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Homogenization of semilinear PDEs with discontinuous averaged coefficients

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Abstract

We study the asymptotic behavior of solutions of semilinear PDEs. Neither periodicity nor ergodicity will be assumed. On the other hand, we assume that the coefficients have averages in the Cesaro sense. In such a case, the averaged coefficients could be discontinuous. We use a probabilistic approach based on weak convergence of the associated backward stochastic differential equation (BSDE) in the Jakubowski S-topology to derive the averaged PDE. However, since the averaged coefficients are discontinuous, the classical viscosity solution is not defined for the averaged PDE. We then use the notion of "L^p-viscosity solution" introduced in [7]. The existence of L^p-viscosity solution to the averaged PDE is proved here by using BSDEs techniques.

Key words: Backward stochastic differential equations (BSDEs), L^p-viscosity solution for PDEs, homogenization, Jakubowski S-topology, limit in the Cesaro sense.

AMS 2000 Subject Classification: Primary 60H20, 60H30, 35K60.

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1 Introduction

Homogenization of a partial differential equation (PDE) is the process of replacing rapidly varying coefficients by new ones such that the solutions are close. Let for example a be a one dimensional periodic function which is positive and bounded away from zero. For $\varepsilon > 0$, we consider the operator

$$L_\varepsilon = div(a(\frac{x}{\varepsilon})\nabla)$$

For small ε, L_ε can be replaced by

$$L = div(\overline{a}\nabla)$$

where $\overline{a}$ is the averaged (or limit, or effective) coefficient associated to a. As ε is small, the solution of the parabolic equation

$$\partial_t u = L_\varepsilon u, \qquad u(0,x) = f(x)$$

is close to the corresponding solution with L_ε replaced by L.

The probabilistic approach to homogenization is one way to prove such results in the periodic or ergodic case. It is based on the asymptotic analysis of the diffusion process associated to the operator L_ε. The averaged coefficient $\overline{a}$ is then determined as a certain "mean" of a with respect to the invariant probability measure of the diffusion process associated to L.

There is a vast literature on the homogenization of PDEs with periodic coefficients, see for example the monographs [3; 12; 21] and the references therein. There also exists a considerable literature on the study of asymptotic analysis of stochastic differential equations (SDEs) with periodic structures and its connection with homogenization of second order partial differential equations (PDEs). In view of the connection between BSDEs and semilinear PDEs, this probabilistic tool has been used in order to prove homogenization results for certain classes of nonlinear PDEs, see in particular [4; 5; 6; 9; 11; 13; 19; 23; 24] and the references therein. The two classical situations which have been mainly studied are the cases of deterministic periodic and random stationary coefficients. This paper is concerned with a different situation, building upon earlier results of Khasminskii and Krylov.

In [15], Khasminskii & Krylov consider the averaging of the following family of diffusions process

$$\begin{cases} U_t^{1,\varepsilon} = \dfrac{x_1}{\varepsilon} + \dfrac{1}{\varepsilon}\displaystyle\int_0^t \varphi(U_s^{1,\varepsilon}, U_s^{2,\varepsilon})dW_s, \\ U_t^{2,\varepsilon} = x_2 + \displaystyle\int_0^t b^{(1)}(U_s^{1,\varepsilon}, U_s^{2,\varepsilon})ds + \int_0^t \sigma^{(1)}(U_s^{1,\varepsilon}, U_s^{2,\varepsilon})d\widetilde{W}_s, \end{cases} \tag{1.1}$$

where for each $\varepsilon > 0$ small, $U_t^{1,\varepsilon}$ is a one-dimensional null-recurrent fast component and $U_t^{2,\varepsilon}$ is a d–dimensional slow component. The function φ (resp. $\sigma^{(1)}$, resp. $b^{(1)}$) is $\mathbb{R}$-valued (resp. $\mathbb{R}^{d\times(k-1)}$-valued, resp. $\mathbb{R}^d$-valued). $(W,\widetilde{W})$ is a k-dimensional standard Brownian motion whose component W (resp. $\widetilde{W}$) is one dimensional (resp. (k-1)-dimensional). Define now $(X^{1,\varepsilon}, X^{2,\varepsilon}) = (\varepsilon U^{1,\varepsilon}, U^{2,\varepsilon})$. The process $\{X_t^\varepsilon := (X_t^{1,\varepsilon}, X_t^{2,\varepsilon}),\ t \geq 0\}$ solves the SDE

$$\begin{cases} X_t^{1,\varepsilon} = x_1 + \displaystyle\int_0^t \varphi\left(\frac{X_s^{1,\varepsilon}}{\varepsilon}, X_s^{2,\varepsilon}\right) dW_s, \\ X_t^{2,\varepsilon} = x_2 + \displaystyle\int_0^t b^{(1)}\left(\frac{X_s^{1,\varepsilon}}{\varepsilon}, X_s^{2,\varepsilon}\right) ds + \int_0^t \sigma^{(1)}\left(\frac{X_s^{1,\varepsilon}}{\varepsilon}, X_s^{2,\varepsilon}\right) d\widetilde{W}_s, \end{cases} \tag{1.2}$$

They define the averaged coefficients as limits in the Cesaro sense. With the additional assumption that the presumed SDE limit is weakly unique, they prove that the process $(X_t^{1,\varepsilon}, X_t^{2,\varepsilon})$ converges in distribution towards a Markov diffusion (X_t^1, X_t^2). As a byproduct, they derive the limit behavior of the linear PDE associated to $(X_t^{1,\varepsilon}, X_t^{2,\varepsilon})$, in the case where weak uniqueness of the limiting PDE holds in the Sobolev space $W^{1,2}_{d+1,\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$ of all funcions $u(t,x)$ defined on $\mathbb{R}_+ \times \mathbb{R}^d$ such that both u and all the generalized derivatives $D_t u$, $D_x u$, and $D^2_{xx} u$ belong to $L^{d+1}_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$.

In the present note, we extend the results of [15] to parabolic semilinear PDEs. Note that the limiting coefficients can be discontinuous. More precisely, we consider the following sequence of semi-linear PDEs, indexed by $\varepsilon > 0$,

$$\begin{cases} \dfrac{\partial v^\varepsilon}{\partial t}(t, x_1, x_2) = (\mathscr{L}^\varepsilon v^\varepsilon)(t, x_1, x_2) + f(\dfrac{x_1}{\varepsilon}, x_2, v^\varepsilon(t, x_1, x_2)), \quad t > 0 \\ \quad v^\varepsilon(0, x_1, x_2) = H(x_1, x_2);\ (x^1, x^2) \in \mathbb{R} \times \mathbb{R}^d. \end{cases} \tag{1.3}$$

$$\mathscr{L}^\varepsilon := a_{00}(\frac{x_1}{\varepsilon}, x_2)\frac{\partial^2}{\partial^2 x_1} + \sum_{i,j=1}^{d} a_{ij}(\frac{x_1}{\varepsilon}, x_2)\frac{\partial^2}{\partial x_{2i}\partial x_{2j}} + \sum_{i=1}^{d} b_i^{(1)}(\frac{x_1}{\varepsilon}, x_2)\frac{\partial}{\partial x_{2i}},$$

where φ, $\sigma^{(1)}$ and $b^{(1)}$ are those defined above in equation (1.1),

$$a_{00} := \frac{1}{2}\varphi^2, \qquad a_{ij} := \frac{1}{2}(\sigma^{(1)}\sigma^{(1)*})_{ij},\ i, j = 1, ..., d,$$

and the real valued measurable functions f and H are defined on $\mathbb{R}^{d+1} \times \mathbb{R}$ and $\mathbb{R}^{d+1}$ respectively.

We put

$$b := \begin{pmatrix} 0 \\ b^{(1)} \end{pmatrix}, \quad a(x) := \frac{1}{2}(\sigma\sigma^*)(x), \quad \text{with } \sigma := \begin{pmatrix} \varphi & 0 \\ 0 & \sigma^{(1)} \end{pmatrix}.$$

We write

$$B := \begin{pmatrix} W \\ \widetilde{W} \end{pmatrix} \quad \text{and} \quad X^\varepsilon := \begin{pmatrix} X^{1,\varepsilon} \\ X^{2,\varepsilon} \end{pmatrix}.$$

The PDE (1.3) is then connected to the system of SDE – BSDE

$$\begin{cases} X_s^\varepsilon = x + \displaystyle\int_0^s b(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon})dr + \int_0^s \sigma(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon})dB_r, \\ Y_s^\varepsilon = H(X_t^\varepsilon) + \displaystyle\int_s^t f(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)dr - \int_s^t Z_r^\varepsilon\, dM_r^{X^\varepsilon},\ \forall s \in [0, t] \end{cases} \tag{1.4}$$

where M^{X^ε} is the martingale part of the process X^ε i. e.

$$M_s^{X^\varepsilon} = \int_0^s \sigma(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon})dB_r, \quad 0 \le s \le t.$$

Note that Y_0^ε does depend upon the pair (t,x) where x is the initial condition of the forward SDE part of (1.4), and t is the final time of the BSDE part of (1.4). It follows from e. g. Remark 2.6 in [22] that under suitable conditions upon the coefficients $\{v^\varepsilon(t, x) := Y_0^\varepsilon, t \ge 0,\ x = (x_1, x_2) \in \mathbb{R}^{d+1}\}$ solves the PDE (1.3).

