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# The Beurling Estimate for a Class of Random Walks

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**Abstract:** An estimate of Beurling states that if $K$ is a curve from 0 to the unit circle in the complex plane, then the probability that a Brownian motion starting at $-\epsilon$ reaches the unit circle without hitting the curve is bounded above by $c\,\epsilon^{1/2}$. This estimate is very useful in analysis of boundary behavior of conformal maps, especially for connected but rough boundaries. The corresponding estimate for simple random walk was first proved by Kesten. In this note we extend this estimate to random walks with zero mean, finite $(3+\delta)-$moment.

**Keywords:** Beurling projection, random walk, Green's function, escape probabilities

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# 1 Introduction

The Beurling projection theorem (see, e.g., [1, Theorem V.4.1]) states that if $K$ is a closed subset of the closed unit disk in $\mathbb{C}$, then the probability that a Brownian motion starting at $-\epsilon$ avoids $K$ before reaching the unit circle is less than or equal to the same probability for the angular projection

$$K' = \{|z| : z \in K\}.$$

If $K' = [0, 1]$, a simple conformal mapping argument shows that the latter probability is comparable to $\epsilon^{1/2}$ as $\epsilon \to 0+$. In particular, if $K$ is a connected set of diameter one at distance $\epsilon$ from the origin the probability that a Brownian motion from the origin to the unit circle avoids $K$ is bounded above by $c\epsilon^{1/2}$.

This estimate, which we will call the Beurling estimate, is very useful in analysis of boundary behavior of conformal maps especially for connected but rough boundaries. A similar estimate for random walks is useful, especially when considering convergence of random walk to Brownian motion near (possibly nonsmooth) boundaries. For simple random walk such an estimate was first established in [5] to derive a discrete harmonic measure estimate for application to diffusion limited aggregation. It has been used since in a number of places, e.g., in deriving "Makarov's Theorem" for random walk [7] or establishing facts about intersections of random walks (see, e.g., [8]). Recently it has been used by the first author and collaborators to analyze the rate of convergence of random walk to Brownian motion in domains with very rough boundaries. Because of its utility, we wish to extend this estimate to walks other than just simple random walk. In this note we extend it to a larger class of random walks.

We state the precise result in the next section, but we will summarize briefly here. As in [5], we start with the estimate for a half-line. We follow the argument in [6]; see [2, 3] for extensions. The argument in [6] strongly uses the time reversibility of simple random walk. In fact, as was noted in [3], the argument really only needs symmetry in $x$ component. We give a proof of this estimate, because we need the result not just for $\mathbb{Z}^+$ but also for $\kappa\mathbb{Z}^+$ where $\kappa$ is a positive integer. The reason is that we establish the Beurling estimate here for "$(1/\kappa)$-dense" sets. One example of such a set that is not connected is the path of a non-nearest neighbor random walk whose increments have finite range; a possible application of our result would be to extend the results of [8] to finite range walks. While our argument is essentially complete for random walks that are symmetric in the $x$ component, for the nonsymmetric case we use a result of Fukai [3] that does the estimate for $\kappa = 1$. Since $\kappa\mathbb{Z}_+ \subset \mathbb{Z}_+$ this gives a lower bound for our case, and our bound for the full line then gives the upper bound.

The final section derives the general result from that for a half-line; this argument closely follows that in [5]. We assume a $(3 + \delta)$-moment for the increments of the random walk in order to ensure that the asymptotics for the potential kernel are sufficiently sharp (see (5)). (We also use the bound for some "overshoot" estimates, but in these cases weaker bounds would suffice.) If one would have in (5) a weaker bound, $c/|z|^b$ for some $b > 1/2$, an analogue of (33) would hold and this would suffice to carry out the argument in section 5. So the method presented here should require only $(2.5 + \delta)$ moment.

## 2 Preliminaries

Denote by $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ the integers, the real numbers and the complex numbers, respectively. We consider $\mathbb{Z}$ and $\mathbb{R}$ as subsets of $\mathbb{C}$. Let $\mathbb{Z}^+ = \{k \in \mathbb{Z} : k > 0\}; \mathbb{N} = \{k \in \mathbb{Z} : k \geq 0\}, \mathbb{Z}^- = \mathbb{Z} \setminus \mathbb{N}$. Let $\mathbb{L}$ denote a discrete two-dimensional lattice (additive subgroup) of $\mathbb{C}$. Let $X_1, X_2, \ldots$ be i.i.d. random variables taking values in $\mathbb{L}$ and let $S_n$ be the corresponding random walk. We say that $X_1, X_2, \ldots$ generates $\mathbb{L}$ if for each $z \in \mathbb{L}$ there is an $n$ with $\mathbf{P}(X_1 + \cdots + X_n = z) > 0$. Let

$$T_B := \inf\{l \geq 1 : S_l \in B\}, \quad T_B^0 := \inf\{l \geq 0 : S_l \in B\},$$

be the first entrance time of $B$ after time 0, and the first entrance time of $B$ including time 0, respectively. We abbreviate $T_{\{b\}}, T_{\{b\}}^0$ by $T_b, T_b^0$ respectively. Denote by $\mathcal{C}_n = \{z \in \mathbb{L} : |z| < n\}$ the discrete open disk of radius $n$, and let $\tau_n := T^0_{\mathcal{C}_n^c}$ be the first time the random walk is not in $\mathcal{C}_n$.

Suppose $\kappa$ is a positive integer and $A$ is a subset of the lattice $\mathbb{L}$. We call $A$ $(1/\kappa)$-*dense (about the origin)* if for every $j \in \mathbb{N}$, $A \cap \{j\kappa \leq |z| < (j+1)\kappa\} \neq \emptyset$. A set of the form $A = \{w_j : j \in \kappa\mathbb{N}\}$ with $j \leq |w_j| < j + \kappa$ for each $j$ will be called a *minimal* $(1/\kappa)$-dense set. Any $(1/\kappa)$-dense set contains a minimal $(1/\kappa)$-dense set. If $0 < j_1 < j_2 < \infty$, we let $A[j_1, j_2] = A \cap (\mathcal{C}_{j_2} \setminus \mathcal{C}_{j_1})$. If $-\infty < j_1 < j_2 < \infty$, we write $[j_1, j_2]_\kappa = \kappa\mathbb{N} \cap [j_1, j_2)$.

The purpose of this paper is the prove the following result.

