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# ON STOCHASTIC EULER EQUATION IN $\mathbf{R}^d$

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**Abstract** Following the Arnold-Marsden-Ebin approach, we prove local (global in 2-D) existence and uniqueness of classical (Hölder class) solutions of stochastic Euler equation with random forcing.

**Keywords** Stochastic partial differential equations , Euler equation

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In [1], [3] and [2] the Euler coordinates were used to study the motion of an incompressible fluid on compact manifolds. Following the ideas of [1], [2] and [3] we find in a small time interval the classical solutions of Euler equation with random forcing in the whole space (We have the first order SPDE in this case). Then we perturb randomly (using an independent Wiener process) the position of individual particles and derive the corresponding second order SPDE. In this case SPDE is completely degenerated in [7] sense. As random perturbation goes to zero we obtain the deterministic velocity fields. In the case $d=2$, classical solutions exist in an infinite time interval. In [8] no random forcing was considered and the second order SPDE was derived by a different method for $d>2$.

# 1 Euler equation with random forcing

Let us consider the stochastic Euler equation in $\mathbf{R}^d$ $(d\geq 2)$

$$\begin{cases} \partial_t u + u^l\partial_l u + \nabla p = \epsilon \dot{W}_t, \text{ in } [0,\,\infty)\times\mathbf{R}^d, \\ \\ u(0,\,x) = h(x), \text{ div } u(t,\,\cdot) = 0, \end{cases} \tag{1}$$

where $W_t = \left(W^k\right)_{1\leq k\leq d}$ is a standard Wiener process. If $u\,(t,x)$ and $p\,(t,x)$ are solutions, then $u\,(t,x)$ represents the velocity of the fluid particle at position $x$, at time $t$, and $p\,(t,x)$ is the pressure of the fluid at the same time and place. The right hand side of (1) represents a random force. Let $\eta\,(t) = \eta\,(t,x)$ be a flow associated to $u$, i.e. $\eta$ is a solution of differential equation

$$\begin{cases} \partial_t\eta^k\,(t) = u^k\,(t,\eta\,(t))\,, \quad k=1,\ldots,d \\ \eta\,(0,x) = x, \end{cases} \tag{2}$$

If $\eta\,(t,x)$ is the solution of (2), then $\eta\,(t,x)$ is the position at time $t$ of the fluid particle which at time zero was at $x$. In the language of fluid mechanics, $\eta\,(t,x)$ is the Euler coordinate of the particle whose Lagrange coordinate is $x$. If for any $t$ the map $x\longmapsto\eta\,(t,x)$ is a diffeomorphism, then the equation (2) can be used to recover $u\,(t,x)$ (see [3]). If $\epsilon=0$, we have

$$u\,(t,x) = \partial_t\eta\,(t,\sigma\,(t,x))\,,$$

where $\sigma\,(t) = \sigma\,(t,x)$ is the inverse of $\eta\,(t) = \eta\,(t,x)$ . Following [3], we will find an equation for $\eta\,(t)$ which is equivalent to (1). In [3] the case of a compact manifold with $\epsilon=0$ was considered.

## 1.1 Function spaces and decomposition of vector fields

### 1.1.1 Function spaces

In order to state our result precisely, we shall introduce some function spaces and their decompositions. For $l=0,\,1,\,\ldots,\,\alpha\in(0,\,1),\,\delta\in\mathbf{R}$ we introduce the Banach spaces $C^{l,\,\alpha}_\delta(\mathbf{R}^d)$ of $l$ times continuously differentiable functions $u$ on $\mathbf{R}^d$ with finite norm

$$|u|_{C^{l,\,\alpha}_\delta} = \sup_{x,y}|x-y|^{-\alpha}\sum_{k=0}^l\left|(1+|x|)^{\delta-l+k}\partial^k u(x) - (1+|y|)^{\delta-l+k}\partial^k u(y)\right| +$$

$$+\sup_x\sum_{k=0}^l(1+|x|)^{\delta-l+k-\alpha}\left|\partial^k u(x)\right|.$$

**Remark 1** The norm $|u|_{C_\delta^{l,\alpha}}$ is equivalent to any of the following norms (cf. Proposition 2.3.16 in [5]):

$$|u|_{l,\,\alpha;\,\delta}^{(1)} = \sup_{x,\,y} \min\{1+|x|\,,\,1+|y|\}^{\delta} \frac{\left|\partial^l u(x)-\partial^l u(y)\right|}{|x-y|^{\alpha}} +$$

$$+\sup_x \sum_{k=0}^{l} (1+|x|)^{\delta-l+k-\alpha} \left|\partial^k u(x)\right|,$$

$$|u|_{l,\,\alpha;\,\delta}^{(1,\gamma)} = \sup_x (1+|x|)^{\delta} \sup_{y\in E(x)} \frac{\left|\partial^l u(x)-\partial^l u(y)\right|}{|x-y|^{\alpha}}$$

$$+\sup_x (1+|x|)^{\delta-l-\alpha} |u(x)|,$$

where $\gamma \in (0,1]$, and

$$E(x) = E_\gamma(x) = \{y \in \mathbf{R}^d : |x-y| < \gamma 1 \vee (|x| \wedge |y|)\}.$$

