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Support theorem for a stochastic Cahn-Hilliard equation *

Lijun Bo[1] Kehua Shi[2,†] and Yongjin Wang[3]

[1]Department of Mathematics, Xidian University, Xi'an 710071, China
bolijunnk@yahoo.com.cn
[2]School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
kehuashink@gmail.com
[3]School of Mathematical Sciences, Nankai University, Tianjin 300071, China
yjwang@nankai.edu.cn

Abstract

In this paper, we establish a Stroock-Varadhan support theorem for the global mild solution to a d ($d \leq 3$)-dimensional stochastic Cahn-Hilliard partial differential equation driven by a space-time white noise.

Key words: Stochastic Cahn-Hilliard equation, Space-time white noise, Stroock-Varadhan support theorem.

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†Corresponding author. Email: kehuashink@gmail.com

1 Introduction and main result

In this paper, we consider the following stochastic Cahn-Hilliard equation:

$$\begin{cases} \partial u/\partial t = -\Delta\left[\Delta u + f(u)\right] + \sigma(u)\dot{W}, & \text{in } [0,T]\times D, \\ u(0) = \psi, & \\ \partial u/\partial \mathbf{n} = \partial\left[\Delta u\right]/\partial \mathbf{n} = 0, & \text{on } [0,T]\times\partial D, \end{cases} \tag{1.1}$$

where Δ denotes the Laplace operator, the domain $D = [0,\pi]^d$ $(d = 1,2,3)$, and $f : \mathbf{R} \to \mathbf{R}$ is a polynomial of degree 3 with positive dominant coefficient (which is due to the background of the equation from material science). Assume that $\sigma : \mathbf{R} \to \mathbf{R}$ is a bounded and Lipschitzian function and $\dot{W}$ is a Gaussian space-time white noise on some complete probability space $(\Omega, \mathscr{F}, \mathbf{P})$ satisfying

$$\mathbf{E}\left[\dot{W}(x,t)\dot{W}(y,s)\right] = \delta(|t-s|)\delta(|x-y|), \quad (t,x),(s,y) \in [0,T]\times D.$$

Here $\delta(\cdot)$ is the Dirac delta function concentrated at the point zero.

The (deterministic) Cahn-Hilliard equation (i.e., $\sigma \equiv 0$ in (1.1)) has been extensively studied (see, e.g., [2; 3; 4; 5; 10; 15; 18]) as a well-known model of the macro-phase separation that occurs in an isothermal binary fluid, when a spatially uniform mixture is quenched below a critical temperature at which it becomes unstable. A stochastic version of the Cahn-Hilliard equation (when $\sigma \equiv 1$ in (1.1)) was first proposed by Da Prato and Debussche [8], and the existence, uniqueness and regularity of the global mild solution were explored. In Cardon-Weber [6], the authors considered this type of stochastic equation in a general case on σ, which is equivalent to the following form:

$$\begin{aligned} u(t,x) &= \int_D G_t(x,y)\psi(y)\mathrm{d}y + \int_0^t\int_D \Delta G_{t-s}(x,y)f(u(s,y))\mathrm{d}y\mathrm{d}s \\ &\quad + \int_0^t\int_D G_{t-s}(x,y)\sigma(u(s,y))W(\mathrm{d}y,\mathrm{d}s), \end{aligned} \tag{1.2}$$

where $G_t(\cdot,*)$ denotes the Green kernel corresponding to the operator $\partial/\partial t + \Delta^2$ with the homogeneous Neumann's boundary condition as in (1.1). Since Stroock and Varadhan [17] established their famous support theorem for diffusion processes, there have been many research works on this issue, for example, a variety of support theorems for 1-dimensional second-order parabolic and hyperbolic stochastic partial differential equations (abbr. SPDEs) have been discussed in the literature (see, e.g., [1; 7; 13; 14]). Millet and Sanz-Solé [13] characterized the support of the law of the solution to a class of hyperbolic SPDEs, which simplified the proof in [17]. In Bally et al. [1], the authors proved a support theorem for a semi-linear parabolic SPDE. Moreover, a support result for a generalized Burgers SPDE (containing a quadratic term) was established in Cardon-Weber and Millet [7]. Herein, we are attempting to establish a support theorem of the law corresponding to the solution to Equation (1.1) in $C([0,T], L^p([D]))$ for $p \geq 4$. The main strategy used in this paper is an approximation procedure by using a space-time polygonal interpolation for the white noise, and we particularly adopt a localization argument, which was used in [7] for studying a support theorem of a Burgers-type equation. However here we need more technical estimates concerning the high-order Green kernel $G_t(\cdot,*)$, which sharp the estimates in [6] (see Appendix).

In what follows, we introduce the main result of this paper. To do it, we define the following Cameron-Martin space $\mathscr{H}$ by

$$\begin{aligned}\mathscr{H} &= \Bigg\{h(t,x)=\int_0^t\int_{\prod_{i=1}^d[0,x_i]}\mathbf{h}(s,y)\mathrm{d}y\mathrm{d}s;\ (t,x)=(t,(x_1,\ldots,x_d))\in[0,T]\times D,\\ &\qquad \mathbf{h}\in L^2([0,T]\times D)\Bigg\},\end{aligned}$$

and the corresponding norm by

$$\|h\|_{\mathscr{H}}=\sqrt{\int_0^T\int_D|\mathbf{h}(s,y)|^2\mathrm{d}y\mathrm{d}s},\qquad \text{for all } \ h\in\mathscr{H}.$$

Let $\mathscr{H}_{\mathrm{b}}$ represent the subset of $\mathscr{H}$, in which the first-order derivative $\mathbf{h}$ of $h\in\mathscr{H}$ is bounded. For $h\in\mathscr{H}$, consider the following skeleton equation:

$$\begin{aligned}S(h)(t,x) &= \int_D G_t(x,y)\psi(y)\mathrm{d}y+\int_0^t\int_D\Delta G_{t-s}(x,y)f(S(h)(s,y))\mathrm{d}y\mathrm{d}s\\ &\quad +\int_0^t\int_D G_{t-s}(x,y)\sigma(S(h)(s,y))\mathbf{h}(s,y)\mathrm{d}y\mathrm{d}s. \end{aligned}\tag{1.3}$$

Recall Equations (1.1) and (1.2). We make the following assumptions throughout the paper:

(H1). Assume that $\sigma:\mathbf{R}\to\mathbf{R}$ is bounded and belongs to $C^3(\mathbf{R})$ with bounded first to third-order partial derivatives, and

(H2). The initial function $\psi\in L^p(D)$ for $p\geq 4$, and ψ is $\varrho\in]0,1]$–Hölder continuous.

Now we are at the position to state the main result of this paper.

Theorem 1.1. *Under the assumptions* (H1) *and* (H2)*, let* $u=(u(t,x))_{(t,x)\in[0,T]\times D}$ *be the unique solution to Equation* (1.2) *in* $C([0,T],L^p(D))$ *with* $p\geq 4$ *and* $\mathbf{P}\circ u^{-1}$ *denote the law* (*a probability measure*) *of the solution u. Recall the skeleton equation* (1.3)*, and set* $\mathbb{S}_{\mathscr{H}}=\{S(h);h\in\mathscr{H}\}$*. Then we have*

(a) *Let* $p>6$*. Then for* $\bar{\alpha}\in]0,\min\{\frac{1}{2}(1-\frac{d}{4}),\frac{\varrho}{4}\}[$ *and* $\bar{\beta}\in]0,\min\{2-\frac{d}{2},\varrho\}[$*, the topological support* $\mathrm{supp}(\mathbf{P}\circ u^{-1})$ *in* $C^{\bar{\alpha},\bar{\beta}}([0,T]\times D)$ *of the law* $\mathbf{P}\circ u^{-1}$ *is the closure of* $\mathbb{S}_{\mathscr{H}}$*.*

(b) *Let* $p\geq 4$*. Then for* $\bar{\alpha}\in]0,\min\{\frac{1}{2}(1-\frac{d}{4}),\frac{\varrho}{4}\}[$*, the topological support* $\mathrm{supp}(\mathbf{P}\circ u^{-1})$ *in* $C^{\bar{\alpha}}([0,T],L^p(D))$ *of the law* $\mathbf{P}\circ u^{-1}$ *is the closure of* $\mathbb{S}_{\mathscr{H}}$*.*

The rest of this paper is organized as follows: In the coming section, we give a difference approximation to the $(d+1)$–dimensional space-time white noise $\dot{W}(x,t)$ and study some concrete properties of the approximating noises. In Section 3, we introduce a localization framework as in [7], and then switch to prove the support theorem by checking the conditions (**C1**) and (**C2**) below (see Section 3). Sections 4 and 5 are devoted to checking the validity of the conditions (**C1**) and (**C2**), respectively. In Section 6, we prove the continuity of the solution $S(h)$ to the skeleton equation (1.3) in $\mathscr{H}_{\mathrm{b}}$ and finally we complete the proof of Theorem 1.1.

2 Difference approximation to white noise

In this section, we give a difference approximation to the $(d+1)$-dimensional space-time white noise $\dot{W}$, which is a space-time polygonal interpolation for $\dot{W}$.

Let $n \in \mathbf{N}$ and $t \in [0, T]$, set

$$\underline{t}_n = \max_{j \in \{0,1,\ldots,2^n\}} \left\{ jT2^{-n}; jT2^{-n} \leq t \right\}, \quad \text{and} \;\; t_n = \left[\underline{t}_n - T2^{-n} \right] \vee 0.$$

Let $\mathbf{k} := (k_1, \ldots, k_d) \in \mathbf{I}_n^d := \{0, 1, \ldots, n-1\}^d$. Define a partition $(\triangle_{j,\mathbf{k}})_{j=0,1,\ldots,2^{n-1},\mathbf{k}\in\mathbf{I}_n^d}$ of $\mathscr{O}_T := [0, T] \times D$ by

$$\triangle_{j,\mathbf{k}} = D_{\mathbf{k}} \times]jT2^{-n}, (j+1)T2^{-n}],$$

where $D_{\mathbf{k}} = \prod_{j=1}^d]k_j \pi n^{-1}, (k_j+1)\pi n^{-1}]$. For $x_j \in]k_j \pi n^{-1}, (k_j+1)\pi n^{-1}]$ with $j = 1, \ldots, d$, we set $D_{\mathbf{k}}(x) = \prod_{j=1}^d]k_j \pi n^{-1}, (k_j+1)\pi n^{-1}]$. Further, for each $(t, x) \in \mathscr{O}_T$, we define the following difference approximation to $\dot{W}$ by

$$\dot{W}_n(x,t) = \begin{cases} \frac{W(\triangle_{j-1,\mathbf{k}})}{|\triangle_{j-1,\mathbf{k}}|}, & (x,t) \in \triangle_{j,\mathbf{k}}, \; j = 1, \ldots, 2^n - 1, \; \mathbf{k} \in \mathbf{I}_n^d, \\ 0, & (x,t) \in \triangle_{0,\mathbf{k}}, \; \mathbf{k} \in \mathbf{I}_n^d, \end{cases} \tag{2.1}$$

where $\left|\triangle_{j,\mathbf{k}}\right| = T\pi^d (n^d 2^n)^{-1}$ is the volume of the partition $\triangle_{j,\mathbf{k}}$ for each $j = 0, 1, \ldots, 2^n - 1$ and $\mathbf{k} \in \mathbf{I}_n^d$.

Next we suppose that

(H3). the mappings $F, H, K : \mathbf{R} \to \mathbf{R}$ are bounded, globally Lipschitzian and $H \in C^3(\mathbf{R})$ with bounded first to third-order derivatives.

We now consider the following equations for $h \in \mathscr{H}_b$,

$$\begin{aligned} X_n(t,x) &= G_t * \psi(x) \\ &+ \int_0^t \int_D G_{t-s}(x,y) \left[F(X_n(s,y)) W(\mathrm{d}y, \mathrm{d}s) + H(X_n(s,y)) W_n(\mathrm{d}y, \mathrm{d}s) \right] \\ &+ \int_0^t \int_D G_{t-s}(x,y) \Big[K(X_n(s,y)) \mathbf{h}(s,y) \\ &- \dot{H}(X_n(s,y))[\alpha_n(s,y) F(X_n(s,y)) + \beta_n(s,y) H(X_n(s,y))] \Big] \mathrm{d}y \mathrm{d}s \\ &+ \int_0^t \int_D \Delta_y G_{t-s}(x,y) f(X_n(s,y)) \mathrm{d}y \mathrm{d}s, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} X(t,x) &= G_t * \psi(x) + \int_0^t \int_D G_{t-s}(x,y) [F + H](X(s,y)) W(\mathrm{d}y, \mathrm{d}s) \\ &+ \int_0^t \int_D G_{t-s}(x,y) K(X(s,y)) \mathbf{h}(s,y) \mathrm{d}y \mathrm{d}s \end{aligned}$$

$$+\int_0^t\int_D \Delta_y G_{t-s}(x,y) f(X(s,y))\mathrm{d}y\mathrm{d}s, \tag{2.3}$$

where

$$G_t * \psi(x) := \int_D G_t(x,y)\psi(y)\mathrm{d}y,$$

and for each $n \in \mathbf{N}$,

$$\begin{aligned}
\alpha_n(t,x) &:= n^d 2^n (T\pi^d)^{-1} \int_{t_n}^{\underline{t}_n}\int_{D_{\mathbf{k}}(x)} G_{t-s}(x,y)\mathrm{d}y\mathrm{d}s,\\
\beta_n(t,x) &:= n^d 2^n (T\pi^d)^{-1} \int_{\underline{t}_n}^{t}\int_{D_{\mathbf{k}}(x)} G_{t-s}(x,y)\mathrm{d}y\mathrm{d}s.
\end{aligned}$$

For $\alpha_n(t,x)$ and $\beta_n(t,x)$, by virtue of (A.4) in Lemma A.1, we claim that

$$\sup_{(t,x)\in\mathcal{O}_T} |\alpha_n(t,x)| \leq Cn^d, \text{ and} \tag{2.4}$$

$$\sup_{(t,x)\in\mathcal{O}_T} |\beta_n(t,x)| \leq Cn^d. \tag{2.5}$$

Indeed, using (A.4), we have for each $t \in [0,T]$,

$$\begin{aligned}
\sup_{x\in D}|\alpha_n(t,x)| &\leq Cn^d 2^n \max_{\mathbf{k}\in\mathbf{I}_n^d}\left\{\mathbf{1}_{D_{\mathbf{k}}}(x)\left|\int_{t_n}^{\underline{t}_n}\int_{D_{\mathbf{k}}} |G_{t-s}(x,y)|\mathrm{d}y\mathrm{d}s\right|\right\}\\
&\leq Cn^d 2^n |\underline{t}_n - t_n|\\
&\leq Cn^d,
\end{aligned}$$

and

$$\sup_{x\in D}|\beta_n(t,x)| \leq Cn^d 2^n |t-\underline{t}_n| \leq Cn^d,$$

follows from the equality (A.19).

In the following, let $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ be the natural filtration generated by W, i.e.,

$$\mathcal{F}_t = \sigma\{W(B\times[0,s]); s\in[0,t],\ B\in\mathcal{B}(D)\}.$$

Then for every $t \in [0,T]$ and $n \in \mathbf{N}$ fixed, $(\dot{W}_n(x,t))_{x\in D}$ given by (2.1) is $\mathcal{F}_t$-adapted. More precisely, it is $\mathcal{F}_{\underline{t}_n}$-adapted and which is independent of the information $\mathcal{F}_{t_n}$.

Lemma 2.1. *For each fixed $n \in \mathbf{N}$ and $p \geq 1$, we have*

$$\sup_{(t,x)\in\mathcal{O}_T} \mathbf{E}\left[|\dot{W}_n(x,t)|^p\right] \leq C_p n^{\frac{dp}{2}} 2^{\frac{np}{2}}.$$

Proof. By virtue of the definition (2.1),

$$\begin{aligned}
&\sup_{(t,x)\in\mathcal{O}_T} \mathbf{E}\left[|\dot{W}_n(x,t)|^p\right] \\
= &\sup_{(t,x)\in\mathcal{O}_T} \mathbf{E}\left|\sum_{j=1}^{2^n-1}\sum_{\mathbf{k}\in\mathbf{I}_n^d} \frac{W(\triangle_{j-1,\mathbf{k}})}{|\triangle_{j-1,\mathbf{k}}|}\mathbf{1}_{\triangle_{j-1,\mathbf{k}}}(x,t)\right|^p \\
\le\ & C_p \max\left\{\mathbf{E}\left|\frac{W(\triangle_{j-1,\mathbf{k}})}{|\triangle_{j-1,\mathbf{k}}|}\right|^p;\ j=1,\ldots,2^n-1,\ \mathbf{k}\in\mathbf{I}_n^d\right\}.
\end{aligned}$$

Note that for each $j=1,\ldots,2^n-1$ and $\mathbf{k}\in\mathbf{I}_n^d$,

$$\frac{W(\triangle_{j-1,\mathbf{k}})}{|\triangle_{j-1,\mathbf{k}}|}\sim N(0,|\triangle_{j-1,\mathbf{k}}|^{-1}).$$

For any random variable $Z\sim N(0,\sigma^2)$, it holds that

$$\mathbf{E}|Z|^p=\frac{1}{\sqrt{\pi}}\sqrt{2^p\sigma^{2p}}\ \Gamma\left(\frac{p}{2}+\frac{1}{2}\right),$$

where Γ denotes the Gamma function. This yields that

$$\sup_{(t,x)\in\mathcal{O}_T} \mathbf{E}\left[|\dot{W}_n(x,t)|^p\right]\le C_p\max\left\{\sqrt{|\triangle_{j-1,\mathbf{k}}|^{-p}};\ j=1,\ldots,2^n-1,\ \mathbf{k}\in\mathbf{I}_n^d\right\},$$

and which proves the lemma. □

Let $n\in\mathbf{N}$ be fixed. For $\alpha>0$ and $t\in]0,T]$, we now define an event $\bar{\Omega}^\alpha_{n,t}$ by

$$\bar{\Omega}^\alpha_{n,t}=\left\{\omega\in\Omega;\ \sup_{(s,y)\in[0,t]\times D}\left|\dot{W}(y,s;\omega)\right|\le\alpha n^d 2^{\frac{n}{2}}\right\}. \tag{2.6}$$

For this event, we have

Lemma 2.2. *If choose $\alpha>2\sqrt{\frac{\log 2}{T\pi^d}}$, then*

$$\lim_{n\to\infty}\mathbf{P}\left(\left[\bar{\Omega}^\alpha_{n,T}\right]^c\right)=0.$$

Proof. Let $Z\sim N(0,1)$ be a standard normal random variable. Then according to the definition (2.1) for $\dot{W}_n(x,t)$,

$$\begin{aligned}
\mathbf{P}\left(\left[\bar{\Omega}^\alpha_{n,T}\right]^c\right) &= \mathbf{P}\left(\max_{(j,\mathbf{k})\in\{1,\ldots,2^n-1\}\times\mathbf{I}_n^d}\left\{\frac{W(\triangle_{j-1,\mathbf{k}})}{|\triangle_{j-1,\mathbf{k}}|}\right\}\ge\alpha n^d 2^{\frac{n}{2}}\right) \\
&\le n^d 2^n\mathbf{P}\left(|Z|\ge\alpha\sqrt{T\pi^d}n^{d/2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= n^d 2^n \mathbf{P}\left(\frac{|Z|^2}{4} \geq \frac{\alpha^2 T\pi^d}{4} n^d\right) \\
&\leq n^d 2^n \exp\left[-\frac{\alpha^2 T\pi^d}{4} n^d\right] \mathbf{E}\left[\exp\left(\frac{|Z|^2}{4}\right)\right]. \qquad (2.7)
\end{aligned}
$$

Note that $\mathbf{E}\left[\exp\left(\frac{|Z|^2}{4}\right)\right] = \sqrt{2}$. Then (2.7) further yields that

$$
\begin{aligned}
0 \leq \mathbf{P}\left(\left[\bar{\Omega}^\alpha_{n,T}\right]^c\right) &\leq \sqrt{2} n^d \exp\left[n \log 2 - \frac{\alpha^2 T\pi^d}{4} n^d\right] \\
&\leq \sqrt{2} n^d \exp\left[n^d \left(\log 2 - \frac{\alpha^2 T\pi^d}{4}\right)\right] \\
&\to 0, \quad \text{as } n \to \infty, \qquad (2.8)
\end{aligned}
$$

if $\alpha > 2\sqrt{\frac{\log 2}{T\pi^d}}$. Thus the proof of the lemma is complete. □

3 Localization framework

In this section, we adopt a localization method used in [7] to deal with Equation (1.1). In addition, we will prove a key proposition, which is useful in the proof of Theorem 1.1.