**Theorem 1** *Suppose $\mathbb{L}$ is a discrete two-dimensional lattice in $\mathbb{C}$ and $X_1, X_2, \ldots$ are i.i.d. random variables that generate $\mathbb{L}$ such that $\mathbf{E}[X_1] = 0$ and for some $\delta > 0$, $\mathbf{E}[|X_1|^{3+\delta}] < \infty$. Then for each positive integer $\kappa$, there exists a $c < \infty$ (depending on $\kappa$ and the distribution of $X_1$) such that for every $(1/\kappa)$-dense set $A$ and every $0 < k < n < \infty$,*

$$\mathbf{P}(\tau_{2n} < T_{A[k,n]}) \leq c\sqrt{k/n}.$$

We start by making some reductions. Since $B \subset A$ clearly implies $\mathbf{P}(\tau_m < T_A^0) \leq \mathbf{P}(\tau_m < T_B^0)$, it suffices to prove the theorem for minimal $(1/\kappa)$-dense sets $A = \{w_j : j \in \kappa\mathbb{N}\}$ and, without loss of generality, we assume that $A$ is of this form. By taking a linear transformation of the lattice if necessary, we may assume that $\mathbb{L}$ is of the form

$$\mathbb{L} = \{j + kz^* : j, k \in \mathbb{Z}\},$$

where $z^* \in \mathbb{C} \setminus \mathbb{R}$ and that the covariance matrix of $X_1$ is a multiple of the identity. (When dealing with mean zero, finite variance lattice random walks, one can always choose the lattice to be the integer lattice in which case one may have a non-diagonal covariance matrix, or one can choose a more general lattice but require the covariance matrix to be a multiple of the identity. We are choosing the latter.) Let $p$ be the (discrete) probability mass function of $X_1$. Then our assumptions are $\{z : p(z) > 0\}$ generates $\mathbb{L}$ and for some $\delta, \sigma^2 > 0$,

$$\sum_z zp(z) = 0, \tag{1}$$

$$\sum_z \mathrm{Re}(z)^2 p(z) = \sum_z \mathrm{Im}(z)^2 p(z) = \sigma^2 > 0, \tag{2}$$

$$\sum_{z} |z|^{3+\delta} p(z) < \infty, \tag{3}$$

Let $p_*(z) = p(z)$ be step probability mass function of the time-reversed walk; and note that $p_*$ also satisfies (1)-(3). We denote by $\mathbf{P}_*(A)$ the probability of $A$ under steps according to $p_*$. We call a function $f$ *$p$-harmonic* at $w$ if

$$\Delta_p f(w) := \sum_{z} p(z)\,[f(z+w) - f(w)] = 0. \tag{4}$$

Let $X_1, X_2, \dots$ be independent $\mathbb{L}$-valued random variables with probability mass function $p$, and let $S_n = S_0 + \sum_{i=1}^{n} X_i, n \geq 0$ be the corresponding random walk. Denote by $\mathbf{P}^x$ (resp., $\mathbf{E}^x$) the law (resp., expectation) of $(S_n, n \geq 0)$ when $S_0 = x$, and we will write $\mathbf{P}, \mathbf{E}$, for $\mathbf{P}^0, \mathbf{E}^0$.

Let $a(z)$ denote the *potential kernel* for $p$,

$$a(z) = \lim_{n\to\infty} \sum_{j=0}^{n} \left[\mathbf{P}(S_j = 0) - \mathbf{P}(S_j = z)\right],$$

and let $a^*(z)$ denote the potential kernel using $p_*$. Note that $a$ is $p_*$-harmonic and $a^*$ is $p$-harmonic for $z \neq 0$ and $\Delta_{p_*} a(0) = \Delta_p a^*(0) = 1$. In [4] it is shown that under the assumptions (1) - (3) there exist constants $\bar{k}, c$ (these constants, like all constants in this paper, may depend on $p$), such that for all $z$,

$$\left| a(z) - \frac{\log|z|}{\pi\,\sigma^2} - \bar{k} \right| \leq \frac{c}{|z|}. \tag{5}$$

Since $a^*(z) = a(-z)$, this also holds for $a^*$.

As mentioned above, $\mathcal{C}_n = \{z \in \mathbb{L} : |z| < n\}$ is the discrete open disk of radius $n$ and $\tau_n := T^0_{\mathcal{C}_n^c}$. Denote by $\mathcal{L}_n$ the discrete open strip $\{x + iy \in \mathbb{L} : |y| < n\}$ of width $2n$ and let $\rho_n := T^0_{\mathcal{L}_n^c}$, i.e., $\tau_n, \rho_n$ are the exit times from the disk and the strip, respectively.

For any proper subset $B$ of $\mathbb{L}$, let $G_B(w,z)$ denote the Green's function of $B$ defined by

$$G_B(w,z) = \sum_{j=0}^{T^0_{B^c}-1} \mathbf{P}^w(S_j = z). \tag{6}$$

This equals zero unless $w, z \in B$. We will write $G_n$ for $G_{\mathcal{C}_n}$. If $w, z \in B$, and $G(w) := G_B(w,z)$, then $\Delta_p G(w) = -\delta(w-z)$ where $\Delta_p$ is as in (4), and where $\delta(\cdot)$ is the Kronecker symbol $\delta(x) = 1$, $x = 0$ and $\delta(x) = 0$, $x \neq 0$. Let $G^*_B(w,z)$ denote the Green's function for $p_*$ and note that $G_B(z,w) = G^*_B(w,z)$. A useful formula for finite $B$ is

$$G_B(w,z) = \mathbf{E}^w[a^*(S_T - z)] - a^*(w-z) = \mathbf{E}^w[a(z - S_T)] - a(z-w), \tag{7}$$

where $T = T^0_{B^c}$. This is easily verified by noting that for fixed $z \in B$, each of the three expressions describes the function $(w)$ satisfying: $f(w) = 0, w \notin B; \Delta_p f(z) = -1; \Delta_p f(w) = 0, w \in B \setminus \{z\}$. The following "last-exit decomposition" relates the Green's function and escape probabilities:

$$\mathbf{P}^z\{T^0_{B'} < T^0_{B^c}\} = \sum_{w\in B'} G_B(z,w)\,\mathbf{P}^w(T_{B'} > T_{B^c}). \tag{8}$$

It is easily derived by focusing on the last visit to $B'$ strictly less than $T^0_{B^c}$.

For the remainder of this paper we fix $p,\kappa$ and allow constants to depend on $p,\kappa$. We assume $k \leq n/2$, for otherwise the inequality is immediate. The values of universal constant may change from line to line without further notice. In the next two sections will prove that

$$\mathbf{P}(\tau_{n^4} < T_{[k,n]}) \leq \frac{1}{\log n}\sqrt{\frac{k}{n}}.$$

(Here, and throughout this paper, we use $\asymp$ to mean that both sides are bounded by constants times the other side where the constants may depend on $p,\kappa$.) In the final section we establish the uniform upper bound for all minimal $(1/\kappa)$-dense sets.

## 3 Green's function estimates

We start with an "overshoot" estimate.

