Electronic Journal of Probability

Vol. 13 (2008), Paper no. 74, pages 2248–2258.

Journal URL
http://www.math.washington.edu/~ejpecp/

Regularity of the density for the stochastic heat equation

Carl Mueller*
Department of Mathematics
University of Rochester
Rochester, NY 15627 USA
email: cmlr@math.rochester.edu

David Nualart†
Department of Mathematics
University of Kansas
Lawrence, Kansas, 66045 USA
email: nualart@math.ku.edu

Abstract

We study the smoothness of the density of a semilinear heat equation with multiplicative space-time white noise. Using Malliavin calculus, we reduce the problem to a question of negative moments of solutions of a linear heat equation with multiplicative white noise. Then we settle this question by proving that solutions to the linear equation have negative moments of all orders.

Key words: heat equation, white noise, Malliavin calculus, stochastic partial differential equations.

AMS 2000 Subject Classification: Primary 60H15; Secondary: 60H07.

Submitted to EJP on September 12, 2008, final version accepted December 11, 2008.

*Supported by NSA and NSF grants.
†Supported by NSF grant DMS-0604207.

1 Introduction

Consider a solution $u(t,x)$ to the one-dimensional stochastic heat equation on $[0,1]$ with Dirichlet boundary conditions $u(t,0)=u(t,1)=0$, driven by a two-parameter white noise, and with initial condition $u(t,x)=u_0(x)$:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+b(t,x,u(t,x))+\sigma(t,x,u(t,x))\frac{\partial^2 W}{\partial t\partial x}. \tag{1}$$

Assume that the coefficients $b(t,x,u),\sigma(t,x,u)$ have linear growth in t,x and are Lipschitz functions of u, uniformly in (t,x).

In [5] Pardoux and Zhang proved that $u(t,x)$ has an absolutely continuous distribution for all (t,x) such that $t>0$ and $x\in(0,1)$, if $\sigma(0,y_0,u_0(y_0))\neq 0$ for some $y_0\in(0,1)$. Bally and Pardoux have studied in [1] the regularity of the law of the solution of Equation (1) with Neumann boundary conditions on $[0,1]$, assuming that the coefficients $b(u)$ and $\sigma(u)$ are infinitely differentiable functions, which are bounded together with their derivatives. They proved that for any $0\le x_1<\cdots<x_d\le 1$, $t>0$, the law of $(u(t,x_1),\dots,u(t,x_d))$ admits a strictly positive infinitely differentiable density on the set $\{\sigma\neq 0\}^d$.

Let $u(t,x)$ be the solution of Equation (1) with Dirichlet boundary conditions on $[0,1]$ and assume that the coefficients b and σ are infinitely differentiable functions of the variable u with bounded derivatives. The aim of this paper is to show that if $\sigma(0,y_0,u_0(y_0))\neq 0$ for some $y_0\in(0,1)$, then $u(t,x)$ has a smooth density for all (t,x) such that $t>0$ and $x\in(0,1)$. Notice that this is exactly the same nondegeneracy condition imposed in [5] to establish the absolute continuity. In order to show this result we make use of a general theorem on the existence of negative moments for the solution of Equation (1) in the case $b(t,x,u)=B(t,x)u$ and $\sigma(t,x,u)=H(t,x)u$, where B and H are some bounded and adapted random fields.

2 Preliminaries

First we define white noise W. Let

$$W=\{W(A), A \text{ a Borel subset of } \mathbb{R}^2, |A|<\infty\}$$

be a Gaussian family of random variables with zero mean and covariance

$$E\left[W(A)W(B)\right]=|A\cap B|,$$

where $|A|$ denotes the Lebesgue measure of a Borel subset of $\mathbb{R}^2$, defined on a complete probability space $(\Omega,\mathscr{F},P)$. Then $W(t,x)=W([0,t]\times[0,x])$ defines a two-parameter Wiener process on $[0,\infty)^2$.