Proposition 3.1. *Under the assumptions* (H1) *and* (H2), *let* $X = (X(t,x))_{(t,x)\in[0,T]\times D}$ *(resp.* X_n*) be the unique solution to Equation* (2.3) *(resp.* (2.2)*) in* $C([0,T], L^p(D))$ *with* $p \geq 4$. *Recall the skeleton equation* (1.3)*, and set* $\mathbb{S}_{\mathscr{H}} = \{S(h); h \in \mathscr{H}\}$. *Then we have*

(i) *Let* $p > 6$. *Then for* $\bar{\alpha} \in]0, \min\{\frac{1}{2}(1-\frac{d}{4}), \frac{\varrho}{4}\}[$ *and* $\bar{\beta} \in]0, \min\{2-\frac{d}{2}, \varrho\}[$, *the sequence* X_n *converges in probability to* X *in* $C^{\bar{\alpha},\bar{\beta}}([0,T]\times D)$.

(ii) *Let* $p \geq 4$. *Then for* $\bar{\alpha} \in]0, \min\{\frac{1}{2}(1-\frac{d}{4}), \frac{\varrho}{4}\}[$, *the sequence* X_n *converges in probability to* X *in* $C^{\bar{\alpha}}([0,T], L^p(D))$.

Next we give a sketch for the proof of the conclusion (ii) in Proposition 3.1. The similar argument can also be used to prove the part (i). For $(t,x) \in \mathscr{O}_T$, set

$$
Y_n(t,x) := X_n(t,x) - X(t,x).
$$

From (2.2) and (2.3), it follows that

$$
Y_n(t,x) = \sum_{i=1}^{3} \Gamma^i_n(t,x) + \Lambda_n(t,x), \qquad (3.1)
$$

where

$$
\begin{aligned}
\Gamma^1_n(t,x) &:= \int_0^t \int_D G_{t-s}(x,y)\left[(F+H)(X_n(s,y)) - (F+H)(X(s,y))\right] W(\mathrm{d}y,\mathrm{d}s), \\
\Gamma^2_n(t,x) &:= \int_0^t \int_D G_{t-s}(x,y)\left[K(X_n(s,y)) - K(X(s,y))\right] \mathbf{h}(s,y)\mathrm{d}y\mathrm{d}s,
\end{aligned}
$$

$$\Gamma_n^3(t,x) := \int_0^t\int_D \Delta_y G_{t-s}(x,y)\left[f(X_n(s,y)) - f(X(s,y))\right] \mathrm{d}y\mathrm{d}s,$$

and

$$\begin{aligned}\Lambda_n(t,x) &:= \int_0^t\int_D G_{t-s}(x,y)H(X_n(s,y))\left[W_n(\mathrm{d}y,\mathrm{d}s) - W(\mathrm{d}y,\mathrm{d}s)\right] \\ &\quad - \int_0^t\int_D G_{t-s}(x,y)\dot{H}(X_n(s,y)) \\ &\quad \times \left[\alpha_n(s,y)F(X_n(s,y)) + \beta_n(s,y)H(X_n(s,y))\right] \mathrm{d}y\mathrm{d}s. \end{aligned} \tag{3.2}$$

Introduce an auxiliary $\mathscr{F}_{t_n}$-adapted process

$$X_n^-(t,x) := G_{t-t_n}\left(x, X_n(t_n,\cdot)\right), \quad \text{for } (t,x)\in\mathscr{O}_T. \tag{3.3}$$

Recall the localization argument adopted in [7]. For $\gamma\in(0,1)$ and $p\geq 4$, define

$$\Phi_n^{p,\gamma}(t) := \sup_{s\in[0,t]}\|Y_n(t,\cdot)\|_p + \sup_{s\neq s'\in[0,t]}\frac{\|Y_n(s,\cdot)-Y_n(s',\cdot)\|_p}{|s-s'|^\gamma},$$

where $\|\cdot\|_p$ corresponds to the norm of $L^p(D)$ and for $\delta>0$,

$$\tau_n^\delta := \inf\left\{t>0;\ \Phi_n^{p,\gamma}(t)\geq\delta\right\}\wedge T.$$

For $M>\delta$, define the following events

$$A_t(M-\delta) := \left\{\omega\in\Omega;\ \sup_{s\in[0,t]}\|X(s,\cdot)\|_p \leq M-\delta\right\}, \tag{3.4}$$

$$A_n^M(t) := \left\{\omega\in\Omega;\ \sup_{s\in[0,t]}\|X_n(s,\cdot)\|_p \vee \sup_{s\in[0,t]}\|X(s,\cdot)\|_p\leq M\right\}. \tag{3.5}$$

Then for $t\in]0,T]$,

$$A_t(M-\delta)\cap\{t\leq\tau_n^\delta\}\subseteq A_n^M(t). \tag{3.6}$$

In fact, from the inequality $|y|\leq|x-y|+|x|$, it follows that

$$\begin{aligned} & A_t(M-\delta)\cap\{t\leq\tau_n^\delta\} \\ \subseteq\ & \left\{\sup_{s\in[0,t]}\|X(s,\cdot)\|_p\leq M-\delta\right\}\cap\left\{\sup_{s\in[0,t]}\|X_n(s,\cdot)-X(s,\cdot)\|_p\leq\delta\right\} \\ \subseteq\ & \left\{\sup_{s\in[0,t]}\|X(s,\cdot)\|_p\leq M\right\}\cap\left\{\sup_{s\in[0,t]}\|X_n(s,\cdot)\|_p\leq M\right\} \\ =\ & A_n^M(t). \end{aligned}$$

Recall the event $\bar{\Omega}_{n,t}^\alpha$ defined by (2.6) in Section 2 and that $\alpha>2\sqrt{\frac{\log 2}{T\pi^d}}$. For each fixed $\delta>0$ and $V\in C^\gamma([0,T];L^p(D))$, set

$$\|V\|_{\gamma,p,\tau_n^\delta} := \sup_{s\in[0,T\wedge\tau_n^\delta]}\|V(s,\cdot)\|_p + \sup_{s\neq s'\in[0,T\wedge\tau_n^\delta]}\frac{\|V(s,\cdot)-V(s',\cdot)\|_p}{|s-s'|^\gamma}.$$

Then for $q \geq 1$,

$$\begin{aligned}
& \mathbf{P}\left(\Phi_n^{p,\gamma}(T) \geq \delta\right) \\
\leq \; & \mathbf{P}\left(\Phi_n^{p,\gamma}(T) \geq \delta,\, A_T(M-\delta) \cap \bar{\Omega}_{n,T}^{\alpha}\right) + \mathbf{P}\left(A_T^c(M-\delta)\right) + \mathbf{P}\left(\left[\bar{\Omega}_{n,T}^{\alpha}\right]^c\right) \\
= \; & \mathbf{P}\left(\|Y_n\|_{\gamma,p,\tau_n^\delta} \geq \delta,\, A_T(M-\delta) \cap \bar{\Omega}_{n,T}^{\alpha}\right) + \mathbf{P}\left(A_T^c(M-\delta)\right) + \mathbf{P}\left(\left[\bar{\Omega}_{n,T}^{\alpha}\right]^c\right) \\
\leq \; & \delta^{-q}\mathbf{E}\left[\mathbf{1}_{A_T(M-\delta)\cap\bar{\Omega}_{n,T}^{\alpha}} \|Y_n\|_{\gamma,p,\tau_n^\delta}^q\right] + \mathbf{P}\left(A_T^c(M-\delta)\right) + \mathbf{P}\left(\left[\bar{\Omega}_{n,T}^{\alpha}\right]^c\right),
\end{aligned}$$

However, Lemma 2.2 and Lemma 3.1 (the latter will be proved below) yield that

$$\mathbf{P}\left(A_T^c(M-\delta)\right) + \mathbf{P}\left(\left[\bar{\Omega}_{n,T}^{\alpha}\right]^c\right) \to 0, \text{ as } M \to \infty,\ n \to \infty.$$

Therefore, by Lemma A.1 in [7] and (3.6), in order to prove

$$\Phi_n^{p,\gamma}(T) \to 0, \quad \text{in probability}, \ \text{ as } n \to \infty, \tag{3.7}$$

it suffices to check that there exist $q \geq p$ and $\theta > q\bar{\alpha}$ (we have set $\gamma = \bar{\alpha}$, where $\bar{\alpha}$ is the exponent presented in Proposition 3.1) such that

$$\begin{aligned}
&(\mathbf{C1}) \quad \forall\ t \in [0,T],\ \lim_{n\to\infty} \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)} \|Y_n(t,\cdot)\|_p^q\right] = 0; \\
&(\mathbf{C2}) \quad \forall\ s < t \in [0,T],\ \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)} \|Y_n(t,\cdot) - Y_n(s,\cdot)\|_p^q\right] \leq C|t-s|^{1+\theta}.
\end{aligned}$$

Here the event $\bar{A}_n^M(t)$ is defined by

$$\bar{A}_n^M(t) = A_n^M(t) \cap \bar{\Omega}_{n,t}^{\alpha}, \qquad t \in [0,T], \tag{3.8}$$

and which satisfies the order relation:

$$\bar{A}_n^M(t) \subset \bar{A}_n^M(r), \ \text{ if } r \leq t.$$

Lemma 3.1. *Let the event $A_T(M)$ be defined by* (3.4). *Then*

$$\lim_{M\to\infty} \mathbf{P}\left(\left[A_T(M)\right]^c\right) = 0.$$

Proof. Note that for $p \geq 4$, $\beta \in [p,\infty[$ and $\beta \in]p, 6p/(6-p)^+[$ (if $d=3$),

$$\mathbf{P}\left(\left[A_T(M)\right]^c\right) = \mathbf{P}\left(\sup_{t\in[0,T]} \|X(t,\cdot)\|_p \geq M\right) \leq M^{-\beta}\mathbf{E}\left[\sup_{t\in[0,T]} \|X(t,\cdot)\|_p^\beta\right].$$

Therefore, it remains to prove

$$\mathbf{E}\left(\sup_{t\in[0,T]} \|X(t,\cdot)\|_p^\beta\right) < \infty. \tag{3.9}$$

Define for $(t,x) \in \mathcal{O}_T$,

$$L_1(u)(t,x) \;:=\; \int_0^t \int_D G_{t-s}(x,y)[F+H](u(s,y))W(\mathrm{d}y,\mathrm{d}s);$$

$$L_2(u)(t,x) \ := \ \int_0^t\int_D G_{t-s}(x,y)K(u(s,y))\mathbf{h}(s,y)\mathrm{d}y\mathrm{d}s,$$

and set $Z = X - L(X)$ with $L = L_1 + L_2$. Then

$$\begin{cases} \frac{\partial Z}{\partial t} + \Delta^2 Z - \Delta f([Z + L(X)]) = 0, \\ Z(0) = \psi, \\ \partial Z/\partial \mathbf{n} = \partial[\Delta Z]/\partial \mathbf{n} = 0, \ \text{ on } \partial D. \end{cases} \tag{3.10}$$

Further, the Garsia-Rodemich-Rumsey lemma (see, e.g., Theorem B.1.1 and Theorem B.1.5 in [9]) yields that, if for any $q, \delta \in]1,\infty[$ and some $\gamma', \gamma'' \in]0,1]$,

$$(\mathbf{a}) \quad \sup_{(t,x)\in\mathcal{O}_T} \mathbf{E}\left[|L(u)(t,x)|^{2q\delta}\right] < \infty,$$

$$(\mathbf{b}) \quad \mathbf{E}\left[|L(u)(t,x) - L(u)(t',x')|^{2q}\right] \le C\left[|t-t'|^{\gamma''} + |x-x'|^{\gamma'}\right]^q, \quad q > 1,$$

then (3.9) holds. So we only need to prove (**a**) and (**b**). Note that K is bounded and $d \le 3$. Then in light of (A.4),

$$\begin{aligned} \sup_{(t,x)\in\mathcal{O}_T} \mathbf{E}\left[\left|L_2(u)(t,x)\right|^{2q\delta}\right] &\le \|K\|_\infty^{2q\delta} \sup_{(t,x)\in\mathcal{O}_T} \left|\int_0^t\int_D |G_{t-s}(x,y)\mathbf{h}(s,y)|\mathrm{d}y\mathrm{d}s\right|^{2q\delta} \\ &\le \|K\|_\infty^{2q\delta}\|h\|_{\mathcal{H}}^{q\delta} T^{(1-\frac{d}{4})q\delta} < \infty. \end{aligned}$$

If set $t > t'$, then by (A.7)–(A.9) in Lemma A.2, we have for $\gamma' \in [0, 1-d/4[$ and $\gamma'' \in [0, 2\wedge(4-d)[$,

$$\mathbf{E}\left[|L_2(u)(t,x) - L_2(u)(t',x')|^{2q}\right] \ \le \ C\left[|t-t'|^{\gamma''} + |x-x'|^{\gamma'}\right]^q.$$

The estimate of L_1 is similar to that of L_2 (or see [6]). Thus the proof the lemma is complete. □

4 Auxiliary lemmas

In this section, we present a sequence of auxiliary lemmas for checking the conditions (**C1**) and (**C2**) under (H1)–(H3) given in Section 1 and 2. Throughout Sections 4–6, (H1)–(H3) are assumed to be satisfied.

The following lemma tells us that, to check (**C1**), it suffices to show (**C1**) holds with $\Lambda_n(t,x)$ instead of $Y_n(t,x)$.

Lemma 4.1. *Assume $p \ge 4$, and $q \ge p$ if $d = 1,2$, and $q \in]p, 6p/(6-p)^+[$ if $d = 3$. Then for each $n \in \mathbf{N}$,*

$$\sup_{t\in[0,T]} \mathbf{E}\left[\mathbf{1}_{\tilde{A}_n^M(t)}\|Y_n(t,\cdot)\|_p^q\right] \le C \sup_{t\in[0,T]} \mathbf{E}\left[\mathbf{1}_{\tilde{A}_n^M(t)}\|\Lambda_n(t,\cdot)\|_p^q\right], \tag{4.1}$$

where $\Lambda_n(t,x)$ is defined by (3.2).

Proof. Note that for each $t \in [0,T]$,

$$\mathbf{E}\left[\|Y_n(t,\cdot)\|_p^q\right] \le C\mathbf{E}\left[\sum_{i=1}^{3} \mathbf{1}_{\bar{A}_n^M(t)}\|\Gamma_n^i(t,\cdot)\|_p^q + \mathbf{1}_{\bar{A}_n^M(t)}\|\Lambda_n(t,\cdot)\|_p^q\right],$$

where $\Gamma^i(t,x)$ $(i=1,2,3)$ are defined in (3.1). Therefore by (A.16) in Lemma A.3, for $\frac{1}{r} = \frac{2}{q} - \frac{2}{p} + 1 \in [0,1]$,

$$\begin{aligned}
& \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)}\|\Gamma_n^1(t,\cdot)\|_p^q\right] \\
\le\ & C\int_D \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)}|\Gamma_n^1(t,x)|^q\right]\mathrm{d}x \\
\le\ & C\mathbf{E}\left[\left\|\int_0^t\int_D G_{t-s}^2(\cdot,y)\mathbf{1}_{\bar{A}_n^M(s)}|Y_n(s,y)|^2\mathrm{d}y\mathrm{d}s\right\|_{\frac{q}{2}}^{\frac{q}{2}}\right] \\
\le\ & C\mathbf{E}\left[\int_0^t (t-s)^{\frac{d}{4r}-\frac{d}{2}}\mathbf{1}_{\bar{A}_n^M(s)}\|Y_n(s,\cdot)\|_p^2\mathrm{d}s\right]^{\frac{q}{2}} \\
\le\ & C\left[\int_0^t (t-s)^{\frac{d}{4r}-\frac{d}{2}}\mathrm{d}s\right]^{\frac{q-2}{2}}\left[\int_0^t (t-s)^{\frac{d}{4r}-\frac{d}{2}}\sup_{r\in[0,s]}\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(r)}\|Y_n(r,\cdot)\|_p^q\right]\mathrm{d}s\right].
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)}\|\Gamma_n^2(t,\cdot)\|_p^q\right] \\
\le\ & C\|h\|_{\mathscr{H}}^q\mathbf{E}\left[\left\|\int_0^t\int_D G_{t-s}^2(\cdot,y)\mathbf{1}_{\bar{A}_n^M(s)}|Y_n(s,y)|^2\mathrm{d}y\mathrm{d}s\right\|_{\frac{q}{2}}^{\frac{q}{2}}\right] \\
\le\ & C\left[\int_0^t (t-s)^{\frac{d}{4r}-\frac{d}{2}}\mathrm{d}s\right]^{\frac{q-2}{2}}\left[\int_0^t (t-s)^{\frac{d}{4r}-\frac{d}{2}}\sup_{r\in[0,s]}\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(r)}\|Y_n(r,\cdot)\|_p^q\right]\mathrm{d}s\right].
\end{aligned}$$

As for $\Gamma_n^3(t,x)$, using (A.13) with $\frac{1}{r_2} = \frac{1}{q} - \frac{1}{\rho} + 1 \in [0,1]$,

$$\begin{aligned}
& \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)}\|\Gamma_n^3(t,\cdot)\|_p^q\right] \\
\le\ & C\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)}\left\|\int_0^t\int_D \Delta_y G(t-s,\cdot,y)[f(X_n(s,\cdot)) - f(X(s,\cdot))]\mathrm{d}y\mathrm{d}s\right\|_q^q\right] \\
\le\ & C\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)}\left[\int_0^t (t-s)^{\frac{d}{4r_2}-\frac{d+2}{4}}\|f(X_n(s,\cdot)) - f(X(s,\cdot))\|_\rho\mathrm{d}s\right]^q\right].
\end{aligned} \tag{4.2}$$

Note that $u(s,\cdot), v(s,\cdot) \in L^p(D)$, for each $s \in [0,T]$, we have for $\rho = \frac{p}{3}$,

$$\begin{aligned}
\|u(s,\cdot) - v(s,\cdot)\|_\rho &\le C\|u(s,\cdot) - v(s,\cdot)\|_p, \\
\|u^2(s,\cdot) - v^2(s,\cdot)\|_\rho &\le C\|u(s,\cdot) - v(s,\cdot)\|_p\left[\|u(s,\cdot)\|_p + \|v(s,\cdot)\|_p\right]
\end{aligned}$$

$$\|u^3(s,\cdot)-v^3(s,\cdot)\|_\rho \le C\|u(s,\cdot)-v(s,\cdot)\|_p \times \left[\|u(s,\cdot)\|_p^2+\|v(s,\cdot)\|_p^2+\|u(s,\cdot)\|_p\|v(s,\cdot)\|_p\right].$$

Hence from (4.2), it follows that

$$\begin{aligned} &\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)}\|\Gamma_n^3(t,\cdot)\|_p^q\right] \\ \le\ & C\mathbf{E}\left[\int_0^t (t-s)^{\frac{d}{4r_2}-\frac{d+2}{4}}\mathbf{1}_{\bar{A}_n^M(s)}\|Y_n(s,\cdot)\|_p \mathrm{d}s\right]^q \\ \le\ & C\left[\int_0^t (t-s)^{\frac{d}{4r_2}-\frac{d+2}{4}}\mathrm{d}s\right]^{q-1}\left[\int_0^t (t-s)^{\frac{d}{4r_2}-\frac{d+2}{4}}\sup_{r\in[0,s]}\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(r)}\|Y_n(r,\cdot)\|_p^q\right]\mathrm{d}s\right]. \end{aligned}$$

Note that the following equivalent relations holds:

$$\frac{d}{4r}-\frac{d}{2}+1>0 \iff \frac{1}{q}>\frac{1}{p}-\frac{1}{6},$$
$$\frac{d}{4r_2}-\frac{d+2}{4}+1>0 \iff \frac{1}{q}>\frac{1}{p}+\frac{1}{2}-\frac{2}{d}.$$

Then the desired result follows from the Gronwall's lemma. □

Recall the $\mathscr{F}_{t_n}$-adapted process $X_n^-(t,x)$ defined by (3.3). Then we have

Lemma 4.2. *Let $q\ge p\ge 6$. Then there exists a constant $C:=C_M>0$ such that*

$$\sup_{(t,x)\in\mathcal{O}_T}\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)}|X_n(t,x)-X_n^-(t,x)|^q\right]\le C2^{-nq\iota}, \tag{4.3}$$

where $\iota:=\frac{1}{2}(1-\frac{d}{4})$.