By interpolation inequalities $|u|_{l,\,\alpha;\,\delta}^{(1)}$ is equivalent to

$$\begin{aligned} |u|_{l,\,\alpha;\,\delta}^{(2)} &= \sup_{x,\,y} \min\{1+|x|\,,\,1+|y|\}^{\delta} \frac{\left|\partial^l u(x)-\partial^l u(y)\right|}{|x-y|^{\alpha}} \\ &\quad + \sup_x (1+|x|)^{\delta-l-\alpha} |u(x)|\,. \end{aligned}$$

**Remark 2** Notice that $|x-y| < \gamma 1 \vee (|x| \wedge |y|)$ implies that for each point $a$ on the segment connecting $x$ and $y$

$$(1+|y|)/4 \le 1+|a| \le 2(1+|y|),$$

$$(1+|x|)/4 \le 1+|a| \le 2(1+|x|).$$

The inequality $|x-y| \geq \gamma 1 \vee (|x| \wedge |y|)$ implies that

$$|x-y| \geq \frac{\gamma}{8}(1+|x|+|y|).$$

**Remark 3** (Proposition 2.3.19 in [5]) For any function $u \in C_\delta^{l,\alpha}$ there exists a sequence $u_n$ of functions from $C_0^\infty(\mathbf{R}^d)$ whose partial derivatives up to the order $l$ converge to the corresponding derivatives of $u$ at every point of $\mathbf{R}^d$, and

$$\lim_{n\to\infty} |u_n|_{C_\delta^{l,\alpha}} = |u|_{C_\delta^{l,\alpha}}.$$

Let $\mathcal{D} = \left\{\phi \in C_0^\infty\left(\mathbf{R}^2\right) \text{ and } \phi = 1 \text{ in some neighborhood of } 0\right\}$. Fix $\delta \in (2+\alpha,\, 3+\alpha)$, $\alpha \in (0,1)$. For $d=2$ and fixed $\phi \in \mathcal{D}$ we introduce the space

$$\tilde{C}_{l+\delta}^{l+2,\,\alpha} = \tilde{C}_{l+\delta}^{l+2,\,\alpha}(\mathbf{R}^2) = \left\{\tilde{u} = u + \lambda\left(1-\phi\left(x\right)\right)\ln|x|\,,\; u \in C_{l+\delta}^{l+2,\,\alpha}(\mathbf{R}^2),\; \lambda \in \mathbf{R}^2\right\}.$$

Notice that $\tilde{C}^{l+2,\alpha}_{l+\delta}$ does not depend on a particular $\phi \in \mathcal{D}$. Also, given

$$\tilde{u} = u + \lambda\,(1 - \phi\,(x))\ln|x| \in \tilde{C}^{l+2,\alpha}_{l+\delta},$$

necessarily

$$\lambda = \frac{1}{2\pi}\int \Delta\tilde{u}(y)\,dy,$$

where $\Delta$ is the Laplace operator. Indeed, applying the Laplace operator to the representation of $\tilde{u}$ and integrating by parts,

$$\int \Delta\tilde{u}(y)\,dy = -\lambda\int \phi_{x_i}(y)\frac{y_i}{|y|^2}\,dy = -\lambda\int_0^{2\pi}\int \frac{d}{dr}\phi(r\cos\tau, r\sin\tau)\,dr\,d\tau = 2\pi\lambda.$$

For fixed $\phi$, we introduce the norm in $\tilde{C}^{l+2,\alpha}_{l+\delta}$ by

$$|\tilde{u}|_{\tilde{C}^{l+2,\alpha}_{l+\delta}} = |\lambda| + |u|_{C^{l+2,\alpha}_{l+\delta}},\ \text{if}\ \ \tilde{u} = u + \lambda\,(1 - \phi\,(x))\ln|x|\,.$$

Notice that the norms corresponding to different $\phi$ are equivalent.

In the Appendix, we prove the following statement, regarding the solutions of the Laplace equation in $\mathbf{R}^d$.

