoch. Proc. Appl.* 72 (1997), 187-204.

[25] Peszat, S. & Zabczyk, J. Nonlinear stochastic wave and heat equations (preprint).

[26] Revuz, D. & Yor, M. *Continuous Martingales and Brownian Motion.* Grundlehren der math. Wissenschaften 293, Springer Verlag (1991).

[27] Schwartz, L. *Théorie des distributions.* Hermann, Paris (1966).

[28] Stein, E.M. *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press (1970).

[29] Treves, F. *Topological Vector Spaces, Distributions and Kernels.* Academic Press, New York (1967).

[30] Treves, F. *Basic Linear Partial Differential Equations.* Academic Press, New York (1975).

[31] Walsh, J.B. *An introduction to stochastic partial differential equations*, Ecole d'Eté de Prob. de St–Flour XIV, 1984, Lect. Notes in Math 1180, Springer Verlag (1986).

[32] Wheeden, R.L. & Zygmund, A. *Measure and Integral: An Introduction to Real Analysis,* Marcel Dekker (1977).