Proof. Recall (2.2) and (3.3), and we get

$$X_n(t,x)-X_n^-(t,x)=\sum_{k=1}^4 T_n^k(t,x),$$

where

$$\begin{aligned} T_n^1(t,x) &:= \int_{t_n}^t\int_D G_{t-s}(x,y)F(X_n(s,y))W(\mathrm{d}y,\mathrm{d}s), \\ T_n^2(t,x) &:= \int_{t_n}^t\int_D G_{t-s}(x,y)H(X_n(s,y))W_n(\mathrm{d}y,\mathrm{d}s), \\ T_n^3(t,x) &:= \int_{t_n}^t\int_D \Delta_y G_{t-s}(x,y)f(X_n(s,y))\mathrm{d}y\mathrm{d}s, \\ T_n^4(t,x) &:= \int_{t_n}^t\int_D G_{t-s}(x,y)K_n(s,y)\mathrm{d}y\mathrm{d}s, \end{aligned}$$

and

$$\begin{aligned} K_n(s,y) &= K(X_n(s,y))\mathbf{h}(s,y) \\ &\quad -(F\dot{H})(X_n(s,y))\alpha_n(s,y) - (H\dot{H})(X_n(s,y))\beta_n(s,y). \end{aligned}$$

From (2.4) and (2.5), it follows that for $h \in \mathcal{H}_{\mathrm{b}}$,

$$\sup_{(s,y)\in\mathcal{O}_T} |K_n(s,y)| \le Cn^d. \tag{4.4}$$

On the other hand, using (A.4) and the boundedness of F,

$$\begin{aligned} \mathbf{E}\left[|T_n^1(t,x)|^q\right] &= \mathbf{E}\left[\left|\int_{t_n}^t\int_D G_{t-s}(x,y)F(X_n(s,y))W(\mathrm{d}y,\mathrm{d}s)\right|^q\right] \\ &\le C\mathbf{E}\left[\left|\int_{t_n}^t\int_D G_{t-s}^2(x,y)F^2(X_n(s,y))\mathrm{d}y\mathrm{d}s\right|^{q/2}\right] \\ &\le C\left[\int_{t_n}^t\int_D G_{t-s}^2(x,y)\mathrm{d}y\mathrm{d}s\right]^{q/2} \\ &\le C\left[\int_{t_n}^t (t-s)^{-\frac{d}{4}}\mathrm{d}s\right]^{q/2} \\ &\le C2^{-\frac{1}{2}nq(1-\frac{d}{4})}. \end{aligned} \tag{4.5}$$

Further, the Hölder inequality, Lemma 2.1 and the boundedness of H jointly imply that

$$\begin{aligned} &\mathbf{E}\left[|T_n^2(t,x)|^q\right] \\ &= \mathbf{E}\left[\left|\int_{t_n}^t\int_D G_{t-s}(x,y)H(X_n(s,y))W_n(\mathrm{d}y,\mathrm{d}s)\right|^q\right] \\ &\le C\mathbf{E}\left[\left|\int_{t_n}^t\int_D |G_{t-s}(x,y)||\dot{W}_n(y,s)|\mathrm{d}y\mathrm{d}s\right|^q\right] \\ &\le C\mathbf{E}\left[\left[\int_{t_n}^t\int_D |G_{t-s}(x,y)||\dot{W}_n(y,s)|^q\mathrm{d}y\mathrm{d}s\right]\cdot\left[\int_{t_n}^t\int_D |G_{t-s}(x,y)|\mathrm{d}y\mathrm{d}s\right]^{q-1}\right] \\ &\le C\left[\int_{t_n}^t\int_D |G_{t-s}(x,y)|\mathrm{d}y\mathrm{d}s\right]^q \sup_{(t,x)\in\mathcal{O}_T}\mathbf{E}\left[|\dot{W}_n(x,t)|^q\right] \\ &\le Cn^{\frac{dq}{2}}2^{\frac{nq}{2}}|t-t_n|^q \\ &\le Cn^{\frac{dq}{2}}2^{\frac{-nq}{2}}. \end{aligned} \tag{4.6}$$

Note that f is a polynomial of degree 3. Then by virtue of (A.13) with $\kappa = \frac{1}{\infty} - \frac{3}{p} + 1 = 1 - \frac{3}{p} \in [0,1]$,

$$\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)}|T_n^3(t,x)|^q\right]$$

$$
\begin{aligned}
&= \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)}\left|\int_{t_n}^t\int_D \Delta_y G_{t-s}(x,y)f(X_n(s,y))\mathrm{d}y\mathrm{d}s\right|^q\right] \\
&\leq \mathbf{E}\left[\left[\int_{t_n}^t (t-s)^{\frac{d}{4}\kappa-\frac{d+2}{4}}\mathbf{1}_{\bar{A}_n^M(s)}\|f(X_n(s,\cdot))\|_{\frac{p}{3}}\mathrm{d}s\right]^q\right] \\
&\leq \mathbf{E}\left[\left[\int_{t_n}^t (t-s)^{\frac{d}{4}(1-\frac{3}{p})-\frac{d+2}{4}}\mathbf{1}_{\bar{A}_n^M(s)}\left[\|X_n(s,\cdot)\|_p+\|X_n(s,\cdot)\|_p^2+\|X_n(s,\cdot)\|_p^3\right]\mathrm{d}s\right]^q\right] \\
&\leq C\left[\int_{t_n}^t (t-s)^{-(\frac{1}{2}+\frac{3d}{4p})}\mathrm{d}s\right]^q \\
&\leq C2^{-nq(\frac{1}{2}-\frac{3d}{4p})}. \qquad (4.7)
\end{aligned}
$$

Thanks to (4.4), we conclude that

$$
\begin{aligned}
\mathbf{E}\left[|T_n^4(t,x)|^q\right] &= \mathbf{E}\left[\left|\int_{t_n}^t\int_D G_{t-s}(x,y)K_n(s,y)\mathrm{d}y\mathrm{d}s\right|^q\right] \\
&\leq Cn^{dq}\left[\int_{t_n}^t\int_D |G_{t-s}(x,y)|\mathrm{d}y\mathrm{d}s\right]^q \\
&\leq Cn^{dq}|t-t_n|^q \\
&\leq Cn^{dq}2^{-nq}. \qquad (4.8)
\end{aligned}
$$

Thus the estimate (4.3) follows from (4.5)–(4.8). □

Remark 4.1. *We can easily check that there exists a constant $C_M > 0$ such that for $q \geq p \geq 4$,*

$$
\sup_{t\in[0,T]}\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)}\|X_n(t,\cdot)-X_n^-(t,\cdot)\|_p^q\right] \leq C_M 2^{-nq\iota}, \qquad (4.9)
$$

where $\iota = \frac{1}{2}(1-\frac{d}{4})$. In fact, for the proof of (4.9)*, the only estimate that needs to be checked is that of T_n^4. Apply* (A.13) *with $\kappa = \frac{1}{p}-\frac{3}{p}+1 = 1-\frac{2}{p} \in [0,1]$ to T_n^4 and we can get* (4.9).

Next we prove a useful lemma, which will be used frequently later. For $(t,x),(s,y) \in \mathcal{O}_T$, set

$$
\left[\eta(t,s,x,y)\right](\cdot,*) := \mathbf{1}_{[0,t]}(\cdot)G_{t-\cdot}(x,*) - \mathbf{1}_{[0,s]}(\cdot)G_{s-\cdot}(y,*), \qquad (4.10)
$$

$$
\left[\widetilde{\eta}(t,s,x,y)\right](\cdot,*) := \mathbf{1}_{[0,t]}(\cdot)\Delta G_{t-\cdot}(x,*) - \mathbf{1}_{[0,s]}(\cdot)\Delta G_{s-\cdot}(y,*). \qquad (4.11)
$$

Let V be an $\mathbb{F} = (\mathscr{F}_t)_{0\leq t\leq T}$-predictable process. For $0 \leq s \leq t \leq T$ and $x, y \in D$, define

$$
\lambda_n^k(V)(t,s,x,y) := \int_{\frac{kT}{2^n}\wedge t}^{\frac{(k+1)T}{2^n}\wedge t}\int_D \left[\eta(t,s,x,y)\right](r,z)V(r,z)W_n(\mathrm{d}z,\mathrm{d}r), \qquad (4.12)
$$

with $k = 0,1,\ldots,2^n-1$. Then we have

Lemma 4.3. *Let $V : \Omega \times \mathscr{O}_T \to \mathbf{R}$ be an $\mathbb{F}$-predictable process. If for each $p \in [1, \infty[$, there exist some constants $\hat{\alpha} > 0$, $a > 0$ and $C > 0$ such that*

$$\sup_{t\in[0,T]} \left\| \|V(t,\cdot)\|_{L^{r_1 p}(D)} \right\|_{L^{r_1 p}(\Omega)} \leq C2^{-n\hat{\alpha}} n^a, \quad \forall \;\; r_1 \geq 2. \tag{4.13}$$

Then for $\theta < 1 - \frac{d}{4}$, $\theta' < 4 - d$ and $\theta' \leq 2$, there exists $C > 0$ such that for $0 \leq s \leq t \leq T$ and $x, y \in D$,

$$\left\| \sum_{k=0}^{[2^n t/T]-1} \left|\lambda_n^k(V)(t,s,x,y)\right|^2 \right\|_{L^p(\Omega)} \leq Cn^{(d+2a)} 2^{-n\alpha'} \left[|t-s|^{\theta} + |x-y|^{\theta'} \right], \tag{4.14}$$

where $\alpha' = 2\hat{\alpha} - 2 + \frac{1}{p} + \frac{2}{\gamma\beta} > 0$ with $\gamma, \beta \geq 1$ such that $1 < \gamma\beta < 2$.

Proof. As in (4.10), define

$$\lambda_n^k(V)(t,x,y) := \int_{\frac{kT}{2^n}}^{\frac{(k+1)T}{2^n}} \int_D \left[G_{t-r}(x,z) - G_{t-r}(y,z)\right] V(r,z) W_n(\mathrm{d}z, \mathrm{d}r). \tag{4.15}$$

Applying Hölder inequality thrice with the respective exponents $(2p\delta, \gamma)$, (δ, δ') and $(\beta, \frac{2p\delta'}{\gamma})$. Then by (4.13) and Lemma 2.1,

$$\begin{aligned}
&\mathbf{E}\left[\left|\lambda_n^k(V)(t,x,y)\right|^{2p}\right] \\
= \;& \mathbf{E}\left[\left|\int_{\frac{T}{2^n}k}^{\frac{T}{2^n}(k+1)} \int_D [G_{t-r}(x,z) - G_{t-r}(y,z)] V(r,z) \dot{W}_n(z,r) \mathrm{d}z\mathrm{d}r\right|^{2p}\right] \\
\leq \;& \mathbf{E}\Bigg[\left[\int_{\frac{T}{2^n}k}^{\frac{T}{2^n}(k+1)} \int_D |V(r,z)|^{2p\delta} \mathrm{d}z\mathrm{d}r\right]^{\frac{1}{\delta}} \\
& \times \left[\int_{\frac{T}{2^n}k}^{\frac{T}{2^n}(k+1)} \int_D |\dot{W}_n(z,r)|^{\gamma} |G_{t-r}(x,z) - G_{t-r}(y,z)|^{\gamma} \mathrm{d}z\mathrm{d}r\right]^{\frac{2p}{\gamma}} \Bigg] \\
\leq \;& \left[\mathbf{E}\int_{\frac{T}{2^n}k}^{\frac{T}{2^n}(k+1)} \int_D |V(r,z)|^{2p\delta} \mathrm{d}z\mathrm{d}r\right]^{\frac{1}{\delta}} \Bigg[\mathbf{E}\Big[\int_{\frac{T}{2^n}k}^{\frac{T}{2^n}(k+1)} \int_D |\dot{W}_n(z,r)|^{\gamma} \\
& \times |G_{t-r}(x,z) - G_{t-r}(y,z)|^{\gamma} \mathrm{d}z\mathrm{d}r\Big]^{\frac{2p\delta'}{\gamma}}\Bigg]^{\frac{1}{\delta'}} \\
\leq \;& \left[\mathbf{E}\int_{\frac{T}{2^n}k}^{\frac{T}{2^n}(k+1)} \int_D |V(r,z)|^{2p\delta} \mathrm{d}z\mathrm{d}r\right]^{\frac{1}{\delta}} \Bigg[\mathbf{E}\left[\int_{\frac{T}{2^n}k}^{\frac{T}{2^n}(k+1)} \int_D |\dot{W}_n(z,r)|^{2p\delta'} \mathrm{d}z\mathrm{d}r\right] \\
& \times \left[\int_{\frac{T}{2^n}k}^{\frac{T}{2^n}(k+1)} \int_D |G_{t-r}(x,z) - G_{t-r}(y,z)|^{\gamma\beta} \mathrm{d}z\mathrm{d}r\right]^{\frac{2p\delta'}{\gamma\beta}}\Bigg]^{\frac{1}{\delta'}}
\end{aligned}$$

$$\leq \quad Cn^{dp+2ap}2^{np-2np\hat{\alpha}-n}\left[\int_{\frac{T}{2^n}k}^{\frac{T}{2^n}(k+1)}\int_D |G_{t-r}(x,z)-G_{t-r}(y,z)|^{\gamma\beta}\mathrm{d}z\mathrm{d}r\right]^{\frac{2p}{\gamma\beta}}. \tag{4.16}$$

In light of (A.24) in Lemma A.4, there exists a $\theta' \in]0,4-d[$ and $\theta' < 2$ such that

$$\left\|\sum_{k=0}^{[2^n t/T]-1}\left|\lambda_n^k(V)(t,x,y)\right|^2\right\|_{L^p(\Omega)} \leq Cn^{d+2a}2^{-n(2\hat{\alpha}-2+\frac{2}{\gamma\beta}+\frac{1}{p})}|x-y|^{\theta'}. \tag{4.17}$$

Next we turn to the time increment. Set

$$\lambda_n^k(V)(t,s,x) := \int_{\frac{kT}{2^n}\wedge s}^{\frac{(k+1)T}{2^n}\wedge t}\int_D [G_{t-r}(x,z)-G_{s-r}(x,z)]V(r,z)W_n(\mathrm{d}z,\mathrm{d}r). \tag{4.18}$$

Then for $0 \leq s \leq t \leq T$ and $x \in D$,

$$\begin{aligned}
&\left|\lambda_n^k(V)(t,s,x)\right| \\
\leq\ &\left|\int_{\frac{Tk}{2^n}\wedge s}^{\frac{T}{2^n}(k+1)\wedge s}\int_D [G_{t-r}(x,z)-G_{s-r}(x,z)]V(r,z)\dot{W}_n(z,r)\mathrm{d}z\mathrm{d}r\right| \\
&+\left|\int_{\frac{Tk}{2^n}\vee s}^{\frac{T}{2^n}(k+1)\wedge t}\int_D G_{t-r}(x,z)V(r,z)\dot{W}_n(z,r)\mathrm{d}z\mathrm{d}r\right| \\
=:\ & A_{1,n}^k(V)(t,s,x)+A_{2,n}^k(V)(t,s,x),
\end{aligned} \tag{4.19}$$

with the definition

$$A_{2,n}^k(V)(t,s,x)=0, \quad \text{whenever} \quad [(k+1)T2^{-n}\wedge t] < [kT2^{-n}\vee s].$$

By the similar proof to that of the space increment, there exists a $\theta \in]0,1-\frac{d}{4}]$ such that

$$\left\|\sum_{k=0}^{[2^n t/T]-1}\left|A_{1,n}^k(V)(t,s,x)\right|^2\right\|_{L^p(\Omega)} \leq Cn^{d+2a}2^{-n(2\hat{\alpha}-2+\frac{1}{p}+\frac{2}{\gamma\beta})}|t-s|^{\theta}. \tag{4.20}$$

Using the proof of (4.15), we have for each $k=0,1,\ldots,[2^n t/T]$,

$$\begin{aligned}
&\mathbf{E}\left[|A_{2,n}^k(V)(t,s,x)|^{2p}\right] \\
\leq\ & Cn^{(d+2a)p}2^{-np(2\hat{\alpha}-1+\frac{1}{p})}\left[\int_{\frac{Tk}{2^n}\vee s}^{\frac{T}{2^n}(k+1)\wedge t}\int_D |G_{t-r}(x,z)|^{\gamma\beta}\mathrm{d}z\mathrm{d}r\right]^{\frac{2p}{\gamma\beta}}.
\end{aligned} \tag{4.21}$$

Then from (A.4) and the Hölder inequality, it follows that

$$\left\|\sum_{k=0}^{[2^n t/T]-1}\left|A_{2,n}^k(V)(t,s,x)\right|^2\right\|_{L^p(\Omega)}$$

$$
\begin{aligned}
&\leq \quad Cn^{d+2a}2^{-n(2\hat{\alpha}-1+\frac{1}{p})}\left[\int_{\frac{Tk}{2^n}\vee s}^{\frac{T}{2^n}(k+1)\wedge t}(t-r)^{-\frac{d}{4}(\gamma\beta-1)}\mathrm{d}r\right]^{\frac{2}{\gamma\beta}} \\
&\leq \quad Cn^{d+2a}2^{-n(2\hat{\alpha}-1+\frac{1}{p})}2^{-n(\frac{2}{\gamma\beta}-1)}\left[\int_s^t(t-r)^{-\frac{d}{4}(\gamma\beta-1)}\mathrm{d}r\right] \\
&\leq \quad Cn^{d+2a}2^{-n(2\hat{\alpha}-2+\frac{1}{p}+\frac{2}{\gamma\beta})}|t-s|^{-\frac{d}{4}(\gamma\beta-1)+1}. \qquad (4.22)
\end{aligned}
$$

Thus from (4.17), (4.20) and (4.22), we get (4.14). Hence the proof of the lemma is complete. □

Remark 4.2. *From the proof of Lemma* 4.3*, the conclusion of the lemma still holds if we replace* $\lambda_n^k(V)(t,s,x,y)$ *by* $\mathbf{E}\left[\lambda_n^k(V)(t,s,x,y)|\mathscr{F}_{\frac{(k-1)T}{2^n}}\right]$.

Lemma 4.4. *For* $(s,y)\in\mathscr{O}_T$*, it holds that*

$$
\begin{aligned}
&\mathbf{E}\left[\dot{W}_n(y,s)\int_{s_n}^s\int_D G_{s-r}(y,z)F(X_n^-(r,z))W(\mathrm{d}z,\mathrm{d}r)|\mathscr{F}_{s_n}\right] \\
=\quad &(T\pi^d)^{-1}n^d2^n\int_{s_n}^{\underline{s}_n}\int_{D_{\mathbf{k}}(y)} G_{s-r}(y,z)F(X_n^-(r,z))\mathrm{d}z\mathrm{d}r.
\end{aligned}
$$

Proof. For $u\geq s_n$, set

$$
\begin{aligned}
N_u(s,y) &:= (T\pi^d)^{-1}n^d2^nW(D_{\mathbf{k}}(y)\times]s_n,u]), \\
M_u(s,y) &:= \int_{s_n}^u\int_D G_{s-r}(y,z)F(X_n^-(r,z))W(\mathrm{d}z,\mathrm{d}r).
\end{aligned}
$$

Then the martingale property of $\{M_u(s,y);u\geq s_n\}$ and Itô formula jointly imply that

$$
\begin{aligned}
&\mathbf{E}\left[\dot{W}_n(y,s)\int_{s_n}^s\int_D G_{s-r}(y,z)F(X_n^-(r,z))W(\mathrm{d}z,\mathrm{d}r)|\mathscr{F}_{s_n}\right] \\
=\quad &(T\pi^d)^{-1}n^d2^n\mathbf{E}\left[N_{\underline{s}_n}(s,y)M_s(s,y)|\mathscr{F}_{s_n}\right] \\
=\quad &(T\pi^d)^{-1}n^d2^n\mathbf{E}\left[\mathbf{E}\left[N_{\underline{s}_n}(s,y)M_s(s,y)|\mathscr{F}_{\underline{s}_n}\right]\mathscr{F}_{s_n}\right] \\
=\quad &(T\pi^d)^{-1}n^d2^n\mathbf{E}\left[N_{\underline{s}_n}(s,y)\mathbf{E}[M_s(s,y)|\mathscr{F}_{\underline{s}_n}]\mathscr{F}_{s_n}\right] \\
=\quad &(T\pi^d)^{-1}n^d2^n\mathbf{E}\left[N_{\underline{s}_n}(s,y)M_{\underline{s}_n}(s,y)|\mathscr{F}_{s_n}\right] \\
=\quad &(T\pi^d)^{-1}n^d2^n\int_{s_n}^{\underline{s}_n}\int_{D_{\mathbf{k}}(y)} G_{s-r}(y,z)F(X_n^-(r,z))\mathrm{d}z\mathrm{d}r,
\end{aligned}
$$

follows from the fact that $F(X_n^-(r,z))$ is $\mathscr{F}_{s_n}$-measurable when $r\leq s$. This proves the lemma. □

The following lemma shows the local Hölder continuity of $\mathbb{F}$-adapted process $X_n(t,x)$ defined by (2.2). Recall the assumption (H2), in which the initial function ψ is ϱ-Hölder continuous ($\varrho\in\,]0,1]$).