We are interested in Equation (1), and we will assume that u_0 is a continuous function which satisfies the boundary conditions $u_0(0)=u_0(1)=0$. This equation is formal because the partial derivative $\frac{\partial^2 W}{\partial t\partial x}$ does not exist, and (1) is usually replaced by the evolution equation

$$\begin{aligned} u(t,x) &= \int_0^1 G_t(x,y)u_0(y)dy+\int_0^t\int_0^1 G_{t-s}(x,y)b(s,y,u(s,y))dyds \\ &\quad +\int_0^t\int_0^1 G_{t-s}(x,y)\sigma(s,t,u(s,y))W(dy,ds), \end{aligned} \tag{2}$$

where $G_t(x,y)$ is the fundamental solution of the heat equation on $[0,1]$ with Dirichlet boundary conditions. Equation (2) is called the mild form of the equation.

If the coefficients b and σ are have linear growth and are Lipschitz functions of u, uniformly in (t,x), there exists a unique solution of Equation (2) (see Walsh [8]).

The Malliavin calculus is an infinite dimensional calculus on a Gaussian space, which is mainly applied to establish the regularity of the law of nonlinear functionals of the underlying Gaussian process. We will briefly describe the basic criteria for existence and smoothness of densities, and we refer to Nualart [3] for a more complete presentation of this subject.

Let $\mathscr{S}$ denote the class of smooth random variables of the the form

$$F = f(W(A_1),\dots,W(A_n)), \tag{3}$$

where f belongs to $C_p^\infty(\mathbb{R}^n)$ (f and all its partial derivatives have polynomial growth order), and $A_1,\dots,A_n$ are Borel subsets of $\mathbb{R}_+^2$ with finite Lebesgue measure. The derivative of F is the two-parameter stochastic process defined by

$$D_{t,x}F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(A_1),\dots,W(A_n))\mathbf{1}_{A_i}(t,x).$$

In a similar way we define the iterated derivative $D^{(k)}F$. The derivative operator D (resp. its iteration $D^{(k)}$) is a closed operator from $L^p(\Omega)$ into $L^p(\Omega; L^2(\mathbb{R}^2))$ (resp. $L^p(\Omega; L^2(\mathbb{R}^{2k}))$) for any $p>1$. For any $p>1$ and for any positive integer k we denote by $\mathbb{D}^{p,k}$ the completion of $\mathscr{S}$ with respect to the norm

$$\|F\|_{k,p} = \left\{ E(|F|^p) + \sum_{j=1}^k E\left[\left(\int_{\mathbb{R}^{2j}} \left(D_{z_1}\cdots D_{z_j}F \right)^2 dz_1\cdots dz_j \right)^{\frac{p}{2}} \right] \right\}^{\frac{1}{p}}.$$

Set $\mathbb{D}^\infty = \cap_{k,p}\mathbb{D}^{k,p}$.

Suppose that $F = (F^1,\dots,F^d)$ is a d-dimensional random vector whose components are in $\mathbb{D}^{1,2}$. Then, we define the Malliavin matrix of F as the random symmetric nonnegative definite matrix

$$\sigma_F = \left(\left\langle DF^i, DF^j \right\rangle_{L^2(\mathbb{R}^2)} \right)_{1\le i,j\le d}.$$

The basic criteria for the existence and regularity of the density are the following:

Theorem 1. *Suppose that $F=(F^1,\dots,F^d)$ is a d-dimensional random vector whose components are in $\mathbb{D}^{1,2}$. Then,*

1. *If $\det\sigma_F > 0$ almost surely, the law of F is absolutely continuous.*
2. *If $F^i \in \mathbb{D}^\infty$ for each $i=1,\dots,d$ and $E\left[(\det\sigma_F)^{-p}\right] < \infty$ for all $p\ge 1$, then the F has an infinitely differentiable density.*

3 Negative moments

Theorem 2. *Let $u(t,x)$ be the solution to the stochastic heat equation*

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + Bu + Hu\frac{\partial^2 W}{\partial t \partial x}, \\ u(0,x) &= u_0(x) \end{aligned} \tag{4}$$

on $x \in [0,1]$ with Dirichlet boundary conditions. Assume that $B = B(t,x)$ and $H = H(t,x)$ are bounded and adapted processes. Suppose that $u_0(x)$ is a nonnegative continuous function not identically zero. Then,

$$E\left[|u(t,x)|^{-p}\right] < \infty$$

for all $p \geq 2$, $t > 0$ and $0 < x < 1$.

For the proof of this theorem we will make use of the following large deviations lemma, which follows from Proposition A.2, page 530, of Sowers [7]. We remark that the proof of this result holds true if we replace the periodic boundary conditions considered in [7] by Dirichlet boundary conditions, and the integrand is just measurable, adapted and bounded.