**Lemma 2** *There is a $c$ such that for all $n$ and all $z$ with $|z|<n$,*

$$\mathbf{E}^z[|S_{\tau_n}|] \leq n + c\,n^{2/3}, \quad \mathbf{E}^z[\log|S_{\tau_n}| - \log n] \leq c\,n^{-1/3}.$$

**Proof.** If $a>0$, since $\{|S_{\tau_n}| - n \geq a\} \subset \{|X_{\tau_n}| \geq a\}$ we have

$$\mathbf{P}^z(|S_{\tau_n}| - n \geq a) \leq \sum_{j=1}^{\infty} \mathbf{P}^z(\tau_n = j, |X_j| \geq a) \leq \sum_{j=1}^{\infty} \mathbf{P}^z(\tau_n > j-1, |X_j| \geq a)$$

$$\leq \sum_{j=1}^{\infty} \mathbf{P}^z(\tau_n > j-1)\,\mathbf{P}(|X_1| \geq a) \leq \mathbf{E}^z(\tau_n)\,\mathbf{P}(|X_1| \geq a).$$

From the central limit theorem, we know that $\mathbf{P}^z\{\tau_n > r + n^2 \mid \tau_n > r\} < \alpha < 1$. Therefore, $\tau_n/n^2$ is stochastically bounded by a geometric random variable with success probability $1-\alpha$, and hence $\mathbf{E}^z[\tau_n] \leq cn^2$. Since $\mathbf{E}[|X_1|^3] < \infty$,

$$\mathbf{P}(|X_1| \geq b) = \mathbf{P}(|X_1|^3 \geq b^3) \leq c\,b^{-3}. \tag{9}$$

Therefore,

$$\mathbf{P}^z(|S_{\tau_n}| - n \geq a\,n^{2/3}) \leq c\,a^{-3}, \tag{10}$$

and

$$\mathbf{E}[|S_{\tau_n}| - n] \leq n^{2/3} + \int_{n^{2/3}}^{\infty} \mathbf{P}^z(|S_{\tau_n}| - n \geq y)\,dy \leq c\,n^{2/3}.$$

The second inequality follows immediately from applying $\log(1+x) \leq x$ to $x = (|S_n|-n)/n$. $\Box$

**Remark.** With a finer argument, we could show, in fact, that $\mathbf{E}^z[|S_{\tau_n}|] \leq n + c$. By doing the more refined estimate we could improve some of the propositions below, e.g., the $O(n^{-1/3})$ error term in the next proposition is actually $O(n^{-1})$. However, since the error terms we have proved here suffices for this paper, we will not prove the sharper estimates.

**Lemma 3**

$$\pi\,\sigma^2\,G_n(0,0) = \log n + O(1).$$

*If* $|z| < n$*,*

$$\pi\,\sigma^2 G_n(0,z) = \log n - \log|z| + O\left(\frac{1}{|z|}\right) + O(n^{-1/3}),$$

$$\pi\,\sigma^2 G_n(z,0) = \log n - \log|z| + O\left(\frac{1}{|z|}\right) + O(n^{-1/3}).$$

*Also, for every* $b < 1$*, there exist* $c > 0$ *and* $N$ *such that for all* $n \geq N$*,*

$$G_n(z,w) \geq c, \quad z, w \in \mathcal{C}_{bn}. \tag{11}$$

**Proof.** The first expression follows from (5), (7) and Lemma 2 since $a(0) = 0$. The next two expressions again use (7), Lemma 2, and (5). For the final expression, first note it is true for $b = 1/4$, since for $0 \leq |z|, |w| < n/4$, $G_n(z,w) \geq G_{3n/4}(0, w-z)$. For $b < 1$, the invariance principle implies that there is a $q = q_b > 0$ such that for all $n$ sufficiently large, with probability at least $q$ the random walk (and reversed random walk) starting at $|z| < bn$ reaches $\mathcal{C}_{n/4}$ before leaving $\mathcal{C}_n$. Hence, by the strong Markov property, if $|z| < bn, |w| < bn$, $G_n(z,w) \geq q\,\inf_{|z'|<n/4} G_n(z',w)$. Similarly, using the reversed random walk, if $|w| < bn, |z'| < n/4$, $G_n(z',w) \geq q\,\inf_{|w'|<n/4} G_n(z',w')$. $\square$

**Lemma 4** *If* $m \geq n^4$ *and* $|z|, |w| \leq n$*,*

$$\pi\,\sigma^2\,G_m(z,w) = \log m - \log|z-w| + O\left(\frac{1}{|z-w|}\right) + O(n^{-4/3}).$$

**Proof.** Since $G_{m-n}(0, w-z) \leq G_m(z,w) \leq G_{m+n}(0, w-z)$, this follows from the previous lemma.

**Lemma 5** *There is a* $c < \infty$ *such that for every* $z \in \mathcal{C}_n$ *and every minimal* $(1/\kappa)$*-dense set* $A$*,*

$$\sum_{w \in A} G_n(z,w) \leq c\,n. \tag{12}$$

**Proof.** By Lemma 3,

$$\pi\,\sigma^2\,G_n(z,w) \leq \pi\,\sigma^2\,G_{2n}(0, w-z) \leq \log n - \log|w-z| + O(1).$$

If $A$ is a minimal $(1/\kappa)$-dense set, then $\#\{w \in A : |z-w| \leq r\} \leq cr$, for some $c$ independent of $z$. Hence,

$$\sum_{w \in A} G_n(z,w) \leq c \sum_{j=1}^{2n} [\log n - \log j + O(1)] = O(n).$$

$\square$

## 4 Escape probability estimates for $[j,k]_\kappa$

The main purpose of this section is to obtain estimates in Proposition 12 and Lemmas 13 and 14 which will be used in the proof of Theorem 1 in section 5.

**Lemma 6**

$$\mathbf{P}(\tau_n < T_{[-n,n]_\kappa}) \asymp \frac{1}{n}. \tag{13}$$

**Proof.** Let $q(n) = \mathbf{P}(\tau_n \le T_{[-n,n]_\kappa})$ and note that if $k \in [-n/2, n/2]_\kappa$, then

$$q(4n) \le \mathbf{P}^k(\tau_n \le T_{[-n,n]_\kappa}) \le q(n/4).$$

The last-exit decomposition (8) tells us

$$\sum_{k\in[-n/2,n/2]_\kappa} G_n(0,k)\,\mathbf{P}^k(\tau_n < T_{[-n,n]_\kappa}) \le \sum_{k\in[-n,n]_\kappa} G_n(0,k)\,\mathbf{P}^k(\tau_n < T_{[-n,n]_\kappa}) = 1.$$

But (11) and (12) imply that

$$\sum_{k\in[-n/2,n/2]_\kappa} G_n(0,k) \asymp n,$$

which gives $q(4n) = O(1/n)$. The lower bound can be obtained by noting $\mathbf{P}(\rho_n < T_{\mathbb{Z}}) \le \mathbf{P}(\tau_n < T_{[-n,n]_\kappa})$ which reduces the estimate to a one-dimensional "gambler's ruin" estimate in the $y$-component. This can be established in a number of ways, e.g., using a martingale argument. $\square$

**Lemma 7** *There exist $c > 0$ and $N < \infty$ such that if $n \ge N$ and $z \in \mathcal{C}_{3n/4}$,*

$$\mathbf{P}^z(T^0_{A[n/4,n]} < \tau_n) \ge \mathbf{P}^z(T^0_{A[n/4,n/2]} < \tau_n) \ge c.$$

