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COMPETING SPECIES SUPERPROCESSES WITH INFINITE VARIANCE

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Abstract We study pairs of interacting measure-valued branching processes (superprocesses) with α–stable migration and $(1+\beta)$–branching mechanism. The interaction is realized via some killing procedure. The collision local time for such processes is constructed as a limit of approximating collision local times. For certain dimensions this convergence holds uniformly over all pairs of such interacting superprocesses. We use this uniformity to prove existence of a solution to a competing species martingale problem under a natural dimension restriction. The competing species model describes the evolution of two populations where individuals of different types may kill each other if they collide. In the case of Brownian migration and finite variance branching, the model was introduced by Evans and Perkins (1994). The fact that now the branching mechanism does not have finite variance requires the development of new methods for handling the collision local time which we believe are of some independent interest.

Keywords Superprocess with killing, competing superprocesses, interactive superprocesses, superprocess with immigration, measure-valued branching, interactive branching, state-dependent branching, collision measure, collision local time, martingale problem.

AMS subject classification (2000) Primary 60K35; Secondary 60G57, 60J80.

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Contents

1. Introduction

1.1. **Background, motivation, and purpose.** Measure-valued Markov branching processes (superprocesses) arise as limits of branching particle systems undergoing random migration and critical (or nearly critical) branching. In the last decade, there has been interest in the study of such processes with interactions. Superprocesses with *finite* variance branching have received much attention in the study of models with interactions (see, e.g., [EP94, Per95, DP98, Myt98b, MP00, FX01, DEF$^+$02a, DEF$^+$02b, DFM$^+$03]). In the present paper, we would like to pay attention to interactive superprocesses with *infinite* variance branching where a killing mechanism is at play (including a point type interaction).

In the case of superprocesses with finite branching mechanism, Evans and Perkins [EP94] initiated the study of a pair of non-supercritical continuous super-Brownian motions in R^d with an additional point interaction (the preparation for that model was done by Barlow et al. [BEP91]). This model is introduced to describe two populations of competing species where inter-species "collisions" result in causalities on either side. To be more precise, when different species come within an infinitesimal distance of each other, then either of the colliding (infinitesimal) individuals is killed with an infinitesimal probability. By this interaction, the basic independence assumption in branching theory is violated, and hence the usual tools (such as log-Laplace transforms and equations) of handling superprocesses break down. On the other hand, a very handy feature of the model is that it is bounded from above by two *independent* critical continuous super-Brownian motions (opposed to the case of the more complicated mutually catalytic branching model of Dawson and Perkins [DP98] and Mytnik [Myt98b]; see [DF02] for a recent survey).

Among the others, the following basic tools were used in [EP94] to construct such a competing species model in dimensions $d<4$:
(i) a Girsanov type theorem of Dawson [Daw78], and
(ii) a Tanaka type formula for collision local times of some interacting continuous super-Brownian motions in $\mathsf{R}^d,\ d<6,$ from Barlow et al. [BEP91, Theorem 5.9].

Our *main purpose* is to construct a competing species model of this type (Theorem 9), while dropping the finite variance branching mechanism assumption. The basic idea is the same as in [EP94]: Prove a continuity result for the collision local times for a large class of processes (this is Theorem 7 below) and then take limits in a mollified equation. The technical ingredients needed to implement this program are necessarily quite different due to the lack of higher moments, the Girsanov formula (i) and the Tanaka formula (ii). In fact, Girsanov's theorem requires continuity in time of the martingale component, and hence finite variance of the branching mechanism, which we want to give up. Moreover, Barlow et al.'s proof of already mentioned Tanaka formula relies on some uniform in time and space bound of continuous super-Brownian motion's mass in small balls (see [BEP91, Theorem 4.7 and Corollary 4.8]). This bound is not anymore true under our assumptions, and validity of the Tanaka formula in the more general setup remains *open.* So, we had to find a different approach to construct the desired competing species model.

To prove convergence of approximating collision local times, we heavily apply a log-Laplace technique, originally used for collision local times in [Myt98a]. In our interacting model, an application of log-Laplace tools is possible since we can represent the collision local time for the pair of interacting superprocesses with killing as a linear combination, see (228) below, of collision local times for independent and conditionally independent measure-valued processes. This representation, which also holds for approximating collision local times, is crucial for our proofs and is based on the domination of interacting pairs by a pair of independent critical superprocesses. The domination property is an extension of an analogous result proved by [BEP91] for finite variance super-Brownian motions.

In this paper, we also allow the two species to have their own branching and motion indices and parameters. But to simplify the exposition, we keep the assumption that in the branching mechanisms no supercritical components are involved.

Here we would like to mention that the questions of non-existence of the model in higher dimensions and uniqueness for the corresponding martingale problem were studied in [EP94], [EP98], and [Myt99] for continuous competing super-Brownian motions. Indeed, it was proved in [EP94] that the model does not exist in dimensions $d \geq 4$, and that there is a unique solution to the corresponding martingale problem in $d = 1$. The uniqueness for the particular symmetric case was proved in [Myt99] for all $d \leq 3$. The historical uniqueness for $d \leq 3$ was proved in [EP98]. In the general model which we consider in this paper, the questions of non-existence in higher dimensions and uniqueness remain as interesting *open problems.*

1.2. **Sketch of main results.** Let us start with a brief modelling. For $i = 1, 2$, fix constants

$$0 < \alpha^i \leq 2, \quad 0 < \beta^i \leq 1, \quad \text{and} \quad \vartheta^i > 0 \tag{1}$$

(please, do not misunderstand frequently appearing right upper indices as powers, which also occur), and introduce the (weighted) fractional Laplacian $\Delta_{\alpha^i} := -\vartheta^i(-\Delta)^{\alpha^i/2}$ in R^d. Moreover, let $\bigl(\xi^i, \Pi^i_{r,x}, r \geq 0, x \in \mathsf{R}^d\bigr)$ denote symmetric α^i–stable processes with generator Δ_{α^i}, semigroup $\mathrm{S}^i = \{\mathrm{S}^i_t : t \geq 0\}$, and continuous transition kernel $\mathrm{p}^i = \{\mathrm{p}^i_t(y) : t > 0, y \in \mathsf{R}^d\}$, where, for convenience, we use a time-inhomogeneous writing of the laws $\Pi^i_{r,x}$, even though the processes are time-homogeneous. The α^i–stable process ξ^i will serve as the motion process of individuals of type i.

Let $\mathcal{M}_\mathrm{f} = \mathcal{M}_\mathrm{f}(\mathsf{R}^d)$ denote the set of finite (non-negative) measures equipped with the weak topology. Write $\langle \mu, f \rangle$ for the integral $\int \mu(\mathrm{d}x)\, f(x)$ and $\|\mu\|$ for the total mass $\langle \mu, 1 \rangle$. If we have pairs $\boldsymbol{\mu} = (\mu^1, \mu^2)$ and $\mathbf{f} = (f^1, f^2)$ instead, by the abuse of notation we write

$$\langle \boldsymbol{\mu}, \mathbf{f} \,\rangle \;:=\; \bigl\langle \mu^1, f^1 \bigr\rangle + \bigl\langle \mu^2, f^2 \bigr\rangle. \tag{2}$$

Such pair $\boldsymbol{\mu}$ will describe a state of the system.

Here is the rough description of a *martingale problem* $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{\mu}}$ for a pair $\mathbf{X} = \bigl(X^1, X^2\bigr)$ of interacting superprocesses in R^d starting from $\mathbf{X}_0 = \boldsymbol{\mu}$, where X^i has the underlying α^i–stable symmetric motion and the branching index $1+\beta^i$. We impose a killing described by a pair $\mathbf{A} = (A^1, A^2)$ of (random, possibly dependent) measures on $\mathsf{R}_+ \times \mathsf{R}^d$, where by the abuse of notation $A^i_t := A^i\bigl([0,t] \times \cdot\,\bigr) < \infty$, $t \geq 0$, are also considered as non-decreasing $\mathcal{M}_\mathrm{f}$–valued processes. (For more precise formulations, see Definition 5 below.) In fact, the pairs $\mathbf{X} = (X^1, X^2)$ and $\mathbf{A} = (A^1, A^2)$ are assumed to be $\mathcal{F}_\cdot$–adapted $\mathcal{M}^2_\mathrm{f}$–valued processes on some stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_\cdot, \mathcal{P})$ starting from $\mathbf{A}_0 = \mathbf{0}$, and such that $t \mapsto \mathbf{A}_t$ is non-decreasing and continuous, and, for each pair $\boldsymbol{\varphi} = (\varphi^1, \varphi^2)$ of suitable non-negative test functions on R^d,

$$t \;\mapsto\; \mathrm{e}^{-\langle \mathbf{X}_t,\boldsymbol{\varphi}\rangle} - \mathrm{e}^{-\langle \boldsymbol{\mu},\boldsymbol{\varphi}\rangle} + \int_0^t \mathrm{d}s\; \mathrm{e}^{-\langle \mathbf{X}_s,\boldsymbol{\varphi}\rangle} \Big\langle \mathbf{X}_s\,,\, \Delta_{\boldsymbol{\alpha}}\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\mathbf{1}+\boldsymbol{\beta}} \Big\rangle \tag{3}$$
$$- \int_0^t \langle \mathbf{A}_{\mathrm{d}s}\,, \boldsymbol{\varphi}\rangle\, \mathrm{e}^{-\langle \mathbf{X}_s,\boldsymbol{\varphi}\rangle}, \qquad t \ge 0,$$

is an $\mathcal{F}_\cdot$–martingale starting from 0 at time $t = 0$. Here, in the spirit of our convention (2),

$$\Big\langle \mathbf{X}_s\,,\, \Delta_{\boldsymbol{\alpha}}\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\mathbf{1}+\boldsymbol{\beta}} \Big\rangle \;:=\; \sum_{i=1,2} \Big\langle X_s^i\,,\, \Delta_{\alpha^i}\varphi^i - (\varphi^i)^{1+\beta^i} \Big\rangle \tag{4}$$

and, if $I \subseteq \mathsf{R}_+$ is an interval,

$$\langle \mathbf{A}, \boldsymbol{\psi}\rangle_I \;:=\; \int_I \langle \mathbf{A}_{\mathrm{d}s}\,, \boldsymbol{\psi}_s\rangle \;:=\; \sum_{i=1,2} \int_{I\times\mathsf{R}^d} A^i\big(\mathrm{d}(s,x)\big)\, \psi_s^i(x), \tag{5}$$

for suitable pairs of time-space functions $\boldsymbol{\psi} = (\psi^1,\psi^2) \ge 0$. We call $\mathbf{X}$ an $(\boldsymbol{\alpha},d,\boldsymbol{\beta})$–*pair of interacting superprocesses* in R^d starting from $\mathbf{X}_0 = \boldsymbol{\mu}$, and with *killing mechanism* $\mathbf{A}$. Note that, in general, due to the interaction via $\mathbf{A}$, neither $\mathbf{X}$ nor the X^i are superprocesses in the original meaning (as, for instance, in Dynkin [Dyn94]). Note also that $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ implies that

$$t \;\mapsto\; \big\langle X_t^i, \varphi^i\big\rangle - \big\langle \mu^i, \varphi^i\big\rangle - \int_0^t \mathrm{d}s\; \big\langle X_s^i, \Delta_{\alpha^i}\varphi^i\big\rangle + \big\langle A_t^i, \varphi^i\big\rangle, \qquad i = 1,2, \tag{6}$$

are $\mathcal{F}_\cdot$–martingales starting from 0 at time $t = 0$, for each choice of the test functions $\boldsymbol{\varphi}$ used in (3) (see Corollary 20.) From (6), the presence of the additional killing described by $\mathbf{A}$ is clear. Actually, we will show that $\mathbf{X}$ is almost surely dominated by a pair $\bar{\mathbf{X}} = (\bar{X}^1, \bar{X}^2)$ of *independent* critical (α^i, d, β^i)–superprocesses $\bar{X}^i$, $i = 1,2$, which uniquely solves the martingale problem $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ if the A^i–terms are set to be zero [see Proposition 21(b)].

Next we recall the notion of *collision local time* $L_{\mathbf{Y}}$ of a pair $\mathbf{Y} = (Y^1, Y^2)$ of $\mathcal{M}_{\mathrm{f}}$–valued processes. Loosely speaking, it is the measure

$$L_{\mathbf{Y}}\big(\mathrm{d}(s,x)\big) \;=\; \mathrm{d}s\; Y_s^1(\mathrm{d}x) \int_{\mathsf{R}^d} Y_s^2(\mathrm{d}y)\; \delta_0(x-y) \tag{7}$$

on $\mathsf{R}_+ \times \mathsf{R}^d$. A bit more carefully, it is the limit in probability

$$\big\langle L_{\mathbf{Y}}(t),\, f\big\rangle \;:=\; \lim_{\varepsilon\downarrow 0} \int_0^t \mathrm{d}s \int_{\mathsf{R}^d} Y_s^1(\mathrm{d}x) \int_{\mathsf{R}^d} Y_s^2(\mathrm{d}y)\; J_\varepsilon(x-y)\, f\big(\tfrac{x+y}{2}\big), \tag{8}$$

if it exists for all $t > 0$ and all bounded continuous functions f on R^d, where J_ε is a regularization of the δ–function δ_0 (for more details, see Definition 1). Note again that most of the time we will consider $L_{\mathbf{Y}}$ as a non-decreasing measure-valued process $\{L_{\mathbf{Y}}(t) : t \ge 0\}$ where, with an abuse of notation, $L_{\mathbf{Y}}(t)\,(B) := L_{\mathbf{Y}}\big([0,t] \times B\big)$.

In the case of the mentioned pair $\bar{\mathbf{X}}$ of *independent* critical (α^i, d, β^i)–superprocesses $\bar{X}^i$, $i = 1,2$, the collision local time $L_{\bar{\mathbf{X}}}$ exists non-trivially, provided that

$$1 \;\le\; d \;<\; \frac{\alpha^1}{\beta^1} + \frac{\alpha^2}{\beta^2} + (\alpha^1 \vee \alpha^2), \tag{9}$$

see Mytnik [Myt98a, Theorem 1(ii)]. Now, at least intuitively, the domination $\mathbf{X} \le \bar{\mathbf{X}}$ suggests that also the collision local time $L_{\mathbf{X}}$ may exist non-trivially for the $(\boldsymbol{\alpha}, d, \boldsymbol{\beta})$–pair $\mathbf{X}$ of interacting superprocesses in these dimensions. But this does not necessarily imply that in these dimensions the desired competing species model exists. Actually, for this model to make sense one expects that a *single* "intrinsic particle" collides with the other population with positive probability. This, in turn, will require a stronger dimension restriction.

Our *main result* will state, roughly speaking, that in dimensions d satisfying

$$1 \le d < \Big(\frac{\alpha^1}{\beta^1} + \alpha^2\Big) \wedge \Big(\frac{\alpha^2}{\beta^2} + \alpha^1\Big), \tag{10}$$

for all solutions $(\mathbf{X}, \mathbf{A})$ of the martingale problem $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ the convergence $L_{\mathbf{X}}^{\varepsilon} \to L_{\mathbf{X}}$ as $\varepsilon \downarrow 0$ holds *uniformly* in $(\mathbf{X}, \mathbf{A})$, provided that the pair $\mathbf{X}_0$ of initial measures satisfies an energy condition (see Theorem 7).

Note that with $\alpha^i \equiv 2$ and $\beta^i \equiv 1$ we recover results of [BEP91, Theorems 5.9 and 5.10], where (9) and (10) are read as

$$1 \le d < 6 \quad \text{and} \quad 1 \le d < 4, \tag{11}$$

respectively.

As an *application* we construct the desired more general competing species model (see Theorem 9). Roughly speaking, for this we will replace both A^1 and A^2 in the martingale problem $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ by a multiple of the continuous collision local time $L_{\mathbf{X}}$. Here the dimension restriction (10) is intuitively clear. In fact, recall that $\alpha^i \wedge d$ is the dimension of the range of an α^i–stable process, and $(\alpha^i/\beta^i) \wedge d$ is the carrying dimension of the non-empty support of an (α^i, d, β^i)–superprocess at fixed times. Hence, for example, in dimensions d satisfying

$$\frac{\alpha^1}{\beta^1} + \alpha^2 \le d < \frac{\alpha^2}{\beta^2} + \alpha^1, \tag{12}$$

an "intrinsic type 2 particle" is *not* expected to collide with the type 1 population, whereas an "intrinsic type 1 particle" *may* collide with the type 2 population. In other words, then in (3) we should effectively have $A^2 = 0$ in the martingale problem $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ (this means, degeneration to a "one-sided interaction"). Note that with $\alpha^i \equiv 2$ and $\beta^i \equiv 1$ we recover the model introduced in [EP94, Theorem 3.6].

2. Statement of results

In this section we will state our main results, Theorems 7 and 9.

2.1. Preliminaries: Notation, collision local time and measure.

With c we always denote a positive constant which might change from place to place. The symbol $c_{\#}$ however refers to a specific constant which occurred first around formula line (#).

If $(\Omega, \mathcal{F})$ is a measurable space, write $\mathrm{b}\mathcal{F}$ for the set of all bounded measurable functions $f : \Omega \to \mathsf{R}$. In particular, $\mathrm{b}\mathcal{B} = \mathrm{b}\mathcal{B}(\mathsf{R}^d)$ denotes the space of all bounded measurable functions $f : \mathsf{R}^d \to \mathsf{R}$, and $\mathrm{b}\mathcal{C} = \mathrm{b}\mathcal{C}(\mathsf{R}^d)$ the subspace of bounded continuous functions. Write $f \in \bar{\mathcal{C}} = \bar{\mathcal{C}}(\mathsf{R}^d)$ if $f \in \mathrm{b}\mathcal{C}$ can be continued to a continuous function on the one-point compactification $\dot{\mathsf{R}}^d$ of R^d. Equipped with the supremum norm $\|\cdot\|_\infty$, the Banach space $\bar{\mathcal{C}}$ is separable. Denote by Φ the subset $\bar{\mathcal{C}}^{(2)} = \bar{\mathcal{C}}^{(2)}(\mathsf{R}^d)$ of all functions $f \in \bar{\mathcal{C}}$ which have the first two derivatives in

$\bar{\mathcal{C}}$. Working again with supremum norms, Φ is a separable Banach space, too. We will take Φ_+^2 (the non-negative cone of Φ^2) as the set of test functions $\boldsymbol{\varphi} = (\varphi^1, \varphi^2)$ in our martingale problems. Note that Φ coincides with the domain $\mathcal{D}(\Delta_\alpha)$ of the fractional Laplacian $-(-\Delta)^{\alpha/2}$, for each $0 < \alpha \leq 2$ (see, for instance, [Yos74, Section 9.11 and Example 9.5.2]). We also need the set Ψ_+^2, where $\Psi = \bar{\mathcal{C}}^{(1,2)} = \bar{\mathcal{C}}^{(1,2)}(\mathsf{R}_+ \times \mathsf{R}^d)$ is the set of all functions on $\mathsf{R}_+ \times \mathsf{R}^d$, which derivatives up to the order $(1,2)$ can be extended to continuous functions on $\mathsf{R}_+ \times \dot{\mathsf{R}}^d$.

For constants $\varrho \geq 1$ and $T > 0$, introduce the Lebesgue spaces

$$\mathcal{L}^\varrho \;:=\; \mathcal{L}^\varrho(\mathsf{R}^d, \mathrm{d}x) \quad \text{and} \quad \mathcal{L}^{\varrho,T} \;:=\; \mathcal{L}^\varrho\big((0,T) \times \mathsf{R}^d, \mathrm{d}s\,\mathrm{d}x\big). \tag{13}$$

Let $\mathcal{D}_{\mathcal{M}_\mathrm{f}} = \mathcal{D}\big(\mathsf{R}_+, \mathcal{M}_\mathrm{f}(\mathsf{R}^d)\big)$ denote the space of all càdlàg paths $\nu : \mathsf{R}_+ \to \mathcal{M}_\mathrm{f}$ equipped with the Skorohod topology. If it is not stated otherwise, under a random process we will understand a random element in $\mathcal{D}_{\mathcal{M}_\mathrm{f}}$ or $\mathcal{D}_{\mathcal{M}_\mathrm{f}}^2$ over some stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_\cdot, \mathcal{P})$ (which we might need to enlarge from time to time). In this case, by a slight abuse of notation, we simply write $Y \in \mathcal{D}_{\mathcal{M}_\mathrm{f}}$ or $\mathbf{Y} \in \mathcal{D}_{\mathcal{M}_\mathrm{f}}^2$, respectively. Sometimes we consider also continuous processes: $Y \in \mathcal{C}_{\mathcal{M}_\mathrm{f}} := \mathcal{C}\big(\mathsf{R}_+, \mathcal{M}_\mathrm{f}(\mathsf{R}^d)\big)$ equipped with the topology of uniform convergence on compact subsets of R_+.

For $Y \in \mathcal{D}_{\mathcal{M}_\mathrm{f}}$, let $\mathcal{F}_t^Y$ denote the completion of the σ–field $\bigcap_{\varepsilon>0} \sigma\{Y_s : \; s \leq t + \varepsilon\}$, $t \geq 0$.

Recall next the notion of collision local time [compare with (7) and (8)]:

Definition 1 (**Collision local time $L_\mathbf{Y}$**)**.** Let $\mathbf{Y} = (Y^1, Y^2)$ be a pair of random processes (in $\mathcal{D}_{\mathcal{M}_\mathrm{f}}$). A non-decreasing random process $t \mapsto L_\mathbf{Y}(t) = L_\mathbf{Y}(t, \cdot\,)$ (in $\mathcal{D}_{\mathcal{M}_\mathrm{f}}$) is called the *collision local time* of the pair $\mathbf{Y}$, if we have the convergence in probability

$$\big\langle L_\mathbf{Y}^\varepsilon(t), f \big\rangle \;\xrightarrow[\varepsilon\downarrow 0]{\mathcal{P}}\; \big\langle L_\mathbf{Y}(t), f \big\rangle, \qquad t \geq 0, \quad f \in \mathrm{b}\mathcal{C}. \tag{14}$$

Here the approximating collision local times $L_\mathbf{Y}^\varepsilon = L_\mathbf{Y}^{\varepsilon,J}$ are defined by

$$\big\langle L_\mathbf{Y}^\varepsilon(t), f \big\rangle \;:=\; \int_0^t \mathrm{d}s \int_{\mathsf{R}^d} Y_s^1(\mathrm{d}x) \int_{\mathsf{R}^d} Y_s^2(\mathrm{d}y)\, J_\varepsilon(x-y)\, f\big(\tfrac{x+y}{2}\big), \tag{15}$$

where

$$J_\varepsilon(x) \;:=\; \varepsilon^{-d} J(x/\varepsilon), \qquad x \in \mathsf{R}^d, \tag{16}$$

and J is a *mollifier*, that is, a non-negative, continuous, radially symmetric function on R^d with support in the unit ball in R^d and total mass $\int_{\mathsf{R}^d} \mathrm{d}x \; J(x) = 1$. (Sometimes we write $L_\mathbf{Y}^{\varepsilon,J}$ instead of $L_\mathbf{Y}^\varepsilon$ to stress the dependence on J within the definition of approximating collision local times.) Moreover, the limit $L_\mathbf{Y}$ is required to be independent of the choice of the mollifier J. ◇

It is also convenient to give up the symmetry in the definition of the collision local time:

Lemma 2 (**Equivalent definition of collision local time**)**.** *In Definition* 1 *of the collision local time one can replace the approximating collision local times* $L_\mathbf{Y}^\varepsilon = L_\mathbf{Y}^{\varepsilon,J}$ *by* $L_\mathbf{Y}^{1,\varepsilon} = L_\mathbf{Y}^{1,\varepsilon,J}$ *or* $L_\mathbf{Y}^{2,\varepsilon} = L_\mathbf{Y}^{2,\varepsilon,J}$ *defined by*

$$\big\langle L_\mathbf{Y}^{1,\varepsilon}(t), f \big\rangle \;:=\; \int_0^t \mathrm{d}s \int_{\mathsf{R}^d} Y_s^1(\mathrm{d}y)\, Y_s^2 {*} J_\varepsilon\,(y)\, f(y), \qquad t \geq 0, \quad f \in \mathrm{b}\mathcal{C}, \tag{17}$$

and

$$L^{2,\varepsilon}_{\mathbf{Y}} \;:=\; L^{1,\varepsilon}_{(Y^2,Y^1)}\,. \tag{18}$$

Proof. Follows easily by the uniform continuity of f on compacts; see, for instance, the proof of Lemma 3.4 in [EP94]. ■

We will also need the notation of collision measure for a pair of random processes in $\mathcal{D}_{\mathcal{M}_\mathrm{f}}$:

Definition 3 (**Collision measure** $K_{\mathbf{Y}}$)**.** Let $\mathbf{Y}=(Y^1,Y^2)$ be a pair of random processes (in $\mathcal{D}_{\mathcal{M}_\mathrm{f}}$). A progressively measurable $\mathcal{M}_\mathrm{f}$–valued process $t\mapsto K_{\mathbf{Y}}(t)=K_{\mathbf{Y}}(t,\,\cdot\,)$ is called the *collision measure* of the pair $\mathbf{Y}$, if we have the convergence in probability

$$\big\langle K^{\varepsilon}_{\mathbf{Y}}(t),f\big\rangle \ \underset{\varepsilon\downarrow 0}{\overset{\mathcal{P}}{\longrightarrow}}\ \big\langle K_{\mathbf{Y}}(t),f\big\rangle, \qquad t>0,\quad f\in \mathrm{b}\mathcal{C}. \tag{19}$$

Here the approximating collision measures $K^{\varepsilon}_{\mathbf{Y}}=K^{\varepsilon,J}_{\mathbf{Y}}$ are defined by

$$\big\langle K^{\varepsilon}_{\mathbf{Y}}(t),f\big\rangle \;:=\; \int_{\mathsf{R}^d} Y^1_t(\mathrm{d}x)\int_{\mathsf{R}^d} Y^2_t(\mathrm{d}y)\,J_\varepsilon(x-y)\,f\big(\tfrac{x+y}{2}\big), \tag{20}$$

with J_ε as in (16). Again, the limit $K_{\mathbf{Y}}$ is required to be independent of the choice of the mollifier J. ◇

Remark 4 (**Open problem**)**.** It seems to be an open problem whether the process $t\mapsto K_{\mathbf{Y}}(t)$ can be realized in $\mathcal{D}_{\mathcal{M}_\mathrm{f}}$. ◇

2.2. $(\boldsymbol{\alpha},d,\boldsymbol{\beta})$**–pair of interacting superprocesses with killing.** First we will make precise the martingale problem $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{\mu}}$ mentioned around (3).

Definition 5 (**Martingale problem** $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{\mu}}$)**.** For pairs $\boldsymbol{\alpha}$,$\boldsymbol{\beta}$, and $\boldsymbol{\vartheta}$ as in (1), and $\boldsymbol{\mu}=(\mu^1,\mu^2)\in\mathcal{M}_\mathrm{f}^2$, let $\mathbf{X}=(X^1,X^2)$ and $\mathbf{A}=(A^1,A^2)$ be $\mathcal{F}_\cdot$–adapted processes (in $\mathcal{D}^2_{\mathcal{M}_\mathrm{f}}$) such that $t\mapsto\mathbf{A}_t$ is non-decreasing, continuous, starting from $\mathbf{A}_0=\mathbf{0}$, and, for each $\boldsymbol{\varphi}=(\varphi^1,\varphi^2)\in\Phi^2_+$,

$$\begin{aligned} t \ \mapsto\ \mathrm{e}^{-\langle \mathbf{X}_t,\boldsymbol{\varphi}\rangle}-\mathrm{e}^{-\langle \boldsymbol{\mu},\boldsymbol{\varphi}\rangle}+\int_0^t \mathrm{d}s\;\mathrm{e}^{-\langle \mathbf{X}_s,\boldsymbol{\varphi}\rangle}\Big\langle \mathbf{X}_s\,,\,\Delta_{\boldsymbol{\alpha}}\boldsymbol{\varphi}-\boldsymbol{\varphi}^{\mathbf{1}+\boldsymbol{\beta}}\Big\rangle \\ -\int_0^t \langle \mathbf{A}_{\mathrm{d}s}\,,\boldsymbol{\varphi}\rangle\,\mathrm{e}^{-\langle \mathbf{X}_s,\boldsymbol{\varphi}\rangle}, \qquad t\ge 0, \end{aligned} \tag{21}$$

is an $\mathcal{F}_\cdot$–martingale starting from 0 at time $t=0$ [where we used obvious conventions as in the formula lines (4) and (5)]. Then we say that $(\mathbf{X},\mathbf{A})$ *solves the martingale problem* $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{\mu}}$. ◇

A solution $(\mathbf{X},\mathbf{A})$ of this martingale problem is called an $(\boldsymbol{\alpha},d,\boldsymbol{\beta})$*–pair* $\mathbf{X}=(X^1,X^2)$ *of interacting superprocesses* in R^d starting from $\mathbf{X}_0=\boldsymbol{\mu}$ and with *killing mechanism* $\mathbf{A}$. (For existence of a solution, think of the case $\mathbf{A}=\mathbf{0}$ of independent critical superprocesses; see, for instance, Lemma 14(a) below with $\boldsymbol{\kappa}=\mathbf{0}={}^{\circ}\!\mathbf{A}$.)

We will use the following terminology.

Definition 6 (**Measures of finite energy**)**.** The pair $\boldsymbol{\mu}=(\mu^1,\mu^2)\in\mathcal{M}_\mathrm{f}^2$ of measures is said to have *finite energy* if

$$\int_0^1 \mathrm{d}s\int_{\mathsf{R}^d}\mathrm{d}x\;\mu^1\!*\mathrm{p}^1_s\,(x)\,\mu^2\!*\mathrm{p}^2_s\,(x)\;<\;\infty. \tag{22}$$

In this case we write $\boldsymbol{\mu}\in\mathcal{M}_{\mathrm{f,e}}$. ◇

Note that $\boldsymbol{\mu}\in\mathcal{M}_{\mathrm{f}}^2$ has certainly finite energy if one of the measures μ^i is absolutely continuous with a bounded density function. On the other hand, the finite energy condition is violated if $d>1$ and there is a $z\in\mathsf{R}^d$ such that $\mu^1\bigl(\{z\}\bigr)\,\mu^2\bigl(\{z\}\bigr)>0$.

Here is our *main result:*

Theorem 7 (**Uniform convergence in approximating** $L_{\mathbf{X}^n}$)**.** *Fix* $\boldsymbol{\alpha},\boldsymbol{\beta}$ *satisfying the dimension restriction* (10), *and* $\boldsymbol{\mu}\in\mathcal{M}_{\mathrm{f,e}}$. *Consider any family* $\bigl\{(\mathbf{X}^n,\mathbf{A}^n):\ n\geq1\bigr\}$ *of solutions to the martingale problem* $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ *on* $(\Omega,\mathcal{F},\mathcal{F}_{\cdot},\mathcal{P})$. *Then, for* n *fixed,* $\mathbf{X}^n$ *has a continuous collision local time* $L_{\mathbf{X}^n}$. *Moreover, for each* $T>0$, $f\in\mathrm{b}\mathcal{C}$, *and* $\delta\in(0,1)$,

$$\lim_{\varepsilon\downarrow0}\sup_{n\geq1}\mathcal{P}\biggl(\sup_{0\leq t\leq T}\Bigl|\bigl\langle L_{\mathbf{X}^n}^{\varepsilon}(t),f\bigr\rangle-\bigl\langle L_{\mathbf{X}^n}(t),f\bigr\rangle\Bigr|>\delta\biggr)\;=\;0. \tag{23}$$

Loosely speaking, for this family $\{\mathbf{X}^n:\ n\geq1\}$, continuous collision local times $L_{\mathbf{X}^n}$ could be defined as a uniform in n limit of approximating collision local times.

2.3. **Existence of an** $(\boldsymbol{\alpha},d,\boldsymbol{\beta})$**–pair of competing superprocesses.** Next we specify the former martingale problem $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ by requiring $A^i=\lambda^iL_{\mathbf{X}}$, $i=1,2$.

Definition 8 (**Martingale problem** $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda}}$)**.** For pairs $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\vartheta}$ as in (1), $\boldsymbol{\mu}$ in $\mathcal{M}_{\mathrm{f}}^2$, and any pair $\boldsymbol{\lambda}=(\lambda^1,\lambda^2)\in\mathsf{R}_+^2$, let $\mathbf{X}=(X^1,X^2)$ be an $\mathcal{F}_{\cdot}$–adapted process (in $\mathcal{D}_{\mathcal{M}_{\mathrm{f}}}^2$) such that, for each pair $\boldsymbol{\varphi}=(\varphi^1,\varphi^2)\in\Phi_+^2$,

$$\begin{aligned}t\;\mapsto\;\mathrm{e}^{-\langle\mathbf{X}_t,\boldsymbol{\varphi}\rangle}-\mathrm{e}^{-\langle\boldsymbol{\mu},\boldsymbol{\varphi}\rangle}+\int_0^t\mathrm{d}s\;\mathrm{e}^{-\langle\mathbf{X}_s,\boldsymbol{\varphi}\rangle}\Bigl\langle\mathbf{X}_s\,,\,\Delta_{\boldsymbol{\alpha}}\boldsymbol{\varphi}-\boldsymbol{\varphi}^{\mathbf{1}+\boldsymbol{\beta}}\Bigr\rangle\qquad\\ -\int_0^t\langle\boldsymbol{\Lambda}_{\mathrm{d}s}\,,\boldsymbol{\varphi}\rangle\;\mathrm{e}^{-\langle\mathbf{X}_s,\boldsymbol{\varphi}\rangle},\qquad t\geq0,\end{aligned} \tag{24}$$

is an $\mathcal{F}_{\cdot}$–martingale starting from 0 at time $t=0$, where

$$\boldsymbol{\Lambda}\;=\;(\Lambda^1,\Lambda^2):=\;(\lambda^1L_{\mathbf{X}}\,,\lambda^2L_{\mathbf{X}}). \tag{25}$$

Then we say that $\mathbf{X}$ *solves the martingale problem* $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda}}$. ◇

Note that the requirement on the existence of the collision local time $L_{\mathbf{X}}$ is an integral part of the martingale problem for $\mathbf{X}$. As an application of Theorem 7 we will derive the following existence statement:

Theorem 9 (**Existence of the competing species model**)**.** *Fix* $\boldsymbol{\alpha},\boldsymbol{\beta}$ *satisfying the dimension restriction* (10) *and* $\boldsymbol{\mu}\in\mathcal{M}_{\mathrm{f,e}}$. *Then, for each* $\boldsymbol{\lambda}\in\mathsf{R}_+^2$, *there is a solution* $\mathbf{X}$ *to the martingale problem* $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda}}$.

Such solution $\mathbf{X}$ we call an $(\boldsymbol{\alpha},d,\boldsymbol{\beta})$*–pair of competing superprocesses* in R^d starting from $\mathbf{X}_0=\boldsymbol{\mu}$, and having *competition rates* $\boldsymbol{\lambda}=(\lambda^1,\lambda^2)$.

To verify Theorem 9, our strategy will be to show that to each $\varepsilon>0$, there is a solution $(\mathbf{X}^\varepsilon,\boldsymbol{\Lambda}^\varepsilon)$ to the martingale problem $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ where $\Lambda^{i,\varepsilon}:=\lambda^iL_{\mathbf{X}}^{i,\varepsilon}$, $i=1,2$, with (asymmetric) approximating collision local times $L_{\mathbf{X}}^{i,\varepsilon}$ from Lemma 2. The construction of $(\mathbf{X}^\varepsilon,\boldsymbol{\Lambda}^\varepsilon)$ is done via a Trotter type scheme: On small time intervals only one population is affected by the killing provided by the other population. The roles of populations are alternated on subsequent intervals. Then the interval length is shrinked to zero. Based on Theorem 7 we then pass to a limit along a suitable subsequence $\varepsilon_n \downarrow 0$ to construct the desired competing species model.