The aim of the present paper is

1. to show that for each $t > 0$, $x \in \mathbb{R}^{d+1}$, the sequence of processes $(X_s^\varepsilon, Y_s^\varepsilon, \int_s^t Z_r^\varepsilon \, dM_r^{X^\varepsilon})_{0\le s\le t}$ converges in law to the process $(X_s, Y_s, \int_s^t Z_r \, dM_r^X)_{0\le s\le t}$ which is the unique solution to the system of SDE – BSDE

$$\begin{cases} X_s = x + \displaystyle\int_0^s \bar{b}(X_r)dr + \int_0^s \bar{\sigma}(X_r)dB_r, \; 0 \le s \le t. \\ Y_s = H(X_t) + \displaystyle\int_s^t \bar{f}(X_r, Y_r)dr - \int_s^t Z_r dM_r^X, 0 \le s \le t, \end{cases} \tag{1.5}$$

where M^X is the martingale part of X and $\bar{\sigma}$, $\bar{b}$ and $\bar{f}$ are respectively the average of σ, b and f, in a sense which will be made precise below;

2. deduce from the first result that for each (t, x), $v^\varepsilon(t, x_1, x_2) \longrightarrow v(t, x_1, x_2)$, where v solves the following averaged PDE in the L^p-viscosity sense

$$\begin{cases} \dfrac{\partial v}{\partial t}(t, x_1, x_2) = (\bar{L}v)(t, x_1, x_2) + \bar{f}(x_1, x_2, v(t, x_1, x_2)) \quad t > 0, \\ \quad v(0, x_1, x_2) = H(x_1, x_2), \end{cases} \tag{1.6}$$

with

$$\bar{L} = \sum_{i,j} \bar{a}_{ij}(x_1, x_2)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i \bar{b}_i(x_1, x_2)\frac{\partial}{\partial x_i}$$

the averaged operator.

The method used to derive the averaged BSDE is based on weak convergence in the **S**-topology and is close to that used in [23] and [24]. In our framework, we show that the limiting system of SDE – BSDE (1.5) has a unique solution. However, due to the discontinuity of the coefficients, the classical viscosity solution is not defined for the averaged PDE (1.6). We then use the notion of "L^p-viscosity solution". We use BSDE techniques to establish the existence of L^p-viscosity solution for the averaged PDE. The notion of L^p-viscosity solution has been introduced by Caffarelli *et al.* in [7] to study fully nonlinear PDEs with measurable coefficients. Note however that although the notion of a L^p-viscosity solution is available for PDEs with merely measurable coefficients, continuity of the solution is required. In our situation, the lack of L^2-continuity property for the flow $X^x := (X^{1,x}, X^{2,x})$ transfers the difficulty to the backward one and hence we cannot prove the L^2-continuity of the process Y. To overcome this difficulty, we establish weak continuity for the flow $x \mapsto (X^{1,x}, X^{2,x})$ and using the fact that Y_0^x is deterministic, we derive the continuity property for Y_0^x.

The paper is organized as follows: In section 2, we make precise some notations and formulate our assumptions. Our main results are stated in section 3. Section 4 and 5 are devoted to the proofs.

2 Notations and assumptions

2.1 Notations

For a given function $g(x_1, x_2)$, we define

$$g^+(x_2) := \lim_{x_1\to+\infty} \frac{1}{x_1}\int_0^{x_1} g(t, x_2)dt$$
$$g^-(x_2) := \lim_{x_1\to-\infty} \frac{1}{x_1}\int_0^{x_1} g(t, x_2)dt$$

The average, in Cesaro sense, of g is defined by

$$g^{\pm}(x_1, x_2) := g^+(x_2)1_{\{x_1>0\}} + g^-(x_2)1_{\{x_1\le 0\}}$$

Let $\rho(x_1, x_2) := a_{00}(x_1, x_2)^{-1}(= [\frac{1}{2}\varphi^2(x_1, x_2)]^{-1})$ and denote by $\bar{b}(x_1, x_2)$, $\bar{a}(x_1, x_2)$ and $\bar{f}(x_1, x_2, y)$, the averaged coefficients defined by

$$\begin{aligned}
\bar{b}_i(x_1, x_2) &= \frac{(\rho b_i)^{\pm}(x_1, x_2)}{\rho^{\pm}(x_1, x_2)}, \qquad i = 1, ..., d\\
\bar{a}_{ij}(x_1, x_2) &= \frac{(\rho a_{ij})^{\pm}(x_1, x_2)}{\rho^{\pm}(x_1, x_2)}, \qquad i, j = 0, 1, ..., d\\
\bar{f}(x_1, x_2, y) &= \frac{(\rho f)^{\pm}(x_1, x_2, y)}{\rho^{\pm}(x_1, x_2)}.\\
\bar{\sigma}(x_1, x_2) &= (\bar{a}(x_1, x_2))^{\frac{1}{2}}
\end{aligned}$$

where $\bar{a}(x_1, x_2)$ denotes the matrix $(\bar{a}_{ij}(x_1, x_2))_{i,j}$.

It is worth noting that $\bar{b}$, $\bar{a}$ and $\bar{f}$ may be discontinuous at $x_1 = 0$.

2.2 Assumptions.

We consider the following conditions.

(A1) The functions $b^{(1)}$, $\sigma^{(1)}$, φ are uniformly Lipschitz in the variables (x_1, x_2).

(A2) For each x_1, the first and second order derivatives with respect to x_2 of these functions are bounded continuous functions of x_2.

(A3) $a^{(1)} := \frac{1}{2}(\sigma^{(1)}\sigma^{(1)*})$ is uniformly elliptic, i. e. $\exists\Lambda > 0; \quad \forall x, \xi \in \mathbb{R}^d, \quad \xi^* a^{(1)}(x)\xi \ge \Lambda|\xi|^2$. Moreover, there exist positive constants C_1, C_2, C_3 such that

$$\begin{cases} (i) & C_1 \le a_{00}(x_1, x_2) \le C_2 \\ (ii) & |a^{(1)}(x_1, x_2)| + |b(x_1, x_2)|^2 \le C_3(1 + |x_2|^2). \end{cases}$$

(B1) Let $D_{x_2}\rho$ and $D^2_{x_2}\rho$ denote respectively the gradient vector and the matrix of second derivatives of ρ with respect to x_2. We assume that uniformly with respect to x_2

$$\frac{1}{x_1}\int_0^{x_1} \rho(t, x_2)dt \longrightarrow \rho^{\pm}(x_2) \qquad \text{as} \quad x_1 \to \pm\infty,$$

$$\frac{1}{x_1}\int_0^{x_1} D_{x_2}\rho(t, x_2)dt \longrightarrow D_{x_2}\rho^{\pm}(x_2) \qquad \text{as} \quad x_1 \to \pm\infty,$$

$$\frac{1}{x_1}\int_0^{x_1} D^2_{x_2}\rho(t, x_2)dt \longrightarrow D^2_{x_2}\rho^{\pm}(x_2) \qquad \text{as} \quad x_1 \to \pm\infty.$$

(B2) For every i and j, the coefficients ρb_i, $D_{x_2}(\rho b_i)$, $D^2_{x_2}(\rho b_i)$, ρa_{ij}, $D_{x_2}(\rho a_{ij})$, $D^2_{x_2}(\rho a_{ij})$ have averages in the Cesaro sense.

(B3) For every function $k \in \{\rho b_i, D_{x_2}(\rho b_i), D^2_{x_2}(\rho b_i), \rho a_{ij}, D_{x_2}(\rho a_{ij}), D^2_{x_2}(\rho a_{ij})\}$, there exists a bounded function $\alpha : \mathbb{R}^{d+1} \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{1}{x_1}\displaystyle\int_0^{x_1} k(t, x_2)dt - k^{\pm}(x_1, x_2) = (1+|x_2|^2)\alpha(x_1, x_2), \\ \displaystyle\lim_{|x_1|\longrightarrow\infty} \sup_{x_2\in\mathbb{R}^d} |\alpha(x_1, x_2)| = 0. \end{cases} \tag{2.1}$$

(C1)

(i) The coefficient f is uniformly Lipschitz in (x_1, x_2, y) and, for each $x_1 \in \mathbb{R}$, its derivatives in (x_2, y) up to and including second order derivatives are bounded continuous functions of (x_2, y).

(ii) There exists positive constant K such that

$$\text{for every } (x_1, x_2, y), \qquad |f(x_1, x_2, y)| \leq K(1+|x_2|+|y|).$$

(iii) H is continuous and bounded.

(C2) ρf has a limit in the Cesaro sense and there exists a bounded measurable function $\beta : \mathbb{R}^{d+2} \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{1}{x_1}\displaystyle\int_0^{x_1} \rho(t, x_2)f(t, x_2, y)dt - (\rho f)^{\pm}(x_1, x_2, y) = (1+|x_2|^2+|y|^2)\beta(x_1, x_2, y) \\ \displaystyle\lim_{|x_1|\to\infty} \sup_{(x_2, y)\in\mathbb{R}^d\times\mathbb{R}} |\beta(x_1, x_2, y)| = 0, \end{cases} \tag{2.2}$$

(C3) For each x_1, ρf has derivatives up to second order in (x_2, y) and these derivatives are bounded and satisfy (C2).

Throughout the paper, **(A)** stands for conditions (A1), (A2), (A3); **(B)** for conditions (B1), (B2), (B3) and **(C)** for (C1), (C2), (C3).

3 The main results

Consider the equation

$$X_t^x = x + \int_0^t \bar{b}(X_s^x)ds + \int_0^t \bar{\sigma}(X_s^x)dB_s,\ t \geq 0. \tag{3.1}$$

Assume that **(A), (B)** hold. Then, from Khasminskii & Krylov [15] and Krylov [18], we deduce that for each fixed, $x \in \mathbb{R}^{d+1}$ *the process* $X^\varepsilon := (X^{1,\varepsilon}, X^{2,\varepsilon})$ *converges in distribution to the process* $X := (X^1, X^2)$ which is the unique weak solution to SDE (3.1).