Lemma 3. *Let $w(t,x)$ be an adapted stochastic process, bounded in absolute value by a constant M. Let $\epsilon > 0$. Then, there exist constants C_0, $C_1 > 0$ such that for all $\lambda > 0$ and all $T > 0$*

$$P\left(\sup_{0\leq t\leq T}\sup_{0\leq x\leq 1}\left|\int_0^t\int_0^1 G_{t-s}(x,y)w(s,y)W(ds,dy)\right| > \lambda\right) \leq C_0 \exp\left(-\frac{C_1\lambda^2}{T^{\frac{1}{2}-\epsilon}}\right).$$

We also need a comparison theorem such as Corollary 2.4 of [6]; see also Theorem 3.1 of Mueller [4] or Theorem 2.1 of Donati-Martin and Pardoux [2]. Shiga's result is for $x \in \mathbb{R}$, but it can easily be extended to the following lemma, which deals with $x \in [0,1]$ and Dirichlet boundary conditions.

Lemma 4. *Let $u_i(t,x) : i = 1,2$ be two solutions of*

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= \frac{\partial^2 u_i}{\partial x^2} + B_i u_i + Hu_i\frac{\partial^2 W}{\partial t \partial x}, \\ u_i(0,x) &= u_0^{(i)}(x) \end{aligned} \tag{5}$$

where $B_i(t,x), H(t,x), u_0^{(i)}(x)$ satisfy the same conditions as in Theorem 2. Also assume that with probability one for all $t \geq 0, x \in [0,1]$

$$\begin{aligned} B_1(t,x) &\leq B_2(t,x) \\ u_0^{(1)}(x) &\leq u_0^{(2)}(x). \end{aligned}$$

Then with probability 1, for all $t \geq 0, x \in [0,1]$

$$u_1(t,x) \leq u_2(t,x).$$

Proof of Theorem 2. We will construct a process $w(t,x)$ satisfying $0 \le w(t,x) \le u(t,x)$, and bound $E\left[w(t,x)^{-p}\right]$. Since $E\left[u(t,x)^{-p}\right] \le E\left[w(t,x)^{-p}\right]$, this will give us a bound on $E\left[u(t,x)^{-p}\right]$.

Suppose that $|B(t,x)| \le K$ almost surely for some constant $K > 0$. By the comparison lemma, Lemma 4, it suffices to consider the solution to the equation

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial^2 w}{\partial x^2} - Kw + Hw\frac{\partial^2 W}{\partial t \partial x} \\ w(0,x) &= u_0(x)\end{aligned} \tag{6}$$

on $x \in [0,1]$ with Dirichlet boundary conditions. Indeed, the comparison lemma implies that a solution $w(t,x)$ of (6) will be less than or equal to a solution $u(t,x)$ of (4), and $w(t,x) \ge 0$. As mentioned in the previous paragraph, Theorem 2 will follow if we can bound $E\left[w(t,x)^{-p}\right]$.

Set $u(t,x) = e^{-Kt}w(t,x)$, where $u(t,x)$ is not the same as earlier in the paper. Simple calculus shows that $u(t,x)$ satisfies

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + Hu\frac{\partial^2 W}{\partial t \partial x} \\ u(0,x) &= u_0(x)\end{aligned} \tag{7}$$

and we have

$$E\left[w(t,x)^{-p}\right] = e^{Ktp}E\left[u(t,x)^{-p}\right].$$

So, we can assume that $K = 0$, that is that $u(t,x)$ satisfies (7). The mild formulation of Equation (7) is

$$u(t,x) = \int_0^1 G_t(x,y)u_0(y)dy + \int_0^t \int_0^1 G_{t-s}(x,y)H(s,y)u(s,y)W(ds,dy).$$