Remark 10 (**Some open problems**). *Uniqueness* for the martingale problem $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda}}$ remains an open problem. We see two different possible ways to attack this. One could try to reformulate the competing species model in the *historical* setting and to use Evans and Perkins [EP98] to get historical uniqueness of the model. The other approach may be to use a *duality* technique (see [Myt99] for the finite variance case) to prove the ordinary uniqueness for $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda}}$, but this approach probably works only in the symmetric case $\alpha^1 = \alpha^2,\ \beta^1 = \beta^2,\ \vartheta^1 = \vartheta^2$, and $\lambda^1 = \lambda^2$.

Another open problem which seems to be interesting is to prove *non-existence* of solutions to $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda}}$ in dimensions

$$d \;\geq\; \Big(\frac{\alpha^1}{\beta^1} + \alpha^2\Big) \wedge \Big(\frac{\alpha^2}{\beta^2} + \alpha^1\Big) \tag{26}$$

for $\lambda^1\lambda^2 > 0$. In the continuous super-Brownian motion case, this was solved in [EP94, Theorem 5.3]. $\diamond$

2.4. **Outline.** The remainder of the paper is organized as follows. In the next section we will show that, on some probability space, interacting superprocesses with killing satisfying the martingale problem $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ can simultaneously be dominated by a pair of independent critical superprocesses. In Section 4 we carry out a delicate analysis of solutions to log-Laplace equations with generalized input data. Properties of solutions to these equations will be crucial for the proof of Theorem 7 which will be provided in the subsequent section. The proof of Theorem 9 is given in Section 6. It is based on Theorem 7 and an analysis of the limiting behavior of approximating competing species models. Some auxiliary results are collected in an appendix.

3. Domination by independent critical superprocesses

In this section we will show that interacting superprocesses with killing can be simultaneously dominated by independent critical superprocesses. These results are generalizations of the case of continuous super-Brownian motions dealt with in [BEP91]. In Subsections 3.1-3.6 we provide auxiliary results mainly related to the martingale properties of the processes. They will be used for the proof of the domination property (see Proposition 21 and Corollary 23 in Subsections 3.7 and 3.8 below). For basic facts on superprocesses, we refer to [Daw93, Dyn94, LG99, Eth00], or [Per02].

3.1. **Extension of the martingale problem** $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$**.** By standard techniques, we get the following extension of the martingale problem $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ (we skip the details, cf., for instance, Lemma 6.1.2 in [Daw93]). Recall our conventions as in (4) and (5), and the set Ψ_+^2 of time-space test functions introduced in the beginning of Subsection 2.1.

Lemma 11 (**Extension of** $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$)**.** *If* $(\mathbf{X},\mathbf{A})$ *is a solution to the martingale problem* $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$, *then for all* $\boldsymbol{\psi}\in\Psi_+^2$,

$$t \ \mapsto\ \mathrm{e}^{-\langle \mathbf{X}_t,\boldsymbol{\psi}_t\rangle} - \mathrm{e}^{-\langle \boldsymbol{\mu},\boldsymbol{\psi}_0\rangle} + \int_0^t \mathrm{d}s\ \mathrm{e}^{-\langle \mathbf{X}_s,\,\boldsymbol{\psi}_s\rangle}\Big\langle \mathbf{X}_s\,,\,\Delta_{\boldsymbol{\alpha}}\boldsymbol{\psi}_s + \frac{\partial}{\partial s}\boldsymbol{\psi}_s - \boldsymbol{\psi}_s^{\mathbf{1}+\boldsymbol{\beta}}\Big\rangle$$
$$- \int_0^t \langle \mathbf{A}_{\mathrm{d}s}\,,\boldsymbol{\psi}_s\rangle\,\mathrm{e}^{-\langle \mathbf{X}_s,\,\boldsymbol{\psi}_s\rangle},\qquad t\geq 0, \tag{27}$$

is an $\mathcal{F}_{\cdot}$*–martingale starting from* 0 *at time* $t=0$.

3.2. $(\boldsymbol{\alpha},d,\boldsymbol{\beta})$–**pair of independent superprocesses.** We start by recalling the notion of an $(\boldsymbol{\alpha},d,\boldsymbol{\beta})$–pair of independent superprocesses $\mathbf{X}$ with killing rate $\boldsymbol{\kappa}$ and immigration processes ${}^{\circ}\!\mathbf{A}$ (no dimension restriction is needed here).

Definition 12 (**Martingale problem** $(\mathbf{MP})_{\boldsymbol{\mu},\boldsymbol{\kappa},{}^{\circ}\!\mathbf{A}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$)**.** For pairs $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\vartheta}$ as in (1), $\boldsymbol{\mu}=(\mu^1,\mu^2)\in\mathcal{M}_{\mathrm{f}}^2$, continuous $\boldsymbol{\kappa}=(\kappa^1,\kappa^2):\mathsf{R}_+\times\mathsf{R}^d\to\mathsf{R}_+\times\mathsf{R}_+$, and (deterministic) non-decreasing ${}^{\circ}\!\mathbf{A}=({}^{\circ}\!A^1,{}^{\circ}\!A^2)\in\mathcal{D}_{\mathcal{M}_{\mathrm{f}}}^2$ with ${}^{\circ}\!\mathbf{A}_0=\mathbf{0}$, let $\mathbf{X}=(X^1,X^2)$ be an $\mathcal{F}_{\cdot}$–adapted process (in $\mathcal{D}_{\mathcal{M}_{\mathrm{f}}}^2$) such that for each pair $\boldsymbol{\varphi}=(\varphi^1,\varphi^2)\in\Phi_+^2$,

$$t \ \mapsto\ \mathrm{e}^{-\langle \mathbf{X}_t,\boldsymbol{\varphi}\rangle} - \mathrm{e}^{-\langle \boldsymbol{\mu},\boldsymbol{\varphi}\rangle} + \int_0^t \mathrm{d}s\ \mathrm{e}^{-\langle \mathbf{X}_s,\boldsymbol{\varphi}\rangle}\Big\langle \mathbf{X}_s\,,\,\Delta_{\boldsymbol{\alpha}}\boldsymbol{\varphi} - \boldsymbol{\kappa}\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\mathbf{1}+\boldsymbol{\beta}}\Big\rangle \tag{28}$$
$$+ \int_{(0,t]} \langle {}^{\circ}\!\mathbf{A}_{\mathrm{d}s}\,,\boldsymbol{\varphi}\rangle\,\mathrm{e}^{-\langle \mathbf{X}_s,\boldsymbol{\varphi}\rangle},\qquad t\geq 0,$$

is an $\mathcal{F}_{\cdot}$–martingale starting from 0 at time $t=0$. Then we say that $\mathbf{X}={}^{{}^{\circ}\!\mathbf{A}}\mathbf{X}$ solves the martingale problem $(\mathbf{MP})_{\boldsymbol{\mu},\boldsymbol{\kappa},{}^{\circ}\!\mathbf{A}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$. ◇

Note the differences with Definition 5. First of all, there we did not allow an immigration by some ${}^{\circ}\!\mathbf{A}$. On the other hand, here the killing $\mathbf{A}$ is of the particular form $A^i\big(\mathrm{d}(s,x)\big)=\kappa_s^i(x)\,X_s^i(\mathrm{d}x)\,\mathrm{d}s$ where $\boldsymbol{\kappa}$ is deterministic, leading to the fact that the pair $\mathbf{X}$ has *independent* marginals X^1,X^2.

Definition 13 (**Pair of log-Laplace functions**)**.** Consider pairs $\boldsymbol{\varphi}=(\varphi^1,\varphi^2)$ in $\mathrm{b}\mathcal{B}_+^2(\mathsf{R}^d)$, $\boldsymbol{\psi}=(\psi^1,\psi^2)\in\mathrm{b}\mathcal{B}_+^2(\mathsf{R}_+\times\mathsf{R}^d)$, and continuous $\boldsymbol{\kappa}=(\kappa^1,\kappa^2):\mathsf{R}_+\times\mathsf{R}^d\to\mathsf{R}_+\times\mathsf{R}_+$. For fixed $i\in\{1,2\}$ and $t\geq 0$, let $u^{i,t}=u^{i,t}(\varphi^i,\psi^i)$ denote the unique non-negative solution to the so-called *log-Laplace equation*

$$u_r^{i,t}(x)\ =\ \Pi_{r,x}^i\bigg(\varphi^i(\xi_t^i)+\int_r^t \mathrm{d}s\ \Big[\psi_s^i(\xi_s^i)-\kappa_s^i(\xi_s^i)\,u_s^{i,t}(\xi_s^i)-\big(u_s^{i,t}(\xi_s^i)\big)^{1+\beta^i}\Big]\bigg)$$
$$=\ \mathrm{S}_{t-r}^i\varphi^i\,(x)+\int_r^t \mathrm{d}s\ \mathrm{S}_{s-r}^i\Big[\psi_s^i-\kappa_s^i\,u_s^{i,t}-(u_s^{i,t})^{1+\beta^i}\Big](x), \tag{29}$$

$(r,x)\in[0,t]\times\mathsf{R}^d$. We call $\mathbf{u}^t=\mathbf{u}^t(\boldsymbol{\varphi},\boldsymbol{\psi}):=\big(u^{1,t}(\varphi^1,\psi^1),u^{2,t}(\varphi^2,\psi^2)\big)$ *the pair of log-Laplace functions* (on $[0,t]$ with killing rate $\boldsymbol{\kappa}$ and with input data $\boldsymbol{\varphi},\boldsymbol{\psi}$). ◇

For the unique existence of solutions, see, for instance, [Dyn02, Theorem 4.1.1]. Note that the $u^{i,t}$ are *continuous* functions on $[0,t)\times\mathsf{R}^d$. Moreover, for $\boldsymbol{\varphi}\in\Phi_+^2$, $\boldsymbol{\psi}\in\Psi_+^2$, and i,t,κ^i fixed, $u^{i,t}=u^{i,t}(\varphi^i,\psi^i)\geq 0$ from the log-Laplace equation (29) is the unique solution to the Φ–valued ordinary differential equation

$$\left.\begin{aligned} -\frac{\partial}{\partial r}u_r^{i,t}\ &=\ \Delta_{\alpha^i}\,u_r^{i,t}+\psi^i-\kappa_r^i\,u_r^{i,t}-(u_r^{i,t})^{1+\beta^i}\quad\text{on }(0,t)\times\mathsf{R}^d\\ &\text{with terminal condition }\ u_{t-}^{i,t}\ =\ \varphi^i.\end{aligned}\right\} \tag{30}$$

In the following lemma we collect some standard facts on superprocesses, see, for instance, Roelly-Coppoletta [RC86], Iscoe [Isc86, Theorems 3.1 and 3.2], Dawson [Daw93, Chapter 6], and Dynkin [Dyn93, Theorems 2.1, 3.1, and 4.1].

Lemma 14 (**Independent superprocesses with immigration**). *Consider* $\boldsymbol{\alpha}$,$\boldsymbol{\beta}$, $\boldsymbol{\vartheta}$, $\boldsymbol{\mu}$, $^{\circ}\mathbf{A}$ *as in Definition* 12. *Then the following statements hold.*

- **(a) (Uniqueness):** *There is a unique (in law) solution* $\mathbf{X}$ *to the martingale problem* $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{\mu},\boldsymbol{\kappa},{}^{\circ}\mathbf{A}}$ *of Definition* 12.
- **(b) (Strong Markov):** *This* $\mathbf{X}$ *is a time-inhomogeneous strong Markov process starting from* $\mathbf{X}_0 = \boldsymbol{\mu}$, *having independent marginal processes.*
- **(c) (Log-Laplace representation):** $\mathbf{X}$ *has the following log-Laplace transition functionals: For* $0 \le r \le t$, *non-negative* $\boldsymbol{\varphi} = (\varphi^1, \varphi^2) \in \mathrm{b}\mathcal{B}^2(\mathsf{R}^d)$, *and non-negative* $\boldsymbol{\psi} = (\psi^1, \psi^2) \in \mathrm{b}\mathcal{B}^2(\mathsf{R}_+ \times \mathsf{R}^d)$,

$$\mathcal{P}\left\{\exp\Big[-\langle \mathbf{X}_t\,, \boldsymbol{\varphi}\rangle - \int_r^t \mathrm{d}s\ \langle \mathbf{X}_s\,, \boldsymbol{\psi}_s\rangle\Big]\ \Big|\ \mathcal{F}_r\right\} = \exp\Big[-\big\langle \mathbf{X}_r\,, \mathbf{u}^t_r\big\rangle - \big\langle {}^{\circ}\mathbf{A}, \mathbf{u}^t\big\rangle_{(r,t]}\Big], \tag{31}$$

  where, for t *fixed,* $\mathbf{u}^t = \mathbf{u}^t(\boldsymbol{\varphi}, \boldsymbol{\psi})$ *is the pair of log-Laplace functions according to Definition* 13.
- **(d) (Expectations):** *For* $r, t, \boldsymbol{\varphi}$ *as in* (c),

$$\mathcal{P}\big\{\langle \mathbf{X}_t\,, \boldsymbol{\varphi}\rangle \,\big|\, \mathcal{F}_r\big\} = \sum_{i=1,2} \big\langle X^i_r, \mathrm{S}^i_{t-r}\varphi^i\big\rangle + \sum_{i=1,2} \int_{(r,t]\times\mathsf{R}^d} {}^{\circ}\!A^i\big(\mathrm{d}(s,x)\big)\, \mathrm{S}^i_{t-r}\varphi^i\,(x). \tag{32}$$

- **(e) (Exponential martingale):** *For fixed* $T \ge 0$ *and in the case* $\boldsymbol{\varphi} \in \Phi^2_+$,

$$t \ \mapsto\ \exp\Big[-\big\langle \mathbf{X}_t\,, \mathbf{u}^T_t(\boldsymbol{\varphi}, \mathbf{0})\big\rangle + \big\langle {}^{\circ}\mathbf{A}, \mathbf{u}^T(\boldsymbol{\varphi}, \mathbf{0})\big\rangle_{(0,t]}\Big], \quad 0 \le t \le T,$$

  is an $\mathcal{F}_{\cdot}$*–martingale.*

We call this process $\mathbf{X}$ the $(\boldsymbol{\alpha}, d, \boldsymbol{\beta})$*–pair of independent superprocesses with killing rate* $\boldsymbol{\kappa}$ *and immigration* $^{\circ}\mathbf{A}$, starting from $\mathbf{X}_0 = \boldsymbol{\mu}$. It is said to be *critical* if $\boldsymbol{\kappa} = \mathbf{0}$. Of course, the X^i are ordinary independent (α^i, d, β^i)–superprocesses with killing rate κ^i and with immigration processes $^{\circ}\!A^i$, $i = 1, 2$. We write

$$\big({}^{\circ}\Omega, {}^{\circ}\mathcal{F}, {}^{\circ}\mathcal{F}_{\cdot}\,, {}^{\circ}\mathcal{P}_{\boldsymbol{\mu},\boldsymbol{\kappa},{}^{\circ}\mathbf{A}}\big) \tag{33}$$

for the *canonical basis* of this process.

3.3. **Properties of log-Laplace functions.** We add here a couple of properties of these log-Laplace functions, the proofs are postponed to Subsection A.4 in the appendix.

Lemma 15 (**First order considerations of log-Laplace functions**). *Consider* $t, \varepsilon > 0$, $(\boldsymbol{\varphi},\boldsymbol{\psi}) \in \Phi^2_+ \times \Psi^2_+$ *and the pairs* $\mathbf{u}^t = \mathbf{u}^t(\boldsymbol{\varphi},\boldsymbol{\psi})$ *and* $\mathbf{u}^{t+\varepsilon} = \mathbf{u}^{t+\varepsilon}(\boldsymbol{\varphi},\boldsymbol{\psi})$ *of log-Laplace functions according to Definition* 13 *with* $\boldsymbol{\kappa} = \mathbf{0}$. *Fix* $i \in \{1, 2\}$. *Then the following statements hold.*

(a) (Bounded pointwise convergence): *For* $\varphi \in \Phi_+$,

$$\frac{1}{\varepsilon}\bigl(u_t^{i,t+\varepsilon}(\varphi,0)-\varphi\bigr) \underset{\varepsilon\downarrow 0}{\longrightarrow} \Delta_{\alpha^i}\varphi - \varphi^{1+\beta^i} \quad \textit{boundedly pointwise on } \mathsf{R}^d.$$

(b) (Uniform boundedness): *For* $\varphi \in \Phi_+$,

$$\sup_{t\le s\le t+\varepsilon} \frac{1}{\varepsilon}\,\bigl\|u_s^{i,t+\varepsilon}(\varphi,0)-\varphi\bigr\|_\infty \;\le\; \|\Delta_{\alpha^i}\varphi\|_\infty + \|\varphi\|_\infty^{1+\beta^i}. \tag{34}$$

(c) (Uniform convergence in time-space): *For* $(\varphi,\psi) \in \Phi_+ \times \Psi_+$,

$$\sup_{0\le r\le t} \Bigl\| \frac{1}{\varepsilon}\, u_r^{i,t}(\varepsilon\varphi,\varepsilon\psi) - \mathrm{S}^i_{t-r}\varphi - \int_r^t \mathrm{d}s\; \mathrm{S}^i_{s-r}\psi_s \Bigr\|_\infty \underset{\varepsilon\downarrow 0}{\longrightarrow} 0. \tag{35}$$

Occasionally we will also need to handle the case of terminal conditions in $\mathcal{L}^1$.

Lemma 16 ($\mathcal{L}^1$ **terminal condition).** *Fix* $i \in \{1,2\}$ *and* $t > 0$.

(a) (Unique existence): *Let* $\varphi \in \mathcal{L}^1_+$. *Then there is a unique non-negative solution* $u^{i,t} = u^{i,t}(\varphi,0)$ *to equation* (29)*, and* $u_r^{i,t}$ *is continuous (on* R^d*), for all* $r \in [0,t)$.

(b) (Convergence): *Let* $\varphi_\varepsilon \in \mathcal{L}^1_+$, $\varepsilon \in [0,1]$, *and assume that* $\varphi_\varepsilon \to \varphi_0$ *in* $\mathcal{L}^1$ *as* $\varepsilon \downarrow 0$ *and*

$$\sup_{0<\varepsilon\le 1}\ \operatorname*{ess\,sup}_{x\in\mathsf{R}^d} \bigl|\varphi_\varepsilon(x)\bigr| \;<\; \infty. \tag{36}$$

Then, for each $r \in [0,t)$,

$$u_r^{i,t}(\varphi_\varepsilon,0) \underset{\varepsilon\downarrow 0}{\longrightarrow} u_r^{i,t}(\varphi_0,0) \quad \textit{uniformly on compacts of } \mathsf{R}^d. \tag{37}$$

3.4. **An exponential martingale for solutions to** $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$**.** ¿From the extended martingale problem in Lemma 11 we will construct another exponential martingale:

Proposition 17 (Exponential martingale related to $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$**).** *Fix* $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\vartheta}$ *as in* (1) *and* $\boldsymbol{\mu} \in \mathcal{M}_\mathrm{f}^2$. *Consider a solution* $(\mathbf{X},\mathbf{A})$ *to the martingale problem* $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ *of Definition* 5*. Then, for each* $T \ge 0$ *and* $\boldsymbol{\varphi}$ *in* Φ_+^2,

$$t \;\mapsto\; \exp\Bigl[-\bigl\langle \mathbf{X}_t\,,\mathbf{u}_t^T(\boldsymbol{\varphi},\mathbf{0})\bigr\rangle - \int_0^t \bigl\langle \mathbf{A}_{\mathrm{d}s}\,,\mathbf{u}_s^T(\boldsymbol{\varphi},\mathbf{0})\bigr\rangle\Bigr], \qquad 0\le t\le T, \tag{38}$$

is an $\mathcal{F}_\cdot$*–martingale. Conversely, let* $\mathbf{X}$ *and* $\mathbf{A}$ *be* $\mathcal{F}_\cdot$*–adapted processes (in* $\mathcal{D}^2_{\mathcal{M}_\mathrm{f}}$*) such that* $t \mapsto \mathbf{A}_t$ *is non-decreasing and continuous. If for each* $T \ge 0$ *and* $\boldsymbol{\varphi}$ *in* Φ_+^2 *the process in* (38) *is an* $\mathcal{F}_\cdot$*–martingale, then* $(\mathbf{X},\mathbf{A})$ *is a solution to the martingale problem* $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$.

To prepare for the proof of this proposition, we deal with the following lemma.

Lemma 18 (Expectation). *Let* $(\mathbf{X},\mathbf{A})$ *be an* $\mathcal{F}_\cdot$*–adapted process in* $\mathcal{D}^4_{\mathcal{M}_\mathrm{f}}$ *with deterministic initial state* $\mathbf{X}_0$ *and such that* $\mathbf{A}$ *is non-decreasing, continuous, starting from* $\mathbf{A}_0 = \mathbf{0}$*, and, for each* $T > 0$, $\boldsymbol{\varphi} \in \Phi_+^2$*, the process in* (38) *is an* $\mathcal{F}_\cdot$*–martingale. Then*

$$\mathcal{P}\bigl(\langle \mathbf{X}_t\,,\mathbf{1}\rangle + \langle \mathbf{A}_t\,,\mathbf{1}\rangle\bigr) \;=\; \langle \mathbf{X}_0\,,\mathbf{1}\rangle, \qquad t\ge 0. \tag{39}$$

Proof. By Lemma 15(c), we have

$$\frac{1}{\varepsilon}\, u_s^{i,T}(\varepsilon 1, 0)\,(x) \underset{\varepsilon\downarrow 0}{\longrightarrow} 1, \quad \text{uniformly in } (s,x)\in[0,T]\times\mathsf{R}^d. \tag{40}$$

Hence,

$$\frac{1}{\varepsilon}\Big(1-\exp\Big[-\big\langle \mathbf{X}_T\,,\varepsilon\mathbf{1}\big\rangle - \int_0^T \big\langle \mathbf{A}_{\mathrm{d}s}\,,\mathbf{u}_s^T(\varepsilon\mathbf{1},\mathbf{0})\big\rangle\Big]\Big) \tag{41}$$
$$\underset{\varepsilon\downarrow 0}{\longrightarrow} \langle \mathbf{X}_T\,,\mathbf{1}\rangle + \langle \mathbf{A}_T\,,\mathbf{1}\rangle\,, \quad \mathcal{P}\text{–a.s.}$$

On the other hand, by (38) and again by Lemma 15(c),

$$\frac{1}{\varepsilon}\,\mathcal{P}\Big(1-\exp\Big[-\big\langle \mathbf{X}_T\,,\varepsilon\mathbf{1}\big\rangle - \int_0^T \big\langle \mathbf{A}_{\mathrm{d}s}\,,\mathbf{u}_s^T(\varepsilon\mathbf{1},\mathbf{0})\big\rangle\Big]\Big) \tag{42}$$
$$= \frac{1}{\varepsilon}\Big(1-\exp\Big[-\big\langle \mathbf{X}_0\,,\mathbf{u}_0^T(\varepsilon\mathbf{1},\mathbf{0})\big\rangle\Big]\Big) \underset{\varepsilon\downarrow 0}{\longrightarrow} \langle \mathbf{X}_0\,,\mathbf{1}\rangle.$$

By Fatou's lemma, we get

$$\mathcal{P}\big(\langle \mathbf{X}_t\,,\mathbf{1}\rangle + \langle \mathbf{A}_t\,,\mathbf{1}\rangle\big) \;\le\; \langle \mathbf{X}_0\,,\mathbf{1}\rangle \;<\; \infty. \tag{43}$$

Now use the above procedure again, domination $\frac{1}{\varepsilon}\,\mathbf{u}_s^T(\varepsilon\mathbf{1},\mathbf{0}) \le \mathbf{1}$, and apply dominated convergence to get the desired result. ■

3.5. **Proof of Proposition 17.** 1° (*First claim*). Let $(\mathbf{X},\mathbf{A})$ be a solution to $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$, and fix $T,\boldsymbol{\varphi}$ as in the proposition. Consider additionally $\boldsymbol{\psi}\in\Psi_+^2$. Recall our convention (5). From the integration by parts formula for semimartingales (see, for instance, Protter [Pro90, Corollary II.22.2]),

$$\mathrm{e}^{-\langle \mathbf{X}_t,\boldsymbol{\psi}_t\rangle}\,\mathrm{e}^{-\langle \mathbf{A},\boldsymbol{\psi}\rangle_{[0,t]}} - \mathrm{e}^{-\langle \mathbf{X}_0\,,\,\boldsymbol{\psi}_0\rangle} \tag{44}$$
$$= -\int_0^t \langle \mathbf{A}_{\mathrm{d}s}\,,\boldsymbol{\psi}_s\rangle\,\exp\Big[-\big\langle \mathbf{X}_s\,,\boldsymbol{\psi}_s\big\rangle - \langle \mathbf{A},\boldsymbol{\psi}\rangle_{[0,s]}\Big]$$
$$+\int_{(0,t]} \mathrm{d}\mathrm{e}^{-\langle \mathbf{X}_s\,,\,\boldsymbol{\psi}_s\rangle}\,\mathrm{e}^{-\langle \mathbf{A},\boldsymbol{\psi}\rangle_{[0,s]}}.$$

In fact,

$$t \;\mapsto\; \mathrm{e}^{-\langle \mathbf{A},\boldsymbol{\psi}\rangle_{[0,t]}}, \qquad 0\le t\le T, \tag{45}$$

is a continuous function of bounded variation, hence its quadratic variation process is constantly 0. But then the bracket process of the two semimartingales

$$t\mapsto \mathrm{e}^{-\langle \mathbf{X}_t,\boldsymbol{\psi}_t\rangle} \quad\text{and}\quad t\mapsto \mathrm{e}^{-\langle \mathbf{A},\boldsymbol{\psi}\rangle_{[0,t]}} \tag{46}$$

vanishes (use, e.g., the Kunita-Watanabe inequality, [Pro90, Theorem II.25]). Note also that by the continuity of $\mathbf{A}$ no left limits appear in (44). Next use the extended martingale problem from Lemma 11 to substitute for $\mathrm{d}\mathrm{e}^{-\langle \mathbf{X}_s\,,\,\boldsymbol{\psi}_s\rangle}$ into (44). This implies that

$$t \;\mapsto\; \exp\Big[-\langle \mathbf{X}_t\,,\,\boldsymbol{\psi}_t\rangle - \langle \mathbf{A},\boldsymbol{\psi}\rangle_{[0,t]}\Big] - \mathrm{e}^{-\langle \mathbf{X}_0\,,\,\boldsymbol{\psi}_0\rangle} \tag{47}$$
$$+\int_0^t \mathrm{d}s\,\exp\Big[-\big\langle \mathbf{X}_s\,,\boldsymbol{\psi}_s\big\rangle - \langle \mathbf{A},\boldsymbol{\psi}\rangle_{[0,s]}\Big]\Big\langle \mathbf{X}_s\,,\,\Delta_{\boldsymbol{\alpha}}\boldsymbol{\psi}_s + \frac{\partial}{\partial s}\boldsymbol{\psi}_s - \boldsymbol{\psi}_s^{\mathbf{1}+\boldsymbol{\beta}}\Big\rangle,$$

$0\le t\le T$, is an $\mathcal{F}_{\cdot}$–martingale. Specializing to $\boldsymbol{\psi} = \mathbf{u}^T(\boldsymbol{\varphi},\mathbf{0})$, the first claim in the lemma follows since this $\boldsymbol{\psi}$ solves the differential equation (30).

2° (*Second claim*). Now let $(\mathbf{X},\mathbf{A})$ and $\boldsymbol{\varphi}$ be such that the process in (38) is a martingale. Assume for the moment that

$$t \ \mapsto\ Z_t + \int_0^t \mathrm{d}s\ Z_s \left\langle \mathbf{X}_s\,,\,\Delta_{\boldsymbol{\alpha}}\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\mathbf{1}+\boldsymbol{\beta}} \right\rangle \text{ is a martingale,} \tag{48}$$

where

$$Z_t \ :=\ \exp\Big[-\langle \mathbf{X}_t\,,\boldsymbol{\varphi}\rangle - \langle \mathbf{A},\boldsymbol{\varphi}\rangle_{[0,t]}\Big], \qquad t\geq 0. \tag{49}$$

By the integration by parts formula (see, for instance, [Pro90, Corollary II.22.2]), applied to the semimartingales

$$t\mapsto Z_t \quad\text{and}\quad t\mapsto \mathrm{e}^{\langle \mathbf{A},\boldsymbol{\varphi}\rangle_{[0,t]}}, \tag{50}$$

we get

$$\begin{aligned}
\mathrm{e}^{-\langle \mathbf{X}_t,\boldsymbol{\varphi}\rangle} \ &=\ Z_t\,\mathrm{e}^{\langle \mathbf{A},\boldsymbol{\varphi}\rangle_{[0,t]}} \ =\ Z_0 + \int_0^t \mathrm{d}\big(\mathrm{e}^{\langle \mathbf{A},\boldsymbol{\varphi}\rangle_{[0,s]}}\big)\ Z_s + \int_{(0,t]} \mathrm{d}(Z_s)\ \mathrm{e}^{\langle \mathbf{A},\boldsymbol{\varphi}\rangle_{[0,s]}} \qquad (51)\\
&=\ Z_0 + \int_0^t \langle \mathbf{A}_{\mathrm{d}s}\,,\boldsymbol{\varphi}\rangle\,\mathrm{e}^{\langle \mathbf{A},\boldsymbol{\varphi}\rangle_{[0,s]}}\,Z_s\\
&\quad -\ \int_0^t \mathrm{d}s\ Z_s \left\langle \mathbf{X}_s\,,\,\Delta_{\boldsymbol{\alpha}}\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\mathbf{1}+\boldsymbol{\beta}} \right\rangle \mathrm{e}^{\langle \mathbf{A},\boldsymbol{\varphi}\rangle_{[0,s]}}\\
&\quad +\ \int_{(0,t]} \mathrm{d}(\text{local martingale})_s\ \mathrm{e}^{\langle \mathbf{A},\boldsymbol{\varphi}\rangle_{[0,s]}},
\end{aligned}$$

where in the last step we used the martingale (48). Hence, by definition (49) of Z,

$$\begin{aligned}
\mathrm{e}^{-\langle \mathbf{X}_t,\boldsymbol{\varphi}\rangle} \ &=\ \mathrm{e}^{-\langle \mathbf{X}_0,\boldsymbol{\varphi}\rangle} \qquad (52)\\
&\quad + \int_0^t \Big(\langle \mathbf{A}_{\mathrm{d}s}\,,\boldsymbol{\varphi}\rangle + \mathrm{d}s \left\langle \mathbf{X}_s\,,\,-\Delta_{\boldsymbol{\alpha}}\boldsymbol{\varphi} + \boldsymbol{\varphi}^{\mathbf{1}+\boldsymbol{\beta}} \right\rangle\Big)\,\mathrm{e}^{-\langle \mathbf{X}_s,\boldsymbol{\varphi}\rangle} +\ M_t^{\varphi},
\end{aligned}$$

where M^{φ} is a local martingale. To get the martingale statement (21), we need to show that M^{φ} is indeed a martingale. But for any $T>0$,

$$\begin{aligned}
\sup_{t\leq T}|M_t^{\varphi}| \ &\leq\ \sup_{t\leq T}\left|\mathrm{e}^{-\langle \mathbf{X}_t,\boldsymbol{\varphi}\rangle} - \mathrm{e}^{-\langle \mathbf{X}_0,\boldsymbol{\varphi}\rangle}\right| \qquad (53)\\
&\quad + \sup_{t\leq T}\int_0^t \left|\langle \mathbf{A}_{\mathrm{d}s}\,,\boldsymbol{\varphi}\rangle + \mathrm{d}s\left\langle \mathbf{X}_s\,,\,-\Delta_{\boldsymbol{\alpha}}\boldsymbol{\varphi} + \boldsymbol{\varphi}^{\mathbf{1}+\boldsymbol{\beta}}\right\rangle\right|\mathrm{e}^{-\langle \mathbf{X}_s,\boldsymbol{\varphi}\rangle}\\
&\leq\ 1 + \|\boldsymbol{\varphi}\|_\infty\,\langle \mathbf{A}_T\,,\mathbf{1}\rangle + \Big(\|\Delta_{\boldsymbol{\alpha}}\boldsymbol{\varphi}\|_\infty + \|\boldsymbol{\varphi}\|_\infty^{\mathbf{1}+\boldsymbol{\beta}}\Big)\int_0^T \mathrm{d}s\ \langle \mathbf{X}_s\,,\mathbf{1}\rangle\,,
\end{aligned}$$

where, by a slight abuse, we use notation as $\|\boldsymbol{\varphi}\|_\infty := \|\varphi^1\|_\infty + \|\varphi^2\|_\infty$. By Lemma 18, the latter expression has finite expectation, hence $\mathcal{P}\sup_{t\leq T}|M_t^{\varphi}|<\infty$. Now the martingale claim on M^{φ} follows, for instance, from [Pro90, Theorem 1.47]. It remains to verify the martingale statement (48).