We now define the notion of L^p-viscosity solution of a parabolic PDE. This notion has been introduced by Caffarelli *et al.* in [7] to study PDEs with measurable coefficients. Presentations of this topic can be found in [7; 8].

Let $g : \mathbb{R}^{d+1} \times \mathbb{R} \longmapsto \mathbb{R}$ be a measurable function and

$$\bar{L} := \sum_{i,j} \bar{a}_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i \bar{b}_i(x)\frac{\partial}{\partial x_i}$$

denote the second order PDE operator associated to the SDE (3.1).

We consider the parabolic equation

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) = (\bar{L}v)(t,x) + g(x, v(t,x)),\ t \geq 0 \\ v(0,x) = H(x). \end{cases} \tag{3.2}$$

Definition 3.1. Let p be an integer such that $p > d+2$.

(a) A function $v \in \mathscr{C}\left([0,T] \times \mathbb{R}^{d+1}, \mathbb{R}\right)$ is a L^p-viscosity sub-solution of the PDE (3.2), if for every $x \in \mathbb{R}^{d+1}$, $v(0,x) \leq H(x)$ and for every $\varphi \in W^{1,2}_{p,loc}\left(\mathbb{R}_+ \times \mathbb{R}^{d+1}, \mathbb{R}\right)$ and $(\widehat{t}, \widehat{x}) \in (0,T] \times \mathbb{R}^{d+1}$ at which $v - \varphi$ has a local maximum, one has

$$\text{ess} \liminf_{(t,x)\to(\widehat{t},\widehat{x})} \left\{ \frac{\partial \varphi}{\partial t}(t,x) - (\bar{L}\varphi)(t,x) - g(x, v(t,x)) \right\} \leq 0.$$

(b) A function $v \in \mathscr{C}\left([0,T] \times \mathbb{R}^{d+1}, \mathbb{R}\right)$ is a L^p-viscosity super-solution of the PDE (3.2), if for every $x \in \mathbb{R}^{d+1}$, $v(0,x) \geq H(x)$ and for every $\varphi \in W^{1,2}_{p,loc}\left(\mathbb{R}_+ \times \mathbb{R}^{d+1}, \mathbb{R}\right)$ and $(\widehat{t}, \widehat{x}) \in (0,T] \times \mathbb{R}^{d+1}$ at which $v - \varphi$ has a local minimum, one has

$$\text{ess} \limsup_{(t,x)\to(\widehat{t},\widehat{x})} \left\{ \frac{\partial \varphi}{\partial t}(t,x) - (\bar{L}\varphi)(t,x) - g(x, v(t,x)) \right\} \geq 0.$$

Here, $G(t, x, \varphi(s,x))$ is merely assumed to be measurable upon the variable $x =: (x_1, x_2)$.

(c) A function $v \in \mathscr{C}\left([0,T] \times \mathbb{R}^{d+1}, \mathbb{R}\right)$ is a L^p-viscosity solution if it is both a L^p-viscosity sub-solution and super-solution.

Remark 3.2. Condition (a) means that for every $\varepsilon > 0$, $r > 0$, there exists a set $A \subset B_r(\widehat{t}, \widehat{x})$ of positive measure such that, for every $(s, x) \in A$,

$$\frac{\partial \varphi}{\partial s}(s, x) - (\bar{L}\varphi)(t, x) - g(x, v(t, x)) \leq \varepsilon.$$

The main results are (the **S**–topology is explained in the Appendix below)

Theorem 3.3. *Assume (A), (B), (C) hold. Then, for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^{d+1}$, there exists a process $(X_s, Y_s, Z_s)_{0 \leq s \leq t}$ such that,*

(i) the sequence of process X^ε converges in law to the continuous process X, which is the unique weak solution to SDE (1.5), in $C([0, t]; \mathbb{R}^{d+1})$ equipped with the uniform topology.

(ii) the sequence of processes $(Y_s^\varepsilon, \int_s^t Z_r^\varepsilon \, dM_r^{X^\varepsilon})_{0 \leq s \leq t}$ converges in law to the process $(Y_s, \int_s^t Z_r \, dM_r^X)_{0 \leq s \leq t}$ in $D([0, t]; \mathbb{R}^2)$, where M^X is the martingale part of X, equipped with the **S***–topology.*

(iii) (Y,Z) is the unique solution to BSDE (1.5) such that,

(a) (Y,Z) is $\mathcal{F}^X$–adapted and $(Y_s, \int_s^t Z_r \, dM_r^X)_{0 \leq s \leq t}$ is continuous.

(b) $\mathbb{E}\big(\sup_{0 \leq s \leq t} |Y_s|^2 + \int_0^t |Z_r \sigma(X_r)|^2 dr\big) < \infty$

The uniqueness means that, if (Y^1, Z^1) and (Y^2, Z^2) are two solutions of BSDE (1.5) satisfying (iii) (a)-(b) then, $\mathbb{E}\left(\sup_{0 \leq s \leq t} \left|Y_s^1 - Y_s^2\right|^2 + \int_0^t \left|Z_r^1 \sigma(X_r) - Z_r^2 \sigma(X_r)\right|^2 dr\right) = 0$, i. e. since $\sigma\sigma^*$ is elliptic (see **(A3)**), $Y_s^1 = Y_s^2 \;\; \forall 0 \leq s \leq t$, $\mathbb{P}$ a. s., and $Z_s^1 = Z_s^2 \; ds \times d\mathbb{P}$ a. e.

Theorem 3.4. *Assume (A), (B), (C) hold. For $\varepsilon > 0$, let v^ε be the unique solution to the problem (1.3). Let $(Y_s^{(t,x)})_s$ be the unique solution of the BSDE (1.5). Then*

(i) Equation (1.6) has a unique L^p-viscosity solution v such that $v(t, x) = Y_0^{(t,x)}$.

(ii) For every $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^{d+1}$, $v^\varepsilon(t, x) \to v(t, x)$, as $\varepsilon \to 0$.

4 Proof of Theorem 3.3.

In all of this section, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^{d+1}$ is arbitrarily fixed with $t > 0$.

Assertion (i) follows from [15] and [18]. Assertion (iii) can be established as in [23; 24]. We shall prove (ii). We first deduce from our assumptions (see in particular **(A3)** which says that the coefficients of the forward SDE part of (1.4) are bounded with respect to their first variable, and grow at most linearly in their second variable)

Lemma 4.1. *For all $p \geq 1$, there exists constant C_p such that for all $\varepsilon > 0$,*

$$\mathbb{E}\left(\sup_{0 \leq s \leq t} [|X_s^{1,\varepsilon}|^p + |X_s^{2,\varepsilon}|^p]\right) \leq C_p.$$

4.1 Tightness and convergence for the BSDE.

Proposition 4.2. *There exists a positive constant C such that for all $\varepsilon > 0$*

$$\mathbb{E}\left(\sup_{0\le s\le t}\left|Y_s^\varepsilon\right|^2+\int_0^t\left|Z_r^\varepsilon\sigma(X_r^\varepsilon)\right|^2dr\right)\le C.$$

Proof. We deduce from Itô's formula (here and below $\bar{X}_r^{1,\varepsilon}=X_r^{1,\varepsilon}/\varepsilon$)

$$|Y_s^\varepsilon|^2+\int_0^t\left|Z_r^\varepsilon\sigma(X_r^\varepsilon)\right|^2dr)\le|H(X_t^\varepsilon)|^2+K\int_s^t|Y_r^\varepsilon|^2dr+\int_s^t|f(\bar{X}_r^{1,\varepsilon},X_r^{2,\varepsilon},0)|^2dr$$
$$-2\int_s^t\langle Y_r^\varepsilon,Z_r^\varepsilon dM_s^{X^\varepsilon}\rangle.$$

It follows from well known results on BSDEs that we can take the expectation in the above identity (see e. g. [22]; note that introducing stopping times as usual and using Fatou's Lemma would yield (4.1) below). We then deduce from Gronwall's lemma that there exists a positive constant C which does not depend on ε, such that for every $s\in[0,t]$,

$$\mathbb{E}\left(|Y_s^\varepsilon|^2\right)\le C\mathbb{E}\left(|H(X_t^\varepsilon)|^2+\int_0^t|f(\bar{X}_r^{1,\varepsilon},X_r^{2,\varepsilon},0)|^2dr\right)$$

and

$$\mathbb{E}\left(\int_0^t\left|Z_r^\varepsilon\sigma(X_r^\varepsilon)\right|^2dr\right)\le C\mathbb{E}\left(|H(X_t^\varepsilon)|^2+\int_0^t|f(\bar{X}_r^{1,\varepsilon},X_r^{2,\varepsilon},0)|^2dr\right).\tag{4.1}$$

Combining the last two estimates and the Burkhölder-Davis-Gundy inequality, we get

$$\mathbb{E}\left(\sup_{0\le s\le t}|Y_s^\varepsilon|^2+\frac{1}{2}\int_0^t\left|Z_r^\varepsilon\sigma(X_r^\varepsilon)\right|^2dr\right)\le C\mathbb{E}\left(|H(X_t^\varepsilon)|^2+\int_0^t|f(\bar{X}_r^{1,\varepsilon},X_r^{2,\varepsilon},0)|^2dr\right)$$

In view of condition $(C1)$ and Lemma 4.1, the proof is complete. ■

We deduce immediately from Proposition 4.2

Corollary 4.3.

$$\sup_{\varepsilon>0}|Y_0^\varepsilon|<\infty.$$

Proposition 4.4. *For $\varepsilon>0$, let Y^ε be the process defined by equation* (1.4) *and M^ε be its martingale part. The sequence $(Y^\varepsilon, M^\varepsilon)_{\varepsilon>0}$ is tight in the space $\mathcal{D}([0,t],\mathbb{R})\times\mathcal{D}([0,t],\mathbb{R})$ endowed with the* **S***-topology.*

Proof. Since M^ε is a martingale, then by [20] or [14], the Meyer-Zheng tightness criteria is fulfilled whenever

$$\sup_\varepsilon\left(CV(Y^\varepsilon)+\mathbb{E}\left(\sup_{0\le s\le t}|Y_s^\varepsilon|+|M_s^\varepsilon|\right)\right)<+\infty.\tag{4.2}$$

where the conditional variation CV is defined in appendix A.