Suppose that $u_0(x) \ge \delta > 0$ for all $x \in [a,b] \subset (0,1)$. Since (7) is linear, we may divide this equation by δ, and assume $\delta = 1$. We also replace u_0 by $\mathbf{1}_{[a,b]}(x)$, using the comparison lemma. Fix $T > 0$, and consider a larger interval $[a,b] \subset [a_1,b_1] \subset (0,1)$ of the form $b_1 = b + \gamma_2 T$ and $a_1 = a - \gamma_1 T$, where $\gamma_1, \gamma_2 > 0$. We are going to show that $E[(u(T,x)^{-p}] < \infty$ for $x \in [a_1,b_1]$ and for any $p \ge 1$. Define

$$c = \frac{1}{2} \inf_{0\le t+s\le T,\, a-\gamma_1(t+s)\le x\le b+\gamma_2(t+s)} \inf \int_{a-\gamma_1 s}^{b+\gamma_2 s} G_t(x,y)dy$$

and note that $0 < c < 1$ for each $0 < \gamma_1 < \frac{a}{T}$, $0 < \gamma_2 < \frac{1-b}{T}$, and $[a,b] \subset (0,1)$. To see that $c > 0$, note that $G_t(x,y)$ is positive and bounded away from 0 except near $t = 0$. Also, considering the restrictions on x in the infimum, the worst situation for a lower bound on c is when x is at one of the endpoints of the interval under the infimum, say $x = a - \gamma_1(t+s)$. The reader can verify that

$$\inf_{a-\gamma_1 T\le x\le b+\gamma_2 T} \inf_{x+2\sqrt{t}\le T} \int_{x+\sqrt{t}}^{x+2\sqrt{t}} G_t(x,y)dy > 0.$$

However, for small t and for $x = a - \gamma_1(t+s)$ we have that $[x+\sqrt{t}, x+2\sqrt{t}] \subset [a-\gamma_1 s, b+\gamma_2 s]$. This verifies that $c > 0$.

Next we inductively define a sequence $\{\tau_n, n \geq 0\}$ of stopping times and a sequence of processes $v_n(t,x)$ as follows. Let $v_0(t,x)$ be the solution of (7) with initial condition $u_0 = \mathbf{1}_{[a,b]}$ and let

$$\tau_0 = \inf\left\{ t > 0 : \inf_{a-\gamma_1 t \leq x \leq b+\gamma_2 t} v_0(t,x) = c \ \text{ or } \sup_{0\leq x\leq 1} v_0(t,x) = c+1 \right\}.$$

Next, assume that we have defined τ_{n-1} and $v_{n-1}(t,x)$ for $\tau_{n-2} \leq t \leq \tau_{n-1}$. Then, $\{v_n(t,x), \tau_{n-1} \leq t\}$ is defined by (7) with initial condition $v_n(\tau_{n-1}, x) = c^n \mathbf{1}_{[a-\gamma_1\tau_{n-1}, b+\gamma_2\tau_{n-1}]}(x)$. Taking into account that the solution satisfy the Dirichlet boundary conditions, by construction we have $0 < a - \gamma_1\tau_{n-1} < b + \gamma_2\tau_{n-1} < 1$. Also, let

$$\tau_n = \inf\Bigg\{ t > \tau_{n-1} : \inf_{a-\gamma_1 t \leq x \leq b+\gamma_2 t} v_n(t,x) = c^{n+1}$$
$$\text{or } \sup_{0\leq x\leq 1} v_n(t,x) = c^n(c+1) \Bigg\}.$$

Notice that

$$\inf_{a-\gamma_1\tau_n \leq x \leq b+\gamma_2\tau_n} v_n(\tau_n, x) \geq c^{n+1},$$

and by the comparison lemma, we have that

$$u(t,x) \geq v_n(t,x) \tag{8}$$

for all $x \in [0,1]$, $t \geq \tau_{n-1}$ and all $n \geq 0$. For all $p \geq 1$ we have

$$\begin{aligned} E\left[u(T,x)^{-p}\right] &\leq P\Big(u(T,x) \geq 1\Big) \\ &\quad + \sum_{n=0}^{\infty} c^{-(n+1)p} P\left(u(T,x) \in [c^{n+1}, c^n)\right) \\ &\leq 2 + \sum_{n=0}^{\infty} c^{-(n+2)p} P\left(u(T,x) < c^{n+1}\right). \end{aligned} \tag{9}$$