3° (*Sufficient condition for* (48)). Note first that

$$t \ \mapsto\ \frac{1}{\varepsilon}\,\mathcal{P}\Big\{\int_0^{\varepsilon}\mathrm{d}s\ Z_{t+s}\ \Big|\ \mathcal{F}_t\Big\} - \frac{1}{\varepsilon}\int_0^t \mathrm{d}s\ \mathcal{P}\left\{Z_{s+\varepsilon} - Z_s\,|\,\mathcal{F}_s\right\} \tag{54}$$

is a martingale, for each $\varepsilon>0$ (see, for instance, [EK86, Proposition 2.7.5]). The first term goes to Z_t in $\mathcal{L}^1 = \mathcal{L}^1(\mathcal{P})$ as $\varepsilon\downarrow 0$, for each $t>0$, by dominated convergence. Now, in order to get (48), it is enough to check that in $\mathcal{L}^1$,

$$\frac{1}{\varepsilon}\int_0^t \mathrm{d}s\, \mathcal{P}\left\{Z_{s+\varepsilon}-Z_s \,|\, \mathcal{F}_s\right\} \underset{\varepsilon\downarrow 0}{\longrightarrow} -\int_0^t \mathrm{d}s\, Z_s \left\langle \mathbf{X}_s\,,\, \Delta_{\boldsymbol{\alpha}}\boldsymbol{\varphi}-\boldsymbol{\varphi}^{\mathbf{1}+\boldsymbol{\beta}}\right\rangle, \tag{55}$$

for each $t>0$. Note that

$$\begin{aligned}&\varepsilon^{-1}\mathcal{P}\left\{Z_{t+\varepsilon}-Z_t \,|\, \mathcal{F}_t\right\}\\ &= \mathrm{e}^{-\langle \mathbf{A},\boldsymbol{\varphi}\rangle_{[0,t]}}\,\varepsilon^{-1}\mathcal{P}\Big\{\exp\Big[-\Big\langle \mathbf{X}_{t+\varepsilon}\,,\,\mathbf{u}^{t+\varepsilon}_{t+\varepsilon}(\boldsymbol{\varphi},\mathbf{0})\Big\rangle-\langle \mathbf{A},\boldsymbol{\varphi}\rangle_{[t,t+\varepsilon]}\Big]-\mathrm{e}^{-\langle \mathbf{X}_t,\boldsymbol{\varphi}\rangle}\;\Big|\;\mathcal{F}_t\Big\}.\end{aligned}$$

We split this into the two terms

$$\begin{aligned}I_{\varepsilon,t} \;:=\; \mathrm{e}^{-\langle \mathbf{A},\boldsymbol{\varphi}\rangle_{[0,t]}}\,\varepsilon^{-1}\mathcal{P}\Big\{&\exp\Big[-\Big\langle \mathbf{X}_{t+\varepsilon}\,,\,\mathbf{u}^{t+\varepsilon}_{t+\varepsilon}(\boldsymbol{\varphi},\mathbf{0})\Big\rangle-\langle \mathbf{A},\boldsymbol{\varphi}\rangle_{[t,t+\varepsilon]}\Big]\\ &-\exp\Big[-\langle \mathbf{X}_{t+\varepsilon}\,,\boldsymbol{\varphi}\rangle-\int_t^{t+\varepsilon}\Big\langle \mathbf{A}_{\mathrm{d}s}\,,\mathbf{u}^{t+\varepsilon}_s(\boldsymbol{\varphi},\mathbf{0})\Big\rangle\Big]\;\Big|\;\mathcal{F}_t\Big\}\end{aligned} \tag{56}$$

and

$$\begin{aligned}I\!I_{\varepsilon,t} \;:=\; \mathrm{e}^{-\langle \mathbf{A},\boldsymbol{\varphi}\rangle_{[0,t]}}\,\varepsilon^{-1}\mathcal{P}\Big\{&\exp\Big[-\langle \mathbf{X}_{t+\varepsilon}\,,\boldsymbol{\varphi}\rangle-\int_t^{t+\varepsilon}\Big\langle \mathbf{A}_{\mathrm{d}s}\,,\mathbf{u}^{t+\varepsilon}_s(\boldsymbol{\varphi},\mathbf{0})\Big\rangle\Big]\\ &-\mathrm{e}^{-\langle \mathbf{X}_t,\boldsymbol{\varphi}\rangle}\;\Big|\;\mathcal{F}_t\Big\}.\end{aligned} \tag{57}$$

4° (*Error term* $I_{\varepsilon,t}$). For the "error term" $I_{\varepsilon,t}$ we use the estimate

$$\begin{aligned}|I_{\varepsilon,t}| \;&\le\; \varepsilon^{-1}\mathcal{P}\Big\{\Big|\mathrm{e}^{-\langle \mathbf{A},\boldsymbol{\varphi}\rangle_{[t,t+\varepsilon]}}-\exp\Big[-\int_t^{t+\varepsilon}\Big\langle \mathbf{A}_{\mathrm{d}s}\,,\mathbf{u}^{t+\varepsilon}_s(\boldsymbol{\varphi},\mathbf{0})\Big\rangle\Big]\Big|\;\Big|\;\mathcal{F}_t\Big\}\\ &\le\; \varepsilon^{-1}\mathcal{P}\Big\{\int_t^{t+\varepsilon}\Big\langle \mathbf{A}_{\mathrm{d}s}\,,\big|\boldsymbol{\varphi}-\mathbf{u}^{t+\varepsilon}_s(\boldsymbol{\varphi},\mathbf{0})\big|\Big\rangle\;\Big|\;\mathcal{F}_t\Big\}.\end{aligned} \tag{58}$$

By Lemma 15(b), we have

$$\varepsilon^{-1}\left\|\varphi^i-u^{i,t+\varepsilon}_s(\varphi^i,0)\right\|_\infty \;\le\; \left\|\Delta_{\alpha^i}\varphi^i\right\|_\infty+\|\varphi^i\|_\infty^{1+\beta^i} \;=:\; c_{59} \;=\; c_{59}(\varphi^i), \tag{59}$$

$t\le s\le t+\varepsilon$. Therefore, with $0<\varepsilon\le T\wedge 1$,

$$\begin{aligned}\mathcal{P}\int_0^T \mathrm{d}t\;|I_{\varepsilon,t}| \;&\le\; c_{59}\int_0^T \mathrm{d}t\;\mathcal{P}\langle \mathbf{A},\mathbf{1}\rangle_{[t,t+\varepsilon]}\\ &=\; c_{59}\,\mathcal{P}\Big(\int_\varepsilon^{T+\varepsilon}\mathrm{d}t\;\langle \mathbf{A}_t\,,\mathbf{1}\rangle-\int_0^T \mathrm{d}t\;\langle \mathbf{A}_t\,,\mathbf{1}\rangle\Big)\\ &\le\; c_{59}\,\mathcal{P}\int_T^{T+\varepsilon}\mathrm{d}t\;\langle \mathbf{A}_t\,,\mathbf{1}\rangle \;\le\; \varepsilon\, c_{59}\,\mathcal{P}\langle \mathbf{A}_{T+1}\,,\mathbf{1}\rangle\\ &\le\; \varepsilon\, c_{59}\;\langle \boldsymbol{\mu},\mathbf{1}\rangle \underset{\varepsilon\downarrow 0}{\longrightarrow} 0,\end{aligned} \tag{60}$$

where the last inequality follows from Lemma 18. Hence, we get that

$$\int_0^T \mathrm{d}t\;|I_{\varepsilon,t}| \underset{\varepsilon\downarrow 0}{\longrightarrow} 0 \quad \text{in } \mathcal{L}^1, \text{ for any } T>0. \tag{61}$$

5° (*Main term* $II_{\varepsilon,t}$). We start with the identity

$$\mathcal{P}\Big\{ \exp\Big[-\langle \mathbf{X}_{t+\varepsilon}, \boldsymbol{\varphi}\rangle - \int_t^{t+\varepsilon} \big\langle \mathbf{A}_{\mathrm{d}s}, \mathbf{u}_s^{t+\varepsilon}(\boldsymbol{\varphi},\mathbf{0})\big\rangle\Big] \,\Big|\, \mathcal{F}_t\Big\} \tag{62}$$
$$= \exp\Big[\int_0^t \big\langle \mathbf{A}_{\mathrm{d}s}, \mathbf{u}_s^{t+\varepsilon}(\boldsymbol{\varphi},\mathbf{0})\big\rangle\Big]\, \mathcal{P}\Big\{\exp\Big[-\langle \mathbf{X}_{t+\varepsilon}, \boldsymbol{\varphi}\rangle - \int_0^{t+\varepsilon} \big\langle \mathbf{A}_{\mathrm{d}s}, \mathbf{u}_s^{t+\varepsilon}(\boldsymbol{\varphi},\mathbf{0})\big\rangle\Big]\,\Big|\,\mathcal{F}_t\Big\}$$
$$= \mathrm{e}^{-\langle \mathbf{X}_t, \mathbf{u}_t^{t+\varepsilon}(\boldsymbol{\varphi},\mathbf{0})\rangle},$$

where in the last step we used the martingale assumption concerning the process in (38). Thus,

$$II_{\varepsilon,t} \;=\; \mathrm{e}^{-\langle \mathbf{A},\boldsymbol{\varphi}\rangle_{[0,t]}}\, \varepsilon^{-1}\Big(\mathrm{e}^{-\langle \mathbf{X}_t,\mathbf{u}_t^{t+\varepsilon}(\boldsymbol{\varphi},\mathbf{0})\rangle} - \mathrm{e}^{-\langle \mathbf{X}_t,\boldsymbol{\varphi}\rangle}\Big). \tag{63}$$

But

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\Big(\mathrm{e}^{-\langle \mathbf{X}_t,\mathbf{u}_t^{t+\varepsilon}(\boldsymbol{\varphi},\mathbf{0})\rangle} - \mathrm{e}^{-\langle \mathbf{X}_t,\boldsymbol{\varphi}\rangle}\Big) \tag{64}$$
$$= -\,\mathrm{e}^{-\langle \mathbf{X}_t,\boldsymbol{\varphi}\rangle} \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\big\langle \mathbf{X}_t, \mathbf{u}_t^{t+\varepsilon}(\boldsymbol{\varphi},\mathbf{0}) - \boldsymbol{\varphi}\big\rangle$$
$$= -\,\mathrm{e}^{-\langle \mathbf{X}_t,\boldsymbol{\varphi}\rangle}\Big\langle \mathbf{X}_t, \Delta_{\boldsymbol{\alpha}}\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\mathbf{1}+\boldsymbol{\beta}}\Big\rangle, \quad t>0, \quad \mathcal{P}\text{–a.s.},$$

where the last limit follows by Lemma 15(a). Combined with (63), this gives

$$\lim_{\varepsilon\downarrow 0} II_{\varepsilon,t} \;=\; -Z_t\Big\langle \mathbf{X}_t, \Delta_{\boldsymbol{\alpha}}\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\mathbf{1}+\boldsymbol{\beta}}\Big\rangle, \quad t>0, \quad \mathcal{P}\text{–a.s.} \tag{65}$$

Now note that

$$|II_{\varepsilon,t}| \;\le\; \frac{1}{\varepsilon}\Big|\mathrm{e}^{-\langle \mathbf{X}_t,\mathbf{u}_t^{t+\varepsilon}(\boldsymbol{\varphi},\mathbf{0})\rangle} - \mathrm{e}^{-\langle \mathbf{X}_t,\boldsymbol{\varphi}\rangle}\Big| \;\le\; \frac{1}{\varepsilon}\Big\langle \mathbf{X}_t, \big|\mathbf{u}_t^{t+\varepsilon}(\boldsymbol{\varphi},\mathbf{0}) - \boldsymbol{\varphi}\big|\Big\rangle \tag{66}$$
$$\le\; c_{59}\,\langle \mathbf{X}_t, \mathbf{1}\rangle, \quad \varepsilon,t>0, \quad \mathcal{P}\text{–a.s.},$$

where the last inequality follows from (59). Hence, since $\int_0^T \mathrm{d}s\,\langle \mathbf{X}_t, \mathbf{1}\rangle < \infty$, by the dominated convergence theorem, we obtain

$$\int_0^T \mathrm{d}t\; II_{\varepsilon,t} \underset{\varepsilon\downarrow 0}{\longrightarrow} -\int_0^T \mathrm{d}t\; Z_t\Big\langle \mathbf{X}_t, \Delta_{\boldsymbol{\alpha}}\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\mathbf{1}+\boldsymbol{\beta}}\Big\rangle, \quad \mathcal{P}\text{–a.s.}, \quad T>0. \tag{67}$$

Apply again (66), Lemma 18, and dominated convergence to conclude that convergence in (67) is also in $\mathcal{L}^1$–space. This gives (55), and hence finishes the proof of Proposition 17. ∎

3.6. **Another martingale.** Proposition 17 and Lemma 18 immediately have the following implication:

Corollary 19 (**Expectation**). *Let* $(\mathbf{X},\mathbf{A})$ *be any solution to the martingale problem* $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$. *Then,*

$$\mathcal{P}\big(\langle \mathbf{X}_t, \mathbf{1}\rangle + \langle \mathbf{A}_t, \mathbf{1}\rangle\big) \;=\; \langle \boldsymbol{\mu}, \mathbf{1}\rangle, \qquad t\ge 0. \tag{68}$$

Another consequence is the following result.

Corollary 20 (**Another martingale**). *Let* $(\mathbf{X},\mathbf{A})$ *be any solution to the martingale problem* $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$. *Then, for* $i\in\{1,2\}$ *and* $\varphi^i \in \Phi_+$,

$$t \;\mapsto\; M_t^i(\varphi^i) \;:=\; \big\langle X_t^i, \varphi^i\big\rangle - \big\langle \mu^i, \varphi^i\big\rangle - \int_0^t \mathrm{d}s\;\big\langle X_s^i, \Delta_{\alpha^i}\varphi^i\big\rangle + \big\langle A_t^i, \varphi^i\big\rangle \tag{69}$$

are $\mathcal{F}_{\cdot}$–martingales starting from 0 *at time* $t = 0$.

Proof. Without loss of generality, take $i = 1$ and $\boldsymbol{\varphi} = (\varphi^1, 0)$. Then

$$\begin{aligned} t \ \mapsto \ M_t^{1,\varepsilon}(\varphi^1) \ :=& \ \frac{1}{\varepsilon}\Big(1 - \mathrm{e}^{-\langle X_t^1, \varepsilon\varphi^1\rangle}\Big) - \frac{1}{\varepsilon}\Big(1 - \mathrm{e}^{-\langle \mu^1, \varepsilon\varphi^1\rangle}\Big) \\ &- \frac{1}{\varepsilon}\int_0^t \mathrm{d}s\ \mathrm{e}^{-\langle X_s^1, \varepsilon\varphi^1\rangle}\big\langle X_s^1, \varepsilon\Delta_{\alpha^1}\varphi^1\big\rangle + \int_0^t \langle A^1_{\mathrm{d}s}, \varphi^1\rangle\, \mathrm{e}^{-\langle X_s^1, \varepsilon\varphi^1\rangle} \\ &+ \frac{1}{\varepsilon}\int_0^t \mathrm{d}s\ \mathrm{e}^{-\langle X_s^1, \varepsilon\varphi^1\rangle}\Big\langle X_s^1, \varepsilon^{1+\beta^1}(\varphi^1)^{1+\beta^1}\Big\rangle \end{aligned} \tag{70}$$

is a martingale. Now let us check that all the terms converge in $\mathcal{L}^1$. Clearly,

$$\frac{1}{\varepsilon}\Big(1 - \mathrm{e}^{-\langle \mu^1, \varepsilon\varphi^1\rangle}\Big) \underset{\varepsilon\downarrow 0}{\longrightarrow} \big\langle \mu^1, \varphi^1\big\rangle. \tag{71}$$

Also,

$$\frac{1}{\varepsilon}\Big(1 - \mathrm{e}^{-\langle X_t^1, \varepsilon\varphi^1\rangle}\Big) \underset{\varepsilon\downarrow 0}{\longrightarrow} \big\langle X_t^1, \varphi^1\big\rangle \quad \text{in } \mathcal{L}^1, \quad t > 0, \tag{72}$$

by Corollary 19 and dominated convergence. The third and fourth terms in (70) are dominated by

$$\int_0^t \mathrm{d}s\ \big\langle X_s^1, |\Delta_{\alpha^1}\varphi^1|\big\rangle \quad \text{and} \quad \big\langle A_t^1, \varphi^1\big\rangle, \tag{73}$$

and hence converge in $\mathcal{L}^1$ to

$$\int_0^t \mathrm{d}s\ \big\langle X_s^1, \Delta_{\alpha^1}\varphi^1\big\rangle \quad \text{and} \quad \big\langle A_t^1, \varphi^1\big\rangle, \tag{74}$$

respectively, again by Corollary 19 and dominated convergence. Similarly, the last term converges to 0 in $\mathcal{L}^1$. Since all these terms converge in $\mathcal{L}^1$, we get the convergence statement

$$M_t^{1,\varepsilon}(\varphi^1) \underset{\varepsilon\downarrow 0}{\longrightarrow} M_t^1(\varphi^1) \quad \text{in } \mathcal{L}^1. \tag{75}$$

Hence $M^1(\varphi^1)$ is a martingale, and we are done. ∎

3.7. **Domination by independent critical superprocesses.** Here we want to make precise the mentioned domination property. For this purpose, let $(\mathbf{X}, \mathbf{A})$ be any solution to the martingale problem $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Conditioned on $(\mathbf{X}, \mathbf{A})$, we will consider the $(\boldsymbol{\alpha}, d, \boldsymbol{\beta})$–pair ${}^{\mathbf{A}}\mathbf{X}$ of independent critical superprocesses with immigration $\mathbf{A}$ and starting from the pair ${}^{\mathbf{A}}\mathbf{X}_0 = \mathbf{0}$ of zero measures. Our purpose is (by using this *random* family ${}^{\mathbf{A}}\mathbf{X}$) to construct an $(\boldsymbol{\alpha}, d, \boldsymbol{\beta})$–pair $\bar{\mathbf{X}}$ of independent critical superprocesses dominating $\mathbf{X}$, by, loosely speaking, adding up $\mathbf{X} + {}^{\mathbf{A}}\mathbf{X} =: \bar{\mathbf{X}}$. In other words, we reintroduce the population masses ${}^{\mathbf{A}}\mathbf{X}$ which had been killed by $\mathbf{A}$ within the process $\mathbf{X}$. Note the different roles of $\mathbf{A}$: within $\mathbf{X}$ it describes the killing, whereas within ${}^{\mathbf{A}}\mathbf{X}$ the immigration.

The following formalism is essentially recalled from [BEP91, Theorem 5.1]. Write now $(\Omega', \mathcal{F}', \mathcal{F}'_{\cdot}, \mathcal{P}')$ for our original stochastic basis entering into the martingale problem $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ of Definition 5. Recall $(\mathbf{X}, \mathbf{A}) = \big(\mathbf{X}(\omega'), \mathbf{A}(\omega')\big)$ is a solution to this martingale problem. Recall also from (33) the canonical basis

$\big({}^\circ\Omega, {}^\circ\mathcal{F}, {}^\circ\mathcal{F}_{\cdot}, {}^\circ\mathcal{P}_{\boldsymbol{\mu},\boldsymbol{\kappa},{}^\circ\mathbf{A}}\big)$ corresponding to the $(\boldsymbol{\alpha}, d, \boldsymbol{\beta})$–pair of independent critical superprocesses with immigration ${}^\circ\mathbf{A}$ of Lemma 14. Redefine

$$\Omega \ := \ \Omega' \times {}^\circ\Omega, \quad \mathcal{F} \ := \ \mathcal{F}' \times {}^\circ\mathcal{F}, \quad \mathcal{F}_{\cdot} \ := \ \mathcal{F}'_{\cdot} \times {}^\circ\mathcal{F}_{\cdot}\,, \tag{76}$$

and set

$$\hat{\mathbf{X}}(\omega) \ := \ {}^\circ\omega, \qquad \omega = (\omega', {}^\circ\omega) \in \Omega. \tag{77}$$

Then we reintroduce a probability measure $\mathcal{P}$ on $(\Omega, \mathcal{F})$ by

$$\mathcal{P}(\mathrm{d}\omega) \ := \ \mathcal{P}'(\mathrm{d}\omega')\ {}^\circ\mathcal{P}_{\mathbf{0},{}^\circ\mathbf{A}}(\mathrm{d}\,{}^\circ\omega), \qquad \omega = (\omega', {}^\circ\omega) \in \Omega, \tag{78}$$

where, according to Lemma 14, ${}^\circ\mathcal{P}_{\mathbf{0},{}^\circ\mathbf{A}}$ is the canonical law of the $(\boldsymbol{\alpha}, d, \boldsymbol{\beta})$–pair ${}^{\mathbf{A}(\omega')}\mathbf{X}$ of independent critical superprocesses with immigration $\mathbf{A}(\omega')$ and starting from ${}^{\mathbf{A}(\omega')}\mathbf{X}_0 = \mathbf{0}$. We will show below that this implies that conditioned on $(\mathbf{X}, \mathbf{A})$ the process $\hat{\mathbf{X}}$ coincides in law with the $(\boldsymbol{\alpha}, d, \boldsymbol{\beta})$–pair ${}^{\mathbf{A}}\mathbf{X}$ of independent critical superprocesses with immigration $\mathbf{A}$ and starting from $\hat{\mathbf{X}}_0 = \mathbf{0}$, just as desired. Finally, write $\pi : \Omega \to \Omega'$ for the projection map $\omega = (\omega', {}^\circ\omega) \mapsto \omega'$, and set

$$\bar{\mathbf{X}} \ := \ \mathbf{X} \circ \pi + \hat{\mathbf{X}}. \tag{79}$$

Proposition 21 (**Domination**). *Fix* $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\vartheta}, \boldsymbol{\mu}$ *as in Definition* 5. *With notation as in* (76)–(79), *the following two statements hold.*

(a) (Interchange): *For any* $F \in \mathrm{b}\mathcal{F}'$ *and* $t \geq 0$,

$$\mathcal{P}\big(F \circ \pi \;\big|\; \mathcal{F}_t\big) \ = \ \mathcal{P}'\big(F \;\big|\; \mathcal{F}'_t\big) \circ \pi, \quad \mathcal{P}\text{–a.s.} \tag{80}$$

(b) (Pair of independent critical superprocesses): $\bar{\mathbf{X}} = \mathbf{X} \circ \pi + \hat{\mathbf{X}}$ *is the* $(\boldsymbol{\alpha}, d, \boldsymbol{\beta})$*–pair of independent critical superprocesses without immigration* (*that is,* $\boldsymbol{\kappa} = \mathbf{0} = {}^\circ\mathbf{A}$ *in Lemma* 14) *starting from* $\bar{\mathbf{X}}_0 = \boldsymbol{\mu}$.

Roughly speaking, by enlarging our stochastic basis we got the almost sure domination $\mathbf{X} \leq \bar{\mathbf{X}}$ of the $(\boldsymbol{\alpha}, d, \boldsymbol{\beta})$–pair $\mathbf{X}$ of interacting superprocesses by the $(\boldsymbol{\alpha}, d, \boldsymbol{\beta})$–pair $\bar{\mathbf{X}}$ of independent critical superprocesses, where $\mathbf{X}_0 = \bar{\mathbf{X}}_0$.

Proof. Part **(a)** follows as in the proof of the corresponding statement in [BEP91, Theorem 5.1], but for **(b)** we need some modifications of their proof as the lack of continuity of the processes induces us to use exponential martingales.

Clearly, $\bar{\mathbf{X}}$ has the required pair $\boldsymbol{\mu}$ of starting measures. Fix $\boldsymbol{\varphi} \in \Phi^2_+$. Set

$$\hat{\mathcal{F}}_t \ := \ \mathcal{F}' \times {}^\circ\mathcal{F}_t\,, \qquad t \geq 0. \tag{81}$$

First we will show that for $0 \leq r \leq t$,

$$\mathcal{P}\big\{\mathrm{e}^{-\langle \hat{\mathbf{X}}_t, \boldsymbol{\varphi}\rangle} \big| \hat{\mathcal{F}}_r\big\} \ = \ \exp\Big[-\big\langle \hat{\mathbf{X}}_r\,, \mathbf{u}^t_r(\boldsymbol{\varphi}, \mathbf{0})\big\rangle - \big\langle \mathbf{A}, \mathbf{u}^t(\boldsymbol{\varphi}, \mathbf{0})\big\rangle_{[r,t]}\Big], \ \ \mathcal{P}\text{–a.s.}, \tag{82}$$

with $\mathbf{u}^t(\boldsymbol{\varphi},\mathbf{0})$ from our Definition 13. In fact, for $B' \in \mathcal{F}'$ and ${}^\circ\!B \in {}^\circ\!\mathcal{F}_r$, by definition (78) of $\mathcal{P}$,

$$\begin{aligned}
&\mathcal{P}\,1_{B'\times{}^\circ\!B}\Big(\mathrm{e}^{-\langle\hat{\mathbf{X}}_t,\,\boldsymbol{\varphi}\rangle} - \exp\Big[-\big\langle\hat{\mathbf{X}}_r\,,\mathbf{u}_r^t(\boldsymbol{\varphi},\mathbf{0})\big\rangle - \big\langle\mathbf{A},\mathbf{u}^t(\boldsymbol{\varphi},\mathbf{0})\big\rangle_{[r,t]}\Big]\Big) \qquad (83)\\
&= \int_{B'}\mathcal{P}'(\mathrm{d}\omega')\ {}^\circ\mathcal{P}_{\mathbf{0},{}^\circ\mathbf{A}}\,1_{{}^\circ\!B}\Big(\exp\Big[-\big\langle\mathbf{A}(\omega'),\mathbf{u}^t(\boldsymbol{\varphi},\mathbf{0})\big\rangle_{[0,t]}\Big]\\
&\qquad {}^\circ\mathcal{P}_{\mathbf{0},{}^\circ\mathbf{A}}\Big\{\exp\Big[-\big\langle\hat{\mathbf{X}}_t\,,\boldsymbol{\varphi}\big\rangle + \big\langle\mathbf{A}(\omega'),\mathbf{u}^t(\boldsymbol{\varphi},\mathbf{0})\big\rangle_{[0,t]}\Big]\,\Big|\,{}^\circ\mathcal{F}_r\Big\}\\
&\qquad\qquad - \exp\Big[-\big\langle\hat{\mathbf{X}}_r\,,\mathbf{u}_r^t(\boldsymbol{\varphi},\mathbf{0})\big\rangle - \big\langle\mathbf{A}(\omega'),\mathbf{u}^t(\boldsymbol{\varphi},\mathbf{0})\big\rangle_{[r,t]}\Big]\Big).
\end{aligned}$$

But by Lemma 14(e), for each fixed t and ω',

$$r\ \mapsto\ \exp\Big[-\big\langle\hat{\mathbf{X}}_r\,,\mathbf{u}_r^t(\boldsymbol{\varphi},\mathbf{0})\big\rangle + \big\langle\mathbf{A}(\omega'),\mathbf{u}^t(\boldsymbol{\varphi},\mathbf{0})\big\rangle_{[0,r]}\Big],\qquad 0\le r\le t, \tag{84}$$

is an $\hat{\mathcal{F}}_\cdot$–martingale. Hence, since $\mathbf{u}_t^t(\boldsymbol{\varphi},\mathbf{0}) = \boldsymbol{\varphi}$, the middle, conditional expectation expression at the right hand side of identity (83) equals

$$\exp\Big[-\big\langle\hat{\mathbf{X}}_r\,,\mathbf{u}_r^t(\boldsymbol{\varphi},\mathbf{0})\big\rangle + \big\langle\mathbf{A}(\omega'),\mathbf{u}^t(\boldsymbol{\varphi},\mathbf{0})\big\rangle_{[0,r]}\Big]. \tag{85}$$

Thus, altogether the right hand side of (83) vanishes, and the claim (82) follows.

Next, from Proposition 17 and the part (a), we get that for fixed $t\ge 0$,

$$r\ \mapsto\ \exp\Big[-\big\langle\mathbf{X}_r\circ\pi\,,\mathbf{u}_r^t(\boldsymbol{\varphi},\mathbf{0})\big\rangle - \big\langle\mathbf{A}\circ\pi,\mathbf{u}^t(\boldsymbol{\varphi},\mathbf{0})\big\rangle_{[0,r]}\Big],\quad 0\le r\le t, \tag{86}$$

is an $\mathcal{F}_\cdot$–martingale. We will use this to show that for $0\le r\le t$,

$$\begin{aligned}
\mathcal{P}\Big\{\exp\Big[-\langle\mathbf{X}_t\circ\pi\,,\,\boldsymbol{\varphi}\rangle - \big\langle\mathbf{A}\circ\pi,\mathbf{u}^t(\boldsymbol{\varphi},\mathbf{0})\big\rangle_{[r,t]}\Big]\Big|\,\mathcal{F}_r\Big\} \qquad (87)\\
= \exp\big\langle\mathbf{X}_r\circ\pi\,,-\mathbf{u}_r^t(\boldsymbol{\varphi},\mathbf{0})\big\rangle.
\end{aligned}$$

Indeed, the left hand side can be written as

$$\exp\Big[\big\langle\mathbf{A}\circ\pi,\mathbf{u}^t(\boldsymbol{\varphi},\mathbf{0})\big\rangle_{[0,r]}\Big]\,\mathcal{P}\Big\{\exp\Big[-\langle\mathbf{X}_t\circ\pi,\boldsymbol{\varphi}\rangle - \big\langle\mathbf{A}\circ\pi,\mathbf{u}^t(\boldsymbol{\varphi},\mathbf{0})\big\rangle_{[0,t]}\Big]\Big|\mathcal{F}_r\Big\}.$$

¿From (86) we may reformulate it as

$$\exp\Big[\big\langle\mathbf{A}\circ\pi,\mathbf{u}^t(\boldsymbol{\varphi},\mathbf{0})\big\rangle_{[0,r]}\Big]\exp\Big[-\big\langle\mathbf{X}_r\circ\pi\,,\mathbf{u}_r^t(\boldsymbol{\varphi},\mathbf{0})\big\rangle - \big\langle\mathbf{A}\circ\pi,\mathbf{u}^t(\boldsymbol{\varphi},\mathbf{0})\big\rangle_{[0,r]}\Big],$$

and (87) follows.

Finally, we will show that for $0\le r\le t$,

$$\mathcal{P}\Big\{\exp\big\langle\mathbf{X}_t\circ\pi + \hat{\mathbf{X}}_t\,,\,-\boldsymbol{\varphi}\big\rangle\,\Big|\,\mathcal{F}_r\Big\}\ =\ \exp\Big\langle\mathbf{X}_r\circ\pi + \hat{\mathbf{X}}_r\,,\,-\mathbf{u}_r^t(\boldsymbol{\varphi},\mathbf{0})\Big\rangle, \tag{88}$$

which by the log-Laplace representation in Lemma 14(c) says that $\bar{\mathbf{X}} = \mathbf{X}\circ\pi + \hat{\mathbf{X}}$ is the desired $(\boldsymbol{\alpha},d,\boldsymbol{\beta})$–pair of independent critical superprocesses without immigration. We start from

$$\mathcal{P}\Big\{\exp\big\langle\mathbf{X}_t\circ\pi + \hat{\mathbf{X}}_t\,,\,-\boldsymbol{\varphi}\big\rangle\,\Big|\,\mathcal{F}_r\Big\}\ =\ \mathcal{P}\Big\{\mathrm{e}^{-\langle\mathbf{X}_t\circ\pi\,,\,\boldsymbol{\varphi}\rangle}\,\mathcal{P}\Big\{\mathrm{e}^{-\langle\hat{\mathbf{X}}_t\,,\,\boldsymbol{\varphi}\rangle}\,\Big|\,\hat{\mathcal{F}}_r\Big\}\,\Big|\,\mathcal{F}_r\Big\}.$$

By (82) we may continue with

$$= \mathcal{P}\Big\{ \mathrm{e}^{-\langle \mathbf{X}_t \circ \pi \,,\, \boldsymbol{\varphi} \rangle} \exp\Big[-\big\langle \hat{\mathbf{X}}_r \,,\, \mathbf{u}_r^t(\boldsymbol{\varphi}, \mathbf{0}) \big\rangle - \big\langle \mathbf{A} \circ \pi, \mathbf{u}^t(\boldsymbol{\varphi}, \mathbf{0}) \big\rangle_{[r,t]} \Big] \,\Big|\, \mathcal{F}_r \Big\}$$

$$= \exp\big\langle \hat{\mathbf{X}}_r \,,\, -\mathbf{u}_r^t(\boldsymbol{\varphi}, \mathbf{0}) \big\rangle \, \mathcal{P}\Big\{ \exp\Big[-\langle \mathbf{X}_t \circ \pi \,,\, \boldsymbol{\varphi} \rangle - \big\langle \mathbf{A} \circ \pi, \mathbf{u}^t(\boldsymbol{\varphi}, \mathbf{0}) \big\rangle_{[r,t]} \Big] \,\Big|\, \mathcal{F}_r \Big\}.$$

By (87) we get

$$= \exp\big\langle \hat{\mathbf{X}}_r \,,\, -\mathbf{u}_r^t(\boldsymbol{\varphi}, \mathbf{0}) \big\rangle \exp\big\langle \mathbf{X}_r \circ \pi \,,\, -\mathbf{u}_r^t(\boldsymbol{\varphi}, \mathbf{0}) \big\rangle \;=\; \exp\big\langle \bar{\mathbf{X}}_r \,,\, -\mathbf{u}_r^t(\boldsymbol{\varphi}, \mathbf{0}) \big\rangle, \tag{89}$$

finishing the proof. ∎

3.8. **Simultaneous domination by independent critical superprocesses.** In the previous subsection, for each solution $(\mathbf{X}, \mathbf{A})$ to the martingale problem $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ we found a stochastic basis such that, on this basis, $\mathbf{X}$ is a.s. dominated by the $(\boldsymbol{\alpha}, d, \boldsymbol{\beta})$–pair of independent critical superprocesses $\bar{\mathbf{X}}$ without immigration. This construction can actually be done *simultaneously* to all solutions to $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$, that is, on a common stochastic basis, with the same $\bar{\mathbf{X}}$. This follows from the following lemma.

Lemma 22 (Simultaneous domination). *Let $\big\{\mathbf{Y}^\upsilon \in \mathcal{D}^2_{\mathcal{M}_\mathrm{f}} : \upsilon \in \Upsilon\big\}$ be a collection of pairs of random processes. Suppose that for each $\upsilon \in \Upsilon$ there exists $\bar{\mathbf{Y}}^\upsilon \in \mathcal{D}^2_{\mathcal{M}_\mathrm{f}}$ such that*

$$\mathbf{Y}^\upsilon \leq \bar{\mathbf{Y}}^\upsilon \;\; a.s., \tag{90}$$

where all the $\bar{\mathbf{Y}}^\upsilon$ are identically distributed. Then there exists a common probability space $\big(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{P}}\big)$ on which some $\big\{\mathbf{Z}^\upsilon \in \mathcal{D}^2_{\mathcal{M}_\mathrm{f}} : \upsilon \in \Upsilon\big\}$ and $\bar{\mathbf{Z}} \in \mathcal{D}^2_{\mathcal{M}_\mathrm{f}}$ are defined such that $\big(\mathbf{Y}^\upsilon, \bar{\mathbf{Y}}^\upsilon\big)$ coincides in law with $\big(\mathbf{Z}^\upsilon, \bar{\mathbf{Z}}\big)$, for each $\upsilon \in \Upsilon$.

In words, there is a $\bar{\mathbf{Z}}$ which equals in law to any of $\bar{\mathbf{Y}}^\upsilon$ and dominates a.s. all the $\mathbf{Z}^\upsilon$ simultaneously.

Proof. Denote by $\bar{Q}$ the law on $\mathcal{D}^2_{\mathcal{M}_\mathrm{f}}$ of $\bar{\mathbf{Y}}^\upsilon$ (recall that $\bar{Q}$ is the same for any $\upsilon \in \Upsilon$). Let $\bar{\mathbf{Z}} \in \mathcal{D}^2_{\mathcal{M}_\mathrm{f}}$ be defined over a probability space $(\Omega_2, \mathcal{F}_2, \mathcal{P}_2)$ with law $\bar{Q}$. Fix an arbitrary $\upsilon \in \Upsilon$. Denote by Q^υ the joint law on $\mathcal{D}^2_{\mathcal{M}_\mathrm{f}} \times \mathcal{D}^2_{\mathcal{M}_\mathrm{f}}$ of $(\mathbf{Y}^\upsilon, \bar{\mathbf{Y}}^\upsilon)$. Also, let Q^υ_y denote the regular conditional distribution on $\mathcal{D}^2_{\mathcal{M}_\mathrm{f}}$ of $\mathbf{Y}^\upsilon$ given $\bar{\mathbf{Y}}^\upsilon = y$. Hence,

$$Q^\upsilon\big(\mathrm{d}(y_1, y_2)\big) \;=\; Q^\upsilon_{y_2}(\mathrm{d}y_1)\, \bar{Q}(\mathrm{d}y_2), \qquad (y_1, y_2) \in \mathcal{D}^2_{\mathcal{M}_\mathrm{f}}\,. \tag{91}$$

Let $(\Omega_1, \mathcal{F}_1, \mathcal{P}_1)$ be another sufficiently rich probability space. Now for $\mathcal{P}_2$–almost all ω_2 we may construct $\mathbf{Z}^\upsilon(\omega_1, \omega_2)$ on

$$(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{P}}) \;:=\; (\Omega_1 \times \Omega_2,\, \mathcal{F}_1 \otimes \mathcal{F}_2,\, \mathcal{P}_1 \times \mathcal{P}_2) \tag{92}$$

with conditional law $Q^\upsilon_{\bar{\mathbf{Z}}(\omega_2)}$. Piecing everything together we get that the pair $\big(\mathbf{Z}^\upsilon, \bar{\mathbf{Z}}\big)$ is defined on $\big(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{P}}\big)$, and its joint distribution is given by (91), which coincides with the law of $\big(\mathbf{Y}^\upsilon, \bar{\mathbf{Y}}\big)$. By repeating this construction on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{P}})$ for all $\upsilon \in \Upsilon$ with the *same* $\mathbf{Z}$, we get the desired result. ∎

Combining Lemma 22 with Proposition 21 we obtain the following result:

Corollary 23 (**Simultaneous domination**). *Fix* $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\vartheta},\boldsymbol{\mu}$ *as in Definition* 5. *Let* $\{(\mathbf{X}^{\upsilon},\mathbf{A}^{\upsilon}):\upsilon\in\Upsilon\}$ *be a family of solutions to the martingale problem* $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$. *Then there exists a probability space* $(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathcal{P}})$ *on which a family*

$$\big\{({}'\mathbf{X}^{\upsilon},{}'\mathbf{A}^{\upsilon},\bar{\mathbf{X}}):\ \upsilon\in\Upsilon\big\} \tag{93}$$

is defined and posses the following properties:

- **(a) (Same laws):** $({}'\mathbf{X}^{\upsilon},{}'\mathbf{A}^{\upsilon})$ *coincides in law with* $(\mathbf{X}^{\upsilon},\mathbf{A}^{\upsilon})$, *for each* υ *in* Υ.
- **(b) (Independent superprocesses):** $\bar{\mathbf{X}}$ *is the* $(\boldsymbol{\alpha},d,\boldsymbol{\beta})$*–pair of independent critical superprocesses without immigration, starting from* $\bar{\mathbf{X}}_0=\boldsymbol{\mu}$.
- **(c) (Domination):** *For any* $\upsilon\in\Upsilon$, *given* $({}'\mathbf{X}^{\upsilon},{}'\mathbf{A}^{\upsilon})$,

$${}'\hat{\mathbf{X}}^{\upsilon}\ :=\ \bar{\mathbf{X}}-{}'\mathbf{X}^{\upsilon} \tag{94}$$

is the $(\boldsymbol{\alpha},d,\boldsymbol{\beta})$*–pair of independent critical superprocesses starting from* ${}'\hat{\mathbf{X}}_0^{\upsilon}=\mathbf{0}$, *and with immigration* ${}'\mathbf{A}^{\upsilon}$.