>From [25], the conditional variation $CV(Y^\varepsilon)$ satisfies

$$CV(Y^\varepsilon)\le\mathbb{E}\left(\int_0^t|f(\bar{X}_s^{1,\varepsilon},X_s^{2,\varepsilon},Y_s^\varepsilon)|ds\right),$$

Now clearly (4.2) follows from $(C1)$, Lemma 4.1 and Proposition 4.2. ■

Proposition 4.5. *There exists (Y, M) and a countable subset D of $[0, t]$ such that along a subsequence $\varepsilon_n \to 0$,*

(i) $(Y^{\varepsilon_n}, M^{\varepsilon_n}) \Longrightarrow (Y, M)$ on $\mathscr{D}([0, t], \mathbb{R}) \times \mathscr{D}([0, t], \mathbb{R})$ endowed with the **S***–topology.*

(ii) The finite dimensional distributions of $\left(Y_s^{\varepsilon_n}, M_s^{\varepsilon_n}\right)_{s \in D^c}$ converge to those of $\left(Y_s, M_s\right)_{s \in D^c}$.

(iii) $(X^{1,\varepsilon_n}, X^{2,\varepsilon_n}, Y^{\varepsilon_n}) \Longrightarrow (X^1, X^2, Y)$, in the sense of weak convergence in $C([0,t], \mathbb{R}^{d+1}) \times D([0,t], \mathbb{R})$, equipped with the product of the uniform convergence and the **S** *topology.*

Proof. (i) From Proposition 4.4, the family $(Y^\varepsilon, M^\varepsilon)_\varepsilon$ is tight in $\mathscr{D}([0, t], \mathbb{R}) \times \mathscr{D}([0, t], \mathbb{R})$ endowed with the **S**-topology. Hence along a subsequence (still denoted by ε), $(Y^\varepsilon, M^\varepsilon)_\varepsilon$ converges in law on $\mathscr{D}([0, t], \mathbb{R}) \times \mathscr{D}([0, t], \mathbb{R})$ towards a càd-làg process (Y, M).

(ii) follows from Theorem 3.1 in Jakubowski [14].

(iii) According to Theorem 3.3 (i), $(X^{1,\varepsilon}, X^{2,\varepsilon}) \Longrightarrow (X^1, X^2)$ in $C([0,t], \mathbb{R}^{d+1})$ equipped with the uniform topology. From assertion (i), $(Y^\varepsilon_\cdot)_{\varepsilon>0}$ is tight in $\mathscr{D}([0, t], \mathbb{R})$ equipped with the **S**–topology. Hence the subsequence ε_n can be chosen in such a way that (iii) holds. ∎

4.2 Identification of the limit finite variation process.

Proposition 4.6. *Let (Y, M) be any limit process as in Proposition 4.5. Then*
(i) for every $s \in [0, t] \setminus \mathrm{D}$,

$$\begin{cases} Y_s = H(X_t) + \displaystyle\int_s^t \bar{f}(X_r^1, X_r^2, Y) dr - (M_t - M_s), \\ \mathbb{E}\big(\displaystyle\sup_{0 \le s \le t} \left[|Y_s|^2 + |X_s^1|^2 + |X_s^2|^2 \right] \big) \le C; \end{cases} \tag{4.3}$$

(ii) M is a $\mathscr{F}_s$-martingale, where $\mathscr{F}_s := \sigma\left\{X_r, Y_r, \quad 0 \le r \le s\right\}$ augmented with the $\mathbb{P}$-null sets.

To prove this proposition, we need the following lemmas.

Lemma 4.7. *Assume* **(A)**, **(B)**, *(C2) and (C3). For $x_2 \in \mathbb{R}^d$, $y \in \mathbb{R}$, let $V^\varepsilon(x_1, x_2, y)$ denote the solution of the following equation:*

$$\begin{cases} a_{00}(\dfrac{x_1}{\varepsilon}, x_2) D_{x_1}^2 V^\varepsilon(x_1, x_2, y) = f(\dfrac{x_1}{\varepsilon}, x_2, y) - \bar{f}(x_1, x_2, y), \quad x_1 \in \mathbb{R}, \\ V^\varepsilon(0, x_2, y) = D_{x_1} V^\varepsilon(0, x_2, y) = 0. \end{cases} \tag{4.4}$$

Then, for some bounded functions β_1 and β_2 satisfying (2.2),
(i) $\quad D_{x_1} V^\varepsilon(x_1, x_2, y) = x_1(1 + |x_2|^2 + |y|^2)\beta_1(\frac{x_1}{\varepsilon}, x_2, y)$,
and the same is true with $D_{x_1} V^\varepsilon$ replaced by $D_{x_1} D_{x_2} V^\varepsilon$ and $D_{x_1} D_y V^\varepsilon$;

(ii) $\quad V^\varepsilon(x_1, x_2, y) = x_1^2(1 + |x_2|^2 + |y|^2)\beta_2(\frac{x_1}{\varepsilon}, x_2, y)$,
and the same is true with V^ε replaced by $D_{x_2} V^\varepsilon$, $D_y V^\varepsilon$, $D_{x_2}^2 V^\varepsilon$, $D_y^2 V^\varepsilon$ and $D_{x_2} D_y V^\varepsilon$.

Proof. We will adapt the idea of [15] to our situation. For $\varepsilon > 0$ and $(z, x_2, y) \in \mathbb{R}^{d+2}$ we set

$$F_\varepsilon(z, x_2, y) := \frac{1}{\varepsilon z}\int_0^{\varepsilon z} \rho(\frac{t}{\varepsilon}, x_2) g(\frac{t}{\varepsilon}, x_2, y) dt$$

where $g(z, x_2, y) := f(z, x_2, y) - \bar{f}(\varepsilon z, x_2, y)$.
We only treat the case where $x_1 > 0$. The same argument can be used in the case $x_1 < 0$. We successively use the definition of $\bar{f}$ and assumptions **(C2)**, to obtain

$$\begin{aligned}
F_\varepsilon(\frac{x_1}{\varepsilon}, x_2, y) &= \frac{1}{x_1}\int_0^{x_1} \rho(\frac{t}{\varepsilon}, x_2) f(\frac{t}{\varepsilon}, x_2, y) dt - (\rho f)^+(x_2, y) \\
&\quad + (\rho f)^+(x_2, y) - \frac{(\rho f)^+(x_2, y)}{\rho^+(x_2)} \frac{1}{x_1}\int_0^{x_1} \rho(\frac{t}{\varepsilon}, x_2) dt \\
&= (1 + |x_2|^2 + |y|^2)\beta(\frac{x_1}{\varepsilon}, x_2, y) \\
&\quad + \frac{(\rho f)^+(x_2, y)}{\rho^+(x_2)}\left[\rho^+(x_2) - \frac{1}{x_1}\int_0^{x_1} \rho(\frac{t}{\varepsilon}, x_2) dt\right] \\
&= (1 + |x_2|^2 + |y|^2)\beta(\frac{x_1}{\varepsilon}, x_2, y) \\
&\quad + (1 + |x_2|^2 + |y|^2)\alpha_1(\frac{x_1}{\varepsilon}, x_2, y)
\end{aligned}$$

where $\alpha_1(\frac{x_1}{\varepsilon}, x_2, y) := \frac{(\rho f)^+(x_2, y)}{(1+|x_2|^2+|y|^2)\rho^+(x_2)}\left[\rho^+(x_2) - \frac{1}{x_1}\int_0^{x_1} \rho(\frac{t}{\varepsilon}, x_2) dt\right]$.
Using assumptions **(B1)** and **(C1-ii)**, one can show that α_1 is a bounded function which satisfies (2.2). Since $D_{x_1} V^\varepsilon(x_1, x_2, y) = x_1 F_\varepsilon(\frac{x_1}{\varepsilon}, x_2, y)$, we derive the result for $D_{x_1} V^\varepsilon(x_1, x_2, y)$. Further, by integrating it, we get

$$V^\varepsilon(x_1, x_2, y) = x_1^2(1 + |x_2|^2 + |y|^2)\Big((\frac{\varepsilon}{x_1})^2 \int_0^{\frac{x_1}{\varepsilon}} t\beta_1(t, x_2, y) dt\Big),$$

where $\beta_1 = \alpha_1 + \beta$.

Clearly, $\beta_2(\frac{x_1}{\varepsilon}, x_2, y) := (\frac{\varepsilon}{x_1})^2 \int_0^{\frac{x_1}{\varepsilon}} t\beta_1(t, x_2, y) dt$ is bounded function which satisfies (2.2). The result for the other quantities can be deduced by similar arguments from assumptions **(B1)**, **(C1)**, **(C2)** and **(C3)**. ∎

Lemma 4.8. *As $\varepsilon \longrightarrow 0$,*

$$\sup_{0\le s\le t}\left|\int_0^s \left(f(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) - \bar{f}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)\right) dr\right| \to 0$$

in probability .