Taking into account (8), the event $\{u(T,x) < c^{n+1}\}$ is included in $\mathscr{A}_n = \{\tau_n < T\}$. Set $\sigma_n = \tau_n - \tau_{n-1}$, for all $n \geq 0$, with the convention $\tau_{-1} = 0$. For any $i \geq 0$ the event $\left\{\sigma_i < \frac{2T}{n}\right\}$ is included into the union $\mathscr{C}_n \cup \mathscr{D}_n$, where

$$\mathscr{C}_n = \left\{ \sup_{\tau_{i-1}\leq t\leq \tau_i} \sup_{0\leq x\leq 1} v_i(t,x) \geq c^i(c+1) \right\}$$

and

$$\mathscr{D}_n = \left\{ \inf_{\tau_{i-1}\leq t\leq \tau_i} \inf_{a-\gamma_1 t\leq x\leq b+\gamma_2 t} v_i(t,x) \leq c^{i+1} \right\}.$$

Notice that, for $\tau_{i-1} < t < \tau_i$ we have

$$c^{-i} v_i(t,x) = \int_{a-\gamma_1\tau_{i-1}}^{b+\gamma_2\tau_{i-1}} G_{t-\tau_{i-1}}(x,y)\,dy$$
$$+ \int_{\tau_{i-1}}^{t} \int_0^1 G_{t-s}(x,y) H(s,y) \left(\left[c^{-i} v_i(s,y)\right] \wedge (c+1) \right) W(ds,dy).$$

By the definition of c it holds that

$$2c \leq \int_{a-\gamma_1\tau_{i-1}}^{b+\gamma_2\tau_{i-1}} G_{t-\tau_{i-1}}(x,y)dy \leq 1.$$

As a consequence, on the events $\mathscr{C}_n$ and $\mathscr{D}_n$ we have

$$\sup_{\tau_{i-1}\leq t\leq\tau_i}\sup_{0\leq x\leq 1}\left|N_i(t,x)\right| > c,$$

where

$$N_i(t,x) = \int_{\tau_{i-1}}^{t}\int_0^1 G_{t-s}(x,y)H(s,y)\left(\left[c^{-i}v_i(s,y)\right]\wedge(c+1)\right)W(ds,dy).$$

Therefore,

$$P\left(\sigma_i < \frac{2T}{n}\middle|\mathscr{F}_{\tau_{i-1}}\right) \leq P\left(\sup_{\tau_{i-1}\leq t\leq\tau_{i-1}+\frac{2T}{n},}\sup_{0\leq x\leq 1}\left|N_i(t,x)\right| > c\middle|\mathscr{F}_{\tau_{i-1}}\right).$$

Then Lemma 3 implies that

$$P\left(\sigma_i < \frac{2T}{n}\middle|\mathscr{F}_{\tau_{i-1}}\right) \leq C_0\exp\left(-C_1 n^{\frac{1}{2}-\epsilon}\right). \tag{10}$$

Next we set up some notation. Let $\mathscr{B}_n$ be the event that at least half of the variables $\sigma_i : i = 0,\ldots,n$ satisfy

$$\sigma_i < \frac{2T}{n}$$

Note that

$$\mathscr{A}_n \subset \mathscr{B}_n$$

since if more than half of the $\sigma_i : i = 1,\ldots,n$ are larger than or equal to $\frac{2T}{n}$ then $\tau_n > T$.
For convenience we assume that $n = 2k$ is even, and leave the odd case to the reader. Let Ξ_n be all the subsets of $\{1,\ldots,n\}$ of cardinality $k = \frac{n}{2}$. The number of such subsets is bounded by the total number of subsets of Ξ_n, which is 2^n. Then,

$$\begin{aligned} P(\mathscr{B}_n) &\leq P\left(\bigcup_{\{i_1,\ldots,i_k\}\in\Xi_n}\bigcap_{j=1}^{k}\left\{\sigma_{i_j} < \frac{2T}{n}\right\}\right) \\ &\leq \sum_{\{i_1,\ldots,i_k\}\in\Xi_n} P\left(\bigcap_{j=1}^{k}\left\{\sigma_{i_j} < \frac{2T}{n}\right\}\right). \end{aligned}$$

We can write for $i_1 < \cdots < i_k$

$$P\left(\bigcap_{j=1}^{k}\left\{\sigma_{i_j} < \frac{2T}{n}\right\}\right) = P\left(\left\{\sigma_{i_1} < \frac{2T}{n}\right\}\right)E\left[\prod_{j=2}^{k}P\left(\sigma_{i_j} < \frac{2T}{n}\middle|\mathscr{F}_{\tau_{i_j-1}}\right)\right].$$