For convenience, we introduce the following convention.

Convention 24 (**Simultaneous domination**). Without loss of generality, from now on we will assume that any family $\{(\mathbf{X}^{\upsilon},\mathbf{A}^{\upsilon}):\upsilon\in\Upsilon\}$ of solutions to $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is defined on a common probability space where they enjoy simultaneous domination by $\bar{\mathbf{X}}$ as described in Corollary 23. ◇

4. Log-Laplace equation involving generalized input data

In this section we establish some properties of solutions to log-Laplace equations involving generalized input data (see, e.g., Proposition 34). The developed framework is used in Subsection 4.4 to show the existence of collision local times and collision measures for pairs of independent critical superprocesses for a more general class of initial measures than was known before. We also give their log-Laplace representations (Proposition 35). The log-Laplace technique will be also crucial in Section 5 for the proof of Theorem 7.

4.1. Energy conditions.

We define the suitable sets of measures and measure-valued paths used as input data for log-Laplace equations.

First we need some more notation. Besides the space $\mathcal{D}_{\mathcal{M}_{\mathrm{f}}}$, for each $T>0$, we introduce the space $\mathcal{D}_{\mathcal{M}_{\mathrm{f}}}^{T}$ of all càdlàg paths $\nu:[0,T]\to\mathcal{M}_{\mathrm{f}}(\mathsf{R}^d)$, equipped with the Skorohod topology. We need also the space $\mathfrak{P}^T$ of all (equivalence classes of) measurable paths $\nu:[0,T]\to\mathcal{M}_{\mathrm{f}}(\mathsf{R}^d)$. Note that to each $\nu\in\mathcal{D}_{\mathcal{M}_{\mathrm{f}}}^{T}$ there is a unique element in $\mathfrak{P}^T$. For this reason, we do not distinguish in notation within this correspondence.

Besides the spaces $\mathrm{b}\mathcal{C}$ and $\bar{\mathcal{C}}$, we need to introduce the spaces $\mathrm{b}\mathcal{C}_{\mathrm{co}}$ which refer to $\mathrm{b}\mathcal{C}$ but equipped with the topology of uniform convergence on compacta (the index "co" stands for "compact open").

For $T>0$ and measurable $g:[0,T]\to\mathsf{R}$, put

$$\|g\|_T\ :=\ \operatorname*{ess\,sup}_{0\le s\le T}|g_s|, \tag{95}$$

whereas for measurable $\psi:[0,T]\times\mathsf{R}^d\to\mathsf{R}$ and $B\subseteq\mathsf{R}^d$ set

$$\|\psi\|_{T,B}\ :=\ \sup_{x\in B}\ \operatorname*{ess\,sup}_{0\le r\le T}\big|\psi_r(x)\big| \tag{96}$$

and write

$$\|\cdot\|_{T,\infty} \ := \ \|\cdot\|_{T,\mathsf{R}^d}\,. \tag{97}$$

Write S and p for the semigroup respectively for the continuous transition kernel of the symmetric α–stable process with generator $\Delta_\alpha := -\vartheta(-\Delta)^{\alpha/2}\Delta_\alpha\,,\ \vartheta > 0$, $0 < \alpha \le 2$. For $\eta \in [0,d)$ (with d the dimension of R^d), set

$$h_{x,\eta}(y) \ := \ |x-y|^{-\eta}, \qquad x,y \in \mathsf{R}^d. \tag{98}$$

Definition 25 (**Energy conditions**)**.** Fix $\eta \in [0,d)$.

(a) (Measures): We say the measure $\mu \in \mathcal{M}_\mathrm{f}$ has *weakly finite η–energy* and write $\mu \in \mathcal{M}_\mathrm{f}^{\eta,\circ}$, if

$$\langle \mu, h_{x,\eta}\rangle \ < \ \infty \quad \textit{for}\, \mu\textit{–almost all}\ x \in \mathsf{R}^d. \tag{99}$$

(b) (Paths): Fix $T > 0$. We say, the deterministic (measurable) path ν in $\mathfrak{P}^T$ has *finite η–energy* and write $\nu \in \mathfrak{P}^{T,\eta}$, if

$$\|\nu\|_{-\eta,T} \ := \ \sup_{x\in\mathsf{R}^d} \big\|\langle \nu_{\cdot}\,, h_{x,\eta}\rangle\big\|_T \ < \ \infty. \tag{100}$$

It is said to have *weakly finite η–energy* and we write $\nu \in \mathfrak{P}^{T,\eta,\circ}$, if

$$\langle \nu_s\,, h_{x,\eta}\rangle < \infty \quad \text{for} \ \ \mathrm{d}s\,\nu_s(\mathrm{d}x)\text{–almost all} \ \ (s,x) \in [0,T]\times \mathsf{R}^d \tag{101}$$

holds. ◇

Example 26 ((α,d,β)**–superprocess**)**.** Fix any $\eta \in \big(0, \min(\alpha/\beta, d)\big)$. Let X denote the critical (α,d,β)–superprocess (without immigration) starting from μ in $\mathcal{M}_\mathrm{f}$. Then

$$\begin{aligned} X_t \in \mathcal{M}_\mathrm{f}^{\eta,\circ}, \quad \mathcal{P}\text{–a.s.}, \quad t > 0, \\ \{X_t:\ t\le T\} \in \mathfrak{P}^{T,\eta,\circ}, \quad \mathcal{P}\text{–a.s.}, \quad T > 0, \end{aligned} \tag{102}$$

(see [Myt98a, Lemma 22 and Corollary 4]). ◇

In $\mathfrak{P}^{T,\eta}$ we introduce a *topology* by saying that $\nu^n \to \nu^0$ in $\mathfrak{P}^{T,\eta}$ if

$$\sup_{n\ge 0}\ \operatorname*{ess\,sup}_{0\le s\le T} \|\nu_s^n\| \ < \ \infty, \tag{103}$$

$$\sup_{n\ge 0} \|\nu^n\|_{-\eta,T} \ < \ \infty, \tag{104}$$

$$\mathrm{d}s\ \nu_s^n(\mathrm{d}x) \ \underset{n\uparrow\infty}{\longrightarrow} \ \mathrm{d}s\ \nu_s^0(\mathrm{d}x) \ \text{ in } \ \mathcal{M}_\mathrm{f}^T := \mathcal{M}_\mathrm{f}\big([0,T]\times\mathsf{R}^d\big) \tag{105}$$

(equipped with the weak topology). Roughly speaking, the ν^n converge, if the measures $\nu_s^n(\mathrm{d}x)$ are uniformly bounded, the paths ν^n have uniformly finite η–energy, and they converge weakly as time-space measures.

Lemma 27 (**Some compact sets**)**.** *Fix $T > 0$. Suppose $\tilde{C}$ is a compact subset of $\mathcal{D}^T_{\mathcal{M}_\mathrm{f}}$. Let*

$$C \ := \ \Big\{\nu \in \mathfrak{P}^{T,\eta}:\ \exists\, \tilde\nu \in \tilde C \ \textit{ with } \ \nu \le \tilde\nu \ \textit{ in } \ \mathfrak{P}^T\Big\}. \tag{106}$$

For $m > 0$ fixed, set

$$C^m \ := \ \big\{\nu \in C:\ \|\nu\|_{-\eta,T} \le m\big\}. \tag{107}$$

Then C^m is a compact subset of $\mathfrak{P}^{T,\eta}$.

Proof. Fix $m,T>0$. Let $\{\nu^n: n\geq 1\}$ be any sequence in C^m. We have to find a subsequence $\{\nu^{n_k}: k\geq 1\}$ converging in $\mathfrak{P}^{T,\eta}$ as $k\uparrow\infty$ to some $\nu\in C^m$.

First of all, to each n there is a $\tilde{\nu}^n\in\tilde{C}$ such that

$$\nu^n \;\leq\; \tilde{\nu}^n. \tag{108}$$

Since $\tilde{C}$ is compact, there is a subsequence $\{\tilde{\nu}^{n_k}: k\geq 1\}$ converging in $\tilde{C}\subset\mathcal{D}^T_{\mathcal{M}_\mathrm{f}}$ as $k\uparrow\infty$. From the domination (108), we get (103) with $\sup_{n\geq 0}$ replaced by $\sup_{n\geq 1}$. Moreover, (104) follows from (107), again excluding $n=0$. Introduce the measures

$$\mu^k\big(\mathrm{d}(s,x)\big) \;:=\; \mathrm{d}s\,\nu^{n_k}_s(\mathrm{d}x) \;\leq\; \mathrm{d}s\,\tilde{\nu}^{n_k}_s(\mathrm{d}x) \;=:\; \tilde{\mu}^k\big(\mathrm{d}(s,x)\big) \tag{109}$$

on $[0,T]\times\mathsf{R}^d$. Since the $\tilde{\nu}^{n_k}$ converge in $\mathcal{D}^T_{\mathcal{M}_\mathrm{f}}$, the measures $\tilde{\mu}^k$ converge in $\mathcal{M}^T_\mathrm{f}$. Then the domination (109) gives the relative compactness of $\big\{\mu^k: k\geq 1\big\}\subset\mathcal{M}^T_\mathrm{f}$. We may assume that this sequence converges as $k\uparrow\infty$ (by taking a subsequence if needed) to some $\mu^0\in\mathcal{M}^T_\mathrm{f}$. It remains to show that μ^0 has the form

$$\mu^0\big(\mathrm{d}(s,x)\big) \;=\; \mathrm{d}s\,\nu^0_s(\mathrm{d}x) \quad\text{with } \nu^0_s\in\mathcal{M}_\mathrm{f} \text{ for almost all } s, \tag{110}$$

and that ν^0 satisfies

$$\operatorname*{ess\,sup}_{0\leq s\leq T}\|\nu^0_s\| \;<\;\infty \quad\text{and}\quad \|\nu^0\|_{-\eta,T}\;<\;\infty. \tag{111}$$

¿From the weak convergence $\mu^k\to\mu^0$ in $\mathcal{M}^T_\mathrm{f}$ we get

$$\int_0^T \mathrm{d}s\;f_s\;\langle\nu^{n_k}_s,g\rangle \;\underset{k\uparrow\infty}{\longrightarrow}\; \int_{[0,T]\times\mathsf{R}^d}\mu^0\big(\mathrm{d}(s,x)\big)\,f_s\,g(x), \tag{112}$$

$f\in\mathcal{C}([0,T])$, $g\in\bar{\mathcal{C}}$. However, by (103),

$$\sup_{k\geq 1}\big\|\langle\nu^{n_k}_\cdot,1\rangle\big\|_T \;<\;\infty, \tag{113}$$

hence, from (112), for any fixed $g\in\mathrm{b}\mathcal{B}$, the (finite) signed measure

$$\Theta^g(\mathrm{d}s) \;:=\; \int_{\mathsf{R}^d}\mu^0\big(\mathrm{d}(s,x)\big)\,g(x) \tag{114}$$

has a total variation bound

$$\big|\Theta^g(\mathrm{d}s)\big| \;\leq\; \|g\|_\infty\,\sup_{k\geq 1}\big\|\langle\nu^{n_k}_\cdot,1\rangle\big\|_T\;\mathrm{d}s. \tag{115}$$

Thus, for any $g\in\mathrm{b}\mathcal{B}$, the signed measure $\Theta^g(\mathrm{d}s)$ is absolutely continuous, that is, by Radon-Nikodym it can be represented as

$$\Theta^g(\mathrm{d}s) \;=\; \Theta^g_s\,\mathrm{d}s \tag{116}$$

(see, for instance, [Doo94, Theorem 10.7]). But as a functional of $g\in\mathrm{b}\mathcal{B}$, the signed density functions $s\mapsto\Theta^g_s$ are almost everywhere non-negative, provided that $g\geq 0$, they are almost linear, that is,

$$\Theta^{af+bg}_s \;=\; a\Theta^f_s+b\Theta^g_s,\quad\text{for a.a. } s,\quad\text{where } a,b\in\mathsf{R},\;\; f,g\in\mathrm{b}\mathcal{B}, \tag{117}$$

and they satisfy

$$\Theta^{g_m}_s \;\underset{m\uparrow\infty}{\nearrow}\; \Theta^g_s \text{ for a.a. } s,\quad\text{if}\quad 0\leq g_m\uparrow g \text{ for a.a. } s. \tag{118}$$

Then by [Get74, Proposition 4.1], there is a bounded kernel from $[0,T]$ to $\mathcal{M}_{\mathrm{f}}$, denoted by ν^0, such that

$$\Theta_s^g \;=\; \langle \nu_s^0, g\rangle, \qquad \text{for almost all } s. \tag{119}$$

Consequently, (110) is true. It remains to show that (111) holds.

Clearly, the first part of (111) follows from the dominations (108) and (115). On the other hand, the η–energy of ν^0 is bounded by m. In fact, define

$$h_{x,\eta}^{\ell} \;:=\; h_{x,\eta} \wedge \ell, \qquad \ell \geq 1, \quad x \in \mathsf{R}^d. \tag{120}$$

Since $\nu^{n_k} \in C^m$ for all k, we have

$$\Theta_s^{h_{x,\eta}^{\ell}}\, \mathrm{d}s \;\leq\; \sup_{k\geq 1} \big\| \langle \nu_\cdot^{n_k}, h_{x,\eta}^{\ell}\rangle \big\|_T \,\mathrm{d}s \;\leq\; m\,\mathrm{d}s, \qquad \ell \geq 1, \quad x \in \mathsf{R}^d. \tag{121}$$

Hence, by (119) and (121),

$$\Theta_s^{h_{x,\eta}^{\ell}} \;=\; \big\langle \nu_s^0, h_{x,\eta}^{\ell} \big\rangle \;\leq\; m, \quad \text{a.a. } s, \qquad \ell \geq 1, \quad x \in \mathsf{R}^d. \tag{122}$$

By monotone convergence,

$$\big\langle \nu_s^0, h_{x,\eta} \big\rangle \;=\; \lim_{\ell \uparrow \infty} \big\langle \nu_s^0, h_{x,\eta}^{\ell} \big\rangle \;\leq\; m, \quad \text{a.a. } s, \qquad x \in \mathsf{R}^d. \tag{123}$$

Thus, $\|\nu^0\|_{-\eta,T} \leq m$, that is, $\nu^0 \in C^m$, and the proof is finished. ∎

4.2. **Log-Laplace equation involving generalized input data.** To describe the log-Laplace functionals of some collision local times and collision measures, we need to allow that generalized input data enter into the log-Laplace equation.

In this subsection, we fix constants

$$\alpha \in (0,2], \quad \beta \in (0,1], \quad \vartheta > 0, \quad \text{and} \quad 0 \leq \eta < d. \tag{124}$$

Combining the proofs of the Theorems 2 and 3 in [Myt98a], we get the following result.

Proposition 28 (**Log-Laplace equation involving generalized input data**).

Suppose $d < \frac{\alpha}{\beta} + \eta$. *Fix* $T > 0$, $\mu \in \mathcal{M}_{\mathrm{f}}^{\eta,\circ}$, *and* $\nu \in \mathfrak{P}^{T,\eta,\circ}$ (*recall Definition* 25).

(a) (Existence): *There exists an element* $w^T = w^T(\mu,\nu) \in \mathcal{L}_+^{1,T} \cap \mathcal{L}_+^{1+\beta,T}$ *satisfying*

$$w_r^T(x) \;=\; \mu\ast \mathrm{p}_{T-r}\,(x) + \int_r^T \mathrm{d}s\; \nu_s \ast \mathrm{p}_{s-r}\,(x) - \int_r^T \mathrm{d}s\; \mathrm{S}_{s-r}\big((w_s^T)^{1+\beta}\big)\,(x), \tag{125}$$

for almost all $(r,x) \in [0,T) \times \mathsf{R}^d$.

(b) (Uniqueness): *For each solution* $w^T \in \mathcal{L}_+^{1,T} \cap \mathcal{L}_+^{1+\beta,T}$ *to* (125),

$$\begin{aligned} &\mu\ast \mathrm{p}_{T-r}\,(\,\cdot\,) + \int_r^T \mathrm{d}s\; \nu_s \ast \mathrm{p}_{s-r}\,(\,\cdot\,) - \int_r^T \mathrm{d}s\; \mathrm{S}_{s-r}\big((w_s^T)^{1+\beta}\big)\,(\,\cdot\,) \\ &=:\; \bar{w}_r^T \;=\; \bar{w}_r^T(\mu,\nu) \;\in\; \mathcal{L}_+^{1} \cap \mathcal{L}_+^{1+\beta} \end{aligned} \tag{126}$$

is well-defined for all $r \in [0,T)$. *Moreover, if* ${}^1w^T, {}^2w^T \in \mathcal{L}_+^{1,T} \cap \mathcal{L}_+^{1+\beta,T}$ *are solutions to* (125) *and* ${}^1\bar{w}^T, {}^2\bar{w}^T$ *are defined as in* (126), *then*

$${}^1\bar{w}_r^T = {}^2\bar{w}_r^T \quad \text{in } \mathcal{L}_+^1, \text{ for all } r \in [0,T). \tag{127}$$

In particular, ${}^1w^T = {}^2w^T$ *in* $\mathcal{L}_+^{1,T} \cap \mathcal{L}_+^{1+\beta,T}$.

Note that heuristically integral equation (125) can be written as

$$\left.\begin{aligned} -\frac{\partial}{\partial r} w_r^T \;=\; \Delta_\alpha w_r^T - (w_r^T)^{1+\beta} + \frac{\nu_r(\mathrm{d}x)}{\mathrm{d}x} \quad \text{on} \quad (0,T)\times \mathsf{R}^d \\ \text{with terminal condition} \quad w_{T-}^T = \frac{\mu(\mathrm{d}x)}{\mathrm{d}x} \end{aligned}\right\} \tag{128}$$

where $\frac{\mu(\mathrm{d}x)}{\mathrm{d}x}$ denotes the *generalized* derivative of the measure $\mu(\mathrm{d}x)$.

Remark 29 (**Extending the dimension range**). If in the previous proposition, μ is absolutely continuous (with respect to Lebesgue measure), then the assumption on the dimension can be weakened to $d < \alpha + \frac{\alpha}{\beta} + \eta$, again by Theorems 2 and 3 (and their proofs) in [Myt98a]. $\diamond$

Further on we will need to define particular *everywhere* non-negative representatives of $\mathcal{L}_+$–solutions introduced in Proposition 28. The next two technical lemmas are a necessary preparation for this.

Lemma 30 (**Continuous representative under truncation**). *Fix $T>0$ and $\nu \in \mathfrak{P}^{T,\eta,\circ}$. Let $\mu \in \mathcal{M}_{\mathrm{f}}^{\eta,\circ}$ and $d<\eta+\frac{\alpha}{\beta}$, or μ as in Remark* 29 *and $d<\alpha+\frac{\alpha}{\beta}+\eta$. For fixed $t\in(0,T)$, define*

$$\nu_s^t(\mathrm{d}x) \;:=\; 1_{[t,T]}(s)\,\nu_s(\mathrm{d}x), \qquad 0\le s\le T, \tag{129}$$

and consider the solution $w^T = w^T(\mu,\nu^t)$ according to Proposition 28*. Then,*

$$\bar{w}_r^T(x) \;:=\; \mu * \mathrm{p}_{T-r}\,(x) + \int_r^T \mathrm{d}s\; \nu_s^t * \mathrm{p}_{s-r}\,(x) - \int_r^T \mathrm{d}s\; \mathrm{S}_{s-r}\big((w_s^T)^{1+\beta}\big)\,(x) \tag{130}$$

$r\in[0,T)$, $x\in\mathsf{R}^d$, defines a representative $\bar{w}^T = \bar{w}^T(\mu,\nu^t)$ of $w^T(\mu,\nu^t)$ which, for each $r\in[0,t)$, is non-negative and continuous on R^d.

Roughly speaking, if on $[0,t]$ the forcing term ν is set to 0, then on $[0,t]$ the corresponding $\mathcal{L}_+$–solution w^T has a non-negative continuous representative $\bar{w}^T$.

Proof. By Proposition 28,

$$w_t^T(\mu,\nu^t) \in \mathcal{L}_+^1\,. \tag{131}$$

Moreover, by definition (129), and the semigroup property of solutions, it is easy to see that

$$\bar{w}_r^T(\mu,\nu^t) \;=\; u_r^t\big(w_t^T(\mu,\nu^t),0\big), \qquad 0\le r<t. \tag{132}$$

Hence, by (131) and Lemma 16(a), we are done. ∎

We will use this lemma to proof the following result.

Lemma 31 (**Non-negative representative**). *Impose the assumptions from the previous lemma. If $w^T = w^T(\mu,\nu)$ solves* (125)*, then the following inequality holds*

$$\int_r^T \mathrm{d}s\; \mathrm{S}_{s-r}\big((w_s^T)^{1+\beta}\big)\,(x) \;\le\; \mu * \mathrm{p}_{T-r}\,(x) + \int_r^T \mathrm{d}s\; \nu_s * \mathrm{p}_{s-r}\,(x) \tag{133}$$

for all $(r,x)\in[0,T)\times\mathsf{R}^d$, for which the integral at the right hand side is finite.

Proof. Fix $r \in [0,T)$ and let n_0 so large that $r+1/n_0 < T$. Then with $\bar{w}^T$ from Proposition 28, define, for $0 \le s \le T$ and $n \ge n_0$,

$$w^{T,n} := w^T(\mu, \nu^{r+1/n}), \tag{134a}$$

$$\bar{w}^{T,n} := \bar{w}^T(\mu, \nu^{r+1/n}), \tag{134b}$$

where

$$\nu_s^{r+1/n} := 1_{[r+1/n,T]}(s)\, \nu_s(\mathrm{d}x), \qquad 0 \le s \le T. \tag{135}$$

By Lemma 30, $\bar{w}_r^{T,n}$ is non-negative and continuous on R^d for all $n \ge n_0$. By monotonicity properties (see Corollary 3 of [Myt98a]), for all $n \ge m \ge n_0$,

$$w_r^{T,n}(x) \ \ge\ w_r^{T,m}(x) \quad \text{for almost all } x \in \mathsf{R}^d, \tag{136}$$

hence, by the continuity of $\bar{w}_r^{T,n}$,

$$\bar{w}_r^{T,n}(x) \ \ge\ \bar{w}_r^{T,m}(x) \quad \text{for all } x \in \mathsf{R}^d. \tag{137}$$

On the other hand, for all $x \in \mathsf{R}^d$,

$$\begin{aligned} 0 \ \le\ \bar{w}_r^{T,n}(x) \ &=\ \mu * \mathrm{p}_{T-r}(x) + \int_{r+1/n}^T \mathrm{d}s\ \nu_s * \mathrm{p}_{s-r}(x) \\ &\qquad - \int_r^T \mathrm{d}s\ \mathrm{S}_{s-r}\big((w_s^{T,n})^{1+\beta}\big)(x). \end{aligned} \tag{138}$$

As $n \uparrow \infty$, by Theorem 3(i) in [Myt98a],

$$w^{T,n} \ \uparrow \ \text{some } w^{T,\infty} \quad \text{in} \quad \mathcal{L}_+^{1,T} \cap \mathcal{L}_+^{1+\beta,T}. \tag{139}$$

Note that $w_s^{T,\infty} = w_s^T$ for $s \in [r,T)$. By monotone convergence theorem, from inequality (138) we get

$$0 \ \le\ \mu * \mathrm{p}_{T-r}(x) + \int_r^T \mathrm{d}s\ \nu_s * \mathrm{p}_{s-r}(x) - \int_r^T \mathrm{d}s\ \mathrm{S}_{s-r}\big((w_s^T)^{1+\beta}\big)(x), \tag{140}$$

for all those $x \in \mathsf{R}^d$ for which the first integral term at the right hand side stays finite. This finishes the proof. ∎

Definition 32 (**Non-negative representative**)**.** Under the assumption in Lemma 30 we now *define* the following representative $u^T = u^T(\mu,\nu)$ *of* $w^T = w^T(\mu,\nu)$ from Proposition 28:

$$u_r^T(x) := \begin{cases} \mu * \mathrm{p}_{T-r}(x) + \int_r^T \mathrm{d}s\ \nu_s * \mathrm{p}_{s-r}(x) - \int_r^T \mathrm{d}s\ \mathrm{S}_{s-r}\big((w_s^T)^{1+\beta}\big)(x) \\ \qquad \text{for all } (r,x) \in [0,T) \times \mathsf{R}^d, \text{ for which the first} \\ \qquad \text{integral term is finite,} \\ +\infty, \quad \text{otherwise.} \end{cases}$$

Note that by Lemma 31, u^T is *non-negative everywhere on* $[0,T) \times \mathsf{R}^d$ and we have the *domination*

$$0 \ \le\ u_r^T(x) \ \le\ \mu * \mathrm{p}_{T-r}(x) + \int_r^T \mathrm{d}s\ \nu_s * \mathrm{p}_{s-r}(x), \qquad (r,x) \in [0,T) \times \mathsf{R}^d. \tag{141}$$

For the further procedure, we adopt the convention to set $F(+\infty) := 0$ in case u^T enters as an argument into a function F on R_+. ◇

Lemma 33 (**Subadditivity**). *Let* μ,ν *and* μ',ν' *satisfy the assumptions in Lemma* 30 (*or Lemma* 31). *Then for the representatives from Definition* 32 *we have*

$$u^T(\mu+\mu',\nu+\nu') \ \leq \ u^T(\mu,\nu)+u^T(\mu',\nu') \quad \textit{everywhere on} \quad [0,T)\times\mathsf{R}^d. \tag{142}$$

Proof. This follows easily from Lemmas 10 and 11, and Corollaries 2 and 3 in [Myt98a] together with the monotone limit construction of $\bar{w}^T$ in the proof of Lemma 31. ■

4.3. **Continuous convergence of log-Laplace functions.** As a preparation for Section 5, we will establish some uniform convergence properties of log-Laplace functions with respect to input data. We need to impose the following parameter restrictions:

$$\alpha\in(0,2],\quad \beta\in(0,1],\quad \vartheta>0,\quad \text{and}\quad \eta\in\bigl((d-\alpha)_+\,,\,d\bigr). \tag{143}$$

Recall notation J_ε from (16) and $\mathfrak{P}^{T,\eta}$ from Definition 25(b).

Proposition 34 (**Continuous convergence**). *Fix* $T>0$ *and a constant* η *satisfying* (143). *In* $\mathfrak{P}^{T,\eta}$, *consider a converging sequence* $\nu^n\to\nu^0$ *as* $n\uparrow\infty$. *Let* J^1 *and* J^2 *be any mollifiers. For* $n\geq0$, $T>0$, *and* $0<\varepsilon^1,\varepsilon^2\leq1$ *fixed, let*

$$u^{n,T,\varepsilon^1,\varepsilon^2} \ := \ u^T\bigl(0,\nu^n\!*\![J^1_{\varepsilon^1}+J^2_{\varepsilon^2}]\bigr) \ \geq \ 0 \tag{144}$$

be the unique solution to the equation

$$u_r^{n,T,\varepsilon^1,\varepsilon^2}(x) \ = \ \int_r^T\!\mathrm{d}s\,\bigl(\nu^n_s\!*\![J^1_{\varepsilon^1}+J^2_{\varepsilon^2}]\bigr)*\mathrm{p}_{s-r}\,(x) \tag{145}$$

$$- \ \int_r^T\!\mathrm{d}s\;\mathrm{S}_{s-r}\bigl((u_s^{n,T,\varepsilon^1,\varepsilon^2})^{1+\beta}\bigr)\,(x)$$

$(r,x)\in[0,T]\times\mathsf{R}^d$ (*recall Definition* 13). *Then, if* $(r_k\,,x_k)\to(r,x)$ *in* $[0,T]\times\mathsf{R}^d$ *as* $k\uparrow\infty$, *the limit*

$$\lim_{n,k\uparrow\infty,\ \varepsilon^1,\varepsilon^2\downarrow0} u_{r_k}^{n,T,\varepsilon^1,\varepsilon^2}(x_k) \ =: \ u_r^T(x) \tag{146}$$

exists, and $u^T=\bigl\{u_r^T(x):\,(r,x)\in[0,T]\times\mathsf{R}^d\bigr\}$ *is the time-space continuous representative* $u^T(0,2\nu^0)$ *of* $w^T=w^T(0,2\nu^0)\in\mathcal{L}_+^{1,T}\cap\mathcal{L}_+^{1+\beta,T}$ (*according to Definition* 32), *the unique solution from Proposition* 28 [*with* (μ,ν) *replaced by* $(0,2\nu^0)$].

Compared with Proposition 28, the novelty of this proposition is that it states the unique existence of a *continuous* solution $u^T=u^T(0,2\nu^0)$.

Proof of Proposition 34. 1° (*Uniform domination*). Fix all the quantities as in the proposition. First of all, note that, for fixed $n,\varepsilon^1,\varepsilon^2$, the functions

$$(s,y) \ \mapsto \ \nu^n_s\!*\![J^1_{\varepsilon^1}+J^2_{\varepsilon^2}]\,(y),\qquad 0\leq s\leq T,\quad y\in\mathsf{R}^d, \tag{147}$$

satisfy

$$\bigl\|\nu^n_\cdot\!*\![J^1_{\varepsilon^1}+J^2_{\varepsilon^2}]\bigr\|_{T,\infty} \ < \ \infty \tag{148}$$

[recall notation (97)]. Therefore $u^{n,T,\varepsilon^1,\varepsilon^2}$ in (144) is well-defined (see Definition 13 and references afterwards). Next, there is a constant $c_{149}=c_{149}(J,\alpha,d)$ such that for $i=1,2$,

$$J^i \ \leq \ c_{149}\,\mathrm{p}_1\,,\quad \text{hence}\quad 0\ \leq\ J^i_\varepsilon\ \leq\ c_{149}\,\mathrm{p}_{\varepsilon^\alpha}\,,\qquad \varepsilon>0. \tag{149}$$

Thus, for $(r,x) \in [0,T] \times \mathsf{R}^d$,

$$0 \;\le\; \int_r^T \mathrm{d}s\; \big(\nu_s^n * [J^1_{\varepsilon^1} + J^2_{\varepsilon^2}]\big) * \mathrm{p}_{s-r}\,(x) \tag{150}$$
$$\le\; c_{149} \int_r^T \mathrm{d}s\; \nu_s^n * \big[\mathrm{p}_{(\varepsilon^1)^\alpha+s-r} + \mathrm{p}_{(\varepsilon^2)^\alpha+s-r}\big]\,(x).$$

But there is a constant $c_{151} = c_{151}(d,\alpha,\eta)$ such that

$$\mathrm{p}_s(x) \;\le\; c_{151}\, s^{-(d-\eta)/\alpha}\, |x|^{-\eta}, \qquad s>0, \quad x \in \mathsf{R}^d, \tag{151}$$

see [Myt98a, Lemma 4]. Note that $(d-\eta)/\alpha < 1$, by assumption (143). Therefore, there is a constant $c_{152} = c_{152}(J,\alpha,d,\eta)$ such that

$$\int_r^T \mathrm{d}s\; \big(\nu_s^n * [J^1_{\varepsilon^1} + J^2_{\varepsilon^2}]\big) * \mathrm{p}_{s-r}\,(x) \tag{152}$$
$$\le\; c_{152} \sup_{n\ge 0} \|\nu^n\|_{-\eta,T} \int_0^{\varepsilon+T-r} \mathrm{d}s\; s^{-(d-\eta)/\alpha} \;<\; \infty$$

with $\varepsilon := (\varepsilon^1)^\alpha + (\varepsilon^2)^\alpha$. In particular,

$$0 \;\le\; u_r^{n,T,\varepsilon^1,\varepsilon^2}(x) \;\le\; c_{152} \sup_{n\ge 0} \|\nu^n\|_{-\eta,T} \int_0^{2+T} \mathrm{d}s\; s^{-(d-\eta)/\alpha} \;<\; \infty. \tag{153}$$

2° (*Decomposition*). Assume additionally that $r<T$, and take $\delta>0$ such that $r+2\delta \le T$. We may also assume in addition that $r_k \le r+\delta$, for all k. Choose a compact $C \subset \mathsf{R}^d$ such that $x_k \in C$ for all k. Then we decompose

$$I \;=\; I_{r_k}^{n,T,\varepsilon^1,\varepsilon^2}(x_k) \;:=\; \int_{r_k}^T \mathrm{d}s\; \big(\nu_s^n * [J^1_{\varepsilon^1} + J^2_{\varepsilon^2}]\big) * \mathrm{p}_{s-r_k}\,(x_k) \;=\; I_1 + I_2 \tag{154}$$

where

$$I_1 \;:=\; \int_{r_k}^{r_k+\delta} \mathrm{d}s\; \big(\nu_s^n * [J^1_{\varepsilon^1} + J^2_{\varepsilon^2}]\big) * \mathrm{p}_{s-r_k}\,(x_k), \tag{155a}$$

$$I_2 \;:=\; \int_{r_k+\delta}^{T} \mathrm{d}s\; \big(\nu_s^n * [J^1_{\varepsilon^1} + J^2_{\varepsilon^2}]\big) * \mathrm{p}_{s-r_k}\,(x_k). \tag{155b}$$

3° (*Error term* I_1). By (152), I_1 is bounded from above uniformly in k and n by

$$c \sup_{n\ge 0} \|\nu^n\|_{-\eta,T} \int_0^{\varepsilon+\delta} \mathrm{d}s\; s^{-(d-\eta)/\alpha} \;\xrightarrow[\varepsilon^1,\varepsilon^2 \downarrow 0]{}\; c \sup_{n\ge 0} \|\nu^n\|_{-\eta,T} \int_0^{\delta} \mathrm{d}s\; s^{-(d-\eta)/\alpha},$$

which goes to 0 as $\delta \downarrow 0$.