Proof. We shall show that for every $s \in [0, t]$, $\left|\int_0^s [f(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) - \bar{f}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)] dr\right|$ tends to zero in probability as ε tends to zero. Let V^ε denote the solution of equation (4.4). Note that

V^ε has first and second derivatives in (x_1, x_2, y) which are possibly discontinuous only at $x_1 = 0$. Then, as in [15], since φ^2 is bounded away from zero, we can use the Itô-Krylov formula to get

$$
\begin{aligned}
V^\varepsilon(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon) &= V^\varepsilon(x_1, x_2, Y_0^\varepsilon) + \int_0^s [f(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) - \bar{f}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)]dr \\
&+ \int_0^s Trace\big[a^{(1)}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}) D_{x_2}^2 V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)\big]dr \\
&+ \int_0^s [D_{x_2} V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) b^{(1)}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}) - D_y V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) f(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)]dr \\
&+ \int_0^s [D_x V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)\sigma(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}) + D_y V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) Z_r^\varepsilon \sigma(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon})]dB_r \\
&+ \frac{1}{2}\int_0^s D_y^2 V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) Z_r^\varepsilon \sigma\sigma^*(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon})(Z_r^\varepsilon)^* dr \\
&+ \frac{1}{2}\int_0^s D_x D_y V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)\sigma\sigma^*(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon})(Z_r^\varepsilon)^* dr
\end{aligned}
\tag{4.5}
$$

In view of Lemma 4.7 and Corollary 4.3, $V^\varepsilon(x_1, x_2, Y_0^\varepsilon)$ tends to zero as $\varepsilon \to 0$.
Using the fact taht $1 = 1_{\{|X_s^{1,\varepsilon}| < \sqrt{\varepsilon}\}} + 1_{\{|X_s^{1,\varepsilon}| \geq \sqrt{\varepsilon}\}}$ and Lemma 4.7, we obtain

$$
\begin{aligned}
\left|V^\varepsilon(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon)\right| &\leq \varepsilon\left[(1 + |X_s^{2,\varepsilon}|^2 + |Y_s^\varepsilon|^2)|\beta_2(\frac{X_s^{1,\varepsilon}}{\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon)|\right] \\
&\quad + 1_{\{|X_s^{1,\varepsilon}| \geq \sqrt{\varepsilon}\}} |X_s^{1,\varepsilon}|^2 \left[(1 + |X_s^{2,\varepsilon}|^2 + |Y_s^\varepsilon|^2)|\beta_2(\frac{X_s^{1,\varepsilon}}{\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon)|\right]
\end{aligned}
$$

From Lemma 4.1 and Proposition 4.2, we deduce that

$$
\mathbb{E}\left(\sup_{0\leq s\leq t} |V^\varepsilon(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon)|\right) \leq K\left(\varepsilon + \sup_{|x_1|\geq\sqrt{\varepsilon}} \sup_{(x_2, y)} |\beta_2(\frac{x^1}{\varepsilon}, x^2, y)|\right)
$$

Then, since β_2 satisfy respectively (2.2), the right hand side of the previous inequality tends to zero as $\varepsilon \longrightarrow 0$. Similarly, one can show that

$$
\begin{aligned}
&\int_0^s Trace\big[a^{(1)}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}) D_{x_2}^2 V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)\big]dr \\
&+ \int_0^s [D_{x_2} V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) b^{(1)}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}) - D_y V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) f(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)]dr \\
&+ \int_0^s [D_x V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)\sigma(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}) + D_y V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) Z_r^\varepsilon \sigma(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon})]dB_r \\
&+ \frac{1}{2}\int_0^s D_y^2 V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) Z_r^\varepsilon \sigma\sigma^*(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon})(Z_r^\varepsilon)^* dr \\
&+ \frac{1}{2}\int_0^s D_x D_y V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)\sigma\sigma^*(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon})(Z_r^\varepsilon)^* dr
\end{aligned}
$$

converges to zero in probability. Let us give an explanation concerning the one but last term, which is the most delicate one.

$$\left|\int_0^s D_y^2 V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) Z_r^\varepsilon \sigma\sigma^*(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon})(Z_r^\varepsilon)^* dr\right|$$
$$\leq C \sup_{0\leq r\leq s}\left|D_y^2 V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)\right| \text{Trace} \int_0^s Z_r^\varepsilon \sigma\sigma^*(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon})(Z_r^\varepsilon)^* dr$$

Since $\{\text{Trace} \int_0^s Z_r^\varepsilon \sigma\sigma^*(\frac{X_r^{1,\varepsilon}}{\varepsilon}, X_r^{2,\varepsilon})(Z_r^\varepsilon)^* dr,\ 0 \leq s \leq t\}$ is the increasing process associated to a martingale which is uniformly $L^p(\mathbb{P})-$integrable for each $p \in \mathbb{N}$, its $L^p(\mathbb{P})$ norm is bounded, for all $p \geq 1$. Finally the same argument as above shows that

$$\sup_{0\leq r\leq s}\left|D_y^2 V^\varepsilon(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)\right| \to 0$$

in probability, as $\varepsilon \to 0$. ∎

Lemma 4.9. $\displaystyle\int_0^\cdot \bar f(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)dr \overset{law}{\Longrightarrow} \int_0^\cdot \bar f(X_r^1, X_r^2, Y_r)dr$ *on* $\mathscr{C}([0, t], \mathbb{R})$ *as* $\varepsilon \longrightarrow 0$.

For the proof of this Lemma, we need the following two results.

Lemma 4.10. *Let* $X_s^1 := x_1 + \displaystyle\int_0^s \bar\varphi(X_r^1, X_r^2)dW_r$, $0 \leq s \leq t$, *and, assume (A2-i), (B1).*
For $\varepsilon > 0$, *let* $D_n^\varepsilon := \left\{s : s \in [0, t]\ /\ |X_s^{1,\varepsilon}| \leq \dfrac{1}{n}\right\}$.
Define also $D_n := \left\{s : s \in [0, t]\ /\ |X_s^1| \leq \dfrac{1}{n}\right\}$.
Then, there exists a constant $c > 0$ *such that for each* $n \geq 1$, $\varepsilon > 0$,

$$\mathbb{E}|D_n^\varepsilon| \leq \frac{c}{n} \qquad \text{and} \qquad \mathbb{E}|D_n| \leq \frac{c}{n},$$

where $|.|$ *denotes the Lebesgue measure.*

Proof. Consider the sequence (Ψ_n) of functions defined as follows,

$$\Psi_n(x) = \begin{cases} -\frac{x}{n} - \frac{1}{2n^2} & if \quad x \leq -\frac{1}{n} \\ \frac{x^2}{2} & if \quad -\frac{1}{n} \leq x \leq \frac{1}{n} \\ \frac{x}{n} - \frac{1}{2n^2} & if \quad x \geq 1/n \end{cases}$$

We put, $\bar\varphi := \bar a_{00} := \rho(x_1, x_2)^{-1}$.

Using Itô's formula, we get

$$\Psi_n(X_s^1) = \Psi_n(X_0^1) + \int_0^s \Psi_n^{'}(X_s^1)\bar\varphi(X_s^1, X_s^2)dW_s + \frac{1}{2}\int_0^s \Psi_n^{"}(X_s^1)\bar\varphi^2(X_s^1, X_s^2)ds,\ s \in [0, t]$$

Since $\bar{\varphi}$ is lower bounded by C_1, taking the expectation, we get

$$C_1\mathbb{E}\int_0^t 1_{[-\frac{1}{n},\frac{1}{n}]}(X_s^1)ds \le \mathbb{E}\int_0^t \Psi_n''(X_s^1)\bar{\varphi}^2(X_s^1, X_s^2)ds$$
$$= 2\mathbb{E}\left[\Psi_n(X_t^1) - \Psi_n(x_1)\right]$$

It follows that $\mathbb{E}(|D_n|) \le 2C_1^{-1}\mathbb{E}\left[\Psi_n(X_t^1) - \Psi_n(x_1)\right] \le c/n$. The same argument, applies to D_n^{ε}, allows us to show the first estimate. ■

Lemma 4.11. *Consider a collection $\{Z^{\varepsilon},\ \varepsilon > 0\}$ of real valued random variables, and a real valued random variable Z. Assume that for each $n \ge 1$, we have the decompositions*

$$Z^{\varepsilon} = Z_n^{1,\varepsilon} + Z_n^{2,\varepsilon}$$
$$Z = Z_n^1 + Z_n^2,$$

such that for each fixed $n \ge 1$,

$$Z_n^{1,\varepsilon} \Rightarrow Z_n^1$$
$$\mathbb{E}|Z_n^{2,\varepsilon}| \le \frac{c}{\sqrt{n}}$$
$$\mathbb{E}|Z_n^2| \le \frac{c}{\sqrt{n}}.$$

Then $Z^{\varepsilon} \Rightarrow Z$, as $\varepsilon \to 0$.