Lemma 4.5. *Let $q \geq p \geq 6$. Then for $\bar{\alpha} \in]0, \min\{\frac{1}{2}(1-\frac{d}{4}), \frac{\varrho}{4}\}[$ and $\bar{\beta} \in]0, \min\{2-\frac{d}{2}, \varrho\}[$, there exists a $C > 0$ such that*

$$\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)}\left|X_n(t,x)-X_n(s,y)\right|^q\right] \leq Cn^{dq}\left[|t-s|^{\bar{\alpha}}+|x-y|^{\bar{\beta}}\right]^q,$$

with $(t,x),(s,y) \in \mathcal{O}_T$. In addition, if $q \geq p \geq 4$, then for $\bar{\alpha} \in]0, \min\{\frac{1}{2}(1-\frac{d}{4}), \frac{\varrho}{4}\}[$, there exists a $C > 0$ such that

$$\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)}\left\|X_n(t,\cdot)-X_n(s,\cdot)\right\|_p^q\right] \leq Cn^{dq}|t-s|^{\bar{\alpha}q}.$$

Proof. Recall (2.2), (4.11) and (4.12). We have for $(t,x),(s,y) \in \mathcal{O}_T$,

$$X_n(t,x)-X_n(s,y) = \sum_{i=1}^{5} J_i(s,t,x,y),$$

where

$$\begin{aligned}
J_1(t,s,x,y) &:= G_t * \psi(x) - G_s * \psi(y),\\
J_2(t,s,x,y) &:= \int_0^t\int_D [\eta(t,s,x,y)](r,z)\Big[F(X_n(r,z))W(\mathrm{d}z,\mathrm{d}r)\\
&\quad +H(X_n^-(r,z))W_n(\mathrm{d}z,\mathrm{d}r)\Big],\\
J_3(t,s,x,y) &:= \int_0^t\int_D [\eta(t,s,x,y)](r,z)\Big[(H(X_n(r,z))-H(X_n^-(r,z)))W_n(\mathrm{d}z,\mathrm{d}r)\\
&\quad -\dot{H}(X_n(r,z))\left[\alpha_n(r,z)F(X_n(r,z))+\beta_n(r,z)H(X_n(r,z))\right]\mathrm{d}z\mathrm{d}r\Big],\\
J_4(t,s,x,y) &:= \int_0^t\int_D [\eta(t,s,x,y)](r,z)K(X_n(r,z))\dot{h}(r,z)\mathrm{d}z\mathrm{d}r,\\
J_5(t,s,x,y) &:= \int_0^t\int_D \left[\tilde{\eta}(t,s,x,y)\right](r,z)f(X_n(r,z))\mathrm{d}z\mathrm{d}r.
\end{aligned}$$

Note that the initial function $\psi(x)$ is ϱ-Hölder continuous in $x \in \mathbf{R}^d$. Then by Lemma 2.2 in [6],

$$\mathbf{E}\left[\left|J_1(s,t,x,y)\right|^q\right] \leq C\left[|t-s|^{\varrho/4}+|x-y|^{\varrho}\right]^q. \tag{4.23}$$

Since F, H are bounded, by Burkhölder's inequality and Lemma A.2, there exist $\gamma' \in]0, 4-d[$, $\gamma' \leq 2$ and $\gamma'' \in]0, 1-d/4[$ such that

$$\mathbf{E}\left[|J_2(t,s,x,y)|^q\right] \leq C\left[|x-y|^{\gamma'}+|t-s|^{\gamma''}\right]^{\frac{q}{2}}. \tag{4.24}$$

On the other hand, since K is bounded, the Schwarz's inequality yields that for $h \in \mathcal{H}_{\mathrm{b}}$,

$$\mathbf{E}\left[|J_4(t,s,x,y)|^q\right] \leq C\left[|x-y|^{\gamma'}+|t-s|^{\gamma''}\right]^{\frac{q}{2}}, \tag{4.25}$$

where γ', γ'' are presented in (4.24). Similar as in the proof of (4.7), by (A.14) and (A.15) with $\frac{1}{r} = \frac{1}{\infty} - \frac{3}{p} + 1 \in [0,1]$, we have for $\alpha \in]0, 1/2 - \frac{3d}{4p}[$ and $\beta \in]0, \min\{2-\frac{3d}{p}, \frac{2}{d}, 1\}]$,

$$\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)}|J_5(t,s,x,y)|^q\right] \leq C\left[|x-y|^{\beta}+|t-s|^{\alpha}\right]^q. \tag{4.26}$$

and using (A.14) with $\kappa = \frac{1}{p} - \frac{3}{p} + 1 = 1 - \frac{2}{p} \in [0,1]$, for $\alpha \in]0, 1/2 - \frac{d}{2p}[$,

$$\mathbf{E}\left[\mathbf{1}_{\tilde{A}_n^M(t)\cap\tilde{A}_n^M(s)}\|J_5(t,s,\cdot,\cdot)\|_p^q\right] \le C\left[|t-s|^{\alpha}\right]^q. \tag{4.27}$$

In the following, we are going to estimate $J_3(t,s,x,y)$. Using the Taylor expansion of H at point $X_n^-(t,x)$,

$$H(X_n(t,x)) - H(X_n^-(t,x)) = \dot{H}(X_n^-(t,x))\left[X_n(t,x) - X_n^-(t,x)\right] + \rho_n(t,x), \tag{4.28}$$

and

$$\rho_n(t,x) \le C|X_n(t,x) - X_n^-(t,x)|^2.$$

Recall the above J_3, and we have

$$J_3(t,s,x,y) = \sum_{i=1}^{5} J_3^i(t,s,x,y),$$

where

$$\begin{aligned}
J_3^1(t,s,x,y) \quad &:= \quad \int_0^t\int_D [\eta(t,s,x,y)](r,z)\Big[T_n^1(r,z)\dot{H}(X_n^-(r,z))\dot{W}_n(z,r) \\
&\qquad -\dot{H}(X_n(r,z))\alpha_n(r,z)F(X_n(r,z))\Big]\mathrm{d}z\mathrm{d}r, \\
J_3^2(t,s,x,y) \quad &:= \quad \int_0^t\int_D [\eta(t,s,x,y)](r,z)\Big[T_n^2(r,z)\dot{H}(X_n^-(r,z))\dot{W}_n(z,r) \\
&\qquad -\dot{H}(X_n(r,z))\beta_n(r,z)F(X_n(r,z))\Big]\mathrm{d}z\mathrm{d}r, \\
J_3^3(t,s,x,y) \quad &:= \quad \int_0^t\int_D [\eta(t,s,x,y)](r,z)\dot{H}(X_n^-(r,z))T_n^3(r,z)\dot{W}_n(z,r)\mathrm{d}z\mathrm{d}r, \\
J_3^4(t,s,x,y) \quad &:= \quad \int_0^t\int_D [\eta(t,s,x,y)](r,z)\dot{H}(X_n^-(r,z))T_n^4(r,z)\dot{W}_n(z,r)\mathrm{d}z\mathrm{d}r, \\
J_3^5(t,s,x,y) \quad &:= \quad \int_0^t\int_D [\eta(t,s,x,y)](r,z)\rho_n(r,z)\dot{W}_n(z,r)\mathrm{d}z\mathrm{d}r.
\end{aligned}$$

Next we decompose $J_3^1(t,s,x,y)$ as follows:

$$J_3^1(t,s,x,y) = \sum_{j=1}^{5} J_3^{1,j}(t,s,x,y),$$

with

$$\begin{aligned}
J_3^{1,1}(t,s,x,y) \quad &= \quad \int_0^t\int_D [\eta(t,s,x,y)](r,z)\dot{H}(X_n^-(r,z))\dot{W}_n(z,r) \\
&\qquad \times\left[\int_{r_n}^r\int_D G_{r-u}(z,v)\left[F(X_n(u,v)) - F(X_n^-(u,v))\right]W(\mathrm{d}v,\mathrm{d}u)\right]\mathrm{d}z\mathrm{d}r,
\end{aligned}$$

$$
\begin{aligned}
J_3^{1,2}(t,s,x,y) &= \int_0^t\int_D [\eta(t,s,x,y)](r,z)\dot{H}(X_n^-(r,z)) \\
&\times\left[\int_{r_n}^r\int_D G_{r-u}(z,v)F(X_n^-(r,z))W(\mathrm{d}v,\mathrm{d}u)\dot{W}_n(z,r)\right. \\
&\left.-\mathbf{E}\left[\int_{r_n}^r\int_D G_{r-u}(z,v)F(X_n^-(u,v))W(\mathrm{d}v,\mathrm{d}u)\dot{W}_n(z,r)|\mathscr{F}_{r_n}\right]\right]\mathrm{d}z\mathrm{d}r \\
J_3^{1,3}(t,s,x,y) &= \int_0^t\int_D [\eta(t,s,x,y)](r,z)\dot{H}(X_n^-(r,z)) \\
&\times\left[\mathbf{E}\left[\int_{r_n}^r\int_D G_{r-u}(z,v)F(X_n^-(u,v))W(\mathrm{d}v,\mathrm{d}u)\dot{W}_n(z,r)|\mathscr{F}_{r_n}\right]\right. \\
&\left.-2^n n^{-d}T^{-1}\pi^{-d}\int_{r_n}^{\underline{r}_n}\int_{D_{\mathbf{k}}(z)} G_{r-u}(z,v)F(X_n^-(u,v))\mathrm{d}v\mathrm{d}u\right]\mathrm{d}z\mathrm{d}r, \\
J_3^{1,4}(t,s,x,y) &= 2^n T^{-1}n^{-d}\pi^{-d}\int_0^t\int_D [\eta(t,s,x,y)](r,z)\dot{H}(X_n^-(r,z)) \\
&\times\left[\int_{r_n}^{\underline{r}_n}\int_{D_{\mathbf{k}}(z)} G_{r-u}(z,v)F(X_n^-(u,v))\mathrm{d}v\mathrm{d}u-\alpha_n(r,z)F(X_n(r,z))\right]\mathrm{d}z\mathrm{d}r, \\
J_3^{1,5}(t,s,x,y) &= \int_0^t\int_D [\eta(t,s,x,y)](r,z)\left[\dot{H}(X_n^-(r,z))-\dot{H}(X_n(r,x))\right] \\
&\times\alpha_n(r,z)F(X_n(r,z))\mathrm{d}z\mathrm{d}r.
\end{aligned}
$$

We first estimate the term $J_3^{1,1}$. Define

$$
\begin{aligned}
V(r,z) &= \mathbf{1}_{\tilde{A}_n^M(r)}\dot{H}(X_n^-(r,z)) \\
&\times\int_{r_n}^r\int_D G_{r-u}(z,v)\left[F(X_n(u,v))-F(X_n^-(u,v))\right]W(\mathrm{d}v,\mathrm{d}u). \qquad (4.29)
\end{aligned}
$$

Then

$$
\begin{aligned}
J_3^{1,1}(t,s,x,y) &= \int_0^t\int_D [\eta(t,s,x,y)](r,z)V(r,z)\dot{W}_n(\mathrm{d}z,\mathrm{d}r) \\
&= \sum_{k=0}^{[2^n t/T]-1}\lambda_n^k(V)(t,s,x,y), \qquad (4.30)
\end{aligned}
$$

where $\lambda_n^k(V)(t,s,x,y)$ is defined by (4.12). By virtue of the Burkhölder inequality, (A.16) with $\kappa=\frac{2}{q}-\frac{2}{q}+1=1$ and Lemma 4.2,

$$
\begin{aligned}
\mathbf{E}\left[\|V(r,\cdot)\|_q^q\right] &\le C2^{-nq(1-\frac{d}{4})}\sup_{r\in[0,T]}\mathbf{E}\left[\mathbf{1}_{\tilde{A}_n^M(r)}\|X_n(r,\cdot)-X_n^-(r,\cdot)\|_q^q\right] \\
&\le C2^{-nq(1-\frac{d}{4}+\sigma)}. \qquad (4.31)
\end{aligned}
$$

Then using Lemma 4.3, there exist $\theta' < 4-d$ and $\theta' \le 2$, $\theta < 1-\frac{d}{4}$ such that

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{1}_{\bar{A}^n_M(t)\cap\bar{A}^n_M(s)}\left|J_3^{1,1}(t,s,x,y)\right|^q\right]\\
\le\ & 2^{\frac{nq}{2}}\mathbf{E}\left[\mathbf{1}_{\bar{A}^n_M(t)\cap\bar{A}^n_M(s)}\left[\sum_{k=0}^{[2^n t/T]-1}|\lambda_n^k(t,s,x,y)|^2\right]^{\frac{q}{2}}\right]\\
\le\ & Cn^{\frac{dq}{2}}2^{-\frac{nq}{2}(1+2\sigma-\frac{d}{2}+\frac{1}{q}+\frac{2}{\gamma\beta})}\left[|x-y|^{\theta'}+|t-s|^{\theta}\right]^{\frac{q}{2}}.
\end{aligned}\tag{4.32}$$

Next we turn to estimate the term $J_3^{1,2}(t,s,x,y)$. If we set

$$V(r,z)=\mathbf{1}_{\bar{A}^M_n(r)}\dot{H}(X_n^-(r,z))\int_{r_n}^r\int_D G_{r-u}(z,v)F(X_n^-(u,v))W(\mathrm{d}v,\mathrm{d}u),\tag{4.33}$$

then

$$J_3^{1,2}(t,s,x,y)=\sum_{k=0}^{[2^n t/T]-1}\left[\lambda_n^k(V)(t,s,x,y)-\mathbf{E}\left[\lambda_n^k(V)(t,s,x,y)|\mathscr{F}_{\frac{(k-1)T}{2^n}}\right]\right].\tag{4.34}$$

Applying the discrete Burkhölder inequality and Jensen's inequality to conclude that

$$\begin{aligned}
&\mathbf{E}\left[|J_3^{1,2}(t,s,x,y)|^q\right]\\
\le\ & C\mathbf{E}\left[\sum_{k=0}^{[2^n t/T]-1}\left|\lambda_n^k(V)(t,s,x,y)-\mathbf{E}\left[\lambda_n^k(V)(t,s,x,y)|\mathscr{F}_{\frac{(k-1)T}{2^n}}\right]\right|^2\right]^{\frac{q}{2}}\\
\le\ & C\mathbf{E}\left[\sum_{k=0}^{[2^n t/T]-1}\left|\lambda_n^k(V)(t,s,x,y)\right|^2\right]^{\frac{q}{2}}\\
&+C\mathbf{E}\left[\sum_{k=0}^{[2^n t/T]-1}\left|\mathbf{E}\left[\lambda_n^k(V)(t,s,x,y)|\mathscr{F}_{\frac{(k-1)T}{2^n}}\right]\right|^2\right]^{\frac{q}{2}}.
\end{aligned}\tag{4.35}$$

According to a similar proof to that of (4.31), for $V(r,z)$ defined by (4.33), one gets

$$\sup_{r\in[0,T]}\mathbf{E}\left[\|V(r,\cdot)\|_q^q\right]\le C2^{-nq(1-d/4)}.$$

Also using Lemma 4.3,

$$\mathbf{E}\left[\mathbf{1}_{\bar{A}^n_M(t)\cap\bar{A}^n_M(s)}\left|J_3^{1,2}(t,s,x,y)\right|^q\right]\le Cn^{\frac{dq}{2}}2^{-\frac{nq\delta'}{2}}\left[|x-y|^{\theta'}+|t-s|^{\theta}\right]^{\frac{q}{2}},\tag{4.36}$$

where $\delta'=-\frac{d}{2}+\frac{1}{q}+\frac{2}{\gamma\beta}>0$. On the other hand, Lemma 4.4 yields that

$$\mathbf{E}\left[\dot{W}_n(z,r)\int_{r_n}^r\int_D G_{r-u}(z,v)F(X_n^-(u,v))W(\mathrm{d}v,\mathrm{d}u)|\mathscr{F}_{r_n}\right]$$

$$= \quad 2^n T^{-1} n^d \pi^{-d} \int_{r_n}^{\underline{r}_n} \int_{D_{\mathbf{k}}(z)} G_{r-u}(z,v) F(X_n^-(u,v)) \mathrm{d}v \mathrm{d}u.$$

This implies that

$$J_3^{1,3}(t,s,x,y) \equiv 0.$$

Since $\dot{H}$ and F are bounded, from (2.4), (A.4) and (A.7)–(A.9), it follows that

$$\begin{aligned} \left| J_3^{1,4}(t,s,x,y) \right|^q &\leq Cn^{dq} \left[\int_0^t \int_D \left| [\eta(t,s,x,y)](r,z) \right| \mathrm{d}z \mathrm{d}r \right]^q \\ &\quad \times \left[2^n \sup_{(r,z)\in\mathcal{O}_T} \int_{r_n}^{\underline{r}_n} \int_{D_{\mathbf{k}}(z)} |G_{r-u}(z,v)| \mathrm{d}v \mathrm{d}u \right]^q \\ &\leq Cn^{dq} \left[|t-s|^{\gamma''} + |x-y|^{\gamma'} \right]^q, \end{aligned} \tag{4.37}$$

where $0 < \gamma' < 2 - \frac{d}{2}$ and $0 < \gamma'' < \frac{1}{2}(1 - \frac{d}{4})$. From (2.4), Lemma 4.2 and (A.11)–(A.12) with $\kappa = \frac{1}{\infty} - \frac{1}{p} + 1 = 1 \in [0,1]$, we conclude that there exist $\alpha \in]0, 1 - \frac{d}{4p}[$ and $\beta \in]0,1[$ such that

$$\begin{aligned} &\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)} \left| J_3^{1,5}(t,s,x,y) \right|^q \right] \\ \leq\ & Cn^{dq} \left[|t-s|^{\alpha} + |x-y|^{\beta} \right]^q \sup_{r\in[0,T]} \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(r)} \|X_n(r,\cdot) - X_n^-(r,\cdot)\|_q^q \right] \\ \leq\ & Cn^{dq} 2^{-nq\sigma} \left[|t-s|^{\alpha} + |x-y|^{\beta} \right]^q. \end{aligned} \tag{4.38}$$

From the above estimations, it follows that there exist $\bar{\alpha} \in]0, \min\{\frac{1}{2} - \frac{3d}{4p}, \frac{1}{2}(1 - \frac{d}{4}), \frac{\rho}{4}\}[$ and $\bar{\beta} \in]0, \min\{2 - \frac{3d}{p}, 2 - \frac{d}{2}, \frac{2}{d}, \rho\}[$ such that for $(t,x),(s,y) \in \mathcal{O}_T$,

$$\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)} \left| J_3^1(t,s,x,y) \right|^q \right] \leq Cn^{dq} \left[|t-s|^{\bar{\alpha}} + |x-y|^{\bar{\beta}} \right]^q. \tag{4.39}$$

Now we turn to estimate the term $J_3^2(t,s,x,y)$. The procedure is similar to that of $J_3^1(t,s,x,y)$. Replace $F(X_n(r,z))$ by $H(X_n(r,z))$, $F(X_n^-(r,z))$ by $H(X_n^-(r,z))$, $W(\mathrm{d}z,\mathrm{d}r)$ by $W_n(\mathrm{d}z,\mathrm{d}r)$, and $\alpha_n(r,z,X_n(r,z))$ by $\beta_n(r,z,X_n(r,z))$, respectively. Then there exist $\bar{\alpha},\bar{\beta}$ presented in (4.39) such that

$$\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)} \left| J_3^2(t,s,x,y) \right|^q \right] \leq Cn^{q} \left[|t-s|^{\bar{\alpha}} + |x-y|^{\bar{\beta}} \right]^q. \tag{4.40}$$