4° (*Convergence of* I_2). First of all, with $G^{T-\delta}$ from (A1) in the appendix,

$$I_2 \;=\; \int_{r_k}^{T-\delta} \mathrm{d}s\; \psi_s^{n,\varepsilon^1,\varepsilon^2} * \mathrm{p}_{s-r_k}\,(x_k) \;=\; (G^{T-\delta}\psi^{n,\varepsilon^1,\varepsilon^2})_{r_k}(x_k), \tag{156}$$

where, for the fixed δ,

$$\psi_s^{n,\varepsilon^1,\varepsilon^2} \;:=\; \big(\nu_{\delta+s}^n * [J^1_{\varepsilon^1} + J^2_{\varepsilon^2}]\big) * \mathrm{p}_\delta\,, \qquad 0 \le s \le T-\delta. \tag{157}$$

Using notation (97), it is easy to see that by (103),

$$\sup_{n\geq 1,\,0<\varepsilon^1,\varepsilon^2\leq 1}\|\psi^{n,\varepsilon^1,\varepsilon^2}\|_{T-\delta,\infty} \;\leq\; \mathrm{p}_\delta(0)\sup_{n\geq 1}\operatorname*{ess\,sup}_{0\leq s\leq T}\|\nu^n_s\| \;<\; \infty, \tag{158a}$$

$$\|\psi\|_{T-\delta,\infty} \;\leq\; \mathrm{p}_\delta(0)\operatorname*{ess\,sup}_{0\leq s\leq T}\|\nu^0_s\| \;<\; \infty, \tag{158b}$$

where

$$\psi_s \;:=\; (2\nu^0_{\delta+s})\ast\mathrm{p}_\delta\,, \qquad 0\leq s\leq T-\delta. \tag{159}$$

Moreover,

$$\lim_{n\uparrow\infty,\,\varepsilon^1,\varepsilon^2\downarrow 0}\psi^{n,\varepsilon^1,\varepsilon^2}_s(x)\,\mathrm{d}s\,\mathrm{d}x \;=\; \psi_s(x)\,\mathrm{d}s\,\mathrm{d}x \quad \text{in } \mathcal{M}^{T-\delta}_{\mathrm{f}}. \tag{160}$$

Then, by Lemma A3 in the appendix, $G^{T-\delta}\psi\in \mathrm{b}\mathcal{C}^+_{\mathrm{co}}\big([0,T-\delta]\times\mathsf{R}^d\big)$, and

$$\lim_{k,n\uparrow\infty,\ \varepsilon^1,\varepsilon^2\downarrow 0}(G^{T-\delta}\psi^{n,\varepsilon^1,\varepsilon^2})_{r_k}(x_k) \;=\; (G^{T-\delta}\psi)_r(x), \tag{161}$$

that is,

$$\lim_{k,n\uparrow\infty,\ \varepsilon^1,\varepsilon^2\downarrow 0} I_2 \;=\; \int_{r+\delta}^{T}\mathrm{d}s\;(2\nu^0_s)\ast\mathrm{p}_{s-r}\,(x). \tag{162}$$

5° (*Convergence of* I). Combining steps 3° and 4° we get

$$\lim_{k,n\uparrow\infty,\ \varepsilon^1,\varepsilon^2\downarrow 0} I^{n,T,\varepsilon^1,\varepsilon^2}_{r_k}(x_k) \;=\; \int_{r}^{T}\mathrm{d}s\;(2\nu^0_s)\ast\mathrm{p}_{s-r}\,(x), \tag{163}$$

that is,

$$\lim_{k,n\uparrow\infty,\ \varepsilon^1,\varepsilon^2\downarrow 0} G^T\big(\nu^n\ast[J^1_{\varepsilon^1}+J^2_{\varepsilon^2}]\big) \;=\; G^T(2\nu^0) \text{ in } \mathrm{b}\mathcal{C}^+_{\mathrm{co}}\big([0,T]\times\mathsf{R}^d\big). \tag{164}$$

6° (*Non-linear term*). By (153), the set

$$\Big\{(u^{n,T,\varepsilon^1,\varepsilon^2})^{1+\beta}:\ n\geq 1,\ 0<\varepsilon^1,\varepsilon^2\leq 1\Big\} \tag{165}$$

of functions on $[0,T]\times\mathsf{R}^d$ is uniformly bounded. Then Lemma A1 implies that

$$\Big\{G^T(u^{n,T,\varepsilon^1,\varepsilon^2})^{1+\beta}:\ n\geq 1,\ 0<\varepsilon^1,\varepsilon^2\leq 1\Big\} \tag{166}$$

is relatively compact in $\mathrm{b}\mathcal{C}^+_{\mathrm{co}}\big([0,T]\times\mathsf{R}^d\big)$. Thus, for each subsequence of $n\uparrow\infty$, $\varepsilon^1,\varepsilon^2\downarrow 0$, there is a further subsequence such that along this subsequence, in $\mathrm{b}\mathcal{C}^+_{\mathrm{co}}\big([0,T]\times\mathsf{R}^d\big)$ we have the convergence

$$G^T(u^{n,T,\varepsilon^1,\varepsilon^2})^{1+\beta} \;\to\; \text{some } F^T. \tag{167}$$

Combining with step 5°, we obtain that along this subsequence,

$$u^{n,T,\varepsilon^1,\varepsilon^2} \;\to\; \text{some } u^T \tag{168}$$

in $\mathrm{b}\mathcal{C}^+_{\mathrm{co}}\big([0,T]\times\mathsf{R}^d\big)$. Then use again (167) and Corollary A2 to see that

$$G^T(u^{n,T,\varepsilon^1,\varepsilon^2})^{1+\beta} \;\to\; G^T(u^T)^{1+\beta} \text{ in } \mathrm{b}\mathcal{C}^+_{\mathrm{co}}\big([0,T]\times\mathsf{R}^d\big) \tag{169}$$

along this subsequence. Putting all together, we obtain that

$$u^T \;=\; G^T(2\nu^0)-G^T(u^T)^{1+\beta} \text{ in } \mathrm{b}\mathcal{C}_{\mathrm{co}}\big([0,T]\times\mathsf{R}^d\big), \tag{170}$$

or equivalently,

$$u_r^T(x) \;=\; \int_r^T \mathrm{d}s\;(2\nu_s^0)\ast \mathrm{p}_{s-r}\,(x) - \int_r^T \mathrm{d}s\;\mathrm{S}_{s-r}\big((u_s^T)^{1+\beta}\big)\,(x), \tag{171}$$

$(r,x) \in [0,T]\times \mathsf{R}^d$. By Proposition 28 and Definition 32, the function u^T is the non-negative (and actually continuous) representative $u^T(0,2\nu^0)$ of the unique solution $w^T(0,2\nu^0) \in \mathcal{L}_+^{1,T} \cap \mathcal{L}_+^{1+\beta,T}$ of (125) [in the case $(\mu,\nu) = (0,2\nu^0)$]. By this uniqueness, $u^T \in \mathrm{b}\mathcal{C}_{\mathrm{co}}^+\big([0,T]\times \mathsf{R}^d\big)$ does not depend on the choice of all the subsequences, and the proof is complete. ■

4.4. **On collision local times and measures of independent superprocesses.** The main result of this subsection, Proposition 35, extends Theorem 1 of [Myt98a] for a more general class of initial measures.

Let $\bar{\mathbf{X}}$ be the $(\boldsymbol{\alpha},d,\boldsymbol{\beta})$–pair of independent critical superprocesses without immigration with parameters $\boldsymbol{\alpha},d,\boldsymbol{\beta}$ satisfying condition (10). If μ is a measure on R^d and $f \geq 0$ a measurable function on R^d, we write $\mu\cdot f$ for the measure $\mu(\mathrm{d}x)\,f(x)$.

Proposition 35 (**Log-Laplace functionals of collision processes**).

(a) (Collision measure): *If*

$$d \;<\; \frac{\alpha^1}{\beta^1} + \frac{\alpha^2}{\beta^2}\,, \tag{172}$$

then for each pair $\boldsymbol{\mu} \in \mathcal{M}_{\mathrm{f}}^2$ *of initial measures, the collision measure* $K_{\bar{\mathbf{X}}}$ (*recall Definition* 3) *exists, and we have the following log-Laplace representation*

$$\mathcal{P}\,\mathrm{e}^{-\langle K_{\bar{\mathbf{X}}}(t),f\rangle} \;=\; \mathcal{P}\,\mathrm{e}^{-\left\langle \mu^1,u_0^{1,t}(\bar{X}_t^2\cdot f,0)\right\rangle}, \qquad t>0, \quad f\in \bar{\mathcal{C}}_+\,, \tag{173}$$

with $u^{1,t}(\bar{X}_t^2\cdot f,0)$ *from Definition* 32.

(b) (Collision local time): *If*

$$d \;<\; \frac{\alpha^1}{\beta^1} + \frac{\alpha^2}{\beta^2} + (\alpha^1\vee\alpha^2), \tag{174}$$

then for each $\boldsymbol{\mu}\in\mathcal{M}_{\mathrm{f,e}}$ *the collision local time* $L_{\bar{\mathbf{X}}}$ *exists, and*

$$\mathcal{P}\,\mathrm{e}^{-\langle L_{\bar{\mathbf{X}}}(t),f\rangle} \;=\; \mathcal{P}\,\mathrm{e}^{-\left\langle \mu^1,u_0^{1,t}(0,\bar{X}^2\cdot f)\right\rangle}, \qquad t>0, \quad f\in \bar{\mathcal{C}}_+\,, \tag{175}$$

with $u^{1,t}(0,\bar{X}^2\cdot f)$ *from Definition* 32, *provided that* $\alpha^1\geq\alpha^2$.

Clearly, the role of the indices 1 and 2 can be interchanged in the previous proposition.

Proof of Proposition 35. Fix $t>0$ and $f\in\bar{\mathcal{C}}_+$.

1° **(a)** Let $0<\varepsilon\leq 1$. Define

$$\psi^\varepsilon(x) \;:=\; \int_{\mathsf{R}^d} \bar{X}_t^2(\mathrm{d}y)\,J_\varepsilon(x-y)\,f\big(\tfrac{x+y}{2}\big), \qquad x\in\mathsf{R}^d. \tag{176}$$

While checking the proof of Theorem 1(i) in [Myt98a] (see p.762 there), we realize that it is enough to show that $\mathcal{P}$–almost surely,

$$u_0^{1,t}(\psi^\varepsilon+\psi^{\varepsilon'},0) \;\xrightarrow[\varepsilon,\varepsilon'\downarrow 0]{}\; u_0^{1,t}(2\bar{X}_t^2\cdot f,0) \quad \text{uniformly on compacts of } \mathsf{R}^d. \tag{177}$$

Note that by our assumption (172) we can choose η satisfying

$$\Big(d-\frac{\alpha^1}{\beta^1}\Big)_+ \ < \ \eta \ < \ \frac{\alpha^2}{\beta^2}\wedge d, \tag{178}$$

and, by Example 26, we have $\bar{X}^2_t \in \mathcal{M}^{\eta,\circ}_{\mathrm{f}}$, $\mathcal{P}$–a.s. Hence $u^{1,t}_0(2\bar{X}^2_t\cdot f,0)$ makes sense. Recall the semigroup property

$$u^{1,t}_0(\psi^\varepsilon+\psi^{\varepsilon'},0) \ = \ u^{1,t-\delta}_0\big(u^{1,\delta}_0(\psi^\varepsilon+\psi^{\varepsilon'},0),0\big) \tag{179}$$

for $\delta\in(0,t)$, domination and boundedness

$$\sup_{0<\varepsilon,\varepsilon'\leq 1}\big\|u^{1,\delta}_0(\psi^\varepsilon+\psi^{\varepsilon'},0)\big\|_\infty \ \leq \ \sup_{0<\varepsilon,\varepsilon'\leq 1}\big\|\mathrm{S}^1_\delta(\psi^\varepsilon+\psi^{\varepsilon'})\big\|_\infty \ < \ \infty, \tag{180}$$

$\mathcal{P}$–a.s., and the convergence

$$u^{1,\delta}_0(\psi^\varepsilon+\psi^{\varepsilon'},0) \ \xrightarrow[\varepsilon,\varepsilon'\downarrow 0]{} \ u^{1,\delta}_0(2\bar{X}^2_t\cdot f,0) \quad \text{in } \mathcal{L}^1(\mathsf{R}^d) \tag{181}$$

(see [Myt98a, 1st display after 3.5]). Therefore, by Lemma 16(b) we get that

$$u^{1,t-\delta}_0\big(u^{1,\delta}_0(\psi^\varepsilon+\psi^{\varepsilon'},0),0\big) \ \xrightarrow[\varepsilon,\varepsilon'\downarrow 0]{} \ u^{1,t-\delta}_0\big(u^{1,\delta}_0(2\bar{X}^2_t\cdot f,0),0\big), \tag{182}$$

uniformly on compacts of R^d, $\mathcal{P}$–a.s. Again by the semigroup property,

$$u^{1,t-\delta}_0\big(u^{1,\delta}_0(2\bar{X}^2_t\cdot f,0),0\big) \ = \ u^{1,t}_0(2\bar{X}^2_t\cdot f,0), \tag{183}$$

and the proof of (a) is finished.

2° **(b)** Suppose $\alpha^1\geq\alpha^2$. By Lemma 2, we may switch to the non-symmetric definition of approximating collision local time. Hence, it is enough to check that $\big\langle L^{1,\varepsilon}_{\bar{\mathbf{X}}}(T),f\big\rangle$ converges in probability as $\varepsilon\downarrow 0$, and that

$$\lim_{\varepsilon\downarrow 0}\mathcal{P}\,\mathrm{e}^{-\left\langle L^{1,\varepsilon}_{\bar{\mathbf{X}}}(t),f\right\rangle} \ = \ \mathcal{P}\,\mathrm{e}^{-\left\langle \mu^1,u^{1,t}_0(0,\bar{X}^2\cdot f)\right\rangle}, \qquad t>0, \quad f\in\bar{\mathcal{C}}_+\,, \tag{184}$$

(in order to identify the limit). Note that by our assumption (174) we can choose η satisfying

$$\Big(d-\alpha^1-\frac{\alpha^1}{\beta^1}\Big)_+ \ < \ \eta \ < \ \frac{\alpha^2}{\beta^2}\wedge d, \tag{185}$$

and, by Example 26, we have

$$\{\bar{X}^2_s:\ 0\leq s\leq t\}\in\mathfrak{P}^{t,\eta,\circ}, \ \ \mathcal{P}\text{-a.s.}, \tag{186}$$

hence the solution $u^{1,t}_0(0,\bar{X}^2\cdot f)$ makes sense. Let us check (184). Choose n_0 such that $1/n_0<t$. For $n\geq n_0$, define

$$L^{1,\varepsilon}_{\bar{\mathbf{X}}}(\tfrac{1}{n},t) \ := \ L^{1,\varepsilon}_{\bar{\mathbf{X}}}(t)-L^{1,\varepsilon}_{\bar{\mathbf{X}}}(\tfrac{1}{n}), \tag{187a}$$

$$\bar{X}^{2,n}_s \ := \ 1_{[1/n,T]}(s)\bar{X}^2_s, \qquad 0\leq s\leq t \tag{187b}$$

$$\psi^{\varepsilon,n}_s(y) \ := \ \bar{X}^{2,n}_s\ast J_\varepsilon(y)\,f(y), \qquad 0\leq s\leq t, \quad y\in\mathsf{R}^d. \tag{187c}$$

Then

$$\begin{aligned}
&\Big|\mathcal{P}\,\mathrm{e}^{-\langle L^{1,\varepsilon}_{\bar{\mathbf{X}}}(t),f\rangle} - \mathcal{P}\,\mathrm{e}^{-\langle \mu^1,u_0^{1,t}(0,\bar{X}^2\cdot f)\rangle}\Big| \qquad (188)\\
&\le\ \Big|\mathcal{P}\,\mathrm{e}^{-\langle L^{1,\varepsilon}_{\bar{\mathbf{X}}}(t),f\rangle} - \mathcal{P}\,\mathrm{e}^{-\big\langle L^{1,\varepsilon}_{\bar{\mathbf{X}}}(1/n,t),f\big\rangle}\Big|\\
&\quad+\ \Big|\mathcal{P}\,\mathrm{e}^{-\big\langle L^{1,\varepsilon}_{\bar{\mathbf{X}}}(1/n,t),f\big\rangle} - \mathcal{P}\,\mathrm{e}^{-\langle \mu^1,u_0^{1,t}(0,\bar{X}^{2,n}\cdot f)\rangle}\Big|\\
&\quad+\ \Big|\mathcal{P}\,\mathrm{e}^{-\langle \mu^1,u_0^{1,t}(0,\bar{X}^{2,n}\cdot f)\rangle} - \mathcal{P}\,\mathrm{e}^{-\langle \mu^1,u_0^{1,t}(0,\bar{X}^2\cdot f)\rangle}\Big|\\
&=:\ I^{\varepsilon,n} + I\!I^{\varepsilon,n} + I\!I\!I^n
\end{aligned}$$

with the obvious correspondence.

3° **(b.I)** It is easy to see that

$$I^{\varepsilon,n}\ \le\ \mathcal{P}\big\langle L^{1,\varepsilon}_{\bar{\mathbf{X}}}(\tfrac1n),f\big\rangle\ =\ \int_0^{1/n}\mathrm{d}s\int_{\mathsf{R}^d}\mathrm{d}y\ \mu^1*\mathrm{p}^1_s(y)\,\mu^2*\mathrm{p}^2_s(y)*J_\varepsilon(y)\,f(y). \qquad (189)$$

Now there is a constant $c_{190}=c_{190}(J,\alpha^1,\alpha^2)$ such that

$$J\ \le\ c_{190}\,\mathrm{p}^1_1*\mathrm{p}^2_{\varepsilon^{\alpha^1-\alpha^2}}\,, \qquad 0<\varepsilon\le 1, \qquad (190)$$

implying

$$\begin{aligned}
J_\varepsilon(x)\ &=\ \varepsilon^{-d}J(x/\varepsilon)\ \le\ c_{190}\,\varepsilon^{-d}\int_{\mathsf{R}^d}\mathrm{d}y\ \mathrm{p}^1_1(y-x/\varepsilon)\,\mathrm{p}^2_{\varepsilon^{\alpha^1-\alpha^2}}(y) \qquad (191)\\
&=\ c_{190}\,\varepsilon^{-2d}\int_{\mathsf{R}^d}\mathrm{d}y\ \mathrm{p}^1_1\big((y-x)/\varepsilon\big)\ \mathrm{p}^2_{\varepsilon^{\alpha^1-\alpha^2}}(y/\varepsilon)\\
&=\ c_{190}\int_{\mathsf{R}^d}\mathrm{d}y\ \mathrm{p}^1_{\varepsilon^{\alpha^1}}(y-x)\,\mathrm{p}^2_{\varepsilon^{\alpha^1}}(y)\ =\ c_{190}\,\mathrm{p}^1_{\varepsilon^{\alpha^1}}*\mathrm{p}^2_{\varepsilon^{\alpha^1}}(x).
\end{aligned}$$

Inserting into (189) gives

$$\begin{aligned}
I^{\varepsilon,n}\ &\le\ c\,\|f\|_\infty\int_0^{1/n}\mathrm{d}s\int_{\mathsf{R}^d}\mathrm{d}y\ \mu^1*\mathrm{p}^1_{s+\varepsilon^{\alpha^1}}(y)\,\mu^2*\mathrm{p}^2_{s+\varepsilon^{\alpha^1}}(y)\\
&=\ c\,\|f\|_\infty\int_{\varepsilon^{\alpha^1}}^{\varepsilon^{\alpha^1}+1/n}\mathrm{d}s\int_{\mathsf{R}^d}\mathrm{d}y\ \mu^1*\mathrm{p}^1_s(y)\,\mu^2*\mathrm{p}^2_s(y). \qquad (192)
\end{aligned}$$

Since $\boldsymbol{\mu}$ belongs to $\mathcal{M}_{\mathrm{f,e}}$, for $\delta>0$ we find $N=N(\delta)$ such that

$$I^{\varepsilon,n}\le\delta \quad \text{for all} \quad \varepsilon\in(0,1] \quad \text{and} \quad n\ge N. \qquad (193)$$

The latter argument also immediately shows that for any $t>0$,

$$\begin{aligned}
&\limsup_{\varepsilon\downarrow 0}\ \mathcal{P}\left\langle L^{1,\varepsilon}_{\bar{\mathbf{X}}}(t),1\right\rangle \qquad (194)\\
&\le\ c\,\limsup_{\varepsilon\downarrow 0}\int_{\varepsilon^{\alpha^1}}^{\varepsilon^{\alpha^1}+t}\mathrm{d}s\int_{\mathsf{R}^d}\mathrm{d}y\ \mu^1*\mathrm{p}^1_s(y)\,\mu^2*\mathrm{p}^2_s(y)\ <\ \infty.
\end{aligned}$$

4° **(b.II)** Let us turn to $I\!I^{\varepsilon,n}$. By the definition of log-Laplace transform, it is easy to see that

$$\mathcal{P}\,\mathrm{e}^{-\big\langle L^{1,\varepsilon}_{\bar{\mathbf{X}}}(1/n,t),f\big\rangle}\ =\ \mathcal{P}\,\mathrm{e}^{-\langle \mu^1,u_0^{1,t}(0,\psi^{\varepsilon,n})\rangle}. \qquad (195)$$

By the semigroup property and the definition (187c) of $\psi^{\varepsilon,n}$ we get

$$u_0^{1,t}(0,\psi^{\varepsilon,n})\ =\ u_0^{1,s_n}\big(u_{s_n}^{1,t}(0,\psi^{\varepsilon,n}),0\big), \qquad (196)$$

where we set $s_n := 1/2n$. By adopting the argument used in the proof of Theorem 1(ii) of [Myt98a], it is easy to get that [recall notation (187b)]

$$u^{1,t}_{s_n}(0,\psi^{\varepsilon,n}) \underset{\varepsilon\downarrow 0}{\longrightarrow} u^{1,t}_{s_n}(0,\bar{X}^{2,n}\cdot f) \quad \text{in } \mathcal{L}^1, \quad \mathcal{P}\text{–a.s.} \tag{197}$$

Moreover, since $\bar{X}^{2,n}$, hence $\psi^{\varepsilon,n}$ is truncated until time $2s_n$, we obtain

$$\sup_{0<\varepsilon\leq 1}\left\|u^{1,t}_{s_n}(0,\psi^{\varepsilon,n})\right\|_\infty \;\leq\; \sup_{0<\varepsilon\leq 1}\left\|\mathrm{S}^1_{s_n}\bigl(u^{1,t}_{2s_n}(0,\psi^{\varepsilon,n})\bigr)\right\|_\infty \;<\; \infty, \tag{198}$$

and hence by Lemma 16(b),

$$u_0^{1,s_n}\bigl(u^{1,t}_{s_n}(0,\psi^{\varepsilon,n}),0\bigr) \underset{\varepsilon\downarrow 0}{\longrightarrow} u_0^{1,s_n}\bigl(u^{1,t}_{s_n}(0,\bar{X}^{2,n}\cdot f),0\bigr) \;=\; u_0^{1,t}(0,\bar{X}^{2,n}\cdot f)$$

uniformly on compacts of R^d, $\mathcal{P}$–a.s. Therefore,

$$\mathit{II}^{\varepsilon,n}\to 0 \quad \text{as} \quad \varepsilon\downarrow 0, \quad \text{for all} \quad n\geq 1. \tag{199}$$

5° **(b.III)** Finally, we deal with III^n. By Lemma 33 and domination (141),

$$\begin{aligned} u_0^{1,t}(0,\bar{X}^2\cdot f)\,(x) - u_0^{1,t}(0,\bar{X}^{2,n}\cdot f)\,(x) \;&\leq\; u_0^{1,t}\bigl(0,(\bar{X}^2-\bar{X}^{2,n})\cdot f\bigr)\,(x) \\ &\leq \int_0^{1/n}\mathrm{d}s\;(\bar{X}^2_s\cdot f)\ast \mathrm{p}^1_s\,(x), \qquad x\in\mathsf{R}^d. \end{aligned} \tag{200}$$

Hence,

$$\begin{aligned} &\mathcal{P}\left|\int_{\mathsf{R}^d}\mu^1(\mathrm{d}x)\Bigl(u_0^{1,t}(0,\bar{X}^2\cdot f)\,(x) - u_0^{1,t}(0,\bar{X}^{2,n}\cdot f)\,(x)\Bigr)\right| \\ &\leq\; \|f\|_\infty\int_0^{1/n}\mathrm{d}s\int_{\mathsf{R}^d}\mathrm{d}y\;(\mu^2\ast\mathrm{p}^2_s)(y)\,(\mu^1\ast\mathrm{p}^1_s)\,(y) \;\leq\; \delta, \qquad n\geq N, \end{aligned} \tag{201}$$

where the last inequality follows from the estimate (193). Combining the statements (193), (199), and (201), we obtain

$$\limsup_{\varepsilon\downarrow 0}\,(I^{\varepsilon,N}+\mathit{II}^{\varepsilon,N}+\mathit{III}^N) \;\leq\; 2\delta, \tag{202}$$

and therefore

$$\limsup_{\varepsilon\downarrow 0}\left|\mathcal{P}\,\mathrm{e}^{-\left\langle L^{1,\varepsilon}_{\bar{\mathbf{X}}}(t),f\right\rangle} - \mathcal{P}\,\mathrm{e}^{-\left\langle \mu^1,u_0^{1,t}(0,\bar{X}^2\cdot f)\right\rangle}\right| \;\leq\; 2\delta. \tag{203}$$

Since δ was arbitrary, claim (184) is verified.

6° (*Conclusion*) By adopting the proof of (184) (see also the proof of Theorem 1(ii) in [Myt98a]), we can easily derive that

$$\mathcal{P}\Bigl(\mathrm{e}^{-\left\langle L^{1,\varepsilon}_{\bar{\mathbf{X}}}(t),f\right\rangle} - \mathrm{e}^{-\left\langle L^{1,\varepsilon'}_{\bar{\mathbf{X}}}(t),f\right\rangle}\Bigr)^2 \underset{\varepsilon,\varepsilon'\downarrow 0}{\longrightarrow} 0. \tag{204}$$

Now claim (b) follows from (204), (194), and (184). ■

Proposition 35 implies the following first moment formulas for $L_{\bar{\mathbf{X}}}$ and $K_{\bar{\mathbf{X}}}$.

Lemma 36 (Expectations in the superprocess case). *For $t>0$ and $f\in\bar{\mathcal{C}}$, the following identities hold.*

(a) (Collision measure): *Under dimension restriction* (172)*, if $\boldsymbol{\mu}\in\mathcal{M}^2_{\mathrm{f}}$,*

$$\mathcal{P}\bigl\langle K_{\bar{\mathbf{X}}}(t),f\bigr\rangle \;=\; \int_{\mathsf{R}^d}\mathrm{d}x\;\mu^1\ast\mathrm{p}^1_t\,(x)\,\mu^2\ast\mathrm{p}^2_t\,(x)\,f(x). \tag{205}$$

(b) (Collision local time): *Under* (174), *if* $\boldsymbol{\mu} \in \mathcal{M}_{\mathrm{f,e}}$,

$$\mathcal{P}\big\langle L_{\bar{\mathbf{X}}}(t), f\big\rangle \;=\; \int_0^t \mathrm{d}s \int_{\mathsf{R}^d} \mathrm{d}x\ \mu^1 * \mathrm{p}_s^1\,(x)\,\mu^2 * \mathrm{p}_s^2\,(x)\,f(x). \tag{206}$$

Proof. We will start with (b) and give afterwards some hints concerning the simpler case (a).

1° **(b)** Without loss of generality, we may assume that $\alpha^1 \geq \alpha^2$. Clearly, for $0 < r < t$ and $f \in \bar{\mathcal{C}}_+$,

$$\mathcal{P}\big\langle L_{\bar{\mathbf{X}}}(t) - L_{\bar{\mathbf{X}}}(r), f\big\rangle \;=\; \lim_{\varepsilon \downarrow 0} \mathcal{P}\,\frac{1}{\varepsilon}\Big(1 - \mathrm{e}^{-\left\langle L_{\bar{\mathbf{X}}}(t) - L_{\bar{\mathbf{X}}}(r), \varepsilon f\right\rangle}\Big). \tag{207}$$

Note that we introduced an $r > 0$ to have later available an additional smoothing which simplifies the proof. By the Markov property and Proposition 35(b),

$$\mathcal{P}\,\mathrm{e}^{-\left\langle L_{\bar{\mathbf{X}}}(t) - L_{\bar{\mathbf{X}}}(r), \varepsilon f\right\rangle} \;=\; \mathcal{P}\,\mathrm{e}^{-\left\langle \bar{X}_r^1, u_r^{1,t}(0, \bar{X}^2 \cdot \varepsilon f)\right\rangle}. \tag{208}$$

In order to interchange limit with expectation, we use the following domination

$$\frac{1}{\varepsilon}\Big(1 - \mathrm{e}^{-\left\langle \bar{X}_r^1, u_r^{1,t}(0, \bar{X}^2 \cdot \varepsilon f)\right\rangle}\Big) \;\leq\; \frac{1}{\varepsilon}\,\big\langle \bar{X}_r^1, u_r^{1,t}(0, \bar{X}^2 \cdot \varepsilon f)\big\rangle \tag{209}$$

$$\leq\; \|f\|_\infty \int_r^t \mathrm{d}s \int_{\mathsf{R}^d} \bar{X}_r^1(\mathrm{d}x)\,\bar{X}_s^2 * \mathrm{p}_{s-r}^1\,(x).$$

But by the expectation formula for $\bar{\mathbf{X}}$ and by independence,

$$\mathcal{P}\int_r^t \mathrm{d}s \int_{\mathsf{R}^d} \bar{X}_r^1(\mathrm{d}x)\,\bar{X}_s^2 * \mathrm{p}_{s-r}^1\,(x) \;=\; \int_r^t \mathrm{d}s \int_{\mathsf{R}^d} \mathrm{d}x\ \mu^1 * \mathrm{p}_r^1\,(x)\ \mu^2 * \mathrm{p}_s^2 * \mathrm{p}_{s-r}^1\,(x)$$

$$=\; \int_r^t \mathrm{d}s \int_{\mathsf{R}^d} \mathrm{d}x\ \mu^1 * \mathrm{p}_s^1\,(x)\ \mu^2 * \mathrm{p}_s^2\,(x) \;<\; \infty, \tag{210}$$

since $\boldsymbol{\mu}$ is assumed to have finite energy. Thus, by dominated convergence,

$$\lim_{\varepsilon \downarrow 0} \mathcal{P}\,\frac{1}{\varepsilon}\Big(1 - \mathrm{e}^{-\left\langle \bar{X}_r^1, u_r^{1,t}(0, \bar{X}^2 \cdot \varepsilon f)\right\rangle}\Big) \;=\; \mathcal{P}\lim_{\varepsilon \downarrow 0}\,\frac{1}{\varepsilon}\Big(1 - \mathrm{e}^{-\left\langle \bar{X}_r^1, u_r^{1,t}(0, \bar{X}^2 \cdot \varepsilon f)\right\rangle}\Big). \tag{211}$$

Again by dominated convergence, $\big\langle \bar{X}_r^1, u_r^{1,t}(0, \bar{X}^2 \cdot \varepsilon f)\big\rangle \to 0$ as $\varepsilon \downarrow 0$, hence, the latter expectation expression equals

$$\mathcal{P}\lim_{\varepsilon \downarrow 0}\,\frac{1}{\varepsilon}\,\big\langle \bar{X}_r^1, u_r^{1,t}(0, \bar{X}^2 \cdot \varepsilon f)\big\rangle \;=\; \lim_{\varepsilon \downarrow 0} \mathcal{P}\,\frac{1}{\varepsilon}\,\big\langle \bar{X}_r^1, u_r^{1,t}(0, \bar{X}^2 \cdot \varepsilon f)\big\rangle$$

$$=\; \lim_{\varepsilon \downarrow 0} \mathcal{P}\,\frac{1}{\varepsilon}\,\big\langle \mu^1, \mathrm{S}_r^1 u_r^{1,t}(0, \bar{X}^2 \cdot \varepsilon f)\big\rangle$$

$$=\; \mathcal{P}\lim_{\varepsilon \downarrow 0}\Big\langle \mu^1, \frac{1}{\varepsilon}\,\mathrm{S}_r^1 u_r^{1,t}(0, \bar{X}^2 \cdot \varepsilon f)\Big\rangle, \tag{212}$$

where in the first and last equality we used dominated convergence once again. Recall that the non-negative function $u^{1,t} = u^{1,t}(0, \bar{X}^2 \cdot \varepsilon f)$ solves

$$u_r^{1,t}(x) \;=\; \int_r^t \mathrm{d}s\ (\bar{X}_s^2 \cdot \varepsilon f) * \mathrm{p}_{s-r}^1\,(x) - \int_r^t \mathrm{d}s\ \mathrm{S}_{s-r}^1\big((u_s^{1,t})^{1+\beta}\big)\,(x). \tag{213}$$

Inserting into (212), ε drops out in the linear term. For the other term we use the domination:

$$u_s^{1,t}(x) \;\leq\; \varepsilon\,\|f\|_\infty \int_s^t \mathrm{d}s'\ \bar{X}_{s'}^2 * \mathrm{p}_{s'-s}^1\,(x), \tag{214}$$

where by Lemma 24 and Corollary 4 in [Myt98a] the latter integral expression belongs to $\mathcal{L}^{1+\beta,t}$, $\mathcal{P}$–a.s. Therefore,

$$\frac{1}{\varepsilon}\Big\langle \mu^1, \int_r^t \mathrm{d}s\ \mathrm{S}_s^1\big((u_s^{1,t})^{1+\beta}\big)\Big\rangle \tag{215}$$

$$\leq\ \varepsilon^\beta\,\|f\|_\infty \left\langle \mu^1, \int_r^t \mathrm{d}s\ \mathrm{S}_s^1\bigg(\Big(\int_s^t \mathrm{d}s'\ \bar{X}_{s'}^2 * \mathrm{p}_{s'-s}^1\Big)^{1+\beta}\bigg)\right\rangle$$

$$\leq\ \varepsilon^\beta\,\|f\|_\infty\,\|\mu^1\| \sup_{r\leq r'\leq t}\|\mathrm{p}_{r'}^1\|_\infty \int_r^t \mathrm{d}s \int_{\mathsf{R}^d}\mathrm{d}x\,\Big(\int_s^t \mathrm{d}s'\ \bar{X}_{s'}^2 * \mathrm{p}_{s'-s}^1\,(x)\Big)^{1+\beta},$$

which $\mathcal{P}$–a.s. converges to 0 as $\varepsilon\downarrow 0$. Hence, the non-linear term is negligible, and we get

$$\mathcal{P}\lim_{\varepsilon\downarrow 0}\Big\langle \mu^1, \frac{1}{\varepsilon}\,\mathrm{S}_r^1 u_r^{1,t}(0,\bar{X}^2\cdot\varepsilon f)\Big\rangle\ =\ \mathcal{P}\Big\langle \mu^1, \int_r^t \mathrm{d}s\ (\bar{X}_s^2\cdot f) * \mathrm{p}_s^1\Big\rangle$$

$$=\ \int_r^t \mathrm{d}s \int_{\mathsf{R}^d}\mathrm{d}y\ (\mu^1 * \mathrm{p}_s^1)(y)\,(\mu^2 * \mathrm{p}_s^2)\,(y)\,f(y). \tag{216}$$

But as $r\downarrow 0,$ the latter double integral converges to the expression claimed in (b). Now it suffices to note that $L_{\bar{\mathbf{X}}}(r)\downarrow 0$ as $r\downarrow 0,$ $\mathcal{P}$–almost surely, to finish the proof of part (b) by monotone convergence.

2° **(a)** In a similar way,

$$\mathcal{P}\big\langle K_{\bar{\mathbf{X}}}(t), f\big\rangle\ =\ \lim_{\varepsilon\downarrow 0}\mathcal{P}\,\frac{1}{\varepsilon}\Big(1-\mathrm{e}^{-\langle K_{\bar{\mathbf{X}}}(t),\varepsilon f\rangle}\Big) \tag{217}$$

$$=\ \lim_{\varepsilon\downarrow 0}\mathcal{P}\,\frac{1}{\varepsilon}\Big(1-\mathrm{e}^{-\langle \mu^1, u_0^{1,t}(\bar{X}_t^2\cdot\varepsilon f, 0)\rangle}\Big),$$

and proceed along the same lines as in the previous step to finish the proof. ■

Lemma 36 will be applied to get the following regularity property of collision local times:

Corollary 37 (**Absolute continuity of collision local time**). *Let* $\boldsymbol{\mu}\in\mathcal{M}_{\mathrm{f,e}}$ *and assume* (172).