**Proposition 1** Let $\delta \in (2+\alpha,\,3+\alpha)$, $\alpha \in (0,1)$. Then for each $l = 0,\,1,\,\ldots$

a) the Laplace operator $\Delta$ is a continuous bijection from $C^{2+l,\alpha}_{l+\delta}$ onto $C^{l,\alpha}_{\delta+l}$, if $d > 2$;

b) the Laplace operator $\Delta$ is a continuous bijection from $\tilde{C}^{2+l,\alpha}_{l+\delta}$ onto $C^{l,\alpha}_{\delta+l}$, if $d = 2$.

**Corollary 1** In the case $d = 2$, the Laplace operator is a continuous bijection from

$$C^{2+l,\alpha}_{l+\delta} = \{u \in \tilde{C}^{2+l,\alpha}_{l+\delta} :\ \int \Delta u\,dx = 0\}\ \text{onto}\ \{f \in C^{l,\alpha}_{l+\delta} :\ \int f\,dx = 0\}.$$

**Remark 4** Given $f \in C^{l,\alpha}_{l+\delta}$,

$$\Delta^{-1}f = u(x) = \int \Gamma(x-y)f(y)\,dy,$$

where $\Gamma$ is the Laplace operator Green's function (see Appendix 1).

### 1.1.2 Decomposition of vector fields

Using Proposition 1 we can decompose a vector field into its divergence free (or solenoidal) and gradient parts.

**Proposition 2** For each $\delta \in (2+\alpha, 3+\alpha)$, $\alpha \in (0,1)$

$$C^{l+1,\alpha}_{\delta+l} = \nabla C^{l+2,\alpha}_{\delta+l} \oplus \left\{h \in C^{l+1,\alpha}_{\delta+l} : \operatorname{div}\,h = 0\right\},\ d > 2,$$

$$C^{l+1,\alpha}_{\delta+l} = \nabla\{u \in \tilde{C}^{l+2,\alpha}_{\delta+l} : \int \Delta u\,dx = 0\} \oplus \left\{h \in C^{l+1,\alpha}_{\delta+l} : \operatorname{div}\,h = 0\right\},\ d = 2.$$

**Proof 1** Indeed, for each $f=\left(f^j\right)_{1\le j\le d}\in C^{l+1,\alpha}_{\delta+l}$ there is a sequence $f^n=(f^{n,j})_{1\le i\le d}\in C^{l+1,\alpha}_{l+1+\delta}$ whose partial derivatives up to the order $l+1$ converge to the corresponding derivatives of $f$ at every point of $\mathbf{R}^d$, and

$$\lim_{n\to\infty}|f^n|_{C^{l+1,\alpha}_{l+\delta}}=|f|_{C^{l+1,\alpha}_{l+\delta}},$$

(see Remark 3). According to Proposition 1, for each $f^n$ there is a unique $v^n=\left(v^{n,j}\right)_{1\le j\le d}\in C^{l+3,\alpha}_{\delta+l+1}$ ($v^n\in\tilde{C}^{l+3,\alpha}_{\delta+l+1}$ in the case $d=2$) such that $\Delta v^{n,j}=f^{n,j}$. Let

$$\mathcal{P}f^n=\left(\sum_i\tfrac{\partial}{\partial x_i}\left(v^{n,j}_{x_i}-v^{n,i}_{x_j}\right)\right)_{1\le j\le d},$$

$$\mathcal{G}f^n=\nabla(\text{div }v^n).$$

Obviously,

$$f^n=\Delta v^n=\mathcal{P}f^n+\mathcal{G}f^n,$$

and

$$\text{div }\mathcal{P}f^n=0.$$

By Remark 4,

$$\begin{aligned}\mathcal{P}f^n&=(\sum_i\frac{\partial}{\partial x_i}\int\Gamma(x-y)(f^{n,j}_{x_i}(y)-f^{n,i}_{x_j}(y))\,dy)_{1\le j\le d}\qquad(3)\\&=(\sum_i\int\Gamma_{x_i}(x-y)(f^{n,j}_{x_i}(y)-f^{n,i}_{x_j}(y))\,dy)_{1\le j\le d}.\end{aligned}$$