Proof. The above assumptions imply that the collection of random variables $\{Z^{\varepsilon},\ \varepsilon > 0\}$ is tight. Hence the result will follow from the fact that

$$\mathbb{E}\Phi(Z^{\varepsilon}) \to \mathbb{E}\Phi(Z), \quad \text{as } \varepsilon \to 0$$

for all $\Phi \in C_b(\mathbb{R})$ which is uniformly Lipschitz. Let Φ be such a function, and denote by K its Lipschitz constant. Then

$$|\mathbb{E}\Phi(Z^{\varepsilon}) - \mathbb{E}\Phi(Z)| \le \mathbb{E}|\Phi(Z^{\varepsilon}) - \Phi(Z_n^{1,\varepsilon})| + +|\mathbb{E}\Phi(Z_n^{1,\varepsilon}) - \mathbb{E}\Phi(Z_n^1)| + \mathbb{E}|\Phi(Z_n^1) - \Phi(Z)|$$
$$\le |\mathbb{E}\Phi(Z_n^{1,\varepsilon}) - \mathbb{E}\Phi(Z_n^1)| + 2K\frac{c}{\sqrt{n}}.$$

Hence

$$\limsup_{\varepsilon\to 0} |\mathbb{E}\Phi(Z^{\varepsilon}) - \mathbb{E}\Phi(Z)| \le 2K\frac{c}{\sqrt{n}},$$

for all $n \ge 1$. The result follows. ■

Proof of Lemma 4.9. For each $n \ge 1$, define a function $\theta_n \in C(\mathbb{R}, [0,1])$ such that $\theta_n(x) = 0$ for $|x| \le \frac{1}{2n}$, and $\theta_n(x) = 1$ for $|x| \ge \frac{1}{n}$. We have

$$\int_0^t \bar{f}(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^{\varepsilon})ds = \int_0^t \bar{f}(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^{\varepsilon})\theta_n(X_s^{1,\varepsilon})ds + \int_0^t \bar{f}(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^{\varepsilon})[1 - \theta_n(X_s^{1,\varepsilon})]ds$$
$$= Z_n^{1,\varepsilon} + Z_n^{2,\varepsilon}$$
$$\int_0^t \bar{f}(X_s^1, X_s^2, Y_s)ds = \int_0^t \bar{f}(X_s^1, X_s^2, Y_s)\theta_n(X_s^1)ds + \int_0^t \bar{f}(X_s^1, X_s^2, Y_s)[1 - \theta_n(X_s^1)]ds$$
$$= Z_n^1 + Z_n^2$$

Note that the mapping

$$(x^1, x^2, y) \longmapsto \int_0^t \bar{f}(x_s^1, x_s^2, y_s)\theta_n(x_s^1)ds$$

is continuous from $C([0,t]) \times D([0,t])$ equipped with the product of the sup–norm and the **S** topologies into $\mathbb{R}$. Hence from Proposition 4.5, $Z_n^{1,\varepsilon} \Longrightarrow Z_n^1$ as $\varepsilon \to 0$, for each fixed $n \geq 1$. Moreover, from Lemma 4.10, the linear growth property of $\bar{f}$, Lemma 4.1 and Proposition 4.2, we deduce that

$$E|Z_n^{2,\varepsilon}| \leq \frac{c}{\sqrt{n}}, \quad E|Z_n^2| \leq \frac{c}{\sqrt{n}}.$$

Lemma 4.9 now follows from Lemma 4.11. ■

Proof of Proposition 4.6 Passing to the limit in the backward component of the equation (1.4) and using Lemmas 4.8 and 4.9, we derive assertion (i).
Assertion (ii) can be proved by using the same arguments as those in section 6 of [24]. ■

4.3 Identification of the limit martingale.

Since $\bar{f}$ is uniformly Lipschitz in y and H is bounded, then standard arguments of BSDEs (see e. g. [23]) show that the BSDE (1.5) has a strongly unique solution and we have,

Proposition 4.12. *Let $(\bar{Y}_s, \bar{Z}_s, 0 \leq s \leq t)$ be the unique solution to BSDE (1.5). Then, for every $s \in [0,\, t]$,*

$$\mathbb{E}|Y_s - \bar{Y}_s|^2 + \mathbb{E}\left([M - \int_0^{\cdot} \bar{Z}_r dM_r^X]_t - [M - \int_0^{\cdot} \bar{Z}_r dM_r^X]_s\right) = 0.$$

Proof. For every $s \in [0,\, t] \setminus \mathrm{D}$, we have

$$\begin{cases} Y_s = H(X_t) + \int_s^t \bar{f}(X_r, Y_r)dr - (M_t - M_s) \\ \bar{Y}_s = H(X_t) + \int_s^t \bar{f}(X_r, \bar{Y}_r)dr - \int_s^t \bar{Z}_r dM_r^X \end{cases}$$

Arguing as in [24], we show that $\bar{M} := \int_s^{\cdot} \bar{Z}_r dM_r^X$ is a $\mathscr{F}_s$-martingale.
Since $\bar{f}$ satisfies condition $(C1)$, we get by Itô's formula, that

$$\mathbb{E}|Y_s - \bar{Y}_s|^2 + \mathbb{E}\left([M - \int_0^{\cdot} \bar{Z}_r dM_r^X]_t - [M - \int_0^{\cdot} \bar{Z}_r dM_r^X]_s\right) \leq C\mathbb{E}\int_s^t |Y_r - \bar{Y}_r|^2 dr.$$

Therefore, Gronwall's lemma yields that $\mathbb{E}|Y_s - \bar{Y}_s|^2 = 0$, $\forall s \in [0,\, t] - \mathrm{D}$.
Since $\bar{Y}$ is continuous, Y is càd-lag and D is countable, then $Y_s = \bar{Y}_s$, $\mathbb{P}$-$a.s$, $\forall s \in [0,\, t]$.
Moreover, we deduce that, $\mathbb{E}\left([M - \int_0^{\cdot} \bar{Z}_r dM_r^X]_t - [M - \int_0^{\cdot} \bar{Z}_r dM_r^X]_s\right) = 0.$ ■

As a consequence of Proposition 4.12, we have

Corollary 4.13. $\left(Y^{\varepsilon}, \int_0^{\cdot} Z_r^{\varepsilon} dM_r^{X^{\varepsilon}}\right) \overset{law}{\Longrightarrow} \left(\bar{Y}, \int_0^{\cdot} \bar{Z}_r dM_r^{X}\right).$

Theorem 3.3 is proved.

5 Proof of Theorem 3.4.

Since the SDE (3.1) is weakly unique ([18]), the martingale problem associated to $X = (X^1, X^2)$ is well posed. We then have the following:

Proposition 5.1. (i) *For any $t > 0$, $x \in \mathbb{R}^d$, the BSDE*

$$Y_s^{t,x} = H(X_t^x) + \int_s^t \bar{f}(X_r^x, Y_r^{t,x})dr - \int_s^t Z_r^{t,x} dM_r^{X^x}, 0 \le s \le t.$$

admits a unique solution $(Y_s^{t,x}, Z_s^{t,x})_{0\le s\le t}$ such that the component $(Y_s^{t,x})_{0\le s\le t}$ is bounded and $Y_0^{t,x}$ is deterministic.

(ii) *If moreover, the deterministic function, $(t, x) \in [0, T] \times \mathbb{R}^{d+1} \longmapsto v(t, x) := Y_0^{t,x}$ belongs to $\mathscr{C}\left([0, T] \times \mathbb{R}^{d+1}, \mathbb{R}\right)$, then it is a L^p-viscosity solution of the PDE (3.2).*

Remark. The continuity of the map $(t, x) \longmapsto v(t, x) := Y_0^{t,x}$, which is assumed in assertion (ii) of Propostion 5.1, will be established in Proposition 5.3 below.

Proof of Proposition 5.1. (i) Thanks to Remark 3.5 of [23], it is enough to prove existence and uniqueness for the BSDE

$$Y_s^{t,x} = H(X_t^x) + \int_s^t \bar{f}(X_r^x, Y_r^{t,x})dr - \int_s^t Z_r^{t,x} dB_r, 0 \le s \le t.$$

Since f satisfies **(C)** and ρ is bounded, one can easily verify that $\bar{f}$ is uniformly Lipschitz in y uniformly with respect to (x_1, x_2) and satisfies (**C1**)-(ii). Existence and uniqueness of solution follow then from standard results for BSDEs, see e. g. [22]. Moreover, since H is uniformly bounded and $\bar{f}$ satisfies the linear growth condition (**C1**)-(ii), one can prove that the solution $Y^{t,x}$ is bounded, see e. g. [1]. Finally, since $(Y_s^{t,x})$ is $\mathscr{F}_s^X-$adapted then $Y_0^{t,x}$ is measurable with respect to a trivial $\sigma-$algebra and hence it is deterministic.

(ii) Assume that the function $v(t, x) := Y_0^{t,x}$ belongs to $\mathscr{C}\left([0, T] \times \mathbb{R}^{d+1}, \mathbb{R}\right)$. We only prove that v is a L^p–viscosity sub–solution. The proof of the super–solution property can be done similarly. Since the coefficient of PDE under consideration are time homogeneous, then $v(t, x)$ is solution to the initial value problem (1.6) if and only if the function $u(t, x) := v(T - t, x)$ is solution to the terminal value problem.

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) = (\bar{L}u)(t, x) + \bar{f}(x, u(t, x)) \quad t \in [0,\ T], \\ \quad u(T, x) = H(x). \end{cases} \tag{5.1}$$

Working with this backward PDE will simplify the details of the proofs below.

Let $X_s^{t,x}$ be the unique weak solution to SDE (3.1). We will establish that the solution Y of the Markovian BSDE

$$Y_s^{t,x} = H(X_T^{t,x}) + \int_s^T \bar{f}(X_r^{t,x}, Y_r^{t,x})dr - \int_s^T Z_r^{t,x} dM_r^{X^{t,x}}, \qquad 0 \le t \le s \le T. \tag{5.2}$$

define a L^p–viscosity sub–solution to the problem (5.1) by puting $u(t,x) := Y_t^{t,x}$.

Let $\varphi \in W_{p,loc}^{1,2}\left([0,T] \times \mathbb{R}^{d+1}, \mathbb{R}\right)$, let $(\widehat{t}, \widehat{x}) \in [0,T] \times \mathbb{R}^{d+1}$ be a point which is a local maximum of $u - \varphi$. Since $p > d+2$, then φ has a continuous version which we consider from now on. We assume without loss of generality that

$$v(\widehat{t}, \widehat{x}) = \varphi(\widehat{t}, \widehat{x}) \tag{5.3}$$

We will argue by contradiction. Assume that there exists $\varepsilon, \alpha > 0$ such that

$$\frac{\partial \varphi}{\partial s}(s,x) + \bar{L}\varphi(s,x) + \bar{f}(x, u(s,x)) < -\varepsilon, \;\; \lambda\text{–}a.e. \text{ in } B_\alpha(\widehat{t}, \widehat{x}). \tag{5.4}$$

where λ denote the Lebesgue measure.