(a) (Representation): *We have*

$$L_{\bar{\mathbf{X}}}(t)\ =\ \int_0^t \mathrm{d}s\ K_{\bar{\mathbf{X}}}(s),\quad t>0,\quad \mathcal{P}\text{–a.s.} \tag{218}$$

(b) (Continuity): *The* $\mathcal{M}_{\mathrm{f}}$*-valued process* $t\mapsto L_{\bar{\mathbf{X}}}(t)$ *is continuous.*

Proof. **(a)** By Proposition 35, a diagonalization argument and the definitions of collision measures and collision locale times, we can easily get that for all $t\in\mathsf{Q}_+$ (the set of all non-negative rational numbers), $f\in\mathrm{b}\mathcal{C},$ and some sequence $\varepsilon_n\downarrow 0$ as $n\uparrow\infty,$

$$\big\langle L_{\bar{\mathbf{X}}}(t), f\big\rangle\ =\ \lim_{n\uparrow\infty}\big\langle L_{\bar{\mathbf{X}}}^{\varepsilon_n}(t), f\big\rangle\ =\ \lim_{n\uparrow\infty}\int_0^t \mathrm{d}s\ \big\langle K_{\bar{\mathbf{X}}}^{\varepsilon_n}(s), f\big\rangle$$

$$\geq\ \int_0^t \mathrm{d}s\ \liminf_{n\uparrow\infty}\big\langle K_{\bar{\mathbf{X}}}^{\varepsilon_n}(s), f\big\rangle\ =\ \int_0^t \mathrm{d}s\ \big\langle K_{\bar{\mathbf{X}}}(s), f\big\rangle,\quad \mathcal{P}\text{–a.s.} \tag{219}$$

On the other hand, from Lemma 36 we get

$$\mathcal{P}\left\langle L_{\bar{\mathbf{X}}}(t), f\right\rangle \;=\; \mathcal{P}\int_0^t \mathrm{d}s\, \left\langle K_{\bar{\mathbf{X}}}(s), f\right\rangle. \tag{220}$$

Therefore the inequality in (219) is in fact an equality.

¿From monotonicity in t, we can remove the restriction to $t \in \mathsf{Q}_+$. Finally, since $f \in \mathrm{b}\mathcal{C}$ and $t \in \mathsf{R}_+$ were arbitrary, claim (a) follows.

(b) is an immediate consequence of (a), finishing the proof. ■

5. Uniform convergence of collision local times

The purpose of this section is to prove the uniform convergence Theorem 7. The key argument is the Cauchy property of approximating collision local times in Lemma 39 below. Its proof starts with a crucial decomposition of collision local time into ones of conditionally independent processes, for which log-Laplace tools prepared in the previous section can be used.

5.1. **Tightness of $\mathbf{A}^n$.** As usual, we say that a family of random objects is *tight in law*, if their laws form a tight family. In the later procedure we will need the following property:

Lemma 38 (Tightness of $\mathbf{A}^n$). *Let $\left\{(\mathbf{X}^n, \mathbf{A}^n) : n \geq 1\right\}$ be a family of solutions to the martingale problem $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{\mu}}$ as in Theorem 7. Fix arbitrary $T > 0$. Then the sequence $\left\{\mathbf{A}^n\big(\mathrm{d}(t,x)\big) :\, n \geq 1\right\}$ of pairs of random measures in $\mathcal{M}_{\mathrm{f}}^T$ is tight in law.*

Proof. Let $\{\varphi_k : k \geq 1\}$ be the sequence of run away functions on R^d defined in Lemma A5. It suffices to show that for $i = 1, 2$,

$$\sup_{n\geq 1} \mathcal{P}\big\langle A_T^{n,i}, \varphi_k\big\rangle \;\underset{k\uparrow\infty}{\longrightarrow}\; 0 \tag{221}$$

[where, of course, $\mathbf{A}^n = (A^{n,1}, A^{n,2})$]. By Corollary 20,

$$\big\langle A_T^{n,i}, \varphi_k\big\rangle \;=\; \big\langle \mu^i, \varphi_k\big\rangle - \big\langle X_T^{n,i}, \varphi_k\big\rangle + \int_0^T \mathrm{d}s\, \big\langle X_s^{n,i},\, \Delta_{\alpha^i}\varphi_k\big\rangle + (\text{martingale})_T\,.$$

Hence,

$$\mathcal{P}\big\langle A_T^{n,i}, \varphi_k\big\rangle \;\leq\; \big\langle \mu^i, \varphi_k\big\rangle + \int_0^T \mathrm{d}s\, \mathcal{P}\big\langle X_s^{n,i}, |\Delta_{\alpha^i}\varphi_k|\big\rangle. \tag{222}$$

But from Corollary 23, $X^{n,i} \leq \bar{X}^i$ with $\bar{X}^i$ the critical (α^i, d, β^i)–superprocess without immigration and starting from μ^i. Moreover, from the expectation formula in Lemma 14(d),

$$\mathcal{P}\big\langle \bar{X}_s^i, \varphi\big\rangle \;=\; \big\langle \mu^i, \mathrm{S}_s^i\varphi\big\rangle, \qquad \varphi \in \mathrm{b}\mathcal{B}. \tag{223}$$

Inserting both into (222) gives

$$\mathcal{P}\big\langle A_T^{n,i}, \varphi_k\big\rangle \;\leq\; \big\langle \mu^i, \varphi_k\big\rangle + \int_0^T \mathrm{d}s\, \big\langle \mu^i, \mathrm{S}_s^i|\Delta_{\alpha^i}\varphi_k|\big\rangle. \tag{224}$$

Clearly, the first term at the right hand side tends to 0 as $k \uparrow \infty$ since μ^i is a finite measure. But also the second term vanishes by Corollary A6. This gives statement (221), and we are done. ■

5.2. **Cauchy property of approximating collision local times.** The purpose of this subsection is to prove the following statement.

Lemma 39 (**Cauchy property of approximating collision local times**). *Let* $\big\{(\mathbf{X}^n,\mathbf{A}^n):n\geq 1\big\}$ *denote any family of solutions to the martingale problem* $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{\mu}}$ *with pairs* $\boldsymbol{\alpha},\boldsymbol{\beta}$ *satisfying* (10) *and with* $\boldsymbol{\mu}\in\mathcal{M}_{\mathrm{f,e}}$. *Fix* $T>0$, *and a function* $f\in \mathrm{b}\mathcal{C}\big([0,T]\times\mathsf{R}^d\big)$. *Then, for any mollifiers* J^1 *and* J^2,

$$\sup_{n\geq 1}\mathcal{P}\bigg(\Big|\big\langle L^{\varepsilon^1,J^1}_{\mathbf{X}^n}(T),f\big\rangle-\big\langle L^{\varepsilon^2,J^2}_{\mathbf{X}^n}(T),f\big\rangle\Big|\geq\delta\bigg)\xrightarrow[\varepsilon^1,\varepsilon^2\downarrow 0]{}0, \tag{225}$$

for all $\delta>0$.

Here, by an abuse of notation, an expression as $\big\langle L_{\mathbf{Y}}(t),f\big\rangle$ in case of a time-space function f means

$$\big\langle L_{\mathbf{Y}}(t),f\big\rangle\ :=\ \int_{[0,t]\times\mathsf{R}^d}L_{\mathbf{Y}}\big(\mathrm{d}(s,x)\big)\,f_s(x). \tag{226}$$

Proof of Lemma 39. 1° (*Decomposition of collision local time*). Recall from Corollary 23 that the $\mathbf{X}^n$ are a.s. dominated by the $(\boldsymbol{\alpha},d,\boldsymbol{\beta})$–pair of independent critical superprocesses $\bar{\mathbf{X}}$ without immigration and with $\bar{\mathbf{X}}_0=\boldsymbol{\mu}$, and that, given $(\mathbf{X}^n,\mathbf{A}^n)$,

$$\bar{\mathbf{X}}-\mathbf{X}^n\ =:\ \hat{\mathbf{X}}^n \tag{227}$$

is the $(\boldsymbol{\alpha},d,\boldsymbol{\beta})$–pair of independent critical superprocesses with immigration $\mathbf{A}^n$ starting from $\hat{\mathbf{X}}^n_0=\mathbf{0}$. Then decomposition (227) yields the following *key identity:*

$$L^{\varepsilon^i,J^i}_{\mathbf{X}^n}\ =\ L^{\varepsilon^i,J^i}_{\bar{\mathbf{X}}}-L^{\varepsilon^i,J^i}_{(\hat{X}^{n,1},\,X^{n,2})}-L^{\varepsilon^i,J^i}_{(\bar{X}^1,\,\hat{X}^{n,2})},\quad n\geq 1,\ \ 0<\varepsilon^i\leq 1,\ \ i=1,2. \tag{228}$$

We want to show that all the terms at the right hand side of (228) satisfy statements as in (225), then yielding (225).

2° (*First term*). Fix $T>0$. First of all, by Lemma 2 and Proposition 35 and its proof, we have

$$L^{\varepsilon^i,J^i}_{\bar{\mathbf{X}}}\ \xrightarrow[\varepsilon^i\downarrow 0]{\mathcal{P}}\ L_{\bar{\mathbf{X}}}\quad\text{in }\mathcal{M}^T_{\mathrm{f}},\quad i=1,2, \tag{229}$$

and hence, the claimed Cauchy property is certainly true for $L^{\varepsilon^i,J^i}_{\bar{\mathbf{X}}}$. Next we want to deal with the second term at the right hand side of (228). The third term will be treated in step 10°.

3° (*Second term: Outline of proof*). Conditioned on $\mathbf{X}^n$, the approximating collision local times $L^{\varepsilon^i,J^i}_{(\hat{X}^{n,1},\,X^{n,2})}$ can be expressed in terms of the weighted occupation time process of $\hat{X}^{n,1}$ (cf. (247) below). Hence, its log-Laplace transform is given by Lemma 14(c). Letting $\varepsilon^i\downarrow 0$, formally we come to a log-Laplace equation involving $X^{n,2}$ as a generalized forcing term as in Proposition 28. But to justify this limit $\varepsilon^i\downarrow 0$, we cannot directly apply Proposition 34 since $X^{n,2}$ belongs to $\mathfrak{P}^{t,\eta,\circ}$ only, and not to $\mathfrak{P}^{t,\eta}$. To overcome this technical difficulty, we will exploit some cutting methods. The details will be given in the steps 4°–9°.

4° (*Second term: Preparation*). In the following procedure, by Lemma 2 we will often replace approximating collision local times L^{ε^i} by their asymmetric versions

L^{2,ε^i} introduced in (18). So we would like to show at this stage that for fixed $f \in \mathrm{b}\mathcal{C}\big([0,T]\times\mathsf{R}^d\big)$,

$$\sup_{n\geq 1}\mathcal{P}\bigg(\Big|\big\langle L^{2,\varepsilon^1,J^1}_{(\hat{X}^{n,1},\,X^{n,2})}(T),f\big\rangle-\big\langle L^{2,\varepsilon^2,J^2}_{(\hat{X}^{n,1},\,X^{n,2})}(T),f\big\rangle\Big|>\delta\bigg)\xrightarrow[\varepsilon^1,\varepsilon^2\downarrow 0]{}0, \tag{230}$$

for all $\delta>0.$

Recall the fact that $(\hat{X}^{n,1},X^{n,2})\leq\bar{\mathbf{X}},$ the $(\boldsymbol{\alpha},d,\boldsymbol{\beta})$–pair of independent critical superprocesses without immigration. This implies that

$$L^{2,\varepsilon^i,J^i}_{(\hat{X}^{n,1},\,X^{n,2})}(T)\ \leq\ L^{2,\varepsilon^i,J^i}_{\bar{\mathbf{X}}}(T),\qquad i=1,2. \tag{231}$$

Combined with (229) we see that the family of random measures

$$\Big\{L^{2,\varepsilon^i,J^i}_{(\hat{X}^{n,1},\,X^{n,2})}\big(\mathrm{d}(t,x)\big)\,1_{[0,T]}(t):\ n\geq 1,\ 0<\varepsilon^i\leq 1,\ i=1,2\Big\}\in\mathcal{M}_{\mathrm{f}}^T \tag{232}$$

is tight in law.

Take

$$\eta\ \in\ \big((d-\alpha_1)_+\,,\,\min(\alpha_2/\beta_2\,,\,d)\big), \tag{233}$$

which is possible by assumption (10). Then we may recall again Example 26 to conclude that

$$\{\bar{X}^2_s:\ 0\leq s\leq t\}\in\mathfrak{P}^{t,\eta,\circ},\ \ \mathcal{P}\text{-a.s.}, \tag{234}$$

For $m\geq 1,$ set

$$B_m\ :=\ \Big\{(s,x)\in[0,T]\times\mathsf{R}^d:\ \big\langle\bar{X}^2_s,h_{x,\eta}\big\rangle\leq m\Big\}, \tag{235}$$

with $h_{x,\eta}$ defined in (98). Put

$$\bar{X}^{2,m}_s(\mathrm{d}x)\ :=\ \bar{X}^2_s(\mathrm{d}x)\,1_{B_m}(s,x),\qquad (s,x)\in[0,T]\times\mathsf{R}^d, \tag{236}$$

and

$$\bar{X}^{2,m,\mathrm{c}}\ :=\ \bar{X}^2-\bar{X}^{2,m}. \tag{237}$$

Note that by (234) and Lemma 3(ii) of [Myt98a], $\bar{X}^{2,m}$ belongs to $\mathfrak{P}^{T,\eta}$, $\mathcal{P}$–a.s. Since

$$B_m\ \uparrow\ B_\infty\ :=\ \Big\{(s,x)\in[0,T]\times\mathsf{R}^d:\ \big\langle\bar{X}^2_s,h_{x,\eta}\big\rangle<\infty\Big\}\quad\text{as}\quad m\uparrow\infty, \tag{238}$$

and

$$\int_{B^{\mathrm{c}}_\infty}\bar{X}^2_s(\mathrm{d}x)\,\mathrm{d}s\ =0 \tag{239}$$

(where B^{c} denotes the complement of the set B), we obtain that

$$\bar{X}^{2,m}_s(\mathrm{d}x)\,\mathrm{d}s\ \uparrow\ \bar{X}^2_s(\mathrm{d}x)\,\mathrm{d}s\quad\text{as}\ m\uparrow\infty. \tag{240}$$

Similarly, for $m,n\geq 1,$ set

$$X^{n,2,m}_s(\mathrm{d}x)\ :=\ X^{n,2}_s(\mathrm{d}x)\,1_{B_m}(s,x),\qquad (s,x)\in[0,T]\times\mathsf{R}^d, \tag{241}$$

and $X^{n,2,m,\mathrm{c}}:=X^{n,2}-X^{n,2,m}$. Note that $X^{n,2,m}\leq\bar{X}^{2,m}\in\mathfrak{P}^{T,\eta}$, and that

$$X^{n,2,m,\mathrm{c}}\ =\ X^{n,2}-X^{n,2,m}\ \leq\ \bar{X}^{2,m,\mathrm{c}}. \tag{242}$$

5° (*Further decomposition*). For fixed f in $\mathrm{b}\mathcal{C}\big([0,T]\times\mathsf{R}^d\big)$, let us bound

$$\Big|\Big\langle L^{2,\varepsilon^1,J^1}_{(\hat{X}^{n,1},\,X^{n,2})}(T),f\Big\rangle-\Big\langle L^{2,\varepsilon^2,J^2}_{(\hat{X}^{n,1},\,X^{n,2})}(T),f\Big\rangle\Big|\;\le\;I^{\varepsilon^1,1}_{n,m}+I\!I^{\varepsilon^1,\varepsilon^2}_{n,m}+I^{\varepsilon^2,2}_{n,m},\tag{243}$$

where

$$I^{\varepsilon^i,i}_{n,m}\;:=\;\Big|\Big\langle L^{2,\varepsilon^i,J^i}_{(\hat{X}^{n,1},\,X^{n,2})}(T),f\Big\rangle-\Big\langle L^{2,\varepsilon^i,J^i}_{(\hat{X}^{n,1},\,X^{n,2,m})}(T),f\Big\rangle\Big|,\qquad i=1,2,\tag{244a}$$

$$I\!I^{\varepsilon^1,\varepsilon^2}_{n,m}\;:=\;\Big|\Big\langle L^{2,\varepsilon^1,J^1}_{(\hat{X}^{n,1},\,X^{n,2,m})}(T),f\Big\rangle-\Big\langle L^{2,\varepsilon^2,J^2}_{(\hat{X}^{n,1},\,X^{n,2,m})}(T),f\Big\rangle\Big|.\tag{244b}$$

6° (*Middle term* $I\!I^{\varepsilon^1,\varepsilon^2}_{n,m}$: *Preparation*). Let us start with $I\!I^{\varepsilon^1,\varepsilon^2}_{n,m}$. We want to show that for fixed m,

$$I\!I^{\varepsilon^1,\varepsilon^2}_{n,m}\;\xrightarrow[\varepsilon^1,\varepsilon^2\downarrow 0]{\mathcal{P}}\;0,\quad\text{uniformly in } n.\tag{245}$$

By dominations as in (231) and tightness as for (232), it suffices to prove

$$\sup_{n\ge1}\mathcal{P}\Big|\exp\Big\langle L^{2,\varepsilon^1,J^1}_{(\hat{X}^{n,1},\,X^{n,2,m})}(T),-f\Big\rangle-\exp\Big\langle L^{2,\varepsilon^2,J^2}_{(\hat{X}^{n,1},\,X^{n,2,m})}(T),-f\Big\rangle\Big|^2\;\xrightarrow[\varepsilon^1,\varepsilon^2\downarrow 0]{}\;0.$$

But for this it is sufficient to show that

$$\lim_{\varepsilon^1,\varepsilon^2\downarrow 0}\;\sup_{n\ge1}\bigg|\mathcal{P}\exp\Big[-2\Big\langle L^{2,\varepsilon^1,J^1}_{(\hat{X}^{n,1},\,X^{n,2,m})}(T),f\Big\rangle\Big]\tag{246}$$
$$-\;\mathcal{P}\exp\Big[-\Big\langle L^{2,\varepsilon^1,J^1}_{(\hat{X}^{n,1},\,X^{n,2,m})}(T),f\Big\rangle-\Big\langle L^{2,\varepsilon^2,J^2}_{(\hat{X}^{n,1},\,X^{n,2,m})}(T),f\Big\rangle\Big]\bigg|\;=\;0.$$

For this, we will use the log-Laplace representation from Lemma 14(c).

7° (*Reformulation of* (246)). In fact, by definition (18) of L^{2,ε^i} (and interchanging the order of integration),

$$\Big\langle L^{2,\varepsilon^1,J^1}_{(\hat{X}^{n,1},\,X^{n,2,m})}(T)+L^{2,\varepsilon^2,J^2}_{(\hat{X}^{n,1},\,X^{n,2,m})}(T),f\Big\rangle\;=\;\int_0^T\mathrm{d}s\,\big\langle\hat{X}^{n,1}_s,\,\psi^{n,m,\varepsilon^1,\varepsilon^2}_s\big\rangle\tag{247}$$

with

$$\psi^{n,m,\varepsilon^1,\varepsilon^2}_s(x)\;:=\;\int_{\mathsf{R}^d}X^{n,2,m}_s(\mathrm{d}y)\,[J^1_{\varepsilon^1}+J^2_{\varepsilon^2}](x-y)\,f_s(y),\tag{248}$$

$0\le s\le T$, $x\in\mathsf{R}^d$. Hence, by Lemma 14(c), applied to $\hat{\mathbf{X}}^n$ given $(\mathbf{X}^n,\mathbf{A}^n)$,

$$\mathcal{P}\Big\{\exp\Big\langle L^{2,\varepsilon^1,J^1}_{(\hat{X}^{n,1},\,X^{n,2,m})}(T)+L^{2,\varepsilon^2,J^2}_{(\hat{X}^{n,1},\,X^{n,2,m})}(T),-f\Big\rangle\;\Big|\;(\mathbf{X}^n,\mathbf{A}^n)\Big\}\tag{249}$$
$$=\;\mathcal{P}\Big\{\exp\Big[-\int_0^T\mathrm{d}s\,\big\langle\hat{X}^{n,1}_s,\,\psi^{n,m,\varepsilon^1,\varepsilon^2}_s\big\rangle\Big]\;\Big|\;(\mathbf{X}^n,\mathbf{A}^n)\Big\}$$
$$=\;\exp\Big[-\int_0^T\Big\langle A^{n,1}_{\mathrm{d}s}\;,\;u^{1,T}_s\big(0,\psi^{n,m,\varepsilon^1,\varepsilon^2}\big)\Big\rangle\Big],$$

with $u^{1,T}\big(0,\psi^{n,m,\varepsilon^1,\varepsilon^2}\big)$ solving (29). Similarly, define

$$\tilde{\psi}^{n,m,\varepsilon^1}_s(x)\;:=\;2\int_{\mathsf{R}^d}X^{n,2,m}_s(\mathrm{d}y)\,J^1_{\varepsilon^1}(x-y)\,f_s(y),\qquad 0\le s\le T,\quad x\in\mathsf{R}^d.\tag{250}$$

Then, (246) (for fixed m) is equivalent to

$$\lim_{\varepsilon^1,\varepsilon^2\downarrow 0}\ \sup_{n\geq 1}\Big|\mathcal{P}\exp\Big[-\int_0^T\Big\langle A^{n,1}_{\mathrm{d}s}\ ,\ u^{1,T}_s\big(0,\tilde{\psi}^{n,m,\varepsilon^1}\big)\Big\rangle\Big] \\ -\ \mathcal{P}\exp\Big[-\int_0^T\Big\langle A^{n,1}_{\mathrm{d}s}\ ,\ u^{1,T}_s\big(0,\psi^{n,m,\varepsilon^1,\varepsilon^2}\big)\Big\rangle\Big]\Big|\ =\ 0. \tag{251}$$

8° (*Proof of* (251)). To verify this we want to apply Proposition 34 with

$$\nu^n_s(\mathrm{d}y)\ =\ X^{n,2,m}_s(\mathrm{d}y)f_s(y). \tag{252}$$

First note that by Lemma 38 to every $\delta\in(0,1)$ there exists a compact subset $K^\delta\subset\mathcal{M}_\mathrm{f}^T$ such that

$$\mathcal{P}\left(A^{n,1}\big(\mathrm{d}(s,x)\big)\in K^\delta\right)\ \geq\ 1-\delta,\qquad n\geq 1. \tag{253}$$

On the other hand, choose a compact subset $\tilde{C}^\delta$ of $\mathcal{D}^T_{\mathcal{M}_\mathrm{f}}$ such that

$$\mathcal{P}(\bar{X}^2\in\tilde{C}^\delta)\ \geq\ 1-\delta. \tag{254}$$

(Here and in similar cases we mean, of course, the restriction of $A^{n,1}$ and $\bar{X}^2$ to the time interval $[0,T]$.) Set

$$C^\delta\ :=\ \Big\{\nu\in\mathfrak{P}^{T,\eta}:\ \exists\,\tilde{\nu}\in\tilde{C}^\delta\ \text{ with }\ \nu\leq\tilde{\nu}\ \text{ in }\ \mathfrak{P}^T\Big\} \tag{255}$$

and, for $k\geq 1$,

$$C^{\delta,k}\ :=\ \big\{\nu\in C^\delta:\ \|\nu\|_{-\eta,T}\leq k\big\}. \tag{256}$$

Note that by Lemma 27, $C^{\delta,k}$ is a compact subset of $\mathfrak{P}^{T,\eta}$.

In the expectation expressions in (251) we now distinguish between two cases. First we assume that $X^{n,2,m}\in C^{\delta,k}$ and $A^{n,1}\in K^\delta$. Then by Proposition 34 we get

$$\lim_{\varepsilon^1,\varepsilon^2\downarrow 0}\ \sup_{n\geq 1}\Big|u^{1,T}_s(0,\tilde{\psi}^{n,m,\varepsilon^1})\,(x)-u^{1,T}_s(0,\psi^{n,m,\varepsilon^1,\varepsilon^2})\,(x)\Big|\ =\ 0, \tag{257}$$

uniformly in $(s,x,X^{n,2,m})\in[0,T]\times B\times C^{\delta,k}$, where B is any compact subset of R^d. Therefore,

$$\lim_{\varepsilon^1,\varepsilon^2\downarrow 0}\ \sup_{n\geq 1}\Big|\int_0^T\mathrm{d}s\ \Big\langle A^{n,1}_{\mathrm{d}s},\ u^{1,T}_s(0,\tilde{\psi}^{n,m,\varepsilon^1})-u^{1,T}_s(0,\psi^{n,m,\varepsilon^1,\varepsilon^2})\Big\rangle\Big|\ =\ 0, \tag{258}$$

uniformly on $\big(X^{n,2,m},\ A^{n,1}\big)\in C^{\delta,k}\times K^\delta$. This gives (251) if we restrict the expectations additionally to the event $\big\{(X^{n,1,m},\ A^{n,1})\in C^{\delta,k}\times K^\delta\big\}$.

But by (253) and (254),

$$\mathcal{P}\left(\big(X^{n,1,m},A^{n,1}\big)\notin C^{\delta,k}\times K^\delta\right)\ \leq\ \mathcal{P}(\bar{X}^2\notin\tilde{C}^\delta)+\mathcal{P}(A^{n,1}\notin K^\delta)\ \leq\ 2\delta. \tag{259}$$

Thus, if in (251) we restrict now to the complementary event, we get the bound 4δ. Since δ was arbitrary, (251) follows immediately.

9° (*Remaining terms* $I^{\varepsilon^i,i}_{n,m}$). By (243), to complete the proof concerning (230), it suffices to show that for $\delta>0$ we can find an $m\geq 1$ such that

$$\sup_{n\geq 1,\ 0<\varepsilon\leq 1}\mathcal{P}\,I^{\varepsilon,i}_{n,m}\ \leq\ \delta,\qquad i=1,2. \tag{260}$$

By (242),

$$I_{n,m}^{\varepsilon,i} \ = \ \Bigl|\bigl\langle L^{2,\varepsilon,J^i}_{(\hat{X}^{n,1},\,X^{n,2,m,\mathrm{c}})}(T), f\bigr\rangle\Bigr| \ \le \ \bigl\langle L^{2,\varepsilon,J^i}_{(\hat{X}^{n,1},\bar{X}^{2,m,\mathrm{c}})}(T), |f|\bigr\rangle. \tag{261}$$

By domination $\hat{X}^{n,1} \le \bar{X}^1$ and definition (18) of asymmetric collision local time, we may continue with

$$\le \ \|f\|_{T,\infty} \int_0^T \mathrm{d}s \int_{\mathsf{R}^d} \bar{X}_s^{2,m,\mathrm{c}}(\mathrm{d}y)\, \bar{X}_s^1 \ast J_\varepsilon^i\,(y), \qquad i = 1,2, \tag{262}$$

where $\|f\|_{T,\infty} < \infty$ by assumption. But $\bar{\mathbf{X}}$ has independent components, hence $\bar{X}^{2,m,\mathrm{c}}$ and $\bar{X}^1$ are independent, and therefore we can built their expectations separately. Now, by the expectation formula in Lemma 14(d),

$$\mathcal{P}\bar{X}_s^1(\mathrm{d}z) \ = \ \mu^1 \ast \mathrm{p}_s^1\,(z)\,\mathrm{d}z, \tag{263}$$

hence,

$$\mathcal{P} I_{n,m}^{\varepsilon,i} \ \le \ \|f\|_{T,\infty}\, \mathcal{P} \int_0^T \mathrm{d}s \int_{\mathsf{R}^d} \bar{X}_s^{2,m,\mathrm{c}}(\mathrm{d}y)\, \mu^1 \ast \mathrm{p}_s^1 \ast J_\varepsilon^i\,(y), \qquad i = 1,2. \tag{264}$$

Take $\delta' \in (0,T)$. First we restrict in the previous integral additionally to $s \le \delta'$. Here, using

$$\bar{X}^{2,m,\mathrm{c}} \ \le \ \bar{X}^2 \quad \text{and} \quad \mathcal{P}\bar{X}_s^2(\mathrm{d}y) \ = \ \mu^2 \ast \mathrm{p}_s^2(y)\,\mathrm{d}y, \tag{265}$$

we come to

$$\int_0^{\delta'} \mathrm{d}s \int_{\mathsf{R}^d} \mathrm{d}y\ \mu^2 \ast \mathrm{p}_s^2(y)\, \mu^1 \ast \mathrm{p}_s^1 \ast J_\varepsilon^i\,(y). \tag{266}$$

Arguing as in step 3° of the proof of Proposition 35, we choose a δ' sufficiently small, such that (264) with the integral restricted to $s \le \delta'$ is smaller than $\delta/2$, uniformly in n, m, ε, and $i = 1,2$.

We restrict now the integral in (264) to $\delta' \le s \le T$. Here we exploit that

$$\sup_{\delta' \le s \le T} \|\mu^1 \ast \mathrm{p}_s\|_\infty < \infty \quad \text{and} \quad \int_{\mathsf{R}^d} \mathrm{d}z\ J_\varepsilon^i(z) = 1. \tag{267}$$

Thus, to complete the proof of (260), it remains to verify that

$$\mathcal{P} \int_0^T \mathrm{d}s\, \|\bar{X}_s^{2,m,\mathrm{c}}\| \ \underset{m\uparrow\infty}{\longrightarrow} \ 0. \tag{268}$$

But by (237) and (240),

$$\|\bar{X}_s^{2,m,\mathrm{c}}\| \ \underset{m\uparrow\infty}{\searrow} \ 0, \quad \mathcal{P} \times \mathrm{d}s\text{–almost everywhere}, \tag{269}$$

and $\|\bar{X}_s^{2,m,\mathrm{c}}\| \le \|\bar{X}_s^2\|$ where $\mathcal{P}\int_0^T \mathrm{d}s\, \|\bar{X}_s^2\| = T\,\|\mu^2\| < \infty$. Hence, by dominated convergence, (268) follows. This finishes the proof of (260), thus the proof concerning (230), and hence of the second term at the right hand side of (228).

10° (*Third term*). It remains to explain the modifications which we need to make to get also the Cauchy property for the third term at the right hand side of (228). Instead of condition (233) we work with

$$\eta \ \in \ \bigl((d-\alpha_2)_+\,,\, \min(\alpha_1/\beta_1\,,\, d)\bigr), \tag{270}$$

which again is based on (10). Define $\bar{X}^{1,m}$ and $\bar{X}^{1,m,\mathrm{c}}$ analogously to (236) etc. Pass once more to the asymmetric version L^{2,ε^i} of the approximating collision local time and bound

$$\left|\left\langle L^{2,\varepsilon^1,J^1}_{(\hat{X}^{n,2},\,\bar{X}^1)}(T),f\right\rangle - \left\langle L^{2,\varepsilon^2,J^2}_{(\hat{X}^{n,2},\,\bar{X}^1)}(T),f\right\rangle\right| \tag{271}$$

analogously to (243). In the definition (248) of the $\psi_s^{n,m,\varepsilon^1,\varepsilon^2}(x)$ now $\bar{X}^{1,m}$ enters instead of $X_s^{n,2,m}$, and for the log-Laplace expression (249) condition additionally on $\bar{X}^1$. The further procedure is even simpler, since we have less n–dependence. We skip any further obvious details.

This finishes the proof of Lemma 39 altogether. ∎

5.3. **Continuous convergence of collision local times.** One can easily derive from Lemma 39 the existence of collision local time for any solution to the martingale problem $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{\mu}}$ with $\boldsymbol{\mu}\in\mathcal{M}_{\mathrm{f,e}}$. The main purpose of this subsection is to show that the family of approximating collision local times corresponding to solutions of $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{\mu}}$ is relatively compact in $\mathcal{C}_{\mathcal{M}_\mathrm{f}}$ (the precise meaning of this will be clear from Proposition 42). But first let us prove the following result.

Lemma 40 (**Existence of collision local times**). *Let* $(\mathbf{X},\mathbf{A})$ *be any solution to the martingale problem* $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{\mu}}$ *with* $\boldsymbol{\alpha},\boldsymbol{\beta}$ *satisfying* (10) *and* $\boldsymbol{\mu}\in\mathcal{M}_{\mathrm{f,e}}$. *Then the local time* $L_\mathbf{X}$ *exists. Moreover, for any sequence* $\{\varepsilon_n:\,n\geq 1\}$ *converging to* 0 *as* $n\uparrow\infty$, *there exists a subsequence* $\{\varepsilon'_n:\,n\geq 1\}$ *such that*

$$L^{\varepsilon'_n}_\mathbf{X}(t)\ \underset{\varepsilon'_n\downarrow 0}{\longrightarrow}\ L_\mathbf{X}(t)\quad \text{in } \mathcal{M}_\mathrm{f} \tag{272}$$

uniformly in t *on any compact interval of continuity of* $t\mapsto L_\mathbf{X}(t)$, $\mathcal{P}$*–almost surely.*

Proof. Fix arbitrary $T>0$ and $\varepsilon_n\downarrow 0$. Let D denote a countable dense subset of $\mathrm{b}\mathcal{C}\big([0,T]\times\mathsf{R}^d\big)$. By Lemma 39, for each $f\in D$, and any mollifier J, the limit in probability

$$\lim_{n\uparrow\infty}\ \big\langle L^{\varepsilon_n,J}_\mathbf{X}(T),f\big\rangle\ =:\ L_\mathbf{X}(T,f) \tag{273}$$

exists and does not depend on the mollifier J. We can use this and a diagonalization argument to show that along some subsequence $\varepsilon'_n\downarrow 0$,

$$\big\langle L^{\varepsilon'_n,J}_\mathbf{X}(T),f\big\rangle\ \underset{n\uparrow\infty}{\longrightarrow}\ L_\mathbf{X}(T,f),\qquad f\in D,\quad \mathcal{P}\text{–a.s.} \tag{274}$$

Define the measures

$$\begin{aligned}\nu_\varepsilon\big(\mathrm{d}(t,x)\big)\ &:=\ \mathcal{P}L^{2,\varepsilon,J}_\mathbf{X}\big(\mathrm{d}(t,x)\big)\,1_{[0,T]}(t) \\ &\leq\ \mu^1\!*\mathrm{p}^1_t\!*J_\varepsilon\,(x)\,\mu^2\!*\mathrm{p}^2_t\,(x)\,1_{[0,T]}(t)\,\mathrm{d}t\,\mathrm{d}x,\end{aligned} \tag{275}$$

where the inequality follows by the domination $\mathbf{X}\leq\bar{\mathbf{X}}$ and Lemma 14(d). Arguing as in step 3° of the proof of Proposition 35, it is easy to check that $\{\nu_\varepsilon:\,0<\varepsilon\leq 1\}$ is tight in $\mathcal{M}_\mathrm{f}^T$. This combined with (274), Lemma A7 (with d replaced by $d+1$) and Lemma 2 implies, that there is a random measure $L_\mathbf{X}\in\mathcal{M}^T_\mathrm{f}$ such that

$$L^{2,\varepsilon'_n}_\mathbf{X}\big(\mathrm{d}(t,x)\big)\,1_{[0,T]}(t)\ \underset{n\uparrow\infty}{\longrightarrow}\ L_\mathbf{X}\big(\mathrm{d}(s,x)\big)\quad \text{in } \mathcal{M}^T_\mathrm{f},\quad \mathcal{P}\text{–a.s.} \tag{276}$$

Now we can define the non-decreasing measure-valued process

$$t \mapsto L_{\mathbf{X}}(t,\mathrm{d}x) := \int_0^t L_{\mathbf{X}}\big(\mathrm{d}(s,x)\big), \qquad 0<t\leq T. \tag{277}$$

Since T was arbitrary, the process $t\mapsto L_{\mathbf{X}}(t)$ can be easily defined for all $t\geq 0$. Moreover, by weak convergence properties of the measures $L_{\mathbf{X}}^{2,\varepsilon'_n}$ we get that $\mathcal{P}$–almost surely,

$$L_{\mathbf{X}}^{2,\varepsilon'_n}(t) \underset{n\uparrow\infty}{\longrightarrow} L_{\mathbf{X}}(t) \quad \text{in } \mathcal{M}_{\mathrm{f}} \tag{278}$$

for all points $t>0$ of continuity of $s\mapsto L_{\mathbf{X}}(s)$. Finally, the uniformity statement on (272) follows easily by a standard theorem on convergence of monotone functions (see e.g. Theorem 10.10 in [Doo94]). ■

Now, for any solution $(\mathbf{X},\mathbf{A})$ to prove the uniform in time convergence of $L_{\mathbf{X}}^{\varepsilon'_n}$ to $L_{\mathbf{X}}$ (as needed for Theorem 7), it is enough to show that $L_{\mathbf{X}}$ is a continuous $\mathcal{M}_{\mathrm{f}}$–valued process. The simplest way to do this is to use the continuity of the $L_{\bar{\mathbf{X}}}$–process (Corollary 37) and the domination $\mathbf{X}\leq\bar{\mathbf{X}}$.