**Proof.** Let

$$V = \sum_{j=0}^{\tau_n - 1} 1\{S_j \in A[n/4,n/2]\},$$

be the number of visits to $A[n/4,n/2]$ before leaving $\mathcal{C}_n$. Then (11) and (12) show that there exist $c_1, c_2$ such that for $n$ sufficiently large,

$$c_1\, n \le \mathbf{E}^w[V] \le c_2\, n, \quad w \in \mathcal{C}_{7n/8}.$$

In particular, if $z \in \mathcal{C}_{3n/4}$,

$$c_1\, n \le \mathbf{E}^z[V] = \mathbf{P}^z(V \ge 1)\, \mathbf{E}^z[V \mid V \ge 1] \le c_2\, n\, \mathbf{P}^z(V \ge 1). \quad \square$$

**Lemma 8** *There exist $0 < c_1 < c_2 < \infty$ and $N < \infty$ such that if $n \ge N$,*

$$\frac{c_1}{\log n} \le \mathbf{P}^z(T_0 < \tau_n) \le \frac{c_2}{\log n}, \quad z \in \mathcal{C}_{9n/10} \setminus \mathcal{C}_{n/10}.$$

**Proof.** This follows immediately from Lemma 3 and $G_n(z,0) = \mathbf{P}^z(T_0 < \tau_n)\, G_n(0,0)$. $\square$

Let $T^+ = T_{\kappa\mathbb{Z}^+}, T^- = T_{\kappa\mathbb{Z}\setminus\mathbb{Z}^+}$. Define

$$E_n^+ = \{\rho_n < T^+\},\ E_n^- = \{\rho_n < T^-\},\ \tilde{E}_n^- = \{\rho_n < T_{\kappa\mathbb{Z}^-}\}$$

and

$$E_n = E_n^+ \cap E_n^- = \{\rho_n < T_{\kappa\mathbb{Z}}\}.$$

Recall that $\mathbf{P}_*$ stands for the probability under step distribution $p_*$.

**Lemma 9** $\mathbf{P}(E_n) = \mathbf{P}_*(\tilde{E}_n^-)\mathbf{P}(E_n^-)$.

**Proof.** Consider $E_n^- \cap (E_n^+)^c = V_1 \cup V_2 \cup \ldots$, where

$$V_m = \{\rho_n < T_{\{\ldots,-\kappa,0,\kappa,\ldots,\kappa(m-1)\}}\} \cap \{\rho_n > T_{\kappa m}\},$$

is the event that integer $\kappa m$ is the smallest integer in $\kappa\mathbb{Z}$ visited by the walk before time $\rho_n$. Clearly $V_1, V_2, \ldots$ are disjoint events. Write

$$V_m = \bigcup_{j=1}^{\infty} V_{m,j},$$

where $V_{m,j} := V_m \cap \{S_j = \kappa m\} \cap \{S_l \neq \kappa m, l = j+1, \ldots, \rho_n\}$ is the intersection of $V_m$ with the event that $\kappa m$ is visited for the last time (before time $\rho_n$) at time $j$. Again, $V_{m,j}$ are mutually disjoint events. Therefore,

$$\mathbf{P}(E_n^- \cap (E_n^+)^c) = \sum_{m=1}^{\infty}\sum_{j=1}^{\infty} \mathbf{P}(V_{m,j}). \tag{14}$$

Note that due to the strong Markov property, and homogeneity of the line and the lattice, we have

$$\begin{aligned}\mathbf{P}(V_{m,j}) &= \mathbf{P}(S_j = \kappa m, j < \rho_n \wedge T_{\kappa\{\ldots,-1,0,1,\ldots,m-1\}})\, \mathbf{P}^{\kappa m}(\rho_n < T_{\kappa\{\ldots,-1,0,1,\ldots,m\}}) \\ &= \mathbf{P}(S_j = \kappa m, j-1 < \rho_n \wedge T_{\kappa\{\ldots,-1,0,1,\ldots,m-1\}})\, \mathbf{P}(E_n^-). \end{aligned}\tag{15}$$

By reversing the path we can see that

$$\begin{aligned}&\mathbf{P}(S_j = \kappa m, j-1 < \rho_n \wedge T_{\kappa\{\ldots,-1,0,1,\ldots,m-1\}}) \\ &\qquad = \mathbf{P}_*^{\kappa m}(S_j = 0, j-1 < \rho_n \wedge T_{\kappa\{\ldots,-1,0,1,\ldots,m-1\}}).\end{aligned}\tag{16}$$

Also note that

$$\begin{aligned}&\mathbf{P}_*^{\kappa m}(S_j = 0, j-1 < \rho_n \wedge T_{\kappa\{\ldots,-1,0,1,\ldots,m-1\}}) = \\ &\qquad \mathbf{P}_*(S_j = -\kappa m, j-1 < \rho_n \wedge T_{\kappa\{\ldots,-2,-1\}})\end{aligned}\tag{17}$$

by translation invariance. Now,

$$\{S_j = -\kappa m, j-1 < \rho_n \wedge T_{\kappa\{\ldots,-2,-1\}}\} = \{\rho_n \wedge T_{\kappa\mathbb{Z}^-} = T_{-\kappa m} = j\}$$

and since

$$\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\mathbf{P}_*(\rho_n \wedge T_{\kappa\mathbb{Z}^-} = T_{-\kappa m} = j) = \sum_{m=1}^{\infty}\mathbf{P}_*(\rho_n \wedge T_{\kappa\mathbb{Z}^-} = T_{-\kappa m}) = \mathbf{P}_*(\rho_n > T_{\kappa\mathbb{Z}^-})$$

relations (15)-(17) imply

$$\sum_{m\geq 1}\sum_{j\geq 1}\mathbf{P}(V_{m,j}) = \mathbf{P}_*(\rho_n > T_{\kappa\mathbb{Z}^-})\,\mathbf{P}(E_n^-) = \mathbf{P}_*((\tilde{E}_n^-)^c)\,\mathbf{P}(E_n^-).$$

This together with (14) implies the lemma. $\square$

**Remark.** The above result implies the following remarkable claim: if the step distribution of the walk is symmetric with respect to $y$-axis then, under $\mathbf{P}$, the events $E_n^+$ and $E_n^-$ are independent.