Since $v^n_{x_i}(x)=\int\Gamma(x-y)f^n_{x_i}(y)\,dy$, we have

$$\Delta(\nabla v^n)=\nabla f^n,\ \ \lim_{n\to\infty}|\nabla f^n|_{C^{l,\alpha}_{l+\delta}}=|\nabla f|_{C^{l,\alpha}_{l+\delta}},$$

and corresponding derivatives of $f^n$ are converging (Notice that $\int\nabla f(y)\,dy=0$, if $d=2$). Therefore by Proposition 1, $\nabla v^n\in C^{l+2}_{l+\delta}$ , $\sup_n|\nabla v^n|_{C^{l+2}_{l+\delta}}<\infty$. Also the corresponding derivatives of $\nabla v^n$ are converging at each point of $\mathbf{R}^d$. So,

$$\begin{aligned}\mathcal{P}f^n&\to\mathcal{P}f=(\sum_i\frac{\partial}{\partial x_i}\int\Gamma(x-y)(f^j_{x_i}(y)-f^i_{x_j}(y))\,dy)_{1\le j\le d}\in C^{l+1,\alpha}_{l+\delta},\\\mathcal{G}f^n&=\nabla(\text{div }v^n)=\nabla\int\Gamma(x-y)\text{div }f^n(y)\,dy\to\mathcal{G}f,\end{aligned}$$

where

$$\mathcal{G}f=\nabla\int\Gamma(x-y)\text{div }f(y)\,dy\in C^{l+1,\alpha}_{l+\delta}$$

Let $d=2$, $g^n(x)=$div $v^n=\int\Gamma(x-y)\text{div }f^n(y)\,dy$. Then, obviously,

$$\int g^n(x)\,dx=0\text{ and }g^n(x)\to g(x)=\int\Gamma(x-y)\text{ div }f(y)\,dy\in C^{l+2,\alpha}_{l+\delta}.$$

Since $\delta\in(2+\alpha,3+\alpha)$ and $\sup_n|g^n|_{C^{l+2,\alpha}_{l+\delta}}<\infty$, we have $\int g(x)\,dx=0$. So, the statement is true.

**Remark 5** It follows from the proof that

$$\mathcal{P}f(x) = \left(\sum_i \frac{\partial}{\partial x_i}\int \Gamma(x-y)\left(f^j_{x_i}(y) - f^i_{x_j}(y)\right) dy\right)_{1\le j\le d}, \tag{4}$$

$$\mathcal{G}f(x) = \nabla \int \Gamma(x-y)\,\mathrm{div}\, f(y)\, dy,$$

where

$$\Gamma(x-y) = \Gamma^d(x-y) = \Gamma^d(|x-y|) = \begin{cases} |x-y|^{2-d}/d(2-d)\omega_d, & d>2 \\ \frac{1}{2\pi}\ln|x-y|, & d=2, \end{cases}$$

and $\omega_d$ is the volume of a unit ball in $\mathbf{R}^d$.

### 1.1.3 Spaces of diffeomorphisms

Fix $l \ge 0$. Let $\mathbf{B}^{l+1,\alpha}_{l+\alpha} = \{\eta \in C^{l+1,\alpha}_{l+\alpha} : \eta$ is a diffeomorphism and $\inf_x |\det \nabla\eta(x)| > 0\}$.

**Remark 6** a) Let $\eta \in C^{l+1,\alpha}_{l+\alpha}$ and $\inf_x |\det \nabla\eta(x)| > 0$. Then there exist some constants $C,\, c > 0$ such that for all $x,\, y \in \mathbf{R}^d$

$$C|x-y| \ge |\eta(x) - \eta(y)| \ge c|x-y|.$$

So, $\eta, \sigma = \eta^{-1} \in \mathbf{B}^{l+1,\alpha}_{l+\alpha}$ and $\mathbf{B}^{l+1,\alpha}_{l+\alpha} = \{\eta \in C^{l+1,\alpha}_{l+\alpha} : \inf_x |\det \nabla\eta(x)| > 0\}$.
b) Obviously, $\mathbf{B}^{l+1,\alpha}_{l+\alpha}$ is an open subset of $C^{l+1,\alpha}_{l+\alpha}$;

**Lemma 1** Let $D = \{\eta \in \mathbf{B}^{l+1,\alpha}_{l+\alpha} : \sup_x |\eta(x)|(1+|x|)^{-1} \le c_0,\ \sup_x |\nabla\eta(x)| \le c_1,\ \inf_x |\det \nabla\eta(x)| \ge c_2 > 0\}$. Then
a) there exists a constant $c = c(c_1,\, c_2)$ such that for all $\eta \in D$

$$c_1|x-y| \ge |\eta(x) - \eta(y)| \ge c|x-y|. \tag{5}$$

b) there exist some constants $\bar{c} = \bar{c}(c_0,\, c_1,\, c_2)$, $\bar{C} = \bar{C}(c_0,\, c_1)$ such that for all $\eta \in D$

$$\bar{c}(1+|x|) \le 1 + |\eta(x)| \le \bar{C}(1+|x|). \tag{6}$$

**Proof 2** Obviously there exists a constant $c = c(c_1,\, c_2)$ such that for all $\eta \in D$ $\sup_x \left|(\nabla\eta)