Corollary 41 (**More on collision local time**). *Let $(\mathbf{X},\mathbf{A})$ be any solution to $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ with $\boldsymbol{\alpha},\boldsymbol{\beta}$ satisfying* (10) *and with $\boldsymbol{\mu}\in\mathcal{M}_{\mathrm{f,e}}$. Then the following statements hold:*

- **(a) (Continuity):** *The $\mathcal{M}_{\mathrm{f}}$–valued process $t\mapsto L_{\mathbf{X}}(t)$ is continuous.*
- **(b) (Convergence on path space):** *For any sequence $\varepsilon_n\downarrow 0$ there is a subsequence $\varepsilon'_n\downarrow 0$ such that*

$$L_{\mathbf{X}}^{\varepsilon'_n} \underset{n\uparrow\infty}{\longrightarrow} L_{\mathbf{X}} \quad \text{in } \mathcal{C}_{\mathcal{M}_{\mathrm{f}}},\ \mathcal{P}\text{–a.s.} \tag{279}$$

Proof. **(a)** Use the domination $\mathbf{X}\leq\bar{\mathbf{X}}$ and the convergence

$$L_{\mathbf{X}}^{\varepsilon}\big(\mathrm{d}(t,x)\big)\,1_{[0,T]}(t) \underset{\varepsilon\downarrow 0}{\longrightarrow} L_{\mathbf{X}}\big(\mathrm{d}(t,x)\big)\,1_{[0,T]}(t) \text{ in } \mathcal{M}_{\mathrm{f}}^T,\ T>0, \text{ in } \mathcal{P}\text{–probability,}$$

to get the domination

$$L_{\mathbf{X}}\big(\mathrm{d}(t,x)\big) \leq L_{\bar{\mathbf{X}}}\big(\mathrm{d}(t,x)\big). \tag{280}$$

By Corollary 37, $L_{\bar{\mathbf{X}}}$ is continuous, non-decreasing, $\mathcal{M}_{\mathrm{f}}$–valued, and hence $L_{\mathbf{X}}$ is also continuous.

(b) follows immediately from (a) and Lemma 40. The proof is complete. ■

Now we are ready to prove relative compactness of approximating collision local times as we announced at the beginning of this subsection.

Proposition 42 (**Relative compactness in $\mathcal{C}_{\mathcal{M}_{\mathrm{f}}}$**). *Let $\big\{(\mathbf{X}^n,\mathbf{A}^n): n\geq 1\big\}$ be a sequence of solutions to $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ as in Theorem 7. Then for each sequence $\varepsilon_k\downarrow 0$ and $n_k\uparrow\infty$ there is a subsequence $\{(\varepsilon'_k,n'_k): k\geq 1\}$ such that the families*

$$\Big\{L_{\mathbf{X}^{n'_k}}^{1,\varepsilon'_k} : k\geq 1\Big\} \quad \text{and} \quad \Big\{L_{\mathbf{X}^{n'_k}} : k\geq 1\Big\} \tag{281}$$

are relatively compact in $\mathcal{C}_{\mathcal{M}_{\mathrm{f}}}$, $\mathcal{P}$–almost surely.

Proof. Recall the domination

$$\mathbf{X}^n \leq \bar{\mathbf{X}}, \quad n \geq 1, \quad \mathcal{P}\text{–a.s.} \tag{282}$$

By Corollary 41(b) and a simple application of Lemma 2, for any $\varepsilon_k \downarrow 0$ we can find a subsequence $\varepsilon'_k \downarrow 0$ such that

$$\left\{L^{1,\varepsilon'_k}_{\bar{\mathbf{X}}} : \, k \geq 1\right\} \text{ converges in } \mathcal{C}_{\mathcal{M}_\mathrm{f}}, \ \mathcal{P}\text{–a.s.} \tag{283}$$

Then, from (282), for $0 \leq s \leq t$ and $f \in \mathrm{b}\mathcal{C}$,

$$\left|\left\langle L^{1,\varepsilon'_k}_{\mathbf{X}^{n'_k}}(t), f\right\rangle - \left\langle L^{1,\varepsilon'_k}_{\mathbf{X}^{n'_k}}(s), f\right\rangle\right| \leq \left|\left\langle L^{1,\varepsilon'_k}_{\bar{\mathbf{X}}}(t), f\right\rangle - \left\langle L^{1,\varepsilon'_k}_{\bar{\mathbf{X}}}(s), f\right\rangle\right|. \tag{284}$$

Hence, by Arcela-Ascoli, from (283) we get that for each $f \in \mathrm{b}\mathcal{C}$,

$$\left\{t \mapsto \left\langle L^{1,\varepsilon'_k}_{\mathbf{X}^{n'_k}}(t), f\right\rangle : \, k \geq 1\right\} \text{ is relatively compact in } \mathcal{C}(\mathsf{R}_+, \mathsf{R}), \ \mathcal{P}\text{–a.s.} \tag{285}$$

Adding the fact that for any $T > 0$ the set

$$\left\{L^{1,\varepsilon}_{\mathbf{X}^n}(s) : \, 0 \leq s \leq T, \ 0 < \varepsilon \leq 1, \ n \geq 1\right\} \tag{286}$$

of measures is $\mathcal{P}$–a.s. tight in $\mathcal{M}_\mathrm{f}$ [use again (282)], we get that

$$\left\{L^{1,\varepsilon'_k}_{\mathbf{X}^{n'_k}} : \, k \geq 1\right\} \text{ is relatively compact in } \mathcal{C}_{\mathcal{M}_\mathrm{f}} \text{ a.s.} \tag{287}$$

The proof of almost sure relative compactness of $\{L_{\mathbf{X}^n} : n \geq 1\}$ goes along the same lines (use the domination $L_{\mathbf{X}^n} \leq L_{\bar{\mathbf{X}}}$, $n \geq 1$), so we omit it. ■

5.4. **Proof of Theorem 7.** Let $\left\{(\mathbf{X}^n, \mathbf{A}^n) : n \geq 1\right\}$ be as in the theorem. Combining Lemmas 39, 40, and a diagonalization argument, we can easily get that for each sequence $\varepsilon_k \downarrow 0$ and $n_k \uparrow \infty$ there is a subsequence $\left\{(\varepsilon'_k, n'_k) : k \geq 1\right\}$ such that for $T > 0$ and $\mathcal{P}$–a.s.,

$$\left|\left\langle L^{1,\varepsilon'_k}_{\mathbf{X}^{n'_k}}(t), f\right\rangle - \left\langle L_{\mathbf{X}^{n'_k}}(t), f\right\rangle\right| \underset{k\uparrow\infty}{\longrightarrow} 0, \quad t \in \mathsf{Q}_+ \cap [0,T], \ f \in \mathrm{b}\mathcal{C}. \tag{288}$$

Moreover, by Proposition 42,

$$\left\{\left(L^{1,\varepsilon'_k}_{\mathbf{X}^{n'_k}}, L_{\mathbf{X}^{n'_k}}\right) : \, k \geq 1\right\} \text{ is relatively compact in } \mathcal{C}^2_{\mathcal{M}_\mathrm{f}}, \ \mathcal{P}\text{–a.s.} \tag{289}$$

Hence, the convergence statement in (288) holds uniformly in t on each compact subset of R_+, that is

$$\sup_{0\leq t\leq T}\left|\left\langle L^{1,\varepsilon'_k}_{\mathbf{X}^{n'_k}}(t), f\right\rangle - \left\langle L_{\mathbf{X}^{n'_k}}(t), f\right\rangle\right| \underset{k\uparrow\infty}{\longrightarrow} 0, \quad T > 0, \ f \in \mathrm{b}\mathcal{C}, \tag{290}$$

$\mathcal{P}$–a.s. Since the sequence $\{(\varepsilon_k, n_k) : k \geq 1\}$ was arbitrary, the proof is finished by a simple application of Lemma 2 and its proof, and by [Kal97, Lemma 3.2]. ■

5.5. **Continuity of the map $\mathbf{X} \mapsto L_{\mathbf{X}}$.** For the proof of Theorem 9, which is based on construction of the converging sequence of approximating competing species models, we will need the following result.

Proposition 43 (**Continuity of the map $\mathbf{X} \mapsto L_{\mathbf{X}}$**). *For each $n \geq 0$, consider solutions $(\mathbf{X}^n, \mathbf{A}^n)$ to $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{\mu}}$ as in Theorem 7. Suppose that $\mathbf{X}^n \to \mathbf{X}^0$ as $n \uparrow \infty$ in $\mathcal{D}^2_{\mathcal{M}_\mathrm{f}}$, $\mathcal{P}$–a.s. Then:*

- **(a)** $L_{\mathbf{X}^n} \to L_{\mathbf{X}^0}$ *as $n \uparrow \infty$, in $\mathcal{C}_{\mathcal{M}_\mathrm{f}}$ in $\mathcal{P}$–probability.*
- **(b)** $L^{\varepsilon_n}_{\mathbf{X}^n} \to L_{\mathbf{X}^0}$ *as $n \uparrow \infty$ and $\varepsilon_n \downarrow 0$, in $\mathcal{C}_{\mathcal{M}_\mathrm{f}}$ in $\mathcal{P}$–probability.*

(c) $L^{i,\varepsilon_n}_{\mathbf{X}^n} \to L_{\mathbf{X}^0}$ *as* $n\uparrow\infty$ *and* $\varepsilon_n\downarrow 0$, *in* $\mathcal{C}_{\mathcal{M}_\mathrm{f}}$ *in* $\mathcal{P}$*–probability,* $i=1,2$.

Proof. **(a)** is proved along the lines of the proof of Lemma 3.5 in [EP94, p.135]. That is, $L_{\mathbf{X}}$ is a uniform limit of continuous maps,

$$L^{\varepsilon}_{\mathbf{X}^n} \xrightarrow[\mathbf{X}^n\to\mathbf{X}^0]{} L^{\varepsilon}_{\mathbf{X}^0}\,, \qquad \varepsilon\geq 0, \tag{291}$$

and by Theorem 7, $L_{\mathbf{X}}$ is the uniform (in probability space) limit of these mappings as $\varepsilon\downarrow 0$. Hence, the limiting mapping is also continuous.

We skip the proof of **(b)** and **(c),** since it uses the same reasoning. ∎

5.6. **A general tightness result.** Here we will prove some tightness criterion that will be useful in the proof of Theorem 9 in Subsection 6.2.

Proposition 44 (**General tightness criterion**)**.** *Let* $(\mathbf{X}^n,\mathbf{A}^n)$ *be any sequence of solutions to* $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{\mu}}$. *Suppose that the family* $\mathbf{A}^n\in\mathcal{C}^2_{\mathcal{M}_\mathrm{f}}$ *is tight in law. Then the family*

$$(\mathbf{X}^n,\mathbf{A}^n)\in\mathcal{D}^2_{\mathcal{M}_\mathrm{f}}\times\mathcal{D}^2_{\mathcal{M}_\mathrm{f}}\,, \quad n\geq 1, \tag{292}$$

is tight in law, too.

Remark 45 (**Tightness in** $\mathcal{C}^2_{\mathcal{M}_\mathrm{f}}$)**.** Note that from Lemma 38 we know the tightness in law of the $\mathbf{A}^n$ as measures on $[0,T]\times\mathsf{R}^d$, for any T. However, this proposition requires more from the $\mathbf{A}^n$: namely, to be tight in $\mathcal{C}^2_{\mathcal{M}_\mathrm{f}}$. ◇

Proof of Proposition 44. For the proof it is enough to check that the sequence $\{\mathbf{X}^n:\,n\geq 1\}\subset\mathcal{D}^2_{\mathcal{M}_\mathrm{f}}$ is tight in law. By the domination

$$\mathbf{X}^n \leq\ \bar{\mathbf{X}}, \qquad n\geq 1, \tag{293}$$

(recall Corollary 23), it suffices to verify that for $(\varphi^1,\varphi^2)\in\Phi^2_+$ the sequence

$$\Bigl\{\bigl(\langle X^{n,1},\varphi^1\rangle,\langle X^{n,2},\varphi^2\rangle\bigr):\ n\geq 1\Bigr\}\subset\mathcal{D}^2(\mathsf{R}_+,\mathsf{R}) \text{ is tight in law.} \tag{294}$$

By Aldous' criterion of tightness (see, for instance, [Wal86, Theorem 6.8]), statement (294) is true if the following two conditions hold:

(i) For each $t\in\mathsf{Q}_+$, the family of random variables $\bigl\{\langle X^{n,i}_t,\varphi^i\rangle:\,n\geq 1,\ i=1,2\bigr\}$ is tight in law.

(ii) For any sequence $\{\tau_n:\,n\geq 1\}$ of bounded stopping times and $\delta_n\downarrow 0$,

$$\Bigl|\bigl\langle\mathbf{X}^n_{\tau_n+\delta_n}\,,\boldsymbol{\varphi}\bigr\rangle-\bigl\langle\mathbf{X}^n_{\tau_n}\,,\boldsymbol{\varphi}\bigr\rangle\Bigr|\ \xrightarrow[n\uparrow\infty]{\mathcal{P}}\ 0.$$

By the domination (293), part (i) follows trivially. Next check (ii). Assume $\tau_n\leq T$, $n\geq 1$. Again by the domination (293),

$$\Bigl\{\bigl\langle\mathbf{X}^n_{\tau_n+\delta_n}\,,\boldsymbol{\varphi}\bigr\rangle,\ \bigl\langle\mathbf{X}^n_{\tau_n}\,,\boldsymbol{\varphi}\bigr\rangle:\ n\geq 1\Bigr\} \text{ is tight in law.} \tag{295}$$

Hence, it is enough to check that

$$\mathcal{P}\Bigl|\exp\bigl\langle\mathbf{X}^n_{\tau_n+\delta_n}\,,-\boldsymbol{\varphi}\bigr\rangle-\exp\bigl\langle\mathbf{X}^n_{\tau_n}\,,-\boldsymbol{\varphi}\bigr\rangle\Bigr|^2\ \xrightarrow[n\uparrow\infty]{}\ 0. \tag{296}$$

For this it suffices to demonstrate that

$$I_n\ :=\ \Bigl|\mathcal{P}\mathrm{e}^{-\langle\mathbf{X}^n_{\tau_n+\delta_n}\,,2\boldsymbol{\varphi}\rangle}-\mathcal{P}\mathrm{e}^{-\langle\mathbf{X}^n_{\tau_n+\delta_n}\,,\boldsymbol{\varphi}\rangle-\langle\mathbf{X}^n_{\tau_n}\,,\boldsymbol{\varphi}\rangle}\Bigr|\ \xrightarrow[n\uparrow\infty]{}\ 0, \tag{297a}$$

$$II_n\ :=\ \Bigl|\mathcal{P}\mathrm{e}^{-\langle\mathbf{X}^n_{\tau_n+\delta_n}\,,\boldsymbol{\varphi}\rangle-\langle\mathbf{X}^n_{\tau_n}\,,\boldsymbol{\varphi}\rangle}-\mathcal{P}\mathrm{e}^{-\langle\mathbf{X}^n_{\tau_n}\,,2\boldsymbol{\varphi}\rangle}\Bigr|\ \xrightarrow[n\uparrow\infty]{}\ 0. \tag{297b}$$

We will verify (297a) since (297b) goes along the same lines with obvious modifications.

We start with comparing the first term in (297a) with

$$\mathcal{P}\exp\Big[-\big\langle \mathbf{X}^n_{\tau_n+\delta_n}\,,2\boldsymbol{\varphi}\big\rangle-\int_{\tau_n}^{\tau_n+\delta_n}\langle \mathbf{A}^n_{\mathrm{d}s}\,,2\boldsymbol{\varphi}\rangle\Big]. \tag{298}$$

Then its absolute difference is bounded by

$$\mathcal{P}\Big|1-\exp\Big[-\int_{\tau_n}^{\tau_n+\delta_n}\langle \mathbf{A}^n_{\mathrm{d}s}\,,2\boldsymbol{\varphi}\rangle\Big]\Big|. \tag{299}$$

By our assumption, the family $\{\mathbf{A}^n : n\geq 1\}\subset \mathcal{C}^2_{\mathcal{M}_\mathrm{f}}$ is tight in law. Therefore, the expectation (299) will vanish as $n\uparrow\infty$.

Similarly, we compare the second term in (297a) with

$$\mathcal{P}\exp\Big[-\big\langle \mathbf{X}^n_{\tau_n+\delta_n}\,,\boldsymbol{\varphi}\big\rangle-\big\langle \mathbf{X}^n_{\tau_n}\,,\boldsymbol{\varphi}\big\rangle-\int_{\tau_n}^{\tau_n+\delta_n}\langle \mathbf{A}^n_{\mathrm{d}s}\,,\boldsymbol{\varphi}\rangle\Big]. \tag{300}$$

By the same argument, the difference will converge to 0 as $n\uparrow\infty$.

Next we compare (298) with

$$\mathcal{P}\exp\Big[-\big\langle \mathbf{X}^n_{\tau_n+\delta_n}\,,2\boldsymbol{\varphi}\big\rangle-\int_{\tau_n}^{\tau_n+\delta_n}\big\langle \mathbf{A}^n_{\mathrm{d}s}\,,\mathbf{u}^{\tau_n+\delta_n}_s(2\boldsymbol{\varphi},\mathbf{0})\big\rangle\Big], \tag{301}$$

with $\mathbf{u}$ from Definition 13. Its absolute difference is bounded by

$$\mathcal{P}\exp\Big[-\int_{\tau_n}^{\tau_n+\delta_n}\Big\langle \mathbf{A}^n_{\mathrm{d}s}\,,\big|2\boldsymbol{\varphi}-\mathbf{u}^{\tau_n+\delta_n}_s(2\boldsymbol{\varphi},\mathbf{0})\big|\Big\rangle\Big]. \tag{302}$$

But by Lemma 15(b),

$$\sup_{\tau_n\leq s\leq\tau_n+\delta_n}\big\|2\varphi^i-u^{i,\tau_n+\delta_n}_s(2\varphi^i,0)\big\|_\infty \underset{n\uparrow\infty}{\longrightarrow} 0, \tag{303}$$

and therefore (302) converges to 0 as $n\uparrow\infty$.

In a similar way, we compare (300) with

$$\mathcal{P}\exp\Big[-\big\langle \mathbf{X}^n_{\tau_n+\delta_n}\,,\boldsymbol{\varphi}\big\rangle-\big\langle \mathbf{X}^n_{\tau_n}\,,\boldsymbol{\varphi}\big\rangle-\int_{\tau_n}^{\tau_n+\delta_n}\big\langle \mathbf{A}^n_{\mathrm{d}s}\,,\mathbf{u}^{\tau_n+\delta_n}_s(\boldsymbol{\varphi},\mathbf{0})\big\rangle\Big]. \tag{304}$$

By the same reasoning, the difference will vanish as $n\uparrow\infty$.

Since $2\varphi^i=u^{i,\tau_n+\delta_n}_{\tau_n+\delta_n}(2\varphi^i,0)$, by the exponential martingale Proposition 17 we can rewrite (301) as

$$\mathcal{P}\exp\Big[-\big\langle \mathbf{X}^n_{\tau_n}\,,\mathbf{u}^{\tau_n+\delta_n}_{\tau_n}(2\boldsymbol{\varphi},\mathbf{0})\big\rangle\Big]. \tag{305}$$

Analogously, (304) equals

$$\mathcal{P}\exp\Big[-\big\langle \mathbf{X}^n_{\tau_n}\,,\mathbf{u}^{\tau_n+\delta_n}_{\tau_n}(\boldsymbol{\varphi},\mathbf{0})+\boldsymbol{\varphi}\big\rangle\Big]. \tag{306}$$

But by time-homogeneity,

$$u^{i,\tau_n+\delta_n}_{\tau_n}(2\varphi^i,0)\;=\;u^{i,\delta_n}_0(2\varphi^i,0), \tag{307}$$

and again by Lemma 15(b),

$$\big\|u^{i,\delta_n}_0(2\varphi^i,0)-2\varphi^i\big\|_\infty \underset{n\uparrow\infty}{\longrightarrow} 0. \tag{308}$$

Recalling domination (293), the difference between (305) and

$$\mathcal{P}\exp\Big[-\big\langle \mathbf{X}^n_{\tau_n}, 2\boldsymbol{\varphi}\big\rangle\Big] \tag{309}$$

will go to 0 as $n\uparrow\infty$.

In the same way, (306) approaches (309) as $n\uparrow\infty$, too.

Consequently, $I_n\to 0$ as $n\uparrow\infty$, that is, (297a) follows, and the proof is complete. ■

6. Construction of the competing species model

Here we want to prove Theorem 9, based on approximating competing species processes which will be constructed afterwards (Subsections 6.3–6.5).

6.1. **Approximating competing species model.** Recall from Lemma 2 the approximating collision local times $L^{1,\varepsilon}_{\mathbf{Y}}$ and $L^{2,\varepsilon}_{\mathbf{Y}}$ of a pair $\mathbf{Y}=(Y^1,Y^2)\in\mathcal{D}^2_{\mathcal{M}_\mathrm{f}}$.

Definition 46 (**Martingale problem** $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda},\varepsilon}_{\boldsymbol{\mu}}$). For fixed pairs $\boldsymbol{\alpha}$,$\boldsymbol{\beta}$, and $\boldsymbol{\vartheta}$ as in (1), $\boldsymbol{\mu}\in\mathcal{M}^2_\mathrm{f}$, a pair $\boldsymbol{\lambda}=(\lambda^1,\lambda^2)\in\mathsf{R}^2_+$, and $\varepsilon\in(0,1]$, let $\mathbf{X}^\varepsilon=(X^{\varepsilon,1},X^{\varepsilon,2})$ be an $\mathcal{F}_\cdot$–adapted process (in $\mathcal{D}^2_{\mathcal{M}_\mathrm{f}}$) such that, for each $\boldsymbol{\varphi}=(\varphi^1,\varphi^2)\in\Phi^2_+$,

$$\begin{aligned} t\ \mapsto\ \mathrm{e}^{-\langle \mathbf{X}^\varepsilon_t,\,\boldsymbol{\varphi}\rangle}-\mathrm{e}^{-\langle \boldsymbol{\mu},\boldsymbol{\varphi}\rangle}+\int_0^t\mathrm{d}s\ \mathrm{e}^{-\langle \mathbf{X}^\varepsilon_s,\,\boldsymbol{\varphi}\rangle}\Big\langle \mathbf{X}^\varepsilon_s,\,\Delta_{\boldsymbol{\alpha}}\boldsymbol{\varphi}-\boldsymbol{\varphi}^{\mathbf{1}+\boldsymbol{\beta}}\Big\rangle \\ -\int_0^t\big\langle \boldsymbol{\Lambda}^\varepsilon_{\mathrm{d}s},\boldsymbol{\varphi}\big\rangle\,\mathrm{e}^{-\langle \mathbf{X}^\varepsilon_s,\,\boldsymbol{\varphi}\rangle},\qquad t\ge 0, \end{aligned} \tag{310}$$

is an $\mathcal{F}_\cdot$–martingale starting from 0 at time $t=0$, where

$$\boldsymbol{\Lambda}^\varepsilon\ =\ (\Lambda^{1,\varepsilon},\Lambda^{2,\varepsilon})\ :=\ \big(\lambda^1L^{1,\varepsilon}_{\mathbf{X}^\varepsilon},\lambda^2L^{2,\varepsilon}_{\mathbf{X}^\varepsilon}\big). \tag{311}$$

Then we say that $\mathbf{X}^\varepsilon$ solves the martingale problem $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda},\varepsilon}_{\boldsymbol{\mu}}$. ◇

In a sense, $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda},\varepsilon}_{\boldsymbol{\mu}}$ describes an approximating competing species model. Heuristically, each particle with path $s\mapsto\xi_s$ in the first population $X^{\varepsilon,1}$ is killed according to the additive functional $t\mapsto\lambda^1\int_0^t\mathrm{d}s\ X^{\varepsilon,2}*J_\varepsilon(\xi_s)$, and conversely for the particles from the second population. Our goal is to show that whenever $\varepsilon\downarrow 0$, then the weak limiting points of the solutions to $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda},\varepsilon}_{\boldsymbol{\mu}}$ satisfy $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda}}_{\boldsymbol{\mu}}$, which in turn will give a proof of Theorem 9. But first we need to state the existence of a solution to $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda},\varepsilon}_{\boldsymbol{\mu}}$.

Proposition 47 (**Approximating competing species model**). *For each choice of our constants* $\boldsymbol{\alpha}$,$\boldsymbol{\beta}$,$\boldsymbol{\vartheta}$,$\boldsymbol{\mu}$,$\boldsymbol{\lambda}$, *and* ε, *there is a solution* $\mathbf{X}^\varepsilon$ *to the martingale problem* $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda},\varepsilon}_{\boldsymbol{\mu}}$.

The proof of this proposition is deferred to Subsections 6.3–6.5.

6.2. **Proof of Theorem 9.** Fix $\boldsymbol{\alpha}$,$\boldsymbol{\beta}$,$\boldsymbol{\vartheta}$,$\boldsymbol{\mu}$,$\boldsymbol{\lambda}$ as in the theorem, and a sequence $\varepsilon_n\downarrow 0$ (as $n\uparrow\infty$). Consider $\mathbf{X}^{\varepsilon_n}$ from Proposition 47 and $\boldsymbol{\Lambda}^{\varepsilon_n}$ as in (311).

Lemma 48 (**Tightness**). *The family* $(\mathbf{X}^{\varepsilon_n},\boldsymbol{\Lambda}^{\varepsilon_n})\in\mathcal{D}^2_{\mathcal{M}_\mathrm{f}}\times\mathcal{C}^2_{\mathcal{M}_\mathrm{f}}$ *is tight in law.*

Proof. By Proposition 44, it is enough to prove that the family $\{\boldsymbol{\Lambda}^{\varepsilon_n}_t : n\ge 1\}$ of processes in $\mathcal{C}^2_{\mathcal{M}_\mathrm{f}}$ is tight in law. But this follows from Proposition 42. ■

Based on this lemma, now we prove the following result.

Lemma 49 (Limit points). *Let* $(\mathbf{X},\boldsymbol{\Lambda})$ *be any weak limit point of* $(\mathbf{X}^{\varepsilon_n},\boldsymbol{\Lambda}^{\varepsilon_n})$. *Then*

$$\boldsymbol{\Lambda} \;=\; (\lambda^1 L_{\mathbf{X}}\,, \lambda^2 L_{\mathbf{X}}), \tag{312}$$

and $\mathbf{X}$ *is a solution to the martingale problem* $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda}}_{\boldsymbol{\mu}}$.

Proof. By Lemma 48, we can choose from $\{(\mathbf{X}^{\varepsilon_n},\boldsymbol{\Lambda}^{\varepsilon_n}) : n\geq 1\}$ a subsequence converging in law on $\mathcal{D}^2_{\mathcal{M}_\mathrm{f}}\times\mathcal{C}^2_{\mathcal{M}_\mathrm{f}}$. By an abuse of notation, we denote this subsequence by the same symbol. Going to the Skorohod space, we may assume that there exists $(\mathbf{X},\boldsymbol{\Lambda})$ such that

$$(\mathbf{X}^{\varepsilon_n},\boldsymbol{\Lambda}^{\varepsilon_n}) \underset{n\uparrow\infty}{\longrightarrow} (\mathbf{X},\boldsymbol{\Lambda}) \quad \text{in } \mathcal{D}^2_{\mathcal{M}_\mathrm{f}}\times\mathcal{C}^2_{\mathcal{M}_\mathrm{f}}\,,\ \ \mathcal{P}\text{–a.s.} \tag{313}$$

But by Proposition 43, convergence of $\mathbf{X}^{\varepsilon_n}$ implies that

$$\Lambda^{i,\varepsilon_n} = \lambda^i L^{i,\varepsilon_n}_{\mathbf{X}^{\varepsilon_n}} \underset{n\uparrow\infty}{\longrightarrow} \lambda^i L_{\mathbf{X}} \quad \text{in } \mathcal{C}_{\mathcal{M}_\mathrm{f}}\,,\ \text{in } \mathcal{P}\text{–probability.} \tag{314}$$

Combining (313) and (314), we get

$$(\mathbf{X}^{\varepsilon_n},\boldsymbol{\Lambda}^{\varepsilon_n}) \underset{n\uparrow\infty}{\longrightarrow} \bigl(\mathbf{X},\lambda^1 L_{\mathbf{X}}\,,\lambda^2 L_{\mathbf{X}}\bigr) \quad \text{in } \mathcal{D}^2_{\mathcal{M}_\mathrm{f}}\times\mathcal{C}^2_{\mathcal{M}_\mathrm{f}}\,, \quad \mathcal{P}\text{–a.s.} \tag{315}$$

In particular, we get this convergence in the weak sense on the original probability space, and (312) follows.

The last thing to check is that the $\mathbf{X}$ we constructed is a solution to $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda}}_{\boldsymbol{\mu}}$. By the exponential martingale Proposition 17, for each $n\geq 1,\ T>0,$ and $\boldsymbol{\varphi}\in\Phi^2_+\,,$

$$t \;\mapsto\; M^{\varepsilon_n}_t(\boldsymbol{\varphi}) \;:=\; \exp\biggl[-\bigl\langle \mathbf{X}^{\varepsilon_n}_t,\mathbf{u}^T_t(\boldsymbol{\varphi},\mathbf{0})\bigr\rangle - \int_0^t \bigl\langle \boldsymbol{\Lambda}^{\varepsilon_n}_{\mathrm{d}s},\mathbf{u}^T_s(\boldsymbol{\varphi},\mathbf{0})\bigr\rangle\biggr], \tag{316}$$

$0\leq t\leq T,$ is a martingale. Switch again to the Skorohod space where (315) holds. Since the martingales $M^{\varepsilon_n}(\boldsymbol{\varphi})$ are bounded uniformly in $n,$ they converge to the martingale

$$t \;\mapsto M_t(\boldsymbol{\varphi}) \;:=\; \exp\biggl[-\bigl\langle \mathbf{X}_t\,,\mathbf{u}^T_t(\boldsymbol{\varphi},\mathbf{0})\bigr\rangle - \int_0^t \bigl\langle \boldsymbol{\Lambda}_{\mathrm{d}s}\,,\mathbf{u}^T_s(\boldsymbol{\varphi},\mathbf{0})\bigr\rangle\biggr], \tag{317}$$

$0\leq t\leq T,$ with $\boldsymbol{\Lambda}$ from (312). Again by Proposition 17, $(\mathbf{X},\boldsymbol{\Lambda})$ solves $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{\mu}}$, and therefore $\mathbf{X}$ is a solution of $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda}}_{\boldsymbol{\mu}}$, finishing the proof. ■

Consequently, based on Proposition 47 (which proof was deferred), we constructed a solution to the martingale problem $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda}}_{\boldsymbol{\mu}}$, that is, *we verified Theorem* 9.

6.3. **Outline of the proof of Proposition 47.** In the finite variance competing species model of [EP94], approximating competing species processes had been constructed by an application of Girsanov's theorem. As we have already mentioned, this "luctionary" tool is not available in our case, so we have to take another root. Actually, we use tools developed in the previous sections. Since the matter here is even simpler, we do not provide all the details.

Fix $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\vartheta},\boldsymbol{\lambda},\boldsymbol{\mu},\varepsilon$ as in the martingale problem $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda},\varepsilon}_{\boldsymbol{\mu}}$ of Definition 46. Loosely speaking, for each $m\geq 1$ we would like to start from a solution $\mathbf{X}^m = (X^{m,1},X^{m,2})$ of the following *martingale problem* $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda},\varepsilon,m}_{\boldsymbol{\mu}}$:

For each $\boldsymbol{\varphi} \in \Phi_+^2$,

$$t \ \mapsto \ \mathrm{e}^{-\langle \mathbf{X}_t^m,\, \boldsymbol{\varphi}\rangle} - \mathrm{e}^{-\langle \boldsymbol{\mu},\boldsymbol{\varphi}\rangle} + \int_0^t \mathrm{d}s\, \mathrm{e}^{-\langle \mathbf{X}_s^m,\, \boldsymbol{\varphi}\rangle} \Big\langle \mathbf{X}_s^m,\, \Delta_{\boldsymbol{\alpha}}\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\mathbf{1}+\boldsymbol{\beta}} \Big\rangle \qquad (318)$$

$$- \int_0^t \big\langle \boldsymbol{\Lambda}_{\mathrm{d}s}^{\varepsilon,m},\, \boldsymbol{\varphi}\big\rangle\, \mathrm{e}^{-\langle \mathbf{X}_s^m,\, \boldsymbol{\varphi}\rangle}, \qquad t \geq 0,$$

is a martingale starting from 0 at time $t = 0$. Here, $\boldsymbol{\Lambda}^{\varepsilon,m} = (\Lambda^{1,\varepsilon,m}, \Lambda^{2,\varepsilon,m})$ is defined as follows:

$$\Lambda_t^{i,\varepsilon,m}(\mathrm{d}x) \ := \ \lambda^i \int_0^t L_{\mathbf{X}^m}^{i,\varepsilon}\big(\mathrm{d}(s,x)\big)\, j_s^{m,i}, \qquad i = 1,2, \qquad (319)$$

where

$$j_s^{m,i} \ := \ \begin{cases} 2 & \text{if } \frac{2k-i}{m} \leq s < \frac{2k+1-i}{m}, \ k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

That is, on time intervals of length $\frac{1}{m}$, starting with an odd multiple of $\frac{1}{m}$, only the first population $X^{m,1}$ is affected by killing provided by $\lambda^1 L_{\mathbf{X}^m}^{1,\varepsilon}$. Conversely, on the remaining time intervals only the second population $X^{m,2}$ is affected by killing using $\lambda^2 L_{\mathbf{X}^m}^{2,\varepsilon}$. Clearly, as $m \uparrow \infty$ we expect the weak limiting points to solve the martingale problem $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda},\varepsilon}$ of Definition 46.

In the following subsections we want to define precisely these processes $\mathbf{X}^m$, to show their tightness in law, and to prove that any weak limit point $\mathbf{X}^\varepsilon$ is a solution to $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda},\varepsilon}$. This will verify Proposition 47.