**Remark.** Versions of this lemma have appeared in a number of places. See [6, 2, 3].

**Lemma 10**

$$\mathbf{P}(\rho_n \leq T_{\kappa\mathbb{N}}) \asymp \mathbf{P}(\rho_n \leq T_{\kappa\mathbb{Z}^-}) \asymp \frac{1}{\sqrt{n}}. \tag{18}$$

**Proof.** In the case $\kappa = 1$, this was essentially proved by Fukai [3]. Theorem 1.1 in [3] states that

$$\mathbf{P}(n^2 < T_{\mathbb{N}}) \asymp \frac{1}{n^{1/2}}. \tag{19}$$

for any zero-mean aperiodic random walk on lattice $\mathbb{Z}^2$ with $2+\delta$ finite moment. Note that we can linearly map $\mathbb{L}$ onto $\mathbb{Z}^2$, and by this cause only multiplicative constant change (depending on $\mathbb{L}$) in the conditions (1)-(3), which imply the assumptions needed for (19) to hold. The conversion from $n^2$ to $\rho_n$ is not difficult and his argument can be extended to give this. Note that this gives a lower bound for other $\kappa$,

$$\mathbf{P}(\tau_n < T_{\kappa\mathbb{N}}) \geq \frac{c}{n^{1/2}}, \tag{20}$$

where $c$ depends on $\mathbb{L}$ and transition probability $p$ only. Hence, the two terms in the product in Lemma 9 are bounded below by $c/\sqrt{n}$ but the product is bounded above by $c_1/n$. Hence, each of the terms is also bounded above by $\tilde{c}/\sqrt{n}$, and this proves the statement. $\square$

**Lemma 11** *There exists $c \in (0,\infty)$ such that*
*(a) $P^{-n}(T_{-n} < T_{\kappa\mathbb{N}}) \leq 1 - \frac{c}{\log n}$,*
*(b) If $|z| \geq n$ then $P^z(T_z < T_{\kappa\mathbb{Z}}) \leq 1 - \frac{c}{\log n}$.*

**Proof.** We prove (a), and note that (b) can be done similarly. It is equivalent to show

$$\mathbf{P}(T_{\kappa\mathbb{N}+n} < T_0) \geq \frac{c}{\log n}$$

Note that since $\tau_n \leq T_{\kappa\mathbb{N}+n}$, Lemma 3 yields the upper bound on the above probability of the same order. For the lower bound note that invariance principle implies

$$\mathbf{P}(\tau_n < T_0, \mathrm{Re}(S_{\tau_n}) \geq 4n/5) \geq \frac{\mathbf{P}(\tau_n < T_0)}{100} \geq \frac{c}{\log n}, \tag{21}$$

by Lemma 3. Use Markov property and Lemma 7 applied to disk centered at $n = (n, 0)$ of radius $9n/10$ to get

$$\mathbf{P}(T_{\kappa\mathbb{N}+n} < T_0 | \tau_n < T_0, \mathrm{Re}(S_{\tau_n}) \geq 4n/5, |S_{\tau_n}| - n \leq n/5) \geq c,$$

uniformly in $n$. An easy overshoot argument yields $\mathbf{P}(\tau_n < T_0, \mathrm{Re}(S_{\tau_n}) \geq 4n/5, |S_{\tau_n}| - n \leq n/5) \asymp \mathbf{P}(\tau_n < T_0, \mathrm{Re}(S_{\tau_n}) \geq 4n/5)$, which implies the lemma. $\square$

**Proposition 12** *If $j, n \in \mathbb{Z}^+$,*

$$(a)\ \ \mathbf{P}(\tau_n \leq T_{\kappa\mathbb{N}}) \asymp \mathbf{P}(\tau_n \leq T_{\kappa\mathbb{Z}^+}) \asymp \frac{1}{\sqrt{n}},$$

$$(b)\ \ \mathbf{P}^{-n}(S_{T_{\kappa\mathbb{N}}} = 0) = O\left(\frac{1}{\sqrt{n}}\right),$$

$$(c)\ \ \mathbf{P}^{n}(S_{T_{\kappa\mathbb{N}}} = 0) = O\left(\frac{1}{n^{3/2}}\right).$$

$$(d)\ \ \mathbf{P}(\tau_n < T_{\kappa(j+\mathbb{N})}) = O\left(\sqrt{\frac{j}{n}}\right),$$

$$(e)\ \ \mathbf{P}(\tau_n < T_{\kappa(-j+\mathbb{N})}) = O\left(\frac{1}{\sqrt{jn}}\right),$$

**Proof.** $(a)$ A simple Markov argument gives

$$\mathbf{P}(\tau_n \leq T_{\kappa\mathbb{N}}) \leq \mathbf{P}(\tau_n \leq T_{\kappa\mathbb{Z}_+}) \leq \mathbf{P}(S_{T_{\kappa\mathbb{N}}} \neq 0)^{-1}\, \mathbf{P}(\tau_n \leq T_{\kappa\mathbb{N}}),$$

and hence the first two quantities are comparable. Since $\tau_n \leq \rho_n$, (18) gives $\mathbf{P}(\tau_n \leq T_{\kappa\mathbb{N}}) \geq c/\sqrt{n}$. For the upper bound, let $A^- = A_n^-$ be the event that $\mathrm{Re}(S_{\tau_n}) \leq 0$. By the invariance principle, $\mathbf{P}(A^-) \geq 1/4$. However, we claim that $\mathbf{P}(A^- \mid \tau_n \leq T_{\kappa\mathbb{N}}) \geq \mathbf{P}(A^-)$. Indeed, by translation invariance, we can see for every $j > 0$, $\mathbf{P}^{j\kappa}(A^-) \leq \mathbf{P}(A^-)$, and hence by the Strong Markov property, $\mathbf{P}(A^- \mid \tau_n > T_{\kappa\mathbb{N}}) \leq \mathbf{P}(A^-)$. Therefore,

$$\mathbf{P}(\tau_n \leq T_{\kappa\mathbb{N}}, \mathrm{Re}(S_{\tau_n}) \leq 0) \geq (1/4)\, \mathbf{P}(\tau_n \leq T_{\kappa\mathbb{N}}).$$

The invariance principle can now be used to see that for some $c$,

$$\mathbf{P}(\rho_n \leq T_{\kappa\mathbb{N}} \mid \tau_n \leq T_{\kappa\mathbb{N}}, \mathrm{Re}(S_{\tau_n}) \leq 0) \geq c,$$

and hence $\mathbf{P}(\rho_n \leq T_{\kappa\mathbb{N}}) \geq (c/4)\, \mathbf{P}(\tau_n \leq T_{\kappa\mathbb{N}})$.

$(b)$ Let $T = T_{-n} \wedge T_{\kappa\mathbb{N}}$. Since $\mathbf{P}^{-n}(S_T \neq -n) \geq c/\log n$ by Lemma 11(a), it suffices by the strong Markov property to show that

$$\mathbf{P}^{-n}(S_T = 0) \leq \frac{c}{(\log n)\, \sqrt{n}}.$$

By considering reversed paths, we see that

$$\mathbf{P}^{-n}(S_T = 0) = \mathbf{P}_*(S_T = -n).$$

But

$$\begin{aligned}\mathbf{P}_*(S_T = -n) &= \mathbf{P}_*(\tau_{n/2} < T_{\kappa\mathbb{N}})\,\mathbf{P}_*(S_T = -n \mid \tau_{n/2} < T_{\kappa\mathbb{N}}) \\ &\leq \mathbf{P}_*(\tau_{n/2} < T_{\kappa\mathbb{N}})\,\mathbf{P}_*(S_T = -n \mid \tau_{n/2} < T_{\kappa\mathbb{N}}, |S(\tau_{n/2})| \leq 3n/4) \\ &\qquad\qquad\qquad\qquad + \mathbf{P}_*(|S(\tau_{n/2})| \geq 3n/4).\end{aligned}$$

To bound the last line, note that by (18), $\mathbf{P}_*(\tau_{n/2} < T_{\kappa\mathbb{N}}) \leq c/\sqrt{n}$ and the conditional probability is bounded by a term of order $1/\log n$ due to Lemmas 7 and 8. Inequality (10) implies that $\mathbf{P}_*(|S(\tau_{n/2})| \geq 3n/4) \leq c/n$.