As for the term $J_3^3(t,s,x,y)$, we have

$$\begin{aligned} &\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)} \left| J_3^3(t,s,x,y) \right|^q \right] \\ =\ & \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)} \left| \int_0^t \int_D [\eta(t,s,x,y)](r,z) \dot{H}(X_n^-(r,z)) T_n^3(r,z) \dot{W}_n(z,r) \mathrm{d}z \mathrm{d}r \right|^q \right] \\ =\ & B_n^1(t,s,x,y) + B_n^2(t,s,x,y), \end{aligned} \tag{4.41}$$

where

$$B_n^1(t,s,x,y) \quad = \quad \mathbf{E}\Bigg[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)} \Bigg| \int_0^t \int_D [\eta(t,s,x,y)](r,z) \dot{H}(X_n^-(r,z)) \dot{W}_n(z,r)$$

$$\times \left[\int_{r_n}^{r} \int_D \Delta_v G_{r-u}(z,v)[f(X_n(u,v)) - f(X_n^-(u,v))] \mathrm{d}v \mathrm{d}u \right] \mathrm{d}z \mathrm{d}r \Big|^q \Bigg],$$

$$B_n^2(t,s,x,y) = \mathbf{E}\Bigg[\mathbf{1}_{\tilde{A}_n^M(t)\cap \tilde{A}_n^M(s)} \Big| \int_0^t \int_D [\eta(t,s,x,y)](r,z) \dot{H}(X_n^-(r,z)) \dot{W}_n(z,r)$$

$$\times \left[\int_{r_n}^{r} \int_D \Delta_v G_{r-u}(z,v) f(X_n^-(u,v)) \mathrm{d}v \mathrm{d}u \right] \mathrm{d}z \mathrm{d}r \Big|^q \Bigg].$$

By virtue of (A.11) and (A.12) with $\kappa = \frac{1}{\infty} - \frac{1}{p} + 1 = 1 - \frac{1}{p} \in [0,1]$, (A.13) with $\kappa_1 = \frac{1}{p} - \frac{1}{p} + 1 = 1$ and (4.9), we have for $\alpha \in]0, 1 - \frac{d}{4p}[$ and $\beta \in]0,1[$,

$$\begin{aligned}
& B_n^1(t,s,x,y) \\
\leq\ & C n^{dq} 2^{\frac{1}{2}nq} \left[|t-s|^\alpha + |x-y|^\beta \right]^q \\
& \times \mathbf{E}\left[\left\| \int_{r_n}^{r} \int_D |\Delta_v G_{r-u}(\cdot,v)| \mathbf{1}_{\tilde{A}_n^M(u)} |X_n(u,v) - X_n^-(u,v)| \mathrm{d}v \mathrm{d}u \right\|_q^p \right] \\
\leq\ & C n^{dq} 2^{\frac{1}{2}nq} \left[|t-s|^\alpha + |x-y|^\beta \right]^q 2^{-\frac{1}{2}nq} \sup_{r\in[0,T]} \mathbf{E}\left[\|X_n(r,\cdot) - X_n^-(r,\cdot)\|_p^q \right] \\
\leq\ & C n^{dq} 2^{-nq\iota} \left[|t-s|^\alpha + |x-y|^\beta \right]^q . \qquad (4.42)
\end{aligned}$$

For $r \leq t$, set

$$V(r,z) := \int_{r_n}^{r} \int_D \Delta_v G_{r-u}(z,v) \mathbf{1}_{\tilde{A}_n^M(u)} f(X_n^-(u,v)) \mathrm{d}v \mathrm{d}u. \qquad (4.43)$$

Using the B-D-G inequality

$$B_n^2(t,s,x,y) \leq \mathbf{E}\left[\sum_{k=0}^{[2^n t/T]-1} \left| \lambda_n^k(V)(t,s,x,y) \right|^2 \right]^{\frac{q}{2}}. \qquad (4.44)$$

Since $X_n^-(u,v) = G_{u-u_n}(v, X_n(u_n,\cdot))$, it is obvious that

$$\|X_n^-(u,\cdot)\|_q \leq C \|X_n(u,\cdot)\|_q .$$

By virtue of (4.7),

$$\sup_{r\in[0,T]} \mathbf{E}\left[\|V(r,\cdot)\|_q^q \right] \leq C 2^{-nq(\frac{1}{2} - \frac{3d}{4q})}. \qquad (4.45)$$

Again by Lemma 4.3, there exist $\theta' < 4 - d$ and $\theta' \leq 2$, $\theta < 1 - \frac{d}{4}$ such that

$$B_n^2(t,s,x,y) \leq C n^{\frac{1}{2}dq} 2^{-nq\alpha'} \left[|t-s|^\theta + |x-y|^{\theta'} \right]^q, \qquad (4.46)$$

where $\alpha' = -1 - \frac{3d}{2q} + \frac{1}{q} + \frac{2}{\gamma\beta}$. So that for $\alpha \in]0, \frac{1}{2}(1-\frac{d}{4})[$ and $\beta \in]0, (2-d/2)\wedge 1[$,

$$\mathbf{E}\left[\mathbf{1}_{\tilde{A}_n^M(t)\cap \tilde{A}_n^M(s)} \left| J_3^3(t,s,x,y) \right|^q \right] \leq C n^{dq} 2^{-nq\delta'} \left[|t-s|^\alpha + |x-y|^\beta \right]^q . \qquad (4.47)$$

with $\delta' = \iota \wedge \frac{\alpha'}{2}$. Finally we consider $J_3^4(t,s,x,y)$ and $J_3^5(t,s,x,y)$. From (A.11) and (A.12) with $\kappa := \frac{1}{\infty} - \frac{1}{p} + 1 \in [0,1]$ and (4.8), it follows that for $\alpha \in]0, 1 - \frac{d}{4p}[$ and $\beta \in]0,1[$,

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)}\left|J_3^4(t,s,x,y)\right|^q\right] \\
\leq\ & Cn^{dq}2^{\frac{1}{2}nq}\left[|t-s|^\alpha + |x-y|^\beta\right]^q \sup_{r\in[0,T]} \mathbf{E}\left[1_{\bar{A}_n^M(r)}\|T_n^4(r,\cdot)\|_p^q\right] \\
\leq\ & Cn^{2dq}2^{-\frac{1}{2}nq}\left[|t-s|^\alpha + |x-y|^\beta\right]^q. \qquad (4.48)
\end{aligned}$$

As for $J_3^5(t,s,x,y)$, using (A.11) and (A.12) with $\kappa := \frac{1}{\infty} - \frac{2}{p} + 1 := 1 - \frac{2}{p} \in [0,1]$ and Lemma 4.2, we get for $\alpha \in]0, 1 - \frac{d}{2p}[$ and $\beta \in]0, \min\{4 - \frac{2d}{p}, 1\}[$,

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)}\left|J_3^5(t,s,x,y)\right|^q\right] \\
=\ & \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)}\left|\int_0^t\int_D [\eta(t,s,x,y)](r,z)\rho_n(r,z)\dot{W}_n(z,r)\mathrm{d}z\mathrm{d}r\right|^q\right] \\
\leq\ & \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)}\left|\int_0^t\int_D [\eta(t,s,x,y)](r,z)|X_n(r,z)-X_n^-(r,z)|^2\dot{W}_n(z,r)\mathrm{d}z\mathrm{d}r\right|^q\right] \\
\leq\ & Cn^{\frac{dq}{2}}2^{(\frac{1}{2}-\frac{d}{4})nq}\left[|t-s|^\alpha + |x-y|^\beta\right]^q.
\end{aligned}$$

When $d = 3$, we can obtain a more precise estimate than the cases of $d = 1, 2$, which will be concluded in Lemma 6.1 of Section 6. Thus we complete the proof of the lemma. □

5 The proof of (C1)

The condition (**C1**) presented by Section 3 shall be verified in this section. By Lemma 4.1, to check (**C1**), we only need to prove the right hand side of (4.1) approaches 0, when $n \to \infty$. Note that for each fixed $n \in \mathbf{N}$,

$$\left\{\frac{\mathbf{1}_{\triangle_{j,\mathbf{k}}}(x,t)}{\sqrt{|\triangle_{j,\mathbf{k}}|}};\ j = 0,1,\ldots,2^n-1,\ \mathbf{k}\in\mathbf{I}_n^d\right\}$$

forms a CONS of $L^2([0,T]\times D)$. Let π_n be the orthogonal projection of above basis and for any mapping $g : \mathbf{R} \to \mathbf{R}$, define

$$\tau_n g(s) = g\left((s + T2^{-n}) \wedge T\right),\ \ s \in [0,T].$$

Then for each $t \in [0,T]$ and $\mathbb{F}$-predictable process $(\psi(t,x); x \in D)_{0\leq t\leq T}$,

$$\int_0^t\int_D \psi(s,y)W_n(\mathrm{d}y,\mathrm{d}s) = \int_0^T\int_D \pi_n\left[\tau_n\left(\mathbf{1}_{[0,t]}(\cdot)\psi(\cdot,*)\right)\right](s,y)W(\mathrm{d}y,\mathrm{d}s).$$

Recall $\Lambda_n(t,x)$ defined by (3.2) and we have

$$\Lambda_n(t,x) = \sum_{i=1}^{3} \tilde{\Lambda}_n^i(t,x),$$

where

$$\begin{aligned}
\tilde{\Lambda}_n^1(t,x) &:= \int_0^t \int_D G_{t-s}(x,y) H(X_n^-(s,y))[W_n(\mathrm{d}y,\mathrm{d}s) - W(\mathrm{d}y,\mathrm{d}s)], \\
\tilde{\Lambda}_n^2(t,x) &:= \int_0^t \int_D G_{t-s}(x,y)[H(X_n^-(s,y)) - H(X_n(s,y))] W(\mathrm{d}y,\mathrm{d}s), \\
\tilde{\Lambda}_n^3(t,x) &:= \int_0^t \int_D G_{t-s}(x,y)[H(X_n(s,y)) - H(X_n^-(s,y))] W_n(\mathrm{d}y,\mathrm{d}s) \\
&\quad - \int_0^t \int_D G_{t-s}(x,y) \dot{H}(X_n(s,y)) \\
&\quad \times \left[\alpha_n(s,y) F(X_n(s,y)) + \beta_n(s,y) H(X_n(s,y))\right] \mathrm{d}y\mathrm{d}s.
\end{aligned}$$

We begin with the estimation of the term $\tilde{\Lambda}_n^1(t,x)$. Note that

$$\begin{aligned}
\tilde{\Lambda}_n^1(t,x) &= \int_0^t \int_D \Big[\pi_n \left[\tau_n \left(\mathbf{1}_{[0,t]}(\cdot) G_{t-\cdot}(x,*) H(X_n^-(\cdot,*))\right)\right](s,y) \\
&\quad - \pi_n \left[\mathbf{1}_{[0,t]}(\cdot) G_{t-\cdot}(x,*) H(X_n^-(\cdot,*))\right](s,y)\Big] W(\mathrm{d}y,\mathrm{d}s), \\
&+ \int_0^t \int_D \Big[\pi_n \left[\mathbf{1}_{[0,t]}(\cdot) G_{t-\cdot}(x,*) H(X_n^-(\cdot,*))\right](s,y) \\
&\quad - \mathbf{1}_{[0,t]}(s) G_{t-s}(x,y) H(X_n^-(s,y))\Big] W(\mathrm{d}y,\mathrm{d}s) \\
&=: \tilde{\Lambda}_n^{1,1}(t,x) + \tilde{\Lambda}_n^{1,2}(t,x).
\end{aligned}$$

Then by the Hölder inequality and Burkhölder's inequality, and note that π_n is an orthogonal projection of $L^2([0,t]\times D)$, we have for $q \geq p$,

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)} \left\|\tilde{\Lambda}_n^{1,1}(t,\cdot)\right\|_p^q\right] \\
\leq\ & C \int_D \mathbf{E}\Bigg[\mathbf{1}_{\bar{A}_n^M(t)} \Bigg| \int_0^T \int_D \Big[\tau_n \left[\mathbf{1}_{[0,t]}(s) G_{t-s}(x,y) H(X_n^-(s,y))\right] \\
& - \mathbf{1}_{[0,t]}(s) G_{t-s}(x,y) H(X_n^-(s,y))\Big] W(\mathrm{d}y,\mathrm{d}s)\Bigg|^q\Bigg] \mathrm{d}x \\
\leq\ & C\mathbf{E}\Bigg[\mathbf{1}_{\bar{A}_n^M(t)} \Bigg\| \int_0^T \int_D \Big[\tau_n \left[\mathbf{1}_{[0,t]}(s) G_{t-s}(\bullet,y) H(X_n^-(s,y))\right] \\
& - \mathbf{1}_{[0,t]}(s) G_{t-s}(\bullet,y) H(X_n^-(s,y))\Big]^2 \mathrm{d}y\mathrm{d}s\Bigg\|_{\frac{q}{2}}^{\frac{q}{2}}\Bigg] \\
=\ & C\mathbf{E}\Bigg[\mathbf{1}_{\bar{A}_n^M(t)} \Bigg\| \int_0^T \int_D \Big[\mathbf{1}_{[0,t]}\left(s + T2^{-n}\right) G_{t-(s+T2^{-n})}(\bullet,y) H\left(X_n^-(s+T2^{-n},y)\right)
\end{aligned}$$

$$
\begin{aligned}
& -\mathbf{1}_{[0,t]}(s)G_{t-s}(\bullet,y)H(X_n^-(s,y))\Big]^2 \mathrm{d}y\mathrm{d}s\Big\|_{\frac{q}{2}}^{\frac{q}{2}}\Big]\\
\le\ & C\mathbf{E}\Big[\mathbf{1}_{\tilde{A}_n^M(t)}\Big\|\int_0^{(t-T2^{-n})^+}\int_D\Big[G_{t-(s+T2^{-n})}(\bullet,y)-G_{t-s}(\bullet,y)\Big]^2\\
& \times H^2\left(X_n^-(s+T2^{-n},y)\right)\mathrm{d}y\mathrm{d}s\Big\|_{\frac{q}{2}}^{\frac{q}{2}}\Big]\\
& +C\mathbf{E}\Big[\mathbf{1}_{\tilde{A}_n^M(t)}\Big\|\int_0^{(t-T2^{-n})^+}\int_D\Big|G_{t-s}(\bullet,y)\Big|^2\\
& \times\Big[H(X_n^-(s+T2^{-n},y))-H(X_n^-(s,y))\Big]^2\mathrm{d}y\mathrm{d}s\Big\|_{\frac{q}{2}}^{\frac{q}{2}}\Big]\\
& +C\mathbf{E}\Big[\mathbf{1}_{\tilde{A}_n^M(t)}\Big\|\int_{(t-T2^{-n})^+}^{t}\int_D G_{t-s}^2(\bullet,y)H^2\left(X_n^-(s,y)\right)\mathrm{d}y\mathrm{d}s\Big\|_{\frac{q}{2}}^{\frac{q}{2}}\Big]\\
\le\ & C\mathbf{E}\Big[\Big\|\int_0^{(t-T2^{-n})^+}\int_D\Big[G_{t-(s+T2^{-n})}(\bullet,y)-G_{t-s}(\bullet,y)\Big]^2\mathrm{d}y\mathrm{d}s\Big\|_{\frac{q}{2}}^{\frac{q}{2}}\Big]\\
& +C\mathbf{E}\Big[\Big\|\int_0^{(t-T2^{-n})^+}\int_D\Big|G_{t-s}(\bullet,y)\Big|^2\\
& \times\mathbf{1}_{\tilde{A}_n^M(s)}\Big[X_n^-\left(s+T2^{-n},y\right)-X_n^-(s,y)\Big]^2\mathrm{d}y\mathrm{d}s\Big\|_{\frac{q}{2}}^{\frac{q}{2}}\Big]\\
& +C\mathbf{E}\Big[\Big\|\int_{(t-T2^{-n})^+}^{t}\int_D G_{t-s}^2(\bullet,y)H^2\left(X_n^-(s,y)\right)\mathrm{d}y\mathrm{d}s\Big\|_{\frac{q}{2}}^{\frac{q}{2}}\Big]\\
=:\ & C\sum_{i=1}^{3}\tilde{\Lambda}_n^{1,1,i}(t).
\end{aligned}
$$

Take the boundedness of the mapping H into account, from (A.8) and (A.9) in Lemma A.2, it follows that for $\gamma''<1-\frac{d}{4}$,

$$
\sup_{t\in[0,T]}\Big[|\tilde{\Lambda}_n^{1,1,1}(t)|+|\tilde{\Lambda}_n^{1,1,3}(t)|\Big]\le C2^{-\frac{1}{2}\gamma''nq}. \tag{5.1}
$$

On the other hand, applying (A.16) with $\kappa=\frac{2}{q}-\frac{2}{q}+1=1$, the Hölder inequality, Lemma 4.2, and Lemma 6.1 (in Section 6) to conclude that

$$
\begin{aligned}
\left|\tilde{\Lambda}_n^{1,1,2}(t)\right| \le\ & C\sup_{s\in[0,T]}\mathbf{E}\Big[\mathbf{1}_{\tilde{A}_n^M(s)\cap\tilde{A}_n^M(s+T/2^n)}\Big\|X_n^-\left(s+T2^{-n},\cdot\right)-X_n^-(s,\cdot)\Big\|_q^q\Big]\\
\le\ & C\sup_{(s,y)\in\mathcal{O}_T}\mathbf{E}\Big[\mathbf{1}_{\tilde{A}_n^M(s+T/2^n)}\Big|X_n^-\left(s+T2^{-n},y\right)-X_n(s+T2^{-n},y)\Big|^q\Big]\\
& +C\sup_{(s,y)\in\mathcal{O}_T}\mathbf{E}\Big[\mathbf{1}_{\tilde{A}_n^M(s)\cap\tilde{A}_n^M(s+T/2^n)}\Big|X_n\left(s+T2^{-n},y\right)-X_n(s,y)\Big|^q\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+C\sup_{(s,y)\in\mathcal{O}_T}\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(s)}\left|X_n(s,y)-X_n^-(s,y)\right|^q\right]\\
\le\;& C\left[2^{-\iota nq}+2^{-\xi nq}+2^{-\iota nq}\right]. \qquad (5.2)
\end{aligned}
$$

Therefore for $q\ge p>\frac{3d}{2}$,

$$
\sup_{t\in[0,T]}\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)}\left\|\tilde{\Lambda}_n^{1,1}(t,\cdot)\right\|_p^q\right]\to 0,\quad n\to\infty. \qquad (5.3)
$$

As for the term $\tilde{\Lambda}_n^{1,2}(t,x)$, note that

$$
\begin{aligned}
&\int_0^t\int_D\pi_n\left[\mathbf{1}_{[0,t]}(\cdot)G_{t-\cdot}(x,*)H(X_n^-(\cdot,*))\right](s,y)W(\mathrm{d}y,\mathrm{d}s)\\
=\;&\sum_{j=0}^{[2^nt]}\sum_{\mathbf{k}\in\mathbf{I}_n^d}\int_0^t\int_D\frac{\mathbf{1}_{\triangle_{j,\mathbf{k}}}(y,s)}{\left|\triangle_{j,\mathbf{k}}\right|}\left[\int_{\triangle_{j,\mathbf{k}}}G_{t-r}(x,z)H(X_n^-(r,z))\mathrm{d}z\mathrm{d}r\right]W(\mathrm{d}y,\mathrm{d}s). \qquad (5.4)
\end{aligned}
$$

Then the B-D-G inequality yields that for $q\ge p$,

$$
\begin{aligned}
&\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)}\left\|\tilde{\Lambda}_n^{1,2}(t,\cdot)\right\|_p^q\right]\\
\le\;& C\int_D\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)}\left|\int_0^T\int_D\sum_{j=0}^{[2^nt]}\sum_{\mathbf{k}\in\mathbf{I}_n^d}\frac{\mathbf{1}_{\triangle_{j,\mathbf{k}}}(y,s)}{\left|\triangle_{j,\mathbf{k}}\right|}\right.\right.\\
&\times\left[\int_{\triangle_{j,\mathbf{k}}}\left[G_{t-r}(x,z)H(X_n^-(r,z))-G_{t-s}(x,y)H(X_n^-(s,y))\right]\mathrm{d}z\mathrm{d}r\right]\\
&\left.\left.\times W(\mathrm{d}y,\mathrm{d}s)\right|^q\mathrm{d}x\right]\\
\le\;& C\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)}\left\|\int_0^T\int_D\sum_{j=0}^{[2^nt]}\sum_{\mathbf{k}\in\mathbf{I}_n^d}\frac{\mathbf{1}_{\triangle_{j,\mathbf{k}}}(y,s)}{\left|\triangle_{j,\mathbf{k}}\right|^2}\right.\right.\\
&\left.\left.\times\left[\int_{\triangle_{j,\mathbf{k}}}\left[G_{t-r}(\bullet,z)-G_{t-s}(\bullet,y)\right]H(X_n^-(r,z))\mathrm{d}z\mathrm{d}r\right]^2\mathrm{d}y\mathrm{d}s\right\|_{\frac{q}{2}}^{\frac{q}{2}}\right]\\
&+C\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)}\left\|\int_0^T\int_D\sum_{j=0}^{[2^nt]}\sum_{\mathbf{k}\in\mathbf{I}_n^d}\frac{\mathbf{1}_{\triangle_{j,\mathbf{k}}}(y,s)}{\left|\triangle_{j,\mathbf{k}}\right|^2}\right.\right.\\
&\left.\left.\times\left[\int_{\triangle_{j,\mathbf{k}}}G_{t-s}(\bullet,y)\left[H(X_n^-(r,z))-H(X_n^-(s,y))\right]\mathrm{d}z\mathrm{d}r\right]^2\mathrm{d}y\mathrm{d}s\right\|_{\frac{q}{2}}^{\frac{q}{2}}\right]\\
=:\;& C\left[H_n^1(t)+H_n^2(t)\right].
\end{aligned}
$$