Since $(\widehat{t}, \widehat{x})$ is a local maximum of $u - \varphi$, we can find a positive number α' (which we can suppose equal to α) such that

$$u(t,x) \le \varphi(t,x) \qquad \text{in} \quad B_\alpha(\widehat{t}, \widehat{x}) \tag{5.5}$$

Define

$$\tau = \inf\left\{s \ge \widehat{t}, ; \quad |X_s^{\widehat{t},\widehat{x}} - \widehat{x}| > \alpha\right\} \wedge (\widehat{t} + \alpha)$$

Since X is a Markov diffusion and $\bar{f}$ is uniformly Lipschitz in y and satisfies condition (**C1**)-(ii), then arguing as in [10], one can show that for every $r \in [\widehat{t}, \, \widehat{t} + \alpha]$, $Y_r^{\widehat{t},\widehat{x}} = u(r, X_r^{\widehat{t},\widehat{x}})$. Hence, the process $(\bar{Y}_s, \bar{Z}_s) := ((Y_{s\wedge\tau}^{\widehat{t},\widehat{x}}), \mathbb{1}_{[0,\tau]}(s)(Z_s^{\widehat{t},\widehat{x}}))_{s\in[\widehat{t}, \, \widehat{t}+\alpha]}$ solves the BSDE

$$\begin{aligned} \bar{Y}_s \quad &= \quad u(\tau, X_\tau^{\widehat{t},\widehat{x}}) + \int_s^{\widehat{t}+\alpha} \mathbb{1}_{[0,\tau]} \bar{f}(r, X_r^{\widehat{t},\widehat{x}}, u(r, X_r^{\widehat{t},\widehat{x}}))dr \\ &- \int_s^{\widehat{t}+\alpha} \bar{Z}_r dM_r^{X^{\widehat{t},\widehat{x}}}, \qquad s \in [\widehat{t}, \, \widehat{t} + \alpha]. \end{aligned}$$

On other hand, by Itô-Krylov formula, the process $(\widehat{Y}_s, \widehat{Z}_s)_{s\in[\widehat{t}, \, \widehat{t}+\alpha]}$, defined by $(\widehat{Y}_s, \widehat{Z}_s) := \Big(\varphi(s\wedge\tau, X_{s\wedge\tau}^{\widehat{t},\widehat{x}}), \; \mathbb{1}_{[0,\tau]}(s)\nabla\varphi(s, X_s^{\widehat{t},\widehat{x}})\Big)$ solves the BSDE

$$\begin{aligned} \widehat{Y}_s \quad &= \quad \varphi(\tau, X_\tau^{\widehat{t},\widehat{x}}) - \int_s^{\widehat{t}+\alpha} \mathbb{1}_{[0,\tau]}[(\frac{\partial \varphi}{\partial r} + \bar{L}\varphi)(r, X_r^{\widehat{t},\widehat{x}})]dr \\ &- \int_s^{\widehat{t}+\alpha} \widehat{Z}_r dM_r^{X^{\widehat{t},\widehat{x}}}. \end{aligned}$$

From the choice of τ, $(\tau, X_\tau^{\widehat{t},\widehat{x}}) \in B_\alpha(\widehat{t}, \widehat{x})$. Therefore, $u(\tau, X_\tau^{\widehat{t},\widehat{x}}) \le \varphi(\tau, X_\tau^{\widehat{t},\widehat{x}})$.

Let $A := \{(t,x) \in B_\alpha(\widehat{t}, \widehat{x}), [\frac{\partial \varphi}{\partial s} + \bar{L}\varphi + \bar{f}(.,u(.))](t,x) < -\varepsilon\}$ and $\bar{A} := B_\alpha(\widehat{t}, \widehat{x}) \setminus A$ the complement of A. By (5.4), $\lambda(\bar{A}) = 0$.

Since the diffusion $\{X_s^{\hat{t},\hat{x}}, s \geq t\}$ is nondegenerate, Krylov's inequality ([17], Ch. 2, Sec. 2 & 3) implies that $1\!\!1_{\bar{A}}(r, X_r^{\widehat{t},\widehat{x}}) = 0 \;\; dr \times d\mathbb{P}-\, a.e.$ It follows that

$$\mathbb{E}\int_{\widehat{t}}^{\widehat{t}+\alpha} -1\!\!1_{[0,\tau]}[(\frac{\partial \varphi}{\partial r} + \bar{L}\varphi)(r, X_r^{\widehat{t},\widehat{x}}) + \bar{f}(r, X_r^{\widehat{t},\widehat{x}}, u(r, X_r^{\widehat{t},\widehat{x}}))]])dr \geq \mathbb{E}(\tau - \widehat{t})\,\varepsilon > 0 \qquad (5.6)$$

This implies that $[-1\!\!1_{[0,\tau]}[(\frac{\partial \varphi}{\partial r} + \bar{L}\varphi)(r, X_r^{\widehat{t},\widehat{x}}) + \bar{f}(r, X_r^{\widehat{t},\widehat{x}}, u(r, X_r^{\widehat{t},\widehat{x}}))]])] > 0$ on a set of $dt \times d\mathbb{P}$ positive measure. Therefore, the strict comparison theorem (Remark 2.5 in [23]) shows that $\bar{Y}_{\widehat{t}} < \widehat{Y}_{\widehat{t}}$, that is $u(\widehat{t}, \widehat{x}) < \varphi(\widehat{t}, \widehat{x})$, which contradicts our assumption (5.3).

Under assumptions **(A), (B)**, the SDE (3.1) has a unique weak solution, see [18]. We then have the following continuity property.

Proposition 5.2. *(Continuity in law of the map $x \mapsto X_.^x$)*

Assume **(A), (B)**. *Let X_s^x be the unique weak solution of the SDE (3.1), and*

$$X_s^n := x_n + \int_0^s \bar{b}(X_r^n)dr + \int_0^s \bar{\sigma}(X_r^n)dB_r, \; 0 \leq s \leq t$$

Assume that $x_n \to x = (x^1, x^2) \in \mathbb{R}^{1+d}$ as $n \to \infty$. Then $X^n \overset{law}{\Longrightarrow} X^x$.

Proof. Since $\bar{b}$ and $\bar{\sigma}$ satisfy **(A), (B)**, one can easily check that the sequence X^n is tight in $\mathscr{C}([0,t] \times \mathbb{R}^{d+1})$. By Prokhorov's theorem, there exists a subsequence (denoted also by X^n) which converges weakly to a process $\widehat{X}$. We shall show that $\widehat{X}$ is a weak solution of SDE (3.1).

• *Step 1:* For every $\varphi \in C_c^\infty(\mathbb{R}^{1+d})$,

$$\forall u \in [0,t], \quad \varphi(\widehat{X}_r) - \int_0^u \bar{L}\varphi(\widehat{X}_v)dv \quad \text{is a } \mathscr{F}^{\widehat{X}}\text{-martingale.}$$

All we need to show is that for every $\varphi \in C_c^\infty(\mathbb{R}^{1+d})$, every $0 \leq s \leq u$ and every function Φ_s of $(X_r^{x_n})_{0 \leq r < s}$ which is bounded and continuous for the topology of uniform convergence, as $n \to \infty$,

$$0 = \mathbb{E}\left\{[\varphi(X_r^{x_n}) - \varphi(X_s^{x_n}) - \int_s^r \bar{L}\varphi(X_\alpha^{x_n})d\alpha]\Phi_s(X_.^{x_n})\right\}$$
$$\longrightarrow \mathbb{E}\left\{[\varphi(\widehat{X}_r) - \varphi(\widehat{X}_s) - \int_s^r \bar{L}\varphi(\widehat{X}_\alpha)d\alpha]\Phi_s(\widehat{X}_.)\right\}$$

Indeed, since φ, Φ are continuous functions and $\bar{L}\varphi$ is continuous away from the set $\{x_1 = 0\}$, similar argument as that developed in the proof of Lemma 4.9 gives

$$[\varphi(X_r^{x_n}) - \varphi(X_s^{x_n}) - \int_s^u \bar{L}\varphi(X_v^{x_n})dv]\Phi_s(X_.^{x_n}) \xrightarrow{law} [\varphi(\widehat{X}_r) - \varphi(\widehat{X}_s) - \int_s^u \bar{L}\varphi(\widehat{X}_v)dv]\Phi_s(\widehat{X}_.)$$

Since φ, Φ are bounded functions, $\bar{L}\varphi$ has at most linear growth at infinity and

$$\sup_n \mathbb{E}(\sup_{s\in[0,t]} |X^{x_n}|^2) < \infty,$$

the result follows by uniform integrability. Hence

$$\mathbb{E}\left\{[\varphi(\widehat{X}_r)-\varphi(\widehat{X}_s)-\int_s^u \bar{L}\varphi(\widehat{X}_v)dv]\Phi_s(\widehat{X}_.)\right\}=0$$

and therefore $\varphi(\widehat{X}_r)-\varphi(\widehat{X}_s)-\int_s^r \bar{L}\varphi(\widehat{X}_v)dv$ is a $\mathscr{F}_r^{\widehat{X}}$–martingale.

•*Step 2:* From *step 1*, there exists a $\mathscr{F}^{\widehat{X}}$-Brownian motion $\widehat{B}$ such that,

$$\widehat{X}_s = x+\int_0^s \bar{b}(\widehat{X}_r)dr+\int_0^s \bar{\sigma}(\widehat{X}_r)d\widehat{B}_r, \quad 0\le s\le t.$$

Weak uniqueness of the SDE (3.1) allows us to deduce that $\widehat{X}=X^x$ in law sense. ■

Proposition 5.3. *Assume* **(A), (B), (C)**. *Then,*

(i) $\lim\limits_{\varepsilon\to 0} Y_0^\varepsilon = Y_0^{t,x}$.