($c$) We will start with the estimate

$$\mathbf{P}^z(S_{T_{\kappa\mathbb{Z}}} = w) \leq \frac{c}{n} \text{ if } |z - w| \geq n. \tag{22}$$

Without loss of generality assume $w = 0$, $|z| \geq n$. As in (b), it suffices to show that $\mathbf{P}^z(S_{T_z \wedge T_{\kappa\mathbb{Z}}} = 0) \leq c/(n \log n)$ due to Lemma 11(b). By using reversed paths, we see that $\mathbf{P}^z(S_{T_z \wedge T_{\kappa\mathbb{Z}}} = 0) = \mathbf{P}_*(S_{T_{\bar{z}} \wedge T_{\kappa\mathbb{Z}}} = \bar{z})$. Hence it suffices to show that for all $|z| \geq n$,

$$\mathbf{P}_*(S_T = z) \leq \frac{c}{n \log n},$$

where $T = T_z \wedge T_{\kappa\mathbb{Z}}$. Similarly to (b), we have $\mathbf{P}_*(\tau_{n/2} < T_{\kappa\mathbb{Z}}) \leq n^{-1}$ and $\mathbf{P}_*(S_T = z \mid \tau_{n/2} < T_{\kappa\mathbb{Z}}, |S_{\tau_{n/2}}| \leq 3n/4) \leq c/\log n$ . We have to be a little more careful with the second term, but

$$\begin{aligned}&\mathbf{P}_*(|S_{\tau_{n/2}}| \geq 3n/4, \tau_{n/2} < T_{\kappa\mathbb{Z}}) \\ &\quad \leq \mathbf{P}_*(|S_{\tau_{\sqrt{n}}}| \geq n/2) + \mathbf{P}_*(\tau_{\sqrt{n}} < \tau_{n/2} \wedge T_{\kappa\mathbb{Z}}, |S_{\tau_{n/2}}| \geq 3n/4\} \\ &\quad \leq O(n^{-2}) + O(n^{-1/2})O(n^{-1}) = O(n^{-3/2}).\end{aligned} \tag{23}$$

Using (b) and (22) and noting $\{S_{T_{\kappa\mathbb{N}}} = 0\} = \cap_{k=0}^{\infty}\{S_{T_{\kappa\mathbb{Z}}} = -k\} \cap \{S_{T_{\kappa\mathbb{N}}} \circ \theta_{T_{\kappa\mathbb{Z}}} = 0\}$, we conclude that

$$\mathbf{P}^z(S_{\kappa\mathbb{N}} = 0) \leq \frac{c}{|z|^{1/2}}, \quad |z| \geq n. \tag{24}$$

The remainder of the argument is done similarly to (b). Namely, use estimate (23) and note that the probability that the random walk starting at $n$ reaches a distance of $n/2$ from its starting point without hitting $\kappa\mathbb{N}$ is $O(n^{-1})$, and, given that $|S_{\tau_{n/2}}| \leq 3n/4$, the probability that it afterwards enters $\kappa\mathbb{N}$ at the origin is $O(n^{-1/2})$ due to (24).

($d$) We may assume $j\kappa \leq n/4$. By the Markov property, translation invariance, (a), and (b), if $l\kappa \leq n/4$,

$$\begin{aligned}&\mathbf{P}(\tau_n < T_{\kappa(l+1+\mathbb{N})}) - \mathbf{P}(\tau_n < T_{\kappa(l+\mathbb{N})}) \\ &\qquad = \mathbf{P}(T_{l\kappa} < T_{\kappa(l+1+\mathbb{N})} \wedge \tau_n)\,\mathbf{P}^{l\kappa}(\tau_n < T_{\kappa(l+1+\mathbb{N})}) \\ &\qquad \leq \mathbf{P}(T_{l\kappa} = T_{\kappa(l+\mathbb{N})})\,\mathbf{P}(\tau_{n/2} < T_{\kappa\mathbb{Z}^+}) \\ &\qquad = \mathbf{P}^{-l\kappa}(S_{\kappa\mathbb{N}} = 0)\,\mathbf{P}(\tau_{n/2} < T_{\kappa\mathbb{Z}^+}) \\ &\qquad \leq c/\sqrt{ln}\end{aligned}$$

If we sum this estimate over $l = 0, \ldots, j$, we get the estimate.

($e$) This is done similarly to (d), using (c) instead of (b). $\square$

**Lemma 13** *There exist $0 < c_1 < c_2 < \infty$ and $N < \infty$, such that if $n \geq N, m = n^4$, and*
*(i) if $w \in \mathcal{C}_{4n} \setminus \mathcal{C}_{3n}$,*

$$\frac{c_1}{\log n} \leq \mathbf{P}^w(\tau_m < \eta_n) \leq \frac{c_2}{\log n},$$

*where $\eta_n = \inf\{j : |S_j| \leq 2n\}$.*
*(ii) if $w \in \mathcal{C}_{4n}$,*

$$\mathbf{P}^w(\tau_m < T_{A[n/2,n]}) \leq \frac{c_2}{\log n}.$$

**Remark.** When $w \in \mathcal{C}_{4n} \setminus \mathcal{C}_{3n}$ (i) implies a lower bound of the same order in (ii).

**Proof.** (i) Let $T = \tau_m \wedge \eta_n$. We will show that

$$\mathbf{P}^w(T = \tau_m) \asymp 1/\log n. \tag{25}$$

Consider the martingale $M_j = \pi\, \sigma^2\, [a^*(S_{j\wedge T}) - \bar{k}] - \log n$, and note that $M_j = \log|S_{j\wedge T}| - \log n + O(|S_{j\wedge T}|^{-1})$. Therefore,

$$\log 3 + O(n^{-1}) \leq M_0 \leq \log 4 + O(n^{-1}). \tag{26}$$

The optional sampling theorem implies that

$$\begin{aligned}\mathbf{E}^w[M_0] = \mathbf{E}^w[M_T] = \\ \mathbf{E}^w[M_T 1_{\{|S_T| \geq m\}}] + \mathbf{E}^w[M_T 1_{\{|S_T| < n\}}] + \mathbf{E}^w[M_T 1_{\{n \leq |S_T| \leq 2n\}}]\end{aligned} \tag{27}$$

(the estimate (10) can be used to show that the optional sampling theorem is valid). Note that

$$\begin{aligned}(\log m)\, \mathbf{P}^w(T = \tau_m) &\leq \mathbf{E}^w[\log|S_T| 1_{\{T=\tau_m\}}] \\ &\leq (\log m)\, \mathbf{P}^w(T = \tau_m) + \mathbf{E}^w[\log|S_{\tau_m}| - \log m] \\ &\leq 4\,(\log n)\, \mathbf{P}^w(T = \tau_m) + O(n^{-4/3}).\end{aligned}$$