By the boundedness of H and Dini's theorem, we have for $t\in[0,T]$,

$$
0\le H_n^1(t)\quad\le\quad C\left\|\int_0^T\int_D\sum_{j=0}^{[2^nt]}\sum_{\mathbf{k}\in\mathbf{I}_n^d}\frac{\mathbf{1}_{\triangle_{j,\mathbf{k}}}(y,s)}{\left|\triangle_{j,\mathbf{k}}\right|^2}\right.
$$

$$
\begin{aligned}
& \times \left[\int_{\triangle_{j,\mathbf{k}}} \left[G_{t-r}(\bullet, z) - G_{t-s}(\bullet, y) \right] \mathrm{d}z \mathrm{d}r \right]^2 \mathrm{d}y \mathrm{d}s \Bigg\|_{\frac{q}{2}}^{\frac{q}{2}} \\
= \;& C \left\| \int_0^t \int_D \left| \pi_n \left[G_{t-\cdot}(\bullet, *) \right](s,y) - G_{t-s}(\bullet, y) \right|^2 \mathrm{d}y \mathrm{d}s \right\|_{\frac{q}{2}}^{\frac{q}{2}} \\
\leq \;& C \sup_{x \in D} \left| \int_0^t \int_D \left| \pi_n [G_{t-\cdot}(x, *)](s,y) - G_{t-s}(x,y) \right|^2 \mathrm{d}y \mathrm{d}s \right|^{\frac{q}{2}} \\
\to \;& 0, \quad n \to \infty,
\end{aligned}
$$

follows from the fact that when $n \to \infty$, it holds that

$$
\left\| \pi_n \left[G_{t-\cdot}(x,*) \right] - G_{t-\cdot}(x,*) \right\|_{L^2([0,T]\times D)} \to 0,
$$

for all $(t,x) \in \mathscr{O}_T$ (a compact set in $\mathbf{R}^{d+1}$). Because H is Lipschitzian continuous, using the Hölder inequality

$$
\begin{aligned}
H_n^2(t) \;\leq\;& C\mathbf{E}\left[\mathbf{1}_{\tilde{A}_n^M(t)} \left\| \int_0^T \int_D \sum_{j=0}^{[2^n t]} \sum_{\mathbf{k}\in\mathbf{I}_n^d} \frac{\mathbf{1}_{\triangle_{j,\mathbf{k}}}(y,s)}{\left|\triangle_{j,\mathbf{k}}\right|^2} \left[\int_{\triangle_{j,\mathbf{k}}} G_{t-s}^2(\bullet, y) \mathrm{d}z \mathrm{d}r \right] \right.\right. \\
& \left.\left. \times \left[\int_{\triangle_{j,\mathbf{k}}} |H(X_n^-(r,z)) - H(X_n^-(s,y))|^2 \mathrm{d}z \mathrm{d}r \right] \mathrm{d}y \mathrm{d}s \right\|_{\frac{q}{2}}^{\frac{q}{2}} \right] \\
\leq\;& C\mathbf{E}\left[\mathbf{1}_{\tilde{A}_n^M(t)} \left\| \int_0^T \int_D \sum_{j=0}^{[2^n t]} \sum_{\mathbf{k}\in\mathbf{I}_n^d} \frac{\mathbf{1}_{\triangle_{j,\mathbf{k}}}(y,s)}{\left|\triangle_{j,\mathbf{k}}\right|^2} \left[\int_{\triangle_{j,\mathbf{k}}} G_{t-s}^2(\bullet, y) \mathrm{d}z \mathrm{d}r \right] \right.\right. \\
& \left.\left. \times \left[\int_{\triangle_{j,\mathbf{k}}} |H(X_n^-(r,z)) - H(X_n^-(s,y))|^2 \mathrm{d}z \mathrm{d}r \right] \mathrm{d}y \mathrm{d}s \right\|_{\frac{q}{2}}^{\frac{q}{2}} \right] \\
\leq\;& C \sup_{(s,y)\in\mathscr{O}_T} \sum_{j=0}^{[2^n t]} \sum_{\mathbf{k}\in\mathbf{I}_n^d} \sup_{(z,r)\in\triangle_{j,\mathbf{k}}} \mathbf{E}\left[\mathbf{1}_{\tilde{A}_n^M(r)\cap\tilde{A}_n^M(s)} \left| X_n^-(r,z)) - X_n^-(s,y) \right|^q \right] \\
& \times \mathbf{1}_{\triangle_{j,\mathbf{k}}}(y,s). \qquad (5.5)
\end{aligned}
$$

Note that

$$
\begin{aligned}
|X_n^-(r,z) - X_n^-(s,y)| \;\leq\;& |X_n^-(r,z) - X_n(r,z)| + |X_n(r,z) - X_n(s,y)| \\
& + |X_n(s,y) - X_n^-(s,y)|.
\end{aligned}
$$

Then from Lemma 4.2, and Lemma 6.1 (in Section 6), it follows that

$$
H_n^2(t) \;\leq\; C\left[2^{-\iota n q} + 2^{-\xi n q} + 2^{-\iota n q} \right] \leq C 2^{-\lambda n q}, \qquad (5.6)
$$

where $\lambda = \min\{\iota, \xi\}$. This further implies that for $q \geq p > \frac{3d}{2}$,

$$
\sup_{t\in[0,T]} \mathbf{E}\left[\mathbf{1}_{\tilde{A}_n^M(t)} \left\| \tilde{\Lambda}_n^{1,2}(t,\cdot) \right\|_p^q \right] \to 0, \quad n \to \infty,
$$

and hence for $q \geq p > \frac{3d}{2}$,

$$\sup_{t\in[0,T]} \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)} \left\|\tilde{\Lambda}_n^1(t,\cdot)\right\|_p^q\right] \to 0, \quad n\to\infty. \tag{5.7}$$

Finally, we turn to the estimation of the term $\tilde{\Lambda}_n^2(t,x)$. In light of the B-D-G inequality and (A.16) with $\kappa = \frac{2}{p} - \frac{2}{p} + 1 = 1$,

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)} \left\|\tilde{\Lambda}_n^2(t,\cdot)\right\|_p^q\right] \\
\leq\ & C\mathbf{E}\left\|\mathbf{1}_{\bar{A}_n^M(t)}\left[\int_0^t \int_D G_{t-s}^2(\cdot,y)|H(X_n(s,y)) - H(X_n^-(s,y))|^2 \mathrm{d}y\mathrm{d}s\right]\right\|_{\frac{p}{2}}^{\frac{q}{2}} \\
\leq\ & C\mathbf{E}\left[\int_0^t (t-s)^{-\frac{d}{4}} \|X_n(s,\cdot) - X_n^-(s,\cdot)\|_q^p \mathrm{d}s\right] \\
\leq\ & C2^{-\iota nq}.
\end{aligned} \tag{5.8}$$

Observe that

$$\tilde{\Lambda}_n^3(t,x) = J_3(t,0,x,0), \tag{5.9}$$

for the term $J_3(t,s,x,y)$ defined in Lemma 4.5. Then there exists a $\lambda > 0$ such that for $q \geq p > \frac{3d}{2}$,

$$\sup_{t\in[0,T]} \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)} \left\|\tilde{\Lambda}_n^3(t,\cdot)\right\|_p^q\right] \leq C2^{-\lambda nq}. \tag{5.10}$$

Thus we prove that the condition (**C1**) holds. □

6 The proof of (C2)

The aim of this section is to check the validity of the condition (**C2**) presented in Section 3. Note that for all $s < t \in [0,T]$, we have for $q \geq p > \frac{3d}{2}$,

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)} \left\|Y_n(t,\cdot) - Y_n(s,\cdot)\right\|_p^q\right] \\
\leq\ & C\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)} \left\|X_n(t,\cdot) - X_n(s,\cdot)\right\|_p^q\right] + C\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)} \|X(t,\cdot) - X(s,\cdot)\|_p^q\right].
\end{aligned}$$

Observe the forms of Equations (2.2) and (2.3), X is a particular case of X_n. Hence in order to prove that (**C2**) holds, it suffices to check that for all $0 \leq s < t \leq T$, there exist $q \geq p$ and $\theta > q\bar{\alpha}$ ($\bar{\alpha}$ is presented in Theorem 1.1) such that for each $n \in \mathbf{N}$,

$$(\mathbf{C2})' \quad \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)} \left\|X_n(t,\cdot) - X_n(s,\cdot)\right\|_p^q\right] \leq C|t-s|^{1+\theta}.$$

From (2.2), it follows that for $(s,y),(t,x) \in \mathcal{O}_T$,

$$X_n(t,x) - X_n(s,y)$$

$$
\begin{aligned}
= \quad & G_t * \psi(x) - G_s * \psi(y) \\
& + \int_0^T \int_D [\eta(s,t,x,y)](r,z) \left[F(X_n(r,z)) W(\mathrm{d}z, \mathrm{d}r) + H(X_n(r,z) W_n(\mathrm{d}z, \mathrm{d}r)) \right] \\
& + \int_0^T \int_D [\eta(s,t,x,y)](r,z) \Big[K(X_n(r,z)) \dot{h}(r,z) \\
& \qquad - \left[\alpha_n(r,z)(F\dot{H})(X_n(r,z)) + \beta_n(r,z)(H\dot{H})(X_n(r,z)) \right] \Big] \mathrm{d}z \mathrm{d}r \\
& + \int_0^T \int_D [\widetilde{\eta}(s,t,x,y)](r,z) f(X_n(r,z)) \mathrm{d}z \mathrm{d}r. \qquad (6.1)
\end{aligned}
$$

To prove $(\mathbf{C2})'$, we will sharp the estimations in Lemma 4.5 by the following lemma.

Lemma 6.1. *It holds that*

(i) *Let $q \geq p > 6$. Then for $\bar{\alpha} \in]0, \min\{\frac{1}{2}(1 - \frac{d}{4}), \frac{\varrho}{4}\}[$ and $\bar{\beta} \in]0, \min\{2 - \frac{d}{2}, \varrho\}[$, there exists a constant $C > 0$ such that*

$$
\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(s) \cap \bar{A}_n^M(t)} \left| X_n(t,x) - X_n(s,y) \right| \right]^q \leq C n^{dq} 2^{-nq\xi} \left[|t-s|^{\bar{\alpha}} + |x-y|^{\bar{\beta}} \right]^q, \qquad (6.2)
$$

for some $\xi \in]0, \min\{\frac{1}{2}, \sigma, \frac{1}{2}(-1 - \frac{3d}{2q} + \frac{1}{q} + \frac{2}{\gamma\beta})[$ with σ, γ, β defined in Lemma 4.2 and Lemma 4.3, respectively.

(ii) *Let $q \geq p \geq 4$. Then for $\bar{\alpha} \in]0, \min\{\frac{1}{2}(1 - \frac{d}{4}), \frac{\varrho}{4}\}[$ there exists a $C > 0$ such that*

$$
\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(s) \cap \bar{A}_n^M(t)} \left\| X_n(t,\cdot) - X_n(s,\cdot) \right\|_p^q \right] \leq C n^{dq} 2^{-nq\xi} |t-s|^{\bar{\alpha} q}. \qquad (6.3)
$$

where ξ is the same as in (i).

Proof. Recall the entire proof of Lemma 4.5. We only need to re-estimate $J_3^1(t,s,x,y)$, $J_3^2(t,s,x,y)$ and $J_3^5(t,s,x,y)$ given in Lemma 4.5. To improve the estimation of $J_3^1(t,s,x,y)$, we only need to estimate the term $J_3^{1,4}(t,s,x,y)$. Note that, for $(s,y),(t,x) \in \mathscr{O}_T$,

$$
\begin{aligned}
& J_3^{1,4}(t,s,x,y) \\
\leq \quad & C 2^{\frac{n}{2}} n^d \int_0^t \int_D [\eta(s,t,x,y)](r,z) \dot{H}(X_n^-(r,z)) \\
& \times \left[\int_{r_n}^{\underline{r}_n} \int_{D_{\mathrm{k}}(z)} G_{r-u}(z,v) \left[F(X_n^-(u,v)) - F(X_n(u,v)) \right] \mathrm{d}v \mathrm{d}u \right] \mathrm{d}z \mathrm{d}r \\
& + C 2^{\frac{n}{2}} n^d \int_0^t \int_D [\eta(s,t,x,y)](r,z) \dot{H}(X_n^-(r,z)) \\
& \times \left[\int_{r_n}^{\underline{r}_n} \int_{D_{\mathrm{k}}(z)} G_{r-u}(z,v) \left[F(X_n(u,v)) - F(X_n(r,z)) \right] \mathrm{d}v \mathrm{d}u \right] \mathrm{d}z \mathrm{d}r \\
=: \quad & K_n^1(t,s,x,y) + K_n^2(t,s,x,y). \qquad (6.4)
\end{aligned}
$$

Using a similar argument to that of (4.42), we get for $\alpha, \beta \in (0,1)$,

$$\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)}\left|K_n^1(t,s,x,y)\right|^q\right] \le Cn^{dq}2^{-nq(\frac{1}{2}+\iota)}\left[|t-s|^\alpha+|x-y|^\beta\right]^q. \tag{6.5}$$

In order to estimate $K_n^2(t,s,x,y)$, we apply the Hölder inequality w.r.t the measures $\mathrm{d}y$ and $G_{r-u}(z,v)\mathrm{d}v\mathrm{d}u$, respectively. Then (A.11), (A.12) and Lemma 4.5 jointly imply that, for $\alpha, \beta \in]0,1[$,

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)}\left|K_n^2(t,s,x,y)\right|^q\right] \\
\le\ & C2^{\frac{nq}{2}}n^{dq}\left[|t-s|^\alpha+|x-y|^\beta\right]^q\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)}\int_0^t\left\|\int_{r_n}^{\underline{r}_n}\int_{D_{\mathbf{k}}(\cdot)}G_{r-u}(\cdot,v)\right.\right. \\
&\left.\left.\times\left[F(X_n(u,v))-F(X_n(r,\cdot))\right]\mathrm{d}v\mathrm{d}u\right\|_q^q\mathrm{d}r\right] \\
\le\ & C2^{\frac{nq}{2}}n^{2dq}2^{-(q-1)n}\left[|t-s|^\alpha+|x-y|^\beta\right]^q \\
&\times\left[\int_0^t\left[\int_{r_n}^{\underline{r}_n}\int_{D_{\mathbf{k}}(z)}G_{r-u}(z,v)\left[|r-u|^{\bar{\alpha}}+|z-v|^{\bar{\beta}}\right]^q\mathrm{d}v\mathrm{d}u\right]\mathrm{d}r\right] \\
\le\ & C2^{n-nq/2}n^{2dq}\left[|t-s|^\alpha+|x-y|^\beta\right]^q \\
&\times\left[\int_0^t\left[\int_{r_n}^{\underline{r}_n}\int_{\mathbf{R}^d}\left[|r-u|^{-\frac{d}{4}}|v|^{\bar{\beta}q}\exp\left(-C\frac{|v|^{4/3}}{|r-u|^{1/3}}\right)\right.\right.\right. \\
&\left.\left.\left.+2^{-n\bar{\alpha}q}G_{r-u}(z,v)\right]\mathrm{d}v\mathrm{d}u\right]\mathrm{d}r\right] \\
\le\ & C2^{n-nq/2}n^{2dq}\left[|t-s|^\alpha+|x-y|^\beta\right]^q\int_0^t\left[\int_{r_n}^{\underline{r}_n}|r-u|^{\frac{\bar{\beta}q}{4}}\mathrm{d}u+2^{-n-\bar{\alpha}qn}\right]\mathrm{d}r \\
\le\ & C2^{-nq(\frac{1}{2}+\alpha')}n^{2dq}\left[|t-s|^\alpha+|x-y|^\beta\right]^q,
\end{aligned} \tag{6.6}$$

where $\alpha' = \min\{\bar{\alpha}, \frac{\bar{\beta}}{4}\}$. The estimation of $J_3^2(t,s,x,y)$ is similar to that of $J_3^1(t,s,x,y)$.

Finally, we improve the estimation of the term $J_3^5(t,s,x,y)$. Using the decomposition of $X_n(r,z) - X_n^-(r,z)$,

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)}\left|J_3^5(t,s,x,y)\right|^q\right] \\
\le\ & \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)}\left|\int_0^t\int_D[\eta(t,s,x,y)](r,z)|X_n(r,z)-X_n^-(r,z)|^2\dot{W}_n(z,r)\mathrm{d}z\mathrm{d}r\right|^q\right] \\
:=\ & \sum_{i=1}^4\sum_{j=1}^4 R_n^{i,j}(t,s,x,y),
\end{aligned} \tag{6.7}$$

where

$$\begin{aligned}
&R_n^{i,j}(t,s,x,y) \\
=\ & \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)}\left|\int_0^t\int_D[\eta(t,s,x,y)](r,z)T_n^i(r,z)T_n^j(r,z)\dot{W}_n(z,r)\mathrm{d}z\mathrm{d}r\right|^q\right].
\end{aligned}$$

Using (A.11) and (A.12) with $\kappa := \frac{1}{\infty} - \frac{2}{p} + 1 = 1 - \frac{2}{p} \in [0,1]$, we get for $\alpha \in]0, 1 - \frac{d}{2p}[$, $\beta \in]0, \min\{4 - \frac{2d}{p}, 1\}[$, and $\gamma > \max\left\{\frac{1}{1+\frac{d}{4}(\kappa-1)-\alpha}, \frac{1}{1+\frac{d}{4}(\kappa-1)-\frac{\beta}{4}}, \frac{1}{1-\frac{d}{4}\beta}\right\}$,

$$\begin{aligned} R_n^{i,j}(t,s,x,y) &\leq Cn^{\frac{dq}{2}} 2^{\frac{nq}{2}} \left[|t-s|^{\alpha} + |x-y|^{\beta}\right]^q \sup_{r\in[0,t]} \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)} \|T_n^i(r,\cdot)\|_p^{q\gamma}\right]^{\frac{1}{\gamma}} \\ &\quad \times \sup_{r\in[0,t]} \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)} \|T_n^j(r,\cdot)\|_p^{q\gamma}\right]^{\frac{1}{\gamma}}. \end{aligned}$$

Thanks to (4.5)–(4.8), we deduce that there exists $\delta > 0$ such that for all (i,j) with either i or j not belonging to $\{1,4\}$,

$$R_n^{i,j}(t,s,x,y) \leq C 2^{-nq\delta} \left[|t-s|^{\alpha} + |x-y|^{\beta}\right]^q. \tag{6.8}$$

Next we improve the estimate of $R_n^{i,j}$ when $i,j \in \{1,4\}$. We first deal with $R_n^{1,4}$. By the Hölder inequality with $\frac{1}{2} + \frac{1}{2} = 1$, for $p > 6$,

$$\begin{aligned} R_n^{i,j}(t,s,x,y) &\leq Cn^{\frac{dq}{2}} 2^{\frac{nq}{2}} \left[|t-s|^{\alpha} + |x-y|^{\beta}\right]^q \sup_{r\in[0,t]} \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)} \|T_n^i(r,\cdot)\|_{2p}^{q\gamma}\right]^{\frac{1}{\gamma}} \\ &\quad \times \sup_{r\in[0,t]} \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)} \|T_n^j(r,\cdot)\|_{2p}^{q\gamma}\right]^{\frac{1}{\gamma}} \\ &\leq C 2^{-nq(\frac{1}{2}-\frac{3d}{4p})} \left[|t-s|^{\alpha} + |x-y|^{\beta}\right]^q. \end{aligned} \tag{6.9}$$

The estimation of the term $R_n^{4,4}$ is similar to that of $R_n^{1,4}$. To improve the estimation of $R_n^{1,1}(t,s,x,y)$, we introduce the process

$$T_n^{1-}(r,z) = \int_{r_n}^{r}\int_D G_{r-u}(z,v) F(X_n^-(u,v)) W(\mathrm{d}v, \mathrm{d}u). \tag{6.10}$$