6.4. **Processes with "one-sided" conditioned killing.** For $i = 1,2$, let

$$\big({}^\circ\Omega, {}^\circ\mathcal{F}, {}^\circ\mathcal{F}_\cdot, Q_{r,\mu^i,\kappa^i}^i\big) \qquad (320)$$

be the canonical stochastic basis of the (α^i, d, β^i)–superprocess $\tilde{X}^i$ with killing rate κ^i, without immigration, and starting at time $r \geq 0$ with measure $\mu^i \in \mathcal{M}_\mathrm{f}$ (recall Subsection 3.2). Now, for fixed $\boldsymbol{\lambda} = (\lambda^1, \lambda^2) \in \mathsf{R}_+^2$ and $\varepsilon \in (0,1]$, let

$$\kappa^1 \ := \ 0 \quad \text{and} \quad \kappa^2 \ := \ 2\lambda^2\, \tilde{X}^1(\omega^1) {*} J_\varepsilon\,. \qquad (321)$$

That is, autonomously $\tilde{X}^1$ is the critical (α^1, d, β^1)–superprocess (without immigration), and, given a realization $\tilde{X}^1(\omega^1)$ of $\tilde{X}^1$, the process $\tilde{X}^2$ is constructed as the (α^2, d, β^2)–superprocess with killing at rate $2\lambda^2\, \tilde{X}_s^1(\omega^1){*}J_\varepsilon\,(x)$ at time s at site x. Write

$${}^1P_{r,\boldsymbol{\mu}}(\mathrm{d}\boldsymbol{\omega}) \ := \ Q_{r,\mu^1,0}^1(\mathrm{d}\omega^1)\, Q_{r,\mu^2,2\lambda^2\tilde{X}^1(\omega^1)*J_\varepsilon}^2(\mathrm{d}\omega^2), \qquad (322)$$

$\boldsymbol{\omega} = (\omega^1, \omega^2) \in {}^\circ\Omega \times {}^\circ\Omega$, for the joint law of $\tilde{\mathbf{X}} : = (\tilde{X}^1, \tilde{X}^2)$ in the present special case of one-sided conditioned killing. For $i = 1,2$ denote by $\mathcal{F}_\cdot^i$ a right-continuous filtration generated by $\tilde{X}^i$. Note that the process $(\mathbf{X}, \mathbf{A})$ defined by

$$X^1 := 0, \ \ A^1 := 0 \quad \text{and} \quad X^2 := \tilde{X}^2, \ \ A^2 := 2\lambda^2 L_{\tilde{\mathbf{X}}}^{2,\varepsilon}, \qquad (323)$$

solves our martingale problem $(\mathbf{MP})_{\boldsymbol{\mu}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ under the law ${}^1P_{0,\boldsymbol{\mu}}\{\,\cdot\,|\, \mathcal{F}_\infty^1\}$ (implying a degenerate first component).

Lemma 50 (**Another exponential martingale**). *Let* $\tilde{\mathbf{X}}$ *have the law* ${}^1P_{0,\boldsymbol{\mu}}$ *as introduced in* (322). *Then for any* $T>0$, *and* $\boldsymbol{\varphi}\in\Phi_+^2$,

$$t\mapsto M_t(\boldsymbol{\varphi}) := \exp\Big[-\big\langle\tilde{\mathbf{X}}_t\,,\mathbf{u}_t^T(\boldsymbol{\varphi},\mathbf{0})\big\rangle-2\lambda^2\!\int_{[0,t]\times\mathsf{R}^d}L^{2,\varepsilon}_{\tilde{\mathbf{X}}}\big(\mathrm{d}(s,x)\big)\,u_s^{2,T}(\varphi^2,0)\,(x)\Big],$$

(*with* $\mathbf{u}^T(\boldsymbol{\varphi},\mathbf{0})$ *from Definition* 13) *is an* $\big(\mathcal{F}^1_\cdot\otimes\mathcal{F}^2_\cdot\big)$*–martingale on* $[0,T]$.

Proof. Fix $r\in[0,T)$ and $D\in\mathcal{F}^1_r\otimes\mathcal{F}^2_r$. Then, for $t\in[r,T]$,

$$\mathcal{P}\,1_D\,M_t(\boldsymbol{\varphi}) = \mathcal{P}\,1_D\,\mathrm{e}^{-\langle\tilde{X}^1_t,u_t^{1,T}(\varphi^1,0)\rangle}\,\mathcal{P}\Big\{\exp\Big[-\big\langle\tilde{X}^2_t,u_t^{2,T}(\varphi^2,0)\big\rangle - 2\lambda^2\int_{[0,t]\times\mathsf{R}^d}L^{2,\varepsilon}_{\tilde{\mathbf{X}}}\big(\mathrm{d}(s,x)\big)\,u_s^{2,T}(\varphi^2,0)(x)\Big]\,\Big|\,\mathcal{F}^1_\infty\otimes\mathcal{F}^2_r\Big\}. \tag{324}$$

Recall that $(\mathbf{X},\mathbf{A})$ defined by (323) solves $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{\mu}}$ under ${}^1P_{0,\boldsymbol{\mu}}\{\,\cdot\,|\,\mathcal{F}^1_\infty\}$. Hence, by Proposition 17, the expression inside the conditional expectation in (324) is an $\mathcal{F}^1_\infty\otimes\mathcal{F}^2_\cdot$–martingale. Thus, (324) equals

$$\begin{aligned}&\mathcal{P}\Big(1_D\,\mathrm{e}^{-\langle\tilde{X}^1_t,u_t^{1,T}(\varphi^1,0)\rangle}\,\times \\ &\qquad\exp\Big[-\big\langle\tilde{X}^2_r,u_r^{2,T}(\varphi^2,0)\big\rangle-2\lambda^2\int_{[0,r]\times\mathsf{R}^d}L^{2,\varepsilon}_{\tilde{\mathbf{X}}}\big(\mathrm{d}(s,x)\big)\;u_s^{2,T}(\varphi^2,0)(x)\Big]\Big)\\ &= \mathcal{P}\Big(1_D\,\exp\Big[-\big\langle\tilde{X}^2_r,u_r^{2,T}(\varphi^2,0)\big\rangle-2\lambda^2\int_{[0,r]\times\mathsf{R}^d}L^{2,\varepsilon}_{\tilde{\mathbf{X}}}\big(\mathrm{d}(s,x)\big)\;u_s^{2,T}(\varphi^2,0)(x)\Big]\\ &\qquad\times\;\mathcal{P}\Big\{\mathrm{e}^{-\langle\tilde{X}^1_t,u_t^{1,T}(\varphi^1,0)\rangle}\,\Big|\,\mathcal{F}^1_r\otimes\mathcal{F}^2_r\Big\}\Big).\end{aligned} \tag{325}$$

Now apply Proposition 17 to $\tilde{X}^1$ alone without killing [recall (321)], to see that the latter conditional expectation equals $\mathrm{e}^{-\langle\tilde{X}^1_r,u_r^{1,T}(\varphi^1,0)\rangle}$. Then we arrive at the expression $\mathcal{P}\,1_D\,M_r(\boldsymbol{\varphi})$. Since r and D were arbitrary, $M(\boldsymbol{\varphi})$ has the claimed martingale property, and the proof is finished. ∎

Next we alternate the previous construction, that is, we let $\tilde{\mathbf{X}}=(\tilde{X}^1,\tilde{X}^2)$ be distributed according to

$${}^2P_{r,\boldsymbol{\mu}}(\mathrm{d}\boldsymbol{\omega}) := Q^2_{r,\mu^2,0}(\mathrm{d}\omega^2)\,Q^1_{r,\mu^1,2\lambda^1\tilde{X}^2(\omega^2)*J_\varepsilon}(\mathrm{d}\omega^1). \tag{326}$$

In other words, $\tilde{X}^2$ is an autonomous critical superprocess under ${}^2P_{r,\mu}$, and then, conditioned on $\tilde{X}^2$, the process $\tilde{X}^1$ is constructed with killing rate $2\lambda^1\,\tilde{X}^2_s{*}J_\varepsilon$.

6.5. **Alternating conditional killing.** Now we put together the pieces for our Trotter type construction of the process $\mathbf{X}^m$. Let $\mathcal{F}^i_\cdot$ denote the right-continuous filtration generated by $X^{m,i}$ (which we want to construct). On the time interval $[0,\frac{1}{m}]$, let $\mathbf{X}^m$ evolve according to the law ${}^1P_{0,\boldsymbol{\mu}}$. Conditioned on $\mathcal{F}^1_{1/m}\otimes\mathcal{F}^1_{1/m}$, starting at time $\frac{1}{m}$ with $\mathbf{X}^m_{1/m}$, let $\mathbf{X}^m$ evolve on $[\frac{1}{m},\frac{2}{m}]$ according to ${}^2P_{\frac{1}{m},\mathbf{X}^m_{1/m}}$. Continuing this way, we define the process $\mathbf{X}^m$.

Recall definition (319) of $\boldsymbol{\Lambda}^{\varepsilon,m}=(\Lambda^{1,\varepsilon,m},\Lambda^{2,\varepsilon,m})$. From the given construction of $(\mathbf{X}^m,\boldsymbol{\Lambda}^{\varepsilon,m})$ and Lemma 50, it is easy to get that

$$t\mapsto M^{m,\varepsilon}_t(\boldsymbol{\varphi}) := \exp\Big[-\big\langle\mathbf{X}^m_t,\mathbf{u}^T_t(\boldsymbol{\varphi},\mathbf{0})\big\rangle-\int_0^t\big\langle\boldsymbol{\Lambda}^{\varepsilon,m}_{\mathrm{d}s},\mathbf{u}^T_s(\boldsymbol{\varphi},\mathbf{0})\big\rangle\Big], \tag{327}$$

is an $\mathcal{F}^1_\cdot \otimes \mathcal{F}^2_\cdot$–martingale on $[0,T]$, for any $T>0$ and $\boldsymbol{\varphi} \in \Phi^2_+$. Hence, by the second part of Proposition 17, $(\mathbf{X}^m, \boldsymbol{\Lambda}^{\varepsilon,m})$ is a solution to the martingale problem $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{\mu}}$, for each $m \geq 1$ (and the fixed ε).

Lemma 51 (**Tightness of** $(\mathbf{X}^m, \boldsymbol{\Lambda}^{\varepsilon,m})$)**.** *The sequence*

$$\big\{(\mathbf{X}^m, \boldsymbol{\Lambda}^{\varepsilon,m}):\ m \geq 1\big\} \in \mathcal{D}^2_{\mathcal{M}_\mathrm{f}} \times \mathcal{C}^2_{\mathcal{M}_\mathrm{f}} \tag{328}$$

is tight in law, and any weak limit point of the $\mathbf{X}^m$ *is a solution to the martingale problem* $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda},\varepsilon}_{\boldsymbol{\mu}}$.

Proof. As we mentioned just before the lemma, $\big\{(\mathbf{X}^m, \boldsymbol{\Lambda}^{\varepsilon,m}):\ m \geq 1\big\}$ is a family of solutions to the martingale problem $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{\mu}}$. Hence, by Proposition 44, to show the tightness in law on $\mathcal{D}^2_{\mathcal{M}_\mathrm{f}} \times \mathcal{C}^2_{\mathcal{M}_\mathrm{f}}$ it is enough to check the tightness of $\{\boldsymbol{\Lambda}^{\varepsilon,m}:\ m \geq 1\}$ in law on $\mathcal{C}^2_{\mathcal{M}_\mathrm{f}}$.

By domination $\mathbf{X}^m \leq \bar{\mathbf{X}}$, we immediately get the domination

$$\boldsymbol{\Lambda}^{\varepsilon,m} \ \leq\ 2\big(\lambda^1 L^{1,\varepsilon}_{\bar{\mathbf{X}}}, \lambda^2 L^{2,\varepsilon}_{\bar{\mathbf{X}}}\big), \qquad m \geq 1. \tag{329}$$

Since $\Lambda^{i,\varepsilon,m}$ is a non-decreasing measure-valued process, for any $m \geq 1$ and $i=1,2$, and $L^{i,\varepsilon}_{\bar{\mathbf{X}}}$ is a non-decreasing, continuous, measure-valued process, we immediately get from the domination (329) the tightness of $\{\boldsymbol{\Lambda}^{\varepsilon,m}:\ m \geq 1\}$ in law on $\mathcal{C}^2_{\mathcal{M}_\mathrm{f}}$. Altogether, by Proposition 44 we get the tightness of $\big\{(\mathbf{X}^m, \boldsymbol{\Lambda}^{\varepsilon,m}):\ m \geq 1\big\}$ in law on $\mathcal{D}^2_{\mathcal{M}_\mathrm{f}} \times \mathcal{C}^2_{\mathcal{M}_\mathrm{f}}$.

Now let $(\mathbf{X}, \mathbf{A}^\varepsilon)$ be any weak limit point of $\big\{(\mathbf{X}^m, \boldsymbol{\Lambda}^{\varepsilon,m}):\ m \geq 1\big\}$. Then on an appropriate Skorohod space,

$$(\mathbf{X}^m, \boldsymbol{\Lambda}^{\varepsilon,m}) \underset{m\uparrow\infty}{\longrightarrow} (\mathbf{X}, \mathbf{A}^\varepsilon), \quad \mathcal{P}\text{–a.s.} \tag{330}$$

(by passing to a subsequence if necessary). From the definition of $(\mathbf{X}^m, \boldsymbol{\Lambda}^{\varepsilon,m})$ it is easy to see that, as $\mathbf{X}^m \longrightarrow_{m\uparrow\infty} \mathbf{X}$, then

$$\boldsymbol{\Lambda}^{\varepsilon,m} \underset{m\uparrow\infty}{\longrightarrow} (\lambda^1 L^{1,\varepsilon}_{\mathbf{X}}, \lambda^2 L^{2,\varepsilon}_{\mathbf{X}}), \quad \mathcal{P}\text{–a.s.}, \tag{331}$$

that is $\mathbf{A}^\varepsilon = \boldsymbol{\Lambda}^\varepsilon$. Also, on the same Skorohod space, the martingales $M^{m,\varepsilon}(\boldsymbol{\varphi})$ defined in (327) converge to the martingale

$$t \ \mapsto\ M^\varepsilon_t(\boldsymbol{\varphi}) \ :=\ \exp\Big[-\big\langle \mathbf{X}_t\,, \mathbf{u}^T_t(\boldsymbol{\varphi},\mathbf{0})\big\rangle - \int_0^t \big\langle \boldsymbol{\Lambda}^\varepsilon_{\mathrm{d}s}\,, \mathbf{u}^T_s(\boldsymbol{\varphi},\mathbf{0})\big\rangle\Big], \tag{332}$$

$0 \leq t \leq T$, $\boldsymbol{\varphi} \in \Phi^2_+$. Then by the second part of Proposition 17, $(\mathbf{X}, \boldsymbol{\Lambda}^\varepsilon)$ is a solution to $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{\mu}}$, and hence, $\mathbf{X}$ solves the martingale problem $(\mathbf{MP})^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda},\varepsilon}_{\boldsymbol{\mu}}$. This finishes the proof of Lemma 51. ■

With Lemma 51 also the *proof of Proposition* 47 *is finished.*

Appendix: Auxiliary facts

A.1. **Some convergences in** $\mathrm{b}\mathcal{C}_{\mathrm{co}}\big([0,T]\times \mathsf{R}^d\big)$**.** For convenience, here we collect some standard facts; cf. for instance with [Myt99, Lemma A.4].

Fix $T \geq 1$. Introduce G^T acting on measurable $\psi : [0,T]\times\mathsf{R}^d \to \mathsf{R}$ by

$$(G^T\psi)_r(x) \ :=\ \int_r^T \mathrm{d}s\ \mathrm{S}_{s-r}\psi_s\,(x), \qquad (r,x) \in [0,T]\times\mathsf{R}^d, \tag{A1}$$

with S and p denoting the semigroup respectively the continuous transition kernel of the symmetric α–stable process with generator $\Delta_\alpha := -\vartheta(-\Delta)^{\alpha/2},\ 0<\alpha\le 2,$ $\vartheta>0.$ Recall our notation $\|\cdot\|_{T,\infty}$ introduced in (97).

Lemma A1 (**Relative compactness**)**.** *For* $m\ge 1,$ *let* ψ^m *be (real-valued) measurable functions defined on* $[0,T]\times\mathsf{R}^d$ *such that*

$$\sup_{m\ge 1}\|\psi^m\|_{T,\infty} < \infty. \tag{A2}$$

Then the set $\{G^T\psi^m : m\ge 1\}$ *is relatively compact in the space* $\mathrm{b}\mathcal{C}_{\mathrm{co}}\big([0,T]\times\mathsf{R}^d\big)$ *of continuous functions on* $[0,T]\times\mathsf{R}^d$ *equipped with the topology of uniform convergence on compacta.*

Proof. First of all, for $0\le r\le r'\le T$ and $x\in\mathsf{R}^d,$

$$\int_r^{r'}\mathrm{d}s\ \mathrm{S}_{s-r}\psi^m_s(x) \ \le\ \|\psi^m\|_{T,\infty}\,(r'-r), \tag{A3}$$

in particular, the set $\{G^T\psi^m : m\ge 1\}$ is uniformly bounded on $[0,T]\times\mathsf{R}^d.$ By Arcela-Ascoli it remains to show its equip-continuity on $[0,T]\times C,$ say, where $C\subset\mathsf{R}^d$ is compact. For this purpose, consider $0\le r\le r'\le T$ and $x,x'\in C.$ Then

$$\begin{aligned}\Big|(G^T\psi^m)_r(x)-(G^T\psi^m)_{r'}(x')\Big| \ &\le\ \int_r^{r'}\mathrm{d}s\ \mathrm{S}_{s-r}|\psi^m_s|(x) \\ &+\int_{r'}^T\mathrm{d}s\int_{\mathsf{R}^d}\mathrm{d}y\ \big|\mathrm{p}_{s-r}(y-x)-\mathrm{p}_{s-r'}(y-x')\big|\ \big|\psi^m_s(y)\big|.\end{aligned} \tag{A4}$$

By (A3), the first term on the right hand side is bounded by $\|\psi^m\|_{T,\infty}\,(r'-r).$ On the other hand, by a change of variables, the second term is bounded from above by

$$\|\psi^m\|_{T,\infty}\int_0^T\mathrm{d}s\int_{\mathsf{R}^d}\mathrm{d}y\ \big|\mathrm{p}_{s+r-r'}(y-x)-\mathrm{p}_s(y-x')\big|. \tag{A5}$$

Fix $0<\varepsilon<1.$ If we restrict the integration in (A5) additionally to $s\le\varepsilon,$ then the restricted double integral is bounded by $2\varepsilon.$ On the other hand, we can find a compact set $K\subset\mathsf{R}^d$ such that

$$\sup_{x\in C}\int_0^{T+1}\mathrm{d}s\int_{K^{\mathrm{c}}}\mathrm{d}y\ \mathrm{p}_s(y-x)\ \le\ \varepsilon. \tag{A6}$$

Now fix $\delta\in(0,\varepsilon)$ such that for all

$$x,x'\in C \quad\text{and}\quad 0\le r\le r'\le T \quad\text{satisfying}\quad |x-x'|+|r-r'|\le\delta, \tag{A7}$$

we have

$$\int_\varepsilon^T\mathrm{d}s\int_K\mathrm{d}y\ \big|\mathrm{p}_{s+r-r'}(y-x)-\mathrm{p}_s(y-x')\big|\ \le\ \varepsilon. \tag{A8}$$

Hence, by (A3)-(A8) we get

$$\big|(G^T\psi^m)_r(x)-(G^T\psi^m)_{r'}(x')\big|\ \le\ 5\ \sup_{m\ge 1}\|\psi^m\|_{T,\infty}\,\varepsilon, \tag{A9}$$

for $x,x'r,r'$ as in (A7). Since ε was arbitrary, equi-continuity follows, and we are done. ■

Corollary A2 (**Convergence**). *If, in addition to the assumptions in Lemma* A1,

$$\lim_{m\uparrow\infty} \|\psi^m\|_{T,K} \;=\; 0, \quad \textit{for each compact set } K \subset \mathsf{R}^d, \tag{A10}$$

then

$$G^T\psi^m \;\underset{m\uparrow\infty}{\longrightarrow}\; 0 \;\; \textit{in } \mathrm{b}\mathcal{C}_{\mathrm{co}}\big([0,T]\times\mathsf{R}^d\big). \tag{A11}$$

Proof. Fix a compact set $C \subset \mathsf{R}^d$. For any $\varepsilon > 0$, take a compact set $K \subset \mathsf{R}^d$, such that (A6) is satisfied. Then,

$$\|G^T\psi^m\|_{T,C} \;\le\; \varepsilon\,\|\psi^m\|_{T,\infty} + T\,\|\psi^m\|_{T,K}\,, \tag{A12}$$

and the claim follows. ■

We need also another version of the previous result:

Lemma A3 (**Convergence**). *Let $\psi^m \ge 0$, $m \ge 0$, be (real-valued) measurable functions defined on $[0,T]\times\mathsf{R}^d$ satisfying*

$$\sup_{m\ge 0} \|\psi^m\|_{T,\infty} \;<\; \infty \tag{A13}$$

and such that in $\mathcal{M}_{\mathrm{f}}^T$,

$$\psi_s^m(x)\,\mathrm{d}s\,\mathrm{d}x \;\underset{m\uparrow\infty}{\longrightarrow}\; \psi_s^0(x)\,\mathrm{d}s\,\mathrm{d}x. \tag{A14}$$

Then,

$$G^T\psi^m \;\underset{m\uparrow\infty}{\longrightarrow}\; G^T\psi^0 \;\; \textit{in } \mathrm{b}\mathcal{C}_{\mathrm{co}}\big([0,T]\times\mathsf{R}^d\big). \tag{A15}$$

Proof. It is easy to check that

$$(G^T\psi^m)_r(x) \;\underset{m\uparrow\infty}{\longrightarrow}\; (G^T\psi^0)_r(x), \qquad (r,x)\in[0,T]\times\mathsf{R}^d. \tag{A16}$$

Since by Lemma A1, $\big\{G^T\psi^m : m \ge 1\big\}$ is relatively compact in $\mathrm{b}\mathcal{C}_{\mathrm{co}}\big([0,T]\times\mathsf{R}^d\big)$, claim (A15) follows. ■

For convenience, we add here another simple statement.

Lemma A4 (**Uniform convergence**). *For $n \ge 1$, consider $f_n \in \bar{\mathcal{C}}$. Suppose*

$$\sup_{n\ge 1} \|f_n\|_\infty \;<\; \infty \tag{A17}$$

and that

$$f_n \to 0 \quad \textit{as } n\uparrow\infty, \textit{ uniformly on compacta of } \mathsf{R}^d. \tag{A18}$$

Then, for all $T > 0$,

$$\sup_{0\le s\le T} \mathrm{S}_s f_n \;\to\; 0 \quad \textit{as } n\uparrow\infty, \textit{ uniformly on compacta of } \mathsf{R}^d. \tag{A19}$$

Proof. Fix $T > 0$ and a compact set $C \subset \mathsf{R}^d$. Then the set of measures

$$\Big\{\mu_{s,x}(\mathrm{d}y) := \mathrm{p}_s(y-x)\,\mathrm{d}y :\; (s,x)\in(0,T]\times C\Big\} \tag{A20}$$

on R^d is tight. Consequently, for each $\delta > 0$ we can find a compact set $K_\delta \subset \mathsf{R}^d$ such that $\mu_{s,x}(K_\delta^{\mathrm{c}}) \le \delta$, for any $(s,x) \in (0,T] \times \mathsf{R}^d$. Therefore it is easy to check that

$$\sup_{0<s\le T,\, x\in C} \big\langle \mu_{s,x}\,, f_n \big\rangle \;\le\; \delta \sup_{n\ge 1} \|f_n\|_\infty + \sup_{y\in K} \big|f_n(y)\big| \;\underset{n\uparrow\infty}{\longrightarrow}\; \delta \sup_{n\ge 1} \|f_n\|_\infty\,, \tag{A21}$$

where in the last step we used (A18). By (A17), the claim follows, since δ was arbitrary. ∎

A.2. **On the fractional Laplacian Δ_α.** This subsection is devoted to some elementary properties of the (weighted) fractional Laplacian $\Delta_\alpha = -\vartheta(-\Delta)^{\alpha/2}$, $0 < \alpha \le 2$, $\vartheta > 0$. Recall first that $\mathcal{D}(\Delta_\alpha) = \Phi$, introduced in the beginning of Subsection 2.1.

Lemma A5 (**Run away functions**). *For $k \ge 1$, let $\varphi_k \in \Phi_+$ satisfy $\varphi_k \le 1$ and*

$$\varphi_k(x) \;=\; \begin{cases} 0 & \text{for} \quad x \in B_k(0), \\ 1 & \text{for} \quad x \in B_{k+1}^{\mathrm{c}}(0), \end{cases} \tag{A22}$$

(*with* $B_k = B_k(0) = \big\{x \in \mathsf{R}^d : |x| < k\big\}$). *Then*

$$|\Delta_\alpha \varphi_k| \;\underset{k\uparrow\infty}{\longrightarrow}\; 0 \quad \textit{uniformly on compacta of } \mathsf{R}^d. \tag{A23}$$

Proof. For the proof, we may assume that $\vartheta = 1$ (otherwise, use scaling). The following representation is well-known, see, e.g., [Yos74, formula (9.11.5)]:

$$\Delta_\alpha \varphi\,(x) \;=\; \frac{1}{\Gamma(-\alpha/2)} \int_0^\infty \mathrm{d}s\; s^{-1-\alpha/2} \big[\varphi(x) - \mathrm{S}_s^{(2)} \varphi\,(x)\big], \qquad \varphi \in \Phi, \tag{A24}$$

where $\mathrm{S}^{(2)}$ denotes the semigroup of Brownian motion in R^d related to Δ, and Γ is Euler's Gamma function.

Fix an $N \ge 1$, and consider $x \in B_N(0)$. Then, for all k sufficiently large we have $\varphi_k(x) = 0$ and $\varphi_k(x+y) = 0$ for all $|y| \le k - N$. Using representation (A24) for φ_k, it suffices to show that

$$\int_0^\infty \mathrm{d}s\; s^{-1-\alpha/2} \int_{|y|\ge k} \mathrm{d}y\; s^{-d/2}\, \mathrm{e}^{-|y|^2/4s} \;\underset{k\uparrow\infty}{\longrightarrow}\; 0, \tag{A25}$$

which follows immediately by dominated convergence. ∎

Corollary A6 (**Run away functions**). *Fix $T > 0$ and $\mu \in \mathcal{M}_{\mathrm{f}}$. Then, with $\{\varphi_k : k \ge 1\} \subset \Phi_+$ from Lemma* A5,

$$\int_0^T \mathrm{d}s\; \big\langle \mu, \mathrm{S}_s |\Delta_\alpha \varphi_k| \big\rangle \;\underset{k\uparrow\infty}{\longrightarrow}\; 0. \tag{A26}$$

Proof. By Lemma A5, $|\Delta_\alpha \varphi_k| \to 0$ as $k \uparrow \infty$, uniformly on compacta. Hence, by Lemma A4,

$$\sup_{0\le s\le T} \mathrm{S}_s |\Delta_\alpha \varphi_k| \;\to\; 0 \quad \text{uniformly on compacta of } \mathsf{R}^d. \tag{A27}$$

Since μ is finite, this gives the claim. ∎

A.3. **On weak convergence in $\mathcal{M}_{\mathrm{f}}$.** We need the following simple fact.

Lemma A7 (**Sufficient criterion for weak convergence in $\mathcal{M}_{\mathrm{f}}$**). *Let D be a countable dense subset of the separable Banach space $\bar{\mathcal{C}}$, and $\{Y_n : n \geq 1\}$ be a sequence of random finite measures on R^d defined on some probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Assume that*

$$\langle Y_n , f\rangle \underset{n\uparrow\infty}{\longrightarrow} \textit{some } Y(f) \in \mathsf{R}_+ , \quad f \in D, \quad \mathcal{P}\text{–a.s.,} \tag{A28}$$

$$\{\mathcal{P}Y_n : n \geq 1\} \textit{ is relatively compact in } \mathcal{M}_{\mathrm{f}} . \tag{A29}$$

Then there is a random finite measure Y on R^d such that

$$Y_n \underset{n\uparrow\infty}{\longrightarrow} Y \textit{ in } \mathcal{M}_{\mathrm{f}} , \quad \mathcal{P}\text{–a.s.,} \tag{A30}$$

$$\langle Y, f\rangle = Y(f), \quad f \in D, \quad \mathcal{P}\text{–a.s.} \tag{A31}$$

Proof. Take $\Omega_0 \in \mathcal{F}$ such that the convergence statement in (A28) holds for all $\omega \in \Omega_0$. Fix such ω. We may think of $Y_n(\omega)$ as measures in $\mathcal{M}_{\mathrm{f}}(\dot{\mathsf{R}}^d)$, where $\dot{\mathsf{R}}^d = \mathsf{R}^d \cup \{\infty\}$ is the one-point compactification of R^d. By [Doo94, Theorems 8.4 and 8.5], it follows from (A28) that there exists a finite random measure $Y(\omega)$ on $\dot{\mathsf{R}}^d$ such that

$$Y_n(\omega) \underset{n\uparrow\infty}{\longrightarrow} Y(\omega) \text{ in } \mathcal{M}_{\mathrm{f}}(\dot{\mathsf{R}}^d). \tag{A32}$$

The proof will be finished, if we show that

$$\big\langle Y(\omega), 1_{\{\infty\}}\big\rangle = 0. \tag{A33}$$

In fact, then

$$Y_n(\omega) \underset{n\uparrow\infty}{\longrightarrow} Y(\omega) \text{ in } \mathcal{M}_{\mathrm{f}} , \tag{A34}$$

implying the claims (A30) and (A31).

Take $\varphi_k := 1_{B_k}$ with $B_k = B_k(0)$. Clearly, $\varphi_k \downarrow 1_{\{\infty\}} =: \varphi$ as $k \uparrow \infty$, pointwise on $\dot{\mathsf{R}}^d$. By monotone convergence

$$\big\langle Y(\omega), \varphi_k\big\rangle \downarrow \langle Y(\omega), \varphi\rangle \text{ as } k \uparrow \infty, \tag{A35}$$

hence

$$\mathcal{P}\,\langle Y, \varphi_k\rangle \downarrow \mathcal{P}\,\langle Y, \varphi\rangle \text{ as } k \uparrow \infty. \tag{A36}$$

Assume for the moment that $\mathcal{P}\,\big\langle Y(\omega), \varphi\big\rangle = 0$, then (A33) follows, restricting the set Ω_0 if needed. But by (A36) and Fatou,

$$\mathcal{P}\,\langle Y, \varphi\rangle = \lim_{k\uparrow\infty} \mathcal{P}\,\langle Y, \varphi_k\rangle \leq \lim_{k\uparrow\infty} \liminf_{n\uparrow\infty} \mathcal{P}\,\langle Y_n , \varphi_k\rangle. \tag{A37}$$

In view of (A29), for any $\delta > 0$ there exists $N_\delta \geq 1$ such that

$$\sup_{n\geq 1} \mathcal{P}\,\langle Y_n , \varphi_k\rangle < \delta, \qquad k \geq N_\delta . \tag{A38}$$

Then (A37) gives $\mathcal{P}\,\langle Y, \varphi\rangle \leq \delta$. Since δ was arbitrary, the proof is finished. ∎

A.4. **Some first order considerations of log-Laplace functions.** First we provide the

Proof of Lemma 15. Fix t, φ, ψ as in the lemma.

(a) From the log-Laplace equation (29),

$$\frac{1}{\varepsilon}\big(u_t^{i,t+\varepsilon}(\varphi,0)\,(x)-\varphi(x)\big) \;=\; \frac{1}{\varepsilon}\big(\mathrm{S}_\varepsilon^i\varphi\,(x)-\varphi(x)\big) - \frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathrm{d}r\;\mathrm{S}_{r-t}^i\big((u_r^{i,t})^{1+\beta^i}\big)\,(x), \qquad x\in\mathsf{R}^d. \tag{A39}$$

By letting ε go to 0 the result follows easily.

(b) By a minor change in (A39), for $s\in[t,t+\varepsilon]$ we have

$$\begin{aligned}
&\frac{1}{\varepsilon}\big\|u_s^{i,t+\varepsilon}(\varphi,0)-\varphi\big\|_\infty \\
&\le\; \frac{1}{\varepsilon}\Big\|\int_0^{t+\varepsilon-s}\mathrm{d}r\;\mathrm{S}_r^i(\Delta_{\alpha^i}\varphi)\Big\|_\infty \;+\; \frac{1}{\varepsilon}\,\Big\|\int_s^{t+\varepsilon}\mathrm{d}r\;\mathrm{S}_{r-s}^i\big((u_r^{i,t})^{1+\beta^i}\big)\Big\|_\infty \\
&\le\; \frac{1}{\varepsilon}\int_0^{t+\varepsilon-s}\mathrm{d}r\;\|\Delta_{\alpha^i}\varphi\|_\infty \;+\; \frac{1}{\varepsilon}\int_s^{t+\varepsilon}\mathrm{d}r\;\|u_r^{i,t}\|_\infty^{1+\beta^i} \\
&\le\; \|\Delta_{\alpha^i}\varphi\|_\infty \;+\; \|\varphi\|_\infty^{1+\beta^i},
\end{aligned} \tag{A40}$$

where for the last inequality we used the *domination*

$$u_r^{i,t}(\varphi,\psi) \;\le\; \mathrm{S}_{t-r}^i\varphi \;+\; \int_r^t\mathrm{d}r'\;\mathrm{S}_{r'-r}^i\psi_{r'}\,, \qquad 0\le r\le t. \tag{A41}$$

(c) From the log-Laplace equation (29),

$$\begin{aligned}
&\frac{1}{\varepsilon}\big(u_r^{i,t}(\varepsilon\varphi,\varepsilon\psi)\big) \;-\; \mathrm{S}_{t-r}^i\varphi \;-\; \int_r^t\mathrm{d}s\;\mathrm{S}_{s-r}^i\psi_s \\
&=\; -\,\frac{1}{\varepsilon}\int_r^t\mathrm{d}s\;\mathrm{S}_{s-r}^i\Big(\big(u_s^{i,t}(\varepsilon\varphi,\varepsilon\psi)\big)^{1+\beta^i}\Big),
\end{aligned} \tag{A42}$$

so we need only to deal with the last term. But by domination (A41),

$$\begin{aligned}
\Big\|\mathrm{S}_{s-r}^i\Big(\big(u_s^{i,t}(\varepsilon\varphi,\varepsilon\psi)\big)^{1+\beta^i}\Big)\Big\|_\infty \;&\le\; \Big\|\big(u_s^{i,t}(\varepsilon\varphi,\varepsilon\psi)\big)^{1+\beta^i}\Big\|_\infty \\
&\le\; \varepsilon^{1+\beta^i}\big(\|\varphi\|_\infty+t\|\psi\|_{t,\infty}\big)
\end{aligned} \tag{A43}$$

[recall notation (97)]. This implies claim (c), finishing the proof of Lemma 15. ∎

Proof of Lemma 16. **(a)** Choose $\varphi_n\in\mathrm{b}\mathcal{B}$ such that $\varphi_n\uparrow\varphi$. By subadditivity (Lemma 33) and domination, for $n\ge m$ we immediately get

$$\big|u_r^{i,t}(\varphi_n\,,0)-u_r^{i,t}(\varphi_m\,,0)\big| \;\le\; u_r^{i,t}(\varphi_n-\varphi_m\,,0) \;\le\; \mathrm{S}_{t-r}^i(\varphi_n-\varphi_m) \underset{n,m\uparrow\infty}{\longrightarrow} 0$$

uniformly on compacts of R^d, for each fixed $r<t$. This gives the existence of a continuous limit

$$u_r^{i,t}(\varphi,0) \;:=\; \lim_{n\uparrow\infty} u_r^{i,t}(\varphi_n\,,0). \tag{A44}$$

The fact that $u^{i,t}$ solves (29) follows easily by monotone convergence. The uniqueness argument is standard.

(b) First of all, $\mathrm{S}^i_{t-r}\varphi_\varepsilon \to \mathrm{S}^i_{t-r}\varphi_0$ uniformly on compacts on R^d. Since

$$\big\|u^{i,t}_r(\varphi_\varepsilon\,,0)\big\|_\infty \;\le\; \big\|\mathrm{S}^i_{t-r}\varphi_\varepsilon\,,0)\big\|_\infty \;\le\; \operatorname*{ess\,sup}_{x\in\mathsf{R}^d}\big|\varphi_\varepsilon(x)\big|\,, \tag{A45}$$

by Lemma A1, setting

$$\psi^\varepsilon_s(x) \;:=\; \big(u^{i,t}_s(\varphi_\varepsilon\,,0)\big)^{1+\beta^i}\,, \qquad 0\le s<t, \quad x\in\mathsf{R}^d, \tag{A46}$$

we have that $\{G^t\psi^\varepsilon : 0<\varepsilon\le 1\}$ is relatively compact in $\mathrm{b}\mathcal{C}_{\mathrm{co}}\big([0,t]\times\mathsf{R}^d\big)$. Arguing as in the proof of Proposition 34, it is easy to check that each limit point solves the required equation, and the convergence is uniform on compacts. ■

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