The last inequality uses Lemma 2. Therefore,

$$\mathbf{E}^w[M_T 1_{\{|S_T| \geq m\}}] = 3\,(\log n)\, \mathbf{P}^w(T = \tau_m) + O(n^{-4/3}),$$

and hence it suffices to show that

$$\mathbf{E}^w[M_T 1_{\{|S_T| \geq m\}}] \asymp 1. \tag{28}$$

Clearly,

$$O(n^{-1}) \leq \mathbf{E}^w[M_T 1_{\{n \leq |S_T| \leq 2n\}}] \leq \log 2 + O(n^{-1}).$$

Also,

$$
\begin{aligned}
\mathbf{P}^w(|S_T|<n) &= \sum_{|z|<n}\sum_{w'\in\mathcal{C}_m\setminus\mathcal{C}_{2n}} G_{\mathcal{C}_m\setminus\mathcal{C}_{2n}}(w,w')\,\mathbf{P}(X_1=z-w') \\
&\le c\,(\log n)\sum_{w'\in\mathcal{C}_m\setminus\mathcal{C}_{2n}}\sum_{|z|<n}\mathbf{P}(X_1=w'-z) \\
&\le c\,n^2\,(\log n)\sum_{|z'|\ge n}\mathbf{P}(X_1=z_1') \\
&\le c\,n^{-1}\,(\log n)\,\mathbf{E}[|X_1|^3]\;\le\; c\,n^{-1}\,\log n,
\end{aligned}
$$

and hence $\mathbf{E}^w[M_T;|S_T|<n]=O(\log^2 n/n)$. Combining these estimates with (26) and (27) gives (28) and therefore (25).

(ii) Let $q=q(n,A)$ be the maximum of $\mathbf{P}^w(\tau_m<T_{A[n/2,n]})$ where the maximum is over all $w\in\mathcal{C}_{4n}$. Let $w=w_n$ be a point obtaining this maximum. Let $\bar\eta_n$ be the first time that a random walk enters $\mathcal{C}_n$ and let $\eta_n^*$ be the first time after this time that the walk leaves $\mathcal{C}_{2n}$. Then by a Markovian argument and an easy overshoot argument we get

$$\mathbf{P}^z(\tau_m<T_{A[n/2,n]};\bar\eta_n<\tau_m)\le\alpha\,q+O(n^{-1}),z\in\mathcal{C}_{4n}$$

where $\alpha=1-c<1$ for $c$ the constant from Lemma 7. The $O(n^{-1})$ error term comes from considering the probability that $|S_{\eta_n^*}|\ge 4n$. By letting $z=w$ we get

$$\mathbf{P}^w(\tau_m<T_{A[n/2,n]})\asymp\mathbf{P}^w(\tau_m<T_{A[n/2,n]},\tau_m<\bar\eta_n)=\mathbf{P}^w(\tau_m<\bar\eta_n)$$

We now show that (i) implies

$$\mathbf{P}^z(\tau_m<\bar\eta_n)\le\frac{c}{\log n},\ \text{ for } z\in\mathcal{C}_{4n}. \tag{29}$$

Namely, by the same argument as in (i), applied to $n/2$ instead of $n$ and $m=n^4$ still, one gets

$$\mathbf{P}^z(\tau_m<\bar\eta_n)\asymp\frac{1}{\log n},\ \text{ for } z\in\mathcal{C}_{2n}\setminus\mathcal{C}_{3n/2}.$$

The uniform upper bound can easily be extended to all $z\in\mathcal{C}_{2n}$ using strong Markov property and overshoot estimate (10). Now for $z\in\mathcal{C}_{4n}\setminus\mathcal{C}_{3n}$ we have

$$\mathbf{P}^z(\tau_m<\bar\eta_n)=\mathbf{P}^z(\tau_m<\eta_n)+\mathbf{P}^z(\eta_n<\tau_m<\bar\eta_n),$$

so that the upper bound in (i) together with strong Markov imply (29) for $z\in\mathcal{C}_{4n}\setminus\mathcal{C}_{3n}$. The remaining case $z\in\mathcal{C}_{3n}\setminus\mathcal{C}_{2n}$ is implied again by strong Markov inequality and an overshoot estimate. $\square$

Recall that we may assume $k\le n/2$.

**Lemma 14** *If $0\le k\le n/2$, and $j\in[k,n]_\kappa$*

$$\mathbf{P}^j(\tau_m<T_{[k,n]_\kappa})\le\frac{c}{\sqrt{n}\log n}\left(\frac{1}{(j-k+1)^{1/2}}+\frac{1}{(n-j+1)^{1/2}}\right).$$

**Proof.** Since $S_0 = j$ the probability of not visiting $[k,n]_\kappa$ during interval $[1, \tau_{2n}]$ is bounded above by a constant times

$$\frac{1}{\sqrt{n}(j-k+1)^{1/2}} + \frac{1}{\sqrt{n}(n-j+1)^{1/2}},$$

due to Proposition 12(d),(e).

Now consider the first time after $\tau_{2n}$ that the random walk either leaves $\mathcal{C}_m$ or enters the disk $\mathcal{C}_n$. Estimate (29) says that the probability of random walk leaving $\mathcal{C}_m$ before entering $\mathcal{C}_n$ is bounded above by $c/\log n$. Hence one expects (also using an easy overshoot argument) $O(\log n)$ "excursions" from $\mathcal{C}_{4n} \setminus \mathcal{C}_{2n}$ into $\mathcal{C}_n$ before leaving $\mathcal{C}_m$, and for each such excursion there is a positive probability, conditioned on the past of the walk, that the random walk visits $[k,n]_\kappa$ during that excursion due to Lemma 7. This gives the extra term $c/\log n$ in the above probability. $\Box$

# 5 Proof of Theorem 1

Without loss of generality we assume $k, n \in \kappa\mathbb{Z}^+$ with $k \leq n/2$. By Lemma 13(i), it suffices to show that

$$\mathbf{P}(\tau_m < T_{A[k,n]}) \leq \frac{c\sqrt{k}}{\sqrt{n}\log n},$$

where, as before, $m = n^4$. The above inequality will then imply

$$\mathbf{P}(\tau_{2n} < T_{A[k,n]}) \leq \frac{\mathbf{P}(\tau_m < T_{A[k,n]})}{\mathbf{P}(\tau_m < T_{A[k,n]} | \tau_{2n} < T_{A[k,n]})} = \frac{c\sqrt{k}}{\sqrt{n}},$$

since by Lemma 13 (i)

$$\mathbf{P}(\tau_m < T_{A[k,n]} | \tau_{2n} < T_{A[k,n]}) \geq \frac{c}{\log n}.$$

Let $T = T_{[k,n]_\kappa}, \hat{T} = T_{A[k,n]}, T^0 = T^0_{[k,n]_\kappa}, \hat{T}^0 = T^0_{A[k,n]}$. Proposition 12(d), Lemma 13(ii), and an easy overshoot estimate give

$$\mathbf{P}(\tau_m < T) \leq \frac{c\sqrt{k}}{\sqrt{n}\log n}.$$

Note that similarly we have

$$\mathbf{P}(\tau_m < T) \geq \frac{c\sqrt{k}}{\sqrt{n}\log n},$$

with a different constant $c > 0$, since Lemma 13(i) is two-sided bound, and the proof of Proposition 12 parts (b) and (d) can be slightly modified to obtain two-sided bound of the same order as the upper bound.