Then

$$\begin{aligned} &R_n^{1,1}(t,s,x,y) \\ \leq\ & \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)} \left|\int_0^t\int_D [\eta(t,s,x,y)](r,z)\left[(T_n^1(r,z))^2 - (T_n^{1-}(r,z))^2\right]\dot{W}_n(z,r)\mathrm{d}z\mathrm{d}r\right|^q\right] \\ &+\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)} \left|\int_0^t\int_D [\eta(t,s,x,y)](r,z)(T_n^{1-}(r,z))^2\dot{W}_n(z,r)\mathrm{d}z\mathrm{d}r\right|^q\right] \\ :=\ & R_n^{1,1,1}(t,s,x,y) + R_n^{1,1,2}(t,s,x,y). \end{aligned} \tag{6.11}$$

Similarly to the proof of the term $J^{1,1}(t,s,x,y)$ in (4.32), we conclude that there exists a $\delta' := 2 - d + \frac{1}{q} + \frac{2}{\gamma\beta} > 0$ with $\gamma\beta \in]1,2[$ such that

$$R_n^{1,1,1}(t,s,x,y) \leq C 2^{-\frac{1}{2}nq\delta'} \left[|t-s|^{\alpha} + |x-y|^{\beta}\right]^q. \tag{6.12}$$

As for the term $R_n^{1,1,2}(t,s,x,y)$, we have

$$
\begin{aligned}
& R_n^{1,1,2}(t,s,x,y) \\
\leq\ & \mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)}\left|\int_0^t\int_D[\eta(t,s,x,y)](r,z)\right.\right. \\
& \left.\left.\times\left[(T_n^{1-}(r,z))^2\dot{W}_n(z,r)-\mathbf{E}[(T_n^{1-}(r,z))^2\dot{W}_n(z,r)|\mathscr{F}_{r_n}]\right]\mathrm{d}z\mathrm{d}r\right|^q\right] \\
& +\mathbf{E}\left[\mathbf{1}_{\bar{A}_n^M(t)\cap\bar{A}_n^M(s)}\left|\int_0^t\int_D[\eta(t,s,x,y)](r,z)\mathbf{E}[(T_n^{1-}(r,z))^2\dot{W}_n(z,r)|\mathscr{F}_{r_n}]\mathrm{d}z\mathrm{d}r\right|^q\right] \\
:=\ & Z_n^1(t,s,x,y)+Z_n^2(t,s,x,y).
\end{aligned}
\tag{6.13}
$$

Similarly to the proof of $J_3^{1,2}(t,s,x,y)$ in (4.36), there exists a $\delta:=-\frac{d}{2}+\frac{1}{q}+\frac{1}{\alpha\beta}>0$ with $\alpha\beta\in]1,2[$ such that

$$
Z_n^1(t,s,x,y)\leq C2^{-\frac{1}{2}nq\delta}\left[|t-s|^\alpha+|x-y|^\beta\right]^q. \tag{6.14}
$$

Since $\dot{W}_n(z,r)$ is $\mathscr{F}_{\underline{r}_n}$-measurable and we have

$$
\begin{aligned}
& \mathbf{E}\left[\dot{W}_n(z,r)\left|\int_{\underline{r}_n}^r\int_D G_{r-u}(z,v)F(X_n^-(u,v))W(\mathrm{d}v,\mathrm{d}u)\right|^2\Big|\mathscr{F}_{r_n}\right] \\
=\ & \mathbf{E}\left[\dot{W}_n(z,r)\mathbf{E}\left[\int_{\underline{r}_n}^r\int_D G_{r-u}^2(z,v)F^2(X_n^-(u,v))\mathrm{d}v\mathrm{d}u\Big|\mathscr{F}_{r_n}\right]|\mathscr{F}_{r_n}\right]=0,
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathbf{E}\left[\dot{W}_n(z,r)\left[\int_{\underline{r}_n}^r\int_D G_{r-u}(z,v)F(X_n^-(u,v))W(\mathrm{d}v,\mathrm{d}u)\right]\right. \\
& \left.\times\left[\int_{r_n}^{\underline{r}_n}\int_D G_{r-u}(z,v)F(X_n^-(u,v))W(\mathrm{d}v,\mathrm{d}u)\right]|\mathscr{F}_{r_n}\right]=0.
\end{aligned}
$$

Applying Itô rule to conclude that

$$
\begin{aligned}
& \mathbf{E}\left[(T_n^{1-}(r,z))^2\dot{W}_n(z,r)|\mathscr{F}_{r_n}\right] \\
\leq\ & Cn^{\frac{1}{2}d}2^{\frac{1}{2}n}\mathbf{E}\left[\int_{r_n}^{\underline{r}_n}\left[\int_{r_n}^w\int_D G_{r-u}(z,v)F(X_n^-(u,v))W(\mathrm{d}v,\mathrm{d}u)\right]\right. \\
& \left.\times\left[\int_{D_{\mathbf{k}}(z)}G_{r-u}(z,v)F(X_n^-(u,v))\mathrm{d}v\right]\mathrm{d}w\Big|\mathscr{F}_{r_n}\right] \\
& +Cn^{\frac{1}{2}d}2^{\frac{1}{2}n}\mathbf{E}\left[\int_{r_n}^{\underline{r}_n}W(D_{\mathbf{k}}(z)\times(r_n,w])\right. \\
& \left.\times\left[\int_D G_{r-u}^2(z,v)F^2(X_n^-(u,v))\mathrm{d}v\right]\mathrm{d}w\Big|\mathscr{F}_{r_n}\right]
\end{aligned}
$$

$$:= \quad Z_n^{2,1}(t,s,x,y)+Z_n^{2,2}(t,s,x,y). \tag{6.15}$$

However the Fubini's theorem yields that

$$\begin{aligned}
& \mathbf{E}\left[\left|Z_n^{2,1}(t,s,x,y)\right|^q\right] \\
\leq\ & \mathbf{E}\left[\left|\int_{r_n}^{\underline{r}_n}\left[\int_{r_n}^{w}\int_D G_{r-u}(z,v)F(X_n^-(u,v))W(\mathrm{d}v,\mathrm{d}u)\right]\mathrm{d}w\right|^q\right] \\
\leq\ & C2^{-nq(\frac{3}{2}-\frac{d}{8})}.
\end{aligned} \tag{6.16}$$

Note that the integral $\int_D G^2_{r-u}(z,v)F^2(X_n^-(u,v))\mathrm{d}v$ is $\mathscr{F}_{s_n}$-measurable. Hence

$$Z_n^{2,2}(t,s,x,y)=0.$$

From (A.11) and (A.12) with $\kappa := \frac{1}{\infty}-\frac{1}{p}+1$, it follows that for $\alpha\in]0,1-\frac{d}{2p}[$ and $\beta\in]0,\min\{4-\frac{2d}{p},1\}[$,

$$Z_n^2(t,s,x,y)\leq C2^{-nq(1-\frac{d}{8})}\left[|t-s|^\alpha+|x-y|^\beta\right]^q. \tag{6.17}$$

Using the estimations (6.11)–(6.14) and (6.17), we conclude that there exists a $\delta>0$ such that

$$R_n^{1,1}(t,s,x,y)\leq C2^{-nq\delta}\left[|t-s|^\alpha+|x-y|^\beta\right]^q. \tag{6.18}$$

Thus we complete the proof of the lemma. □

Now we prove the skeleton process $S(h)$ to Equation (1.3) is continuous for all $h\in\mathscr{H}$.

Lemma 6.2. *Let $\alpha\in]0,1[$ and $q\geq 4$. Then for each $a\geq 0$, there exists a $C>0$ such that when $\|h_1\|_{\mathscr{H}}\vee\|h_2\|_{\mathscr{H}}\leq a$,*

$$\left\|S(h_1)-S(h_2)\right\|_{\alpha,q}\leq C\left\|h_1-h_2\right\|_{\mathscr{H}}.$$

Proof. A similar proof to that of Theorem 1.4 in [6] yields that, for $q\geq 4$,

$$\sup_{h\in\{h\in\mathscr{H};\|h\|_{\mathscr{H}}\leq a\}}\ \sup_{t\in[0,T]}\|S(h)(t,\cdot)\|_q=C_a<\infty.$$

Recall the skeleton equation (1.3) and we have

$$\begin{aligned}
& S(h_2)(t,x)-S(h_1)(t,x) \\
=\ & \int_0^t\int_D \Delta G_{t-s}(x,y)\left[f(S(h_2)(s,y))-f(S(h_1)(s,y))\right]\mathrm{d}y\mathrm{d}s \\
& +\int_0^t\int_D G_{t-s}(x,y)\sigma(S(h_2)(s,y))\left[\mathbf{h}_2(s,y)-\mathbf{h}_1(s,y)\right]\mathrm{d}y\mathrm{d}s \\
& +\int_0^t\int_D G_{t-s}(x,y)\left[\sigma(S(h_2)(s,y))-\sigma(S(h_1)(s,y))\right]\mathbf{h}_1(s,y)\mathrm{d}y\mathrm{d}s
\end{aligned}$$

$$=: \quad \sum_{i=1}^{3} R_i(t,x)$$

Note that $u(s,\cdot), v(s,\cdot) \in L^q(D)$ for each $s \in [0,T]$, we have for $\rho = q/3$,

$$\begin{aligned}
\|u(s,\cdot)-v(s,\cdot)\|_\rho &\leq C\|u(s,\cdot)-v(s,\cdot)\|_q, \\
\|u^2(s,\cdot)-v^2(s,\cdot)\|_\rho &\leq C\|u(s,\cdot)-v(s,\cdot)\|_q \left[\|u(s,\cdot)\|_q + \|v(s,\cdot)\|_q\right] \\
\|u^3(s,\cdot)-v^3(s,\cdot)\|_\rho &\leq C\|u(s,\cdot)-v(s,\cdot)\|_q \\
&\quad \times \left[\|u(s,\cdot)\|_q^2 + \|v(s,\cdot)\|_q^2 + \|u(s,\cdot)\|_q \|v(s,\cdot)\|_q\right].
\end{aligned}$$

Then by virtue of (A.13) with $\kappa = \frac{1}{q} - \frac{3}{q} + 1 = 1 - \frac{2}{q} \in [0,1]$,

$$\left\|R_1(t,\cdot)\right\|_q \leq \int_0^t (t-s)^{-\frac{1}{2}-\frac{d}{2q}} \|S(h_2)(s,y)-S(h_1)(s,y)\|_q \mathrm{d}s.$$

On the other hand, under (H1), the Schwarz's inequality implies that

$$\left\|R_2(t,\cdot)\right\|_q \leq C\left\|h_2-h_1\right\|_{\mathscr{H}} \left\|\int_0^t \int_D [G_{t-s}(\cdot,y)]^2 \mathrm{d}y\mathrm{d}s\right\|_q \leq C\|h_2-h_1\|_{\mathscr{H}}.$$

Again by the Schwarz's inequality and (A.10) with $\kappa = \frac{1}{q} - \frac{1}{q} + 1 = 1 \in [0,1]$,

$$\|R_3(t,\cdot)\|_q \leq C\int_0^t (t-s)^{-\frac{d}{4}} \|S(h_2)(s,y)-S(h_1)(s,y)\|_q \mathrm{d}s.$$

Hence the Gronwall's lemma yields that

$$\|S(h_2)-S(h_1)\|_q \leq C\|h_2-h_1\|_{\mathscr{H}}.$$

While for $(t,x) \in \mathscr{O}_T$ and $t' \in [0,T]$,

$$\begin{aligned}
& S(h_2)(t,x) - S(h_1)(t,x) - S(h_2)(t',x) + S(h_1)(t',x) \\
= & \int_0^t \int_D [\tilde{\eta}(t,t',x,x)](s,y)\left[f(S(h_2)(s,y)) - f(S(h_1)(s,y))\right] \mathrm{d}y\mathrm{d}s \\
& + \int_0^t \int_D [\eta(t,t',x,x)(s,y)\sigma(S(h_2)(s,y))\left[\mathbf{h}_2(s,y) - \mathbf{h}_1(s,y)\right] \mathrm{d}y\mathrm{d}s \\
& + \int_0^t \int_D [\eta(t,t',x,x)](s,y)\left[\sigma(S(h_2)(s,y)) - \sigma(S(h_1)(s,y))\right] \mathbf{h}_1(s,y)\mathrm{d}y\mathrm{d}s,
\end{aligned}$$

and then by (A.11) and (A.14) with $\rho = q$ and $\rho = \frac{q}{2}$, we get for $\alpha \in]0, \frac{1}{2}[$,

$$\left\|S(h_2)(t,\cdot) - S(h_1)(t,\cdot) - S(h_2)(t',\cdot) + S(h_1)(t',\cdot)\right\|_{\alpha,q} \leq C\|h_2-h_1\|_{\mathscr{H}} |t-t'|^\alpha,$$

where the Hölder norm $\|\cdot\|_{\alpha,q} := \|\cdot\|_{\alpha,q,\infty}$, which is defined in Section 3. Thus we complete the proof of the lemma. □

Now we are at the position to prove Theorem 1.1.

Proof of Theorem 1.1. We adopt the method used in Theorem 2.1 of [1]. We only provide a sketch for this proof. For parsimony, we only prove the part (b) of Theorem 1.1, since the proof of the part (a) is similar. Given $h \in \mathscr{H}_b$, let X_n^h be the solution to Equation (2.2) with $h = F = K = 0$ and $H = \sigma$. Define $H_n : \Omega \to \mathscr{H}_b$ by

$$\dot{H}_n(s, y) = \dot{W}_n(y, s) - \dot{\sigma}(X_n(s, y))\beta_n(s, y).$$

By virtue of the uniqueness of the solution to (2.2), we have $X_n = S(H_n)$. Moreover, Proposition 3.1 yields that $X_n \to u$ in probability in $C^{\bar{\alpha},0}([0, T], L^p(D))$. Using Proposition 2.1-(i) in [13], it holds that

$$\text{supp } (\mathbf{P} \circ u^{-1}) \subset \overline{S(\mathscr{H}_b)}.$$

On the other hand, we fix $h \in \mathscr{H}_b$ and let X_n be the solution to (2.2) with $F = \sigma$, $H = -\sigma$ and $K = \sigma$. Set

$$\dot{K}_n(s, y) := \dot{h}(s, y) - \dot{\sigma}(X_n(s, y))(\alpha_n(s, y) - \beta_n(s, y)),$$

where α_n, β_n are defined in (2.3). Let $\Gamma_n^h : \Omega \to \Omega$ and

$$\Gamma_n^h(\omega) = \omega - \omega_n + K_n(\omega).$$

Then the Girsanov's theorem implies that $\mathbf{P} \circ (\Gamma_n^h)^{-1} \ll \mathbf{P}$. Note that $X_n = u \circ \Gamma_n^h$ and $X = S(h)$ and then Proposition 3.1 concludes that $u \circ \Gamma_n^h$ converges in probability to $S(h)$ in $C^{\bar{\alpha},0}([0, T]; L^p(D))$. By Proposition 2.2-(ii) in [13],

$$\overline{S(\mathscr{H}_b)} \subset \text{supp } (\mathbf{P} \circ u^{-1}).$$

Next we have to check that $\overline{S(\mathscr{H}_b)} = \overline{S(\mathscr{H})}$. Since $\mathscr{H}_b$ is dense in $\mathscr{H}$, it suffices to check that for any $0 < M < \infty$ and $\bar{\alpha} \in]0, \min\{\frac{1}{2}(1 - \frac{d}{4}), \frac{\varrho}{4}\}[$, there exists a $C > 0$ such that

$$\|S(h_2) - S(h_1)\|_{\bar{\alpha},p} \leq C\|h_1 - h_2\|_{\mathscr{H}}, \tag{6.19}$$

when $\|h_1\|_{\mathscr{H}} \vee \|h_2\|_{\mathscr{H}} < M$. Hence the desired result follows from Lemma 6.2. Thus we finish the entire proof of Theorem 1.1. □

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Appendix

In this section, we present some new estimates on the Green kernel $G_t(\cdot, *)$ corresponding to the operator $\partial/\partial t + \Delta^2$ with the homogeneous Neumann's boundary condition in (1.1). As in Da Prato and Debussche [8], the Green kernel admits the following expansion.

Let $A=-\Delta$ be defined on $\mathscr{D}(A)=\{u\in H^2(D);\ \partial u/\partial \mathbf{n}=0,\ \text{on}\ \partial D\}$ and $\{\Theta_k\}_{k\in\mathbf{N}^d}$ be the basis of eigenfunctions of A in $L^2(D)$, which can be written as

$$\Theta_k(x)=\prod_{i=1}^{d}\theta_{k_i}(x_i), \tag{A.1}$$

where $k=(k_1,\dots,k_d)\in\mathbf{N}^d,\ x=(x_1,\dots,x_d)\in D$ and

$$\begin{cases}\theta_{k_i}(x_i)=\sqrt{\frac{2}{\pi}}\cos(k_ix_i), & k_i\neq 0,\\ \theta_0(x_i)\ =\ \frac{1}{\sqrt{\pi}}, & k_i=0,\end{cases} \tag{A.2}$$

for $i=1,\dots,d$. Here $\{\lambda_k=\sum_{i=1}^{d}k_i^2\}_{k\in\mathbf{N}^d}$ are the eigenvalues corresponding to the eigenfunctions. Hence the Green kernel G on $[0,\infty)\times D$ can be expressed as

$$G_t(x,y)=\sum_{k\in\mathbf{N}^d}e^{-\lambda_k^2t}\Theta_k(x)\Theta_k(y), \tag{A.3}$$

with $(t,x,y)\in[0,\infty)\times D^2$.

The following estimates concerning the Green kernel G are quoted from [6].

Lemma A.1 *There exist $C,K>0$ such that for $t\in]0,T]$ and $x,y\in D$,*

$$\left|G_t(x,y)\right| \le \frac{K}{t^{\frac{d}{4}}}\exp\left(-C\frac{|x-y|^{\frac{4}{3}}}{|t|^{\frac{1}{3}}}\right), \tag{A.4}$$

$$\left|\Delta_y G_t(x,y)\right| \le \frac{K}{t^{\frac{d+2}{4}}}\exp\left(-C\frac{|x-y|^{\frac{4}{3}}}{|t|^{\frac{1}{3}}}\right), \tag{A.5}$$

$$\left|\frac{\partial G_t(x,y)}{\partial t}\right| \le \frac{K}{t^{\frac{d+4}{4}}}\exp\left(-C\frac{|x-y|^{\frac{4}{3}}}{|t|^{\frac{1}{3}}}\right). \tag{A.6}$$

Lemma A.2 *For $\gamma'<4-d$ and $\gamma'\le 2$, $\gamma''<1-\frac{d}{4}$, there exists $C>0$ such that for $0\le s<t\le T$ and $x,z\in D$,*

$$\int_0^t\int_D|G_{t-u}(x,y)-G_{t-u}(z,y)|^2\mathrm{d}y\mathrm{d}u \le C|x-z|^{\gamma'}, \tag{A.7}$$

$$\int_0^s\int_D|G_{t-u}(x,y)-G_{s-u}(x,y)|^2\mathrm{d}y\mathrm{d}u \le C|t-s|^{\gamma''}, \tag{A.8}$$

$$\int_s^t\int_D|G_{t-u}(x,y)|^2\mathrm{d}y\mathrm{d}u \le C|t-s|^{\gamma''}. \tag{A.9}$$

The proofs of the following lemma are similar to that of Lemma 3.1 in Gyöngy [11], which further improve the estimates in [6].

Lemma A.3 *Assume that $\rho\in[1,\infty[$, $p\in[\rho,\infty[$, $\gamma>1$ and $\kappa:=\frac{1}{p}-\frac{1}{\rho}+1\in[0,1]$. For $v\in L^\gamma([0,T],L^\rho(D))$, $0\le t_0\le t\le T$ and $x\in D$, define*

$$J(v)(t_0,t,x):=\int_{t_0}^t\int_D H(t-r,x,y)v(r,y)\mathrm{d}y\mathrm{d}r.$$

Then J is a bounded operator from $L^\gamma([0,T],L^\rho(D))$ into $L^\infty([0,T],L^p(D))$ such that the following conclusions hold.

(a) *Let $H(t,x,y)=G_t(x,y)$. Then there exists $C>0$ such that*

(1) *For all* $\gamma > \frac{1}{1+\frac{d}{4}(\kappa-1)}$,

$$\begin{aligned}\left\|J(v)(t_0,t,*)\right\|_p &\le C\int_{t_0}^t (t-r)^{\frac{d}{4}(\kappa-1)}\|V(r,*)\|_\rho \mathrm{d}r\\ &\le Ct^{1+\frac{d}{4}(\kappa-1)-\frac{1}{\gamma}}\,\|v(\cdot,*)\|_{L^\gamma([t_0,T],L^\rho(D))}\,.\end{aligned} \tag{A.10}$$

(2) *Let* $\alpha\in]0,1+\frac{d}{4}(\kappa-1)[$. *For all* $\gamma > \frac{1}{1+\frac{d}{4}(\kappa-1)-\alpha}$,

$$\|J(v)(0,t,*)-J(v)(0,s,*)\|_p \le C|t-s|^\alpha\,\|v(\cdot,*)\|_{L^\gamma([0,T],L^\rho(D))}\,, \tag{A.11}$$

with $0\le s\le t\le T$.