Using the estimate (10) and the fact that we are summing over at most 2^n sets yields

$$\begin{aligned} P(\mathscr{B}_n) &\leq C_0 2^n \exp\left(-C_1 n^{\frac{1}{2}-\varepsilon}\right)^{\frac{n}{2}} \\ &\leq C_0 \exp\left(-C_1 n^{\frac{3}{2}-\varepsilon} + C_2 n\right) \\ &\leq C_0 \exp\left(-C_1 n^{\frac{3}{2}-\varepsilon}\right), \end{aligned}$$

where the constants C_0, C_1 may have changed from line to line. Hence,

$$P\left(u(T,x) < c^{n+1}\right) \leq C_0 \exp\left(-C_1 n^{\frac{3}{2}-\varepsilon}\right) \tag{11}$$

Finally, substituting (11) into (9) yields $E\left[u(T,x)^{-p}\right] < \infty$. □

4 Smoothness of the density

Let $u(t,x)$ be the solution to Equation (1). Assume that the coefficients b and σ are continuously differentiable with bounded derivatives. Then $u(t,x)$ belongs to the Sobolev space $\mathbb{D}^{1,p}$ for all $p > 1$, and the derivative $D_{\theta,\xi}u(t,x)$ satisfies the following evolution equation

$$\begin{aligned} D_{\theta,\xi}u(t,x) &= \int_\theta^t \int_0^1 G_{t-s}(x,y) \frac{\partial b}{\partial u}(s,y,u(s,y)) D_{\theta,\xi}u(s,y)\,dy\,ds \\ &\quad + \int_\theta^t \int_0^1 G_{t-s}(x,y) \frac{\partial \sigma}{\partial u}(s,y,u(s,y)) D_{\theta,\xi}u(s,y) W(dy,ds) \\ &\quad + \sigma(u(\theta,\xi)) G_{t-\theta}(x,\xi), \end{aligned} \tag{12}$$

if $\theta < t$ and $D_{\theta,\xi}u(t,x) = 0$ if $\theta > t$. That is, $D_{\theta,\xi}u(t,x)$ is the solution of the stochastic partial differential equation

$$\frac{\partial D_{\theta,\xi}u}{\partial t} = \frac{\partial^2 D_{\theta,\xi}u}{\partial x^2} + \frac{\partial b}{\partial u}(t,x,u(t,x)) D_{\theta,\xi}u + \frac{\partial \sigma}{\partial u}(t,x,u(t,x)) D_{\theta,\xi}u \frac{\partial^2 W}{\partial t \partial x}$$

on $[\theta,\infty) \times [0,1]$, with Dirichlet boundary conditions and initial condition $\sigma(u(\theta,\xi))\delta_0(x-\xi)$.

Theorem 5. *Let $u(t,x)$ be the solution of Equation (1) with initial condition $u(0,x) = u_0(x)$, and Dirichlet boundary conditions $u(t,0) = u(t,1) = 0$. We will assume that u_0 is an α-Hölder continuous function for some $\alpha > 0$, which satisfies the boundary conditions $u_0(0) = u_0(1) = 0$. Assume that the coefficients b and σ are infinitely differentiable functions with bounded derivatives. Then, if $\sigma(0, y_0, u_0(y_0)) \neq 0$ for some $y_0 \in (0,1)$, $u(t,x)$ has a smooth density for all (t,x) such that $t > 0$ and $x \in (0,1)$.*

Proof. From the results proved by Bally and Pardoux in [1] we know that $u(t,x)$ belongs to the space $\mathbb{D}^\infty$ for all (t,x). Set

$$C_{t,x} = \int_0^t \int_0^1 \left(D_{\theta,\xi}u(t,x)\right)^2 d\xi\, d\theta.$$

Then, by Theorem 1 it suffices to show that $E(C_{t,x}^{-p}) < \infty$ for all $p \geq 2$.