(ii) *The map* $(t,x)\longmapsto Y_0^{t,x}$ *is continuous.*

(iii) *For* $p>d+2$*, the function* $v(t,x):=Y_0^{t,x}$ *is a* L^p*-viscosity solution to the PDE (1.6).*

Proof. (i) Let $Y^{t,x}$ be the limit process defined in Proposition 4.5. We have

$$\begin{cases} Y_0^\varepsilon = H(X_t^\varepsilon)+\displaystyle\int_0^t f(\frac{X_r^{1,\varepsilon}}{\varepsilon},X_r^{2,\varepsilon},Y_r^\varepsilon)dr-M_t^\varepsilon \\ \\ Y_0^{t,x}=H(X_t^x)+\displaystyle\int_0^t \bar{f}(X_r^x,Y_r^{t,x})dr-M_t \end{cases}$$

From Jakubowski [14], the projection: $y\mapsto y_t$ is continuous from $\mathscr{D}([0,\,t];\mathbb{R})$ into $\mathbb{R}$ for the **S**–topology. We then deduce from the convergence of the above right–hand sides that Y_0^ε converges towards Y_0 in distribution. Since Y_0^ε and Y_0 are deterministic, this means exactly that $Y_0^\varepsilon\to Y_0$

(ii) Let $(t_n,\,x_n)\to(t,\,x)$. We assume that $t>t_n>0$. We have,

$$Y_s^{t_n,x_n} \;=\; H(X_{t_n}^{x_n})+\int_s^{t_n}\bar{f}(X_r^{x_n},\,Y_r^{t_n,x_n})dr-\int_s^{t_n} Z_r^{t_n,x_n}dM_r^{X^{x_n}},\ 0\le s\le t_n, \tag{5.7}$$

where $X^{x_n}\overset{law}{\Rightarrow}X^x$.
Since H is a bounded continuous function and $\bar{f}$ satisfies $(C1)$, one can easily show that the sequence $\{(Y^{t_n,x_n},\int_0^{\cdot} 1_{[s,t_n]}(u)Z_r^{x_n}dM_r^{X^{x_n}})\}_{n\in\mathbb{N}^*}$ is tight in $\mathscr{D}([0,\,t];\mathbb{R}^2)$.
Let us rewrite the equation (5.7) as follows

$$\begin{aligned} Y_s^{t_n,x_n} &= H(X_{t_n}^{x_n})+\int_s^t \bar{f}(X_r^{x_n},\,Y_r^{t_n,x_n})dr-\int_s^t 1_{[s,t_n]}(u)Z_r^{t_n,x_n}dM_r^{X^{x_n}} \qquad (5.8)\\ &- \int_{t_n}^t \bar{f}(X_r^{x_n},\,Y_r^{t_n,x_n})dr,\ 0\le s\le t.\\ &= A_n^1+A_n^2 \end{aligned}$$

• *Convergence of* A_n^2

Since $\bar{f}$ is bounded, $\mathbb{E}\left|\int_{t_n}^{t} \bar{f}(X_r^{x_n}, Y_r^{t_n,x_n})dr\right| \leq K|t-t_n|$. Hence A_n^2 tends to zero in probability.

• *Convergence of* A_n^1

Denote by (Y', M') the weak limit of $\{(Y^{t_n,x_n}, \int_0^{\cdot} 1_{[s,t_n]}(u)Z_r^{x_n}dM_r^{X^{x_n}})\}_{n\in\mathbb{N}^*}$. The same proof as that of Lemma 4.9 establishes that $\int_s^t \bar{f}(X_r^{x_n}, Y_r^{t_n,x_n})dr \overset{law}{\Longrightarrow} \int_s^t \bar{f}(X_r^{x}, Y_r')dr$.

Passing to the limit in (5.8), we obtain that

$$Y_s' \quad = \quad H(X_t^x) + \int_s^t \bar{f}(X_r^x, Y_r')dr - (M_t' - M_s'),\ s \in [0, t] \cap D^c.$$

The uniqueness of the considered BSDE ensures that $\forall s \in [0, t]$, $Y_s' = Y_s^{t,x}$ $\mathbb{P}$-ps. Hence $Y^{t_n,x_n} \overset{law}{\Rightarrow} Y^{t,x}$. As in (i), one derive that $Y_0^{t_n,x_n} \overset{law}{\Longrightarrow} Y_0^{t,x}$ which yields to the continuity of $Y_0^{t,x}$.

Assertion (iii) follows from (ii) and the second statement of Proposition 5.1. ■

A Appendix: S-topology

The **S**–topology has been introduced by Jakubowski ([14], 1997) as a topology defined on the Skorohod space of càdlàg functions: $\mathcal{D}([0, T]; \mathbb{R})$. This topology is weaker than the Skorohod topology but tightness criteria are easier to establish. These criteria are the same as the one used in Meyer-Zheng [20].

Let $N^{a,b}(z)$ denotes the number of up-crossing of the function $z \in \mathcal{D}([0, T]; \mathbb{R})$ from level a to level b $(a < b)$. We recall some facts about the **S**–topology.

Proposition A.1. *(A criteria for S-tight). A sequence $(Y^\varepsilon)_{\varepsilon>0}$ is said to be* **S***–tight if and only if it is relatively compact for the* **S***–topology.*

Let $(Y^\varepsilon)_{\varepsilon>0}$ be a family of stochastic processes in $\mathcal{D}([0, T]; \mathbb{R})$. Then this family is tight for the **S***–topology if and only if $(\|Y^\varepsilon\|_\infty)_{\varepsilon>0}$ and $(N^{a,b}(Y^\varepsilon))_{\varepsilon>0}$ are tight for each $a < b$.*

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ be a stochastic basis. If $(Y)_{0\leq t\leq T}$ is a process in $\mathcal{D}([0, T]; \mathbb{R})$ such that Y_t is integrable for any t, *the conditional variation of* Y is defined by

$$CV(Y) = \sup_{n\geq 1,\ 0\leq t_1<\ldots<t_n=T} \sum_{i=1}^{n-1} \mathbb{E}[|\mathbb{E}[Y_{t_{i+1}} - Y_{t_i} \mid \mathcal{F}_{t_i}]|].$$

The process Y is called a *quasimartingale* if $CV(Y) < +\infty$. When Y is a $\mathcal{F}_t$-martingale, $CV(Y) = 0$. A variation of Doob's inequality (cf. lemma 3, p. 359 in Meyer and Zheng [20], where it is assumed that $Y_T = 0$) implies that

$$\mathbb{P}\left[\sup_{t\in[0,T]} |Y_t| \geq k\right] \leq \frac{2}{k}\left(CV(Y) + \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|\right]\right),$$

$$\mathbb{E}\left[N^{a,b}(Y)\right] \leq \frac{1}{b-a}\left(|a| + CV(Y) + \mathbb{E}\left[\sup_{t\in[0,T]}|Y_t|\right]\right).$$

It follows that a sequence $(Y^\varepsilon)_{\varepsilon>0}$ is **S**-tight whenever

$$\sup_{\varepsilon>0}\left(CV(Y^\varepsilon) + \mathbb{E}\left[\sup_{t\in[0,T]}|Y_t^\varepsilon|\right]\right) < +\infty.$$

Theorem A.2. *Let $(Y^\varepsilon)_{\varepsilon>0}$ be a **S**-tight family of stochastic process whose trajectories belong to $\mathcal{D}([0,T];\mathbb{R})$. Then there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ decreasing to zero, some process $Y \in \mathcal{D}([0,T];\mathbb{R})$ and a countable subset $D \in [0,T]$ such that for any $n \geq 1$ and any $(t_1, ..., t_n) \in [0,T]\backslash D$,*

$$(Y_{t_1}^{\varepsilon_k}, ..., Y_{t_n}^{\varepsilon_k}) \xrightarrow{\mathcal{D}ist} (Y_{t_1}, ..., Y_{t_n})$$

Remark A.3. The projection $\pi_T : y \in (\mathcal{D}([0,T];\mathbb{R}), \mathbf{S}) \mapsto y(T)$ is continuous (see Remark 2.4, p.8 in Jakubowski [14]), but $y \mapsto y(t)$ is not continuous for each $0 \leq t \leq T$.

Lemma A.4. *Let $(U^\varepsilon, M^\varepsilon)$ be a multidimensional process in $\mathcal{D}([0,T];\mathbb{R}^p)(p \in \mathbb{N}^*)$ converging to (U, M) in the S-topology. Let $(\mathcal{F}_t^{U^\varepsilon})_{t\geq 0}$ (resp. $(\mathcal{F}_t^{U})_{t\geq 0}$) be the minimal complete admissible filtration generated by U^ε (resp. U). We assume moreover that for every $T > 0$, $\sup_{\varepsilon>0}\mathbb{E}\left[\sup_{0\leq t\leq T}|M_t^\varepsilon|^2\right] < C_T$.*
If M^ε is a $\mathcal{F}^{U^\varepsilon}$-martingale and M is $\mathcal{F}^U$-adapted, then M is a $\mathcal{F}^U$-martingale.

Lemma A.5. *Let $(Y^\varepsilon)_{\varepsilon>0}$ be a sequence of process converging weakly in $\mathcal{D}([0,T];\mathbb{R}^p)$ to Y. We assume that $\sup_{\varepsilon>0}\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t^\varepsilon|^2\right] < +\infty$. Then for any $t \geq 0$, $E\left[\sup_{0\leq t\leq T}|Y_t|^2\right] < +\infty$.*

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