(3) *Let* $\beta\in]0,\min\{4-(1-\kappa)d,1\}[$. *For all* $\gamma>\max\left\{\frac{1}{1+\frac{d}{4}(\kappa-1)-\frac{\beta}{4}},\frac{1}{1-\frac{d}{4}\beta}\right\}$,

$$\|J(v)(0,t,*)-J(v)(0,t,*+z)\|_p \le C|z|^\beta\,\|v(\cdot,*)\|_{L^\gamma([0,T],L^\rho(D))}\,, \tag{A.12}$$

with $t\in[0,T]$, *where we set* $J(v)(t,y):=0$ *when* $y\in D^c$.

(b) *If* $H(t,x,y)=\Delta_y G_t(x,y)$ *(if* $d=3$, *we also need* $\kappa^{-1}<3$; *if* $d=2$, $\kappa^{-1}\neq\infty$*), there exists* $C>0$ *such that*

(1) *For all* $\gamma > \frac{1}{\frac{1}{2}+\frac{d}{4}(\kappa-1)}$,

$$\begin{aligned}\|J(v)(t_0,t,*)\|_p &\le C\int_{t_0}^t (t-r)^{-\frac{1}{2}+\frac{d}{4}(\kappa-1)}\|v(r,*)\|_\rho \mathrm{d}r\\ &\le Ct^{\frac{1}{2}+\frac{d}{4}(\kappa-1)-\frac{1}{\gamma}}\,\|v(\cdot,*)\|_{L^\gamma([t_0,T],L^\rho(D))}\,.\end{aligned} \tag{A.13}$$

(2) *Let* $\alpha\in]0,\frac{1}{2}+\frac{d}{4}(\kappa-1)[$. *For all* $\gamma > \frac{1}{\frac{1}{2}+\frac{d}{4}(\kappa-1)-\alpha}$,

$$\|J(v)(0,t,*)-J(v)(0,s,*)\|_p \le C|t-s|^\alpha\,\|v(\cdot,*)\|_{L^\gamma([t_0,T],L^\rho(D))}\,, \tag{A.14}$$

with $0\le s\le t\le T$.

(3) *Let* $\beta\in]0,\min\{2-(1-\kappa)d,\frac{2}{d},1\}[$. *For all* $\gamma>\max\left\{\frac{1}{\frac{1}{2}+\frac{d}{4}(\kappa-1)-\frac{\beta}{4}},\frac{1}{\frac{1}{2}-\frac{d}{4}\beta}\right\}$,

$$\|J(v)(0,t,*)-J(v)(0,t,*+z)\|_p \le C|z|^\beta\,\|v(\cdot,*)\|_{L^\gamma([t_0,T],L^\rho(D))}\,, \tag{A.15}$$

with $t\in[0,T]$, *where we set* $J(v)(t,y):=0$ *when* $y\in D^c$.

(c). *If* $H(t,x,y)=G^2(t,x,y)$ *(if* $d=3$, *we also need* $\kappa^{-1}<\frac{3}{2}$; *if* $d=2$, $\kappa^{-1}\neq\infty$*), there is a constant* $C>0$ *such that*

(1) *For all* $\gamma>\frac{1}{1+\frac{d}{4}(\kappa-2)}$,

$$\begin{aligned}\|J(v)(t_0,t,\cdot)\|_p &\le C\int_{t_0}^t (t-r)^{\frac{d}{4}(\kappa-2)}\|V(r,*)\|_\rho \mathrm{d}r\\ &\le Ct^{1+\frac{d}{4}(\kappa-2)-\frac{1}{\gamma}}\,\|v(\cdot,*)\|_{L^\gamma([t_0,T],L^\rho(D))}\,.\end{aligned} \tag{A.16}$$

(2) *Let* $\alpha\in]0,1+\frac{d}{4}(\kappa-2)[$, *for all* $\gamma>\frac{1}{1+\frac{d}{4}(\kappa-2)-\alpha}$,

$$\|J(v)(0,t,\cdot)-J(v)(0,s,\cdot)\|_p \le C|t-s|^\alpha\,\|v(\cdot,*)\|_{L^\gamma([t_0,T],L^\rho(D))}\,, \tag{A.17}$$

for all $0\le s\le t\le T$.

(3) *Let* $\beta\in]0,\min\{4-(2-\kappa)d,\frac{4-d}{d},1\}[$, *for all* $\gamma>\max\left\{\frac{1}{1+\frac{d}{4}(\kappa-2)-\frac{\beta}{4}},\frac{1}{1-\frac{d}{4}(\beta+1)}\right\}$,

$$\|J(v)(0,t,\cdot)-J(v)(0,t,\cdot+z)\|_p \le C|z|^\beta\,\|v(\cdot,*)\|_{L^\gamma([t_0,T],L^\rho(D))}\,, \tag{A.18}$$

for all $t\in[0,T]$, *where we set* $J(v)(t,y):=0$ *when* $y\in D^c$.

Proof. We only prove the case (**a**), since the proofs of the cases (**b**) and (**c**) are similar. We first treat (A.10). By (A.4) and the Minkowski's inequality

$$\begin{aligned}\|J(v)(t_0,t,*)\|_p &\leq \int_{t_0}^t \left\|G(t-r,*,y)v(r,y)\mathrm{d}y\right\|_p \mathrm{d}r \\ &\leq C\int_{t_0}^t |t-r|^{-d/4}\left\|\left[\exp\left(-C\frac{|\cdot|^{4/3}}{|t-r|^{1/3}}\right)*|v(r,\cdot)|\right](\cdot)\right\|_p \mathrm{d}r.\end{aligned}$$

Let $\kappa=\frac{1}{p}-\frac{1}{\rho}+1\in[0,1]$. From the Young's inequality and Hölder's inequality, it follows that, for $\gamma>\frac{1}{1+\frac{d}{4}(\kappa-1)}$,

$$\|J(v)(t_0,t,*)\|_p \leq C\int_{t_0}^t |t-r|^{\frac{d}{4}(\kappa-1)}\|v(r,*)\|_\rho \mathrm{d}r \leq Ct^{1+\frac{d}{4}(\kappa-1)-\frac{1}{\gamma}}\left[\int_{t_0}^t \|v(r,*)\|_\rho^\gamma \mathrm{d}r\right]^{1/\gamma},$$

where we have used the equality

$$\int_{\mathbf{R}^d}\exp\left(-C\frac{|x|^{\frac{4}{3}}}{t^{\frac{1}{3}}}\right)\mathrm{d}x = Kt^{\frac{d}{4}}, \tag{A.19}$$

for some constant $K>0$. This proves (A.10). Next we turn to the proof of (A.11). For any $0\leq s\leq t\leq T$,

$$\|J(v)(0,t,*)-J(v)(0,s,*)\|_p \leq A_1+A_2,$$

where

$$\begin{aligned}A_1 &= \left\|\int_0^s\int_D \left(G_{t-r}(*,y)-G_{s-r}(*,y)\right)v(r,y)\mathrm{d}y\mathrm{d}r\right\|_p, \\ A_2 &= \left\|\int_s^t\int_D G_{t-r}(*,y)v(r,y)\mathrm{d}y\mathrm{d}r\right\|_p.\end{aligned}$$

Note that for $\alpha\in]0,1[$ and $h_1,h_2\in\mathbf{R}$, it holds that

$$|h_1-h_2|\leq\left[|h_1-h_2|^\alpha\right]\left[|h_1|^{1-\alpha}+|h_2|^{1-\alpha}\right].$$

Then by the Mean-Value theorem, for $\theta\in[s,t]$,

$$\begin{aligned}A_1 &\leq \int_0^s\left\|\int_D |G_{t-r}(*,y)-G_{s-r}(*,y)|v(r,y)\mathrm{d}y\right\|_p \mathrm{d}r \\ &\leq C|t-s|^\alpha\int_0^s\left\|\int_D \frac{1}{(\theta-r)^{(\frac{d}{4}+1)\alpha}}\exp\left(-C\frac{|*-y|^{\frac{4}{3}}}{|\theta-r|^{\frac{1}{3}}}\right)\frac{v(r,y)}{(s-r)^{\frac{d}{4}(1-\alpha)}}\mathrm{d}y\right\|_p \mathrm{d}r.\end{aligned}$$

For any $\ell\in\mathbf{R}^+$ and $\epsilon>0$, we have $\exp\{-\ell\}\leq\left(\frac{\epsilon}{\ell}\right)^\epsilon$. Then

$$\begin{aligned}A_1 &\leq |t-s|^\alpha\int_0^s\left\|\int_D \frac{1}{(s-r)^{\alpha(1+\frac{d}{4})+\frac{d}{4}(1-\alpha)}}\frac{|\theta-r|^{\frac{1}{3}\epsilon}}{|*-y|^{\frac{4}{3}\epsilon}}v(r,y)\mathrm{d}y\right\|_p \mathrm{d}r \\ &\leq |t-s|^\alpha\int_0^s\left\|\int_D \frac{1}{(s-r)^{\alpha+\frac{d}{4}-\frac{1}{3}\epsilon}}\frac{1}{|*-y|^{\frac{4}{3}\epsilon}}v(r,y)\mathrm{d}y\right\|_p \mathrm{d}r.\end{aligned}$$

Apply the Young's inequality for $\kappa=\frac{1}{p}-\frac{1}{\rho}+1\in[0,1]$ to conclude that

$$A_1\leq|t-s|^\alpha\int_0^s\frac{1}{(s-r)^{\alpha+\frac{d}{4}-\frac{1}{3}\epsilon}}\left[\int_D|y|^{-\frac{4}{3\kappa}\epsilon}\mathrm{d}y\right]^\kappa\|v(r,*)\|_\rho \mathrm{d}r.$$

Let $\epsilon \in]0, \frac{3\kappa d}{4}[$ be sufficiently close to $\frac{3\kappa d}{4}$. For $\alpha \in]0, 1+\frac{d}{4}(\kappa-1)[$ and $\gamma > \frac{1}{1+\frac{d}{4}(\kappa-1)-\alpha}$, we have

$$A_1 \le |t-s|^\alpha \int_0^s \frac{1}{(s-r)^{\alpha+\frac{d}{4}-\frac{1}{3}\epsilon}} \|v(r,*)\|_\rho \mathrm{d}r \le C|t-s|^\alpha \left[\int_0^s \|v(r,*)\|_\rho^\gamma \mathrm{d}r\right]^{\frac{1}{\gamma}}. \tag{A.20}$$

As for A_2, if $\gamma > \frac{1}{\frac{d}{4}(\kappa-1)+1}$, then

$$\begin{aligned} A_2 &\le \int_s^t \left\| \int_D G_{t-r}(*,y)v(r,y)\mathrm{d}y \right\|_p \mathrm{d}r \le C\int_s^t |t-r|^{\frac{d}{4}(\kappa-1)} \|v(r,*)\|_\rho \mathrm{d}r \\ &\le C|t-s|^{1+\frac{d}{4}(\kappa-1)-\frac{1}{\gamma}} \left[\int_0^t \|v(r,*)\|_\rho^\gamma \mathrm{d}r\right]^{\frac{1}{\gamma}}. \end{aligned} \tag{A.21}$$

The estimates (A.20) and (A.21) imply (A.11). Finally we prove (A.12). Note that

$$\|J(v)(t,*) - J(v)(t,*+z)\|_p \le B_1 + B_2 + R,$$

where R admits the similar expression as that on the right hand side of (A.10) and

$$\begin{aligned} B_1 &= \int_0^t \left[\int_D \mathbf{1}_{\{x+z\in D\}} \left(\int_D \mathbf{1}_{\{|x-y|\le|z|\}} |G_{t-r}(x+z,y) - G_{t-r}(z,y)| v(r,y) dy\right)^p \mathrm{d}x\right]^{\frac{1}{p}} \mathrm{d}r, \\ B_2 &= \int_0^t \left[\int_D \mathbf{1}_{\{x+z\in D\}} \left(\int_D \mathbf{1}_{\{|x-y|>|z|\}} |G_{t-r}(x+z,y) - G_{t-r}(z,y)| v(r,y) \mathrm{d}y\right)^p \mathrm{d}x\right]^{1/p} dr. \end{aligned}$$

By (A.4),

$$\begin{aligned} B_1 \le\ & C\int_0^t \left[\int_D \left(\int_D \frac{1}{(t-r)^{d/4}} \mathbf{1}_{\{|x-y|\le|z|\}} \left[\exp\left(-C\frac{|x-y|^{\frac{4}{3}}}{|t-r|^{\frac{1}{3}}}\right)\right.\right.\right. \\ &\left.\left.\left. + \exp\left(-C\frac{|x+z-y|^{\frac{4}{3}}}{|t-r|^{\frac{1}{3}}}\right)\right] |v(r,y)| \mathrm{d}y\right)^p \mathrm{d}x\right]^{1/p} \mathrm{d}r. \end{aligned}$$

The Young's inequality for $\kappa = \frac{1}{p} - \frac{1}{\rho} + 1 \in [0,1]$ further yields that

$$B_1 \le C\int_0^t \frac{1}{(t-r)^{\frac{d}{4}}} \left[\int_0^{|z|} \left[\exp\left(-C\frac{|x-y|^{\frac{4}{3}}}{|t-r|^{\frac{1}{3}}}\right) + \exp\left(-C\frac{|x+z-y|^{\frac{4}{3}}}{|t-r|^{\frac{1}{3}}}\right)\right]^{\kappa^{-1}} \mathrm{d}y\right]^\kappa \|v(r,\cdot)\|_\rho \mathrm{d}r.$$

Let $r_1, r_2 \ge 1$ satisfy $\frac{1}{r_1} + \frac{1}{r_2} = 1$ and set $\beta = \frac{d\kappa}{r_1}$. Then

$$\begin{aligned} B_1 &\le C\int_0^t \frac{1}{(t-r)^{\frac{d}{4}}} |z|^\beta \left[\int_0^{|z|} \left[\exp\left(-C\frac{|x-y|^{\frac{4}{3}}}{|t-r|^{\frac{1}{3}}}\right) + \exp\left(-C\frac{|x+z-y|^{\frac{4}{3}}}{|t-r|^{\frac{1}{3}}}\right)\right]^{\frac{r_2}{\kappa}} \mathrm{d}y\right]^{\frac{\kappa}{r_2}} \|v(r,*)\|_\rho \mathrm{d}r \\ &\le C|z|^\beta \int_0^t \frac{1}{(t-r)^{\frac{d}{4}}} |t-r|^{\frac{d}{4}(1-\beta)} \|v(r,*)\|_\rho \mathrm{d}r \le C|z|^\beta \int_0^t \frac{1}{(t-r)^{\frac{d}{4}\beta}} \|v(r,*)\|_\rho \mathrm{d}r \\ &\le C|z|^\beta \left[\int_0^t \|v(r,*)\|_\rho^\gamma \mathrm{d}r\right]^{\frac{1}{\gamma}}, \end{aligned} \tag{A.22}$$

for $\gamma > \frac{1}{1-\frac{d}{4}\beta}$. Using the Mean-Value theorem, for $\beta < 1$ and $\phi \in [x, x+z]$,

$$\begin{aligned} B_2 = &\int_0^t \left[\int_D \mathbf{1}_{\{x+z\in D\}} \left(\int_D \mathbf{1}_{\{|x-y|>|z|\}} |G_{t-r}(x+z,y) - G_{t-r}(z,y)|^\beta \right.\right. \\ &\left.\left. \times |G_{t-r}(x+z,y) - G_{t-r}(z,y)|^{1-\beta} |v(r,y)| dy\right)^p \mathrm{d}x\right]^{\frac{1}{p}} \mathrm{d}r \end{aligned}$$

$$\leq \quad C\int_0^t \left[\int_D \left(\int_D \mathbf{1}_{\{|x-y|>|z|\}} \frac{1}{|t-r|^{\frac{\beta(d+1)}{4}}} \exp\left(-C\frac{|\phi-y|^{4/3}}{|t-r|^{1/3}}\right)|z|^\beta \right.\right.$$

$$\left.\left. \times|t-r|^{\frac{d}{4}(1-\beta)}|v(r,y)|dy\right)^p \mathrm{d}x\right]^{\frac{1}{p}} \mathrm{d}r$$

$$\leq \quad C|z|^\beta \int_0^t \frac{1}{|t-r|^{\frac{d}{4}+\frac{\beta}{4}}} \left[\int_D \left(\int_D \mathbf{1}_{\{|x-y|>|z|\}} \exp\left(-C\frac{|\phi-y|^{\frac{4}{3}}}{|t-r|^{\frac{1}{3}}}\right)|v(r,y)|\mathrm{d}y\right)^p \mathrm{d}x\right]^{\frac{1}{p}} \mathrm{d}r.$$

Note that if $|x-y|>|z|$ and $|x|\leq|\phi|\leq|x+z|$, then

$$|\phi-y|\leq\frac{1}{\sqrt{2}}|x-z+y|, \ \text{ or } \ |\phi-y|\leq\frac{1}{\sqrt{2}}|x+z+y|.$$

Hence for $\beta\in]0,\min\{4-(1-\kappa)d,1\}[$ and $\gamma>\max\left\{\frac{1}{1+\frac{d}{4}(\kappa-1)-\frac{\beta}{4}},\frac{1}{1-\frac{d}{4}\beta}\right\}$,

$$B_2\leq C|z|^\beta\int_0^t\frac{1}{|t-r|^{\frac{d}{4}(1-\kappa)+\frac{\beta}{4}}}\|v(r,*)\|_\rho \mathrm{d}r\leq C|z|^\beta\left[\int_0^t\|v(r,*)\|_\rho^\gamma \mathrm{d}r\right]^{\frac{1}{\gamma}}. \tag{A.23}$$

This further yields (A.12). □

Lemma A.4 *Let $1<\gamma<2$ and ℓ denote the positive integer such that $\ell 2^{-n}\leq t\leq(\ell+1)2^{-n}$. Then for $\theta'<4-d$ and $\theta'\leq 2$, $\theta<1-\frac{d}{4}$, there exists $C>0$ such that for $0\leq s<t\leq T$ and $x,z\in D$,*

$$\sum_{k=0}^{\ell}\left(\int_{\frac{kT}{2^n}}^{\frac{(k+1)T}{2^n}}\int_D|G_{t-r}(x,z)-G_{t-r}(y,z)|^\gamma \mathrm{d}z\mathrm{d}r\right)^{\frac{2}{\gamma}}\leq C2^{-n(\frac{2}{\gamma}-1)}|x-y|^{\theta'} \tag{A.24}$$

and

$$\sum_{k=0}^{\ell}\left(\int_{\frac{kT}{2^n}}^{\frac{(k+1)T}{2^n}}\int_D|G_{t-r}(x,z)-G_{s-r}(x,z)|^\gamma \mathrm{d}z\mathrm{d}r\right)^{\frac{2}{\gamma}}\leq C2^{-n(\frac{2}{\gamma}-1)}|t-s|^{\theta}. \tag{A.25}$$

Proof. We only prove (A.24), since the proof of (A.25) is very similar. By (A.8), (A.9) and the Hölder inequality, we have for $\theta'<4-d$ and $\theta'\leq 2$,

$$\begin{aligned}
&\sum_{k=0}^{\ell}\left(\int_{\frac{kT}{2^n}}^{\frac{(k+1)T}{2^n}}\int_D|G_{t-r}(x,z)-G_{t-r}(y,z)|^\gamma \mathrm{d}z\mathrm{d}r\right)^{\frac{2}{\gamma}}\\
\leq\ & C2^{-n(\frac{2}{\gamma}-1)}\left(\sum_{k=0}^{\ell}\int_{\frac{kT}{2^n}}^{\frac{(k+1)T}{2^n}}\int_D|G_{t-r}(x,z)-G_{t-r}(y,z)|^2 \mathrm{d}z\mathrm{d}r\right)\\
\leq\ & C2^{-n(\frac{2}{\gamma}-1)}\left(\int_0^t\int_D|G_{t-r}(x,z)-G_{t-r}(y,z)|^2 \mathrm{d}z\mathrm{d}r\right)\\
\leq\ & C2^{-n(\frac{2}{\gamma}-1)}|x-y|^{\theta'}.
\end{aligned} \tag{A.26}$$

Thus we finish the proof of the lemma. □

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