Suppose that $\sigma(0, y_0, u_0(y_0)) > 0$. By continuity we have that $\sigma(0, y, u(0, y)) \geq \delta > 0$ for all $y \in [a, b] \subset (0, 1)$. Then

$$C_{t,x} \geq \int_0^t \int_a^b \left(D_{\theta,\xi} u(t,x)\right)^2 d\xi d\theta \geq \int_0^t \left(\int_a^b D_{\theta,\xi} u(t,x) d\xi\right)^2 d\theta.$$

Set $Y_{t,x}^{\theta} = \int_a^b D_{\theta,\xi} u(t,x) d\xi$. Fix $r < 1$ and $\varepsilon > 0$ such that $\varepsilon^r < t$. From

$$\varepsilon^r \left(Y_{t,x}^0\right)^2 \leq \int_0^{\varepsilon^r} \left|\left(Y_{t,x}^0\right)^2 - \left(Y_{t,x}^{\theta}\right)^2\right| d\theta + C_{t,x}$$

we get

$$\begin{aligned} P(C_{t,x} < \varepsilon) \quad &\leq \quad P\left(\int_0^{\varepsilon^r} \left|\left(Y_{t,x}^0\right)^2 - \left(Y_{t,x}^{\theta}\right)^2\right| d\theta > \varepsilon\right) \\ &\quad + P\left(\left|Y_{t,x}^0\right| < \sqrt{2}\varepsilon^{\frac{1-r}{2}}\right) \\ &= \quad P(A_1) + P(A_2). \end{aligned}$$

Integrating equation (12) in the variable ξ yields the following equation for the process $\{Y_{t,x}^{\theta}, t \geq \theta, x \in [0,1]\}$

$$\begin{aligned} Y_{t,x}^{\theta} \quad &= \quad \int_{\theta}^t \int_0^1 G_{t-s}(x,y) \frac{\partial b}{\partial u}(s, y, u(s,y)) Y_{s,y}^{\theta} dy ds \\ &\quad + \int_{\theta}^t \int_0^1 G_{t-s}(x,y) \frac{\partial \sigma}{\partial u}(s, y, u(s,y)) Y_{s,y}^{\theta} W(dy, ds) \\ &\quad + \int_a^b \sigma(u(\theta,\xi)) G_{t-\theta}(x,\xi) d\xi. \end{aligned} \tag{13}$$

In particular, for $\theta = 0$, the initial condition is $Y_{0,\xi}^0 = \sigma(0, \xi, u(0,\xi)) \mathbf{1}_{[a,b]}(\xi)$, and by Theorem 2 the random variable $Y_{t,x}^0$ has negative moments of all orders. Hence, for all $p \geq 1$,

$$P(A_2) \leq C_{p,r} \varepsilon^p$$

if $\varepsilon \leq \varepsilon_0$. In order to handle the probability $P(A_1)$ we write

$$P(A_1) \leq \varepsilon^{(r-1)q} \sup_{0 \leq \theta \leq \varepsilon^r} \left(E\left[\left|Y_{t,x}^{\theta} - Y_{t,x}^0\right|^{2q}\right] E\left[\left|Y_{t,x}^{\theta} + Y_{t,x}^0\right|^{2q}\right]\right)^{1/2}.$$

We claim that

$$\sup_{0 \leq \theta \leq t} E\left[\left|Y_{x,t}^{\theta}\right|^{2q}\right] < \infty, \tag{14}$$

and

$$\sup_{0 \leq \theta \leq \varepsilon^r} E\left[\left|Y_{t,x}^{\theta} - Y_{t,x}^0\right|^{2q}\right] < \varepsilon^{r\eta q}, \tag{15}$$

for some $\eta > 0$. Property (14) follows easily from Equation (13). On the other hand, the difference $Y^{\theta}_{t,x} - Y^{0}_{t,x}$ satisfies

$$
\begin{aligned}
Y^{\theta}_{t,x} - Y^{0}_{t,x} &= \int_{\theta}^{t}\int_{0}^{1} G_{t-s}(x,y)\frac{\partial b}{\partial u}(s,y,u(s,y))(Y^{\theta}_{s,y} - Y^{0}_{s,x})dyds \\
&+ \int_{\theta}^{t}\int_{0}^{1} G_{t-s}(x,y)\frac{\partial \sigma}{\partial u}(s,y,u(s,y))(Y^{\theta}_{s,y} - Y^{0}_{s,x})W(dy,ds) \\
&+ \int_{0}^{\theta}\int_{0}^{1} G_{t-s}(x,y)\frac{\partial b}{\partial u}(s,y,u(s,y))Y^{0}_{s,y}dyds \\
&+ \int_{0}^{\theta}\int_{0}^{1} G_{t-s}(x,y)\frac{\partial \sigma}{\partial u}(s,y,u(s,y))Y^{0}_{s,y}W(dy,ds) \\
&+ \int_{a}^{b} (\sigma(u(\theta,\xi))G_{t-\theta}(x,\xi) - \sigma(u_0(\xi))G_t(x,\xi))d\xi \\
&= \sum_{i=1}^{5}\Psi_i(\theta).
\end{aligned}
$$