We will show, in fact, that

$$\mathbf{P}(\tau_m < \hat{T}) - \mathbf{P}(\tau_m < T) \leq \frac{c}{\sqrt{n}\log n}.$$

Note that $\mathbf{P}(\tau_m < \hat{T}) - \mathbf{P}(\tau_m < T) = \mathbf{P}(T < \tau_m) - \mathbf{P}(\hat{T} < \tau_m)$ which equals, by (8) (note that under $\mathbf{P}$, $T^0 = T$ and $\hat{T}^0 = \hat{T}$),

$$\sum_{j\in[k,n]_\kappa} G_m(0,j)\mathbf{P}^j(\tau_m < T) - \sum_{j\in[k,n]_\kappa} G_m(0,w_j)\mathbf{P}^{w_j}(\tau_m < \hat{T}) \quad =$$

$$\sum_{j\in[k,n]_\kappa} [G_m(0,j) - G_m(0,w_j)]\mathbf{P}^j(\tau_m < T) \ + \tag{30}$$

$$\sum_{j\in[k,n]_\kappa} G_m(0,w_j)[\mathbf{P}^j(\tau_m < T) - \mathbf{P}^{w_j}(\tau_m < \hat{T})]. \tag{31}$$

We will show that the sum in (30) is bounded above in absolute value by $c/(\sqrt{n}\log n)$ and that the sum in (31) is bounded above by $c/(\sqrt{n}\log n)$. We will not bound the absolute value in (31).

Lemma 4 gives

$$|G_m(0,j) - G_m(0,w_j)| \le \frac{C}{j}. \tag{32}$$

Lemma 14 gives

$$\mathbf{P}^j(\tau_m < T) = O\left(\frac{1}{\sqrt{n}}\left(\frac{1}{(j-k+1)^{1/2}} + \frac{1}{(n-j+1)^{1/2}}\right)\frac{1}{\log n}\right).$$

The term in (30) is therefore bounded in absolute value by

$$\sum_{j\in[k,n]_\kappa} \frac{C}{j}\left[\frac{1}{\sqrt{n}}\left(\frac{1}{(j-k+1)^{1/2}} + \frac{1}{(n-j+1)^{1/2}}\right)\frac{1}{\log n}\right] \le \frac{c}{\sqrt{n}\ \log n}.$$

(Here and below we use the easy estimate:

$$\sum_{j=-\infty}^{\infty} \frac{1}{|j-k|+1}\,\frac{1}{(|j-l|+1)^{1/2}} \le 2\sum_{j=-\infty}^{\infty}\frac{1}{(|j|+1)^{3/2}} < \infty. \quad ) \tag{33}$$

To estimate the term (31) define the function $f$ from $\mathbb{L}$ to $\mathbb{R}$ by

$$f(z) := \sum_{j\in[k,n]_\kappa} G_m(z,w_j)\ [\mathbf{P}^j(\tau_m < T) - \mathbf{P}^{w_j}(\tau_m < \hat{T})],$$

and note that (31) equals $f(0)$. Since $G_m(\cdot,w_j)$ is $p$-harmonic on $\{z : |z| < m\} \setminus \{w_j\}$, $f$ is $p$-harmonic on $\{z : |z| < m\} \setminus A[k,n]$, and therefore it attains its maximum on $\{z : |z| \ge m\} \cup A[k,n]$. However, $f(z) = 0$ for $z \ge m$, so it suffices to show

$$\max_{\ell\in[k,n]_\kappa} f(w_\ell) \le \frac{c}{\sqrt{n}\ \log n}. \tag{34}$$

Fix $\ell \in [k,n]_\kappa$ and note by (8)

$$\sum_{j\in[k,n]_\kappa} G_m(w_\ell,w_j)\mathbf{P}^{w_j}(\tau_m < \hat{T}) = \mathbf{P}^{w_\ell}(\hat{T}^0 < \tau_m) = 1,$$

and

$$\sum_{j\in[k,n]_\kappa} G_m(\ell,j)\mathbf{P}^j(\tau_m < T) = \mathbf{P}^\ell(T^0 < \tau_m) = 1.$$

Hence,

$$\sum_{j\in[k,n]_\kappa} G_m(w_\ell,w_j)\,[\mathbf{P}^j(\tau_m < T) - \mathbf{P}^{w_j}(\tau_m < \hat{T})] =$$

$$\sum_{j\in[k,n]_\kappa} [G_m(w_\ell,w_j) - G_m(\ell,j)]\,\mathbf{P}^j(\tau_m < T). \tag{35}$$

Since $|w_\ell - w_j| \geq |\ell - j| - \kappa$, Lemma 4 gives

$$G_m(w_\ell,w_j) - G_m(\ell,j) \leq \frac{c}{|j-\ell|+1}$$

(note that we are not bounding the absolute value).

Hence,

$$\begin{aligned}
&\sum_{j\in[k,n]_\kappa} (G_m(w_\ell,w_j) - G_m(\ell,j))\mathbf{P}^j(\tau_m < T)\\
\leq\quad &\sum_{j\in[k,n]_\kappa} \frac{1}{\sqrt{n}}\,\frac{1}{|j-\ell|+1}\left(\frac{1}{(j-k+1)^{1/2}} + \frac{1}{(n-j+1)^{1/2}}\right)\frac{1}{\log n}\\
\leq\quad &\frac{c}{\sqrt{n}\log n}. \quad \square
\end{aligned}$$

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## References

[1] R. Bass (1995), *Probabilistic Techniques in Analysis*, Springer-Verlag.

[2] M. Bousquet-Mélou (2001), Walks on the slit plane: other approaches, Adv. Appl. Math **27**, 242–288.

[3] Y. Fukai (2004), Hitting time of a half line by two-dimensional random walk. *Probab. Theo. Rel. Fields*, **128**, 323-346.

[4] Y. Fukai, K. Uchiyama (1996). Potential kernel for two-dimensional random walk, Annals of Probab. **24**, 1979–1992.

[5] H. Kesten (1987) Hitting probabilities of random walks on $\mathbb{Z}^d$, Stoc. Proc. and App;. **25**, 165–184.

[6] G. Lawler (1991), *Intersections of Random Walks*, Birkhäuser.

[7] G. Lawler (1993), A discrete analogue of a theorem of Makarov, Combin. Probab. and Comp. **2**, 181–200.

[8] G. Lawler, E. Puckette (2000). The intersection exponent for simple random walk, Combin. Probab. and Comp. **9**, 441–464.