Applying Gronwall's lemma and standard estimates, to show (15) it suffices to prove that

$$\sup_{0\le\theta\le\varepsilon^r} E\left(\left|\Psi_i(\theta)\right|^{2q}\right) < \varepsilon^{r\eta q}, \tag{16}$$

for $i = 3,4,5$ and for some $\eta > 0$. The estimate (16) for $i = 4$ follows from Burkholder's inequality for two-parameter stochastic integrals and (14) as follows,

$$
\begin{aligned}
E\left(\left|\Psi_4(\theta)\right|^{2q}\right) &\le c_q \left\|\frac{\partial \sigma}{\partial u}\right\|_{\infty}^{2q} E\left(\left|\int_0^{\theta}\int_0^1 G^2_{t-s}(x,y)|Y^0_{s,y}|^2 dyds\right|^q\right) \\
&\le c_q \left\|\frac{\partial \sigma}{\partial u}\right\|_{\infty}^{2q} \left(\int_0^{\theta}\int_0^1 G^2_{t-s}(x,y)\left\|Y^0_{s,y}\right\|^2_{L^{2q}} dyds\right)^q \\
&\le C\theta^{\frac{q}{2}}.
\end{aligned}
$$

This implies (16) with $\eta = \frac{1}{2}$. The term $i = 3$ can be estimated in the same way. For $i = 5$ we write

$$\Psi_5(\theta) = \int_a^b G_{t-\theta}(x,y)\int_0^1 G_{\theta}(y,\xi)\left[\sigma(u(\theta,y)) - \sigma(u_0(\xi))\right] d\xi dy. \tag{17}$$

The Hölder continuity of u_0 yields

$$E(|u(\theta,y) - u_0(\xi)|^{2q}) \le C(|\xi - y|^{2\alpha q} + E(|u(\theta,y) - u_0(y)|^{2q}))$$

and we know that $E(|u(\theta,y) - u_0(y)|^{2q})$ can be estimated by $C\theta^{\beta q}$ where $\beta = \inf(\alpha, \frac{1}{2})$ (see [8]). Substituting these estimates in (17) we get the desired bound (16) for $i = 5$.

Finally, it suffices to choose $r \in (0,1)$ such that $r(1 + \frac{\eta}{2}) > 1$. Then for all $q \ge 1$ we get the estimate

$$P(A_1) \le C_q \varepsilon^q,$$

for any $\varepsilon \le \varepsilon_0$. The proof is now complete. □

References

[1] V. Bally and E. Pardoux: Malliavin calculus for white noise driven Parabolic SPDEs. *Potential Analysis* **9**(1998) 27–64.

[2] C. Donati-Martin and E. Pardoux: White noise driven SPDEs with reflection. *Probab. Theory Related Fields* **95** (1993) 1–24.

[3] D. Nualart: *The Malliavin Calculus and related topics.* 2nd edition. Springer-Verlag 2006.

[4] C. Mueller: On the support of solutions to the heat equation with noise. *Stochastics Stochastics Rep.* **37** (1991) 225–245.

[5] E. Pardoux and T. Zhang: Absolute continuity of the law of the solution of a parabolic SPDE. *J. Functional Anal.* **112** (1993) 447–458.

[6] T. Shiga: Two contrasting properties of solutions for one-dimensional stochastic partial differential equations. *Can. J. Math.* **46** (1994) 415–437.

[7] R. Sowers: Large deviations for a reaction-diffusion equation with non-Gaussian perturbations. *Ann. Probab.* **20** (1992) 504–537.

[8] J. B. Walsh: An introduction to stochastic partial differential equations. In: *Ecole d'Ete de Probabilites de Saint Flour XIV,* Lecture Notes in Mathematics **1180** (1986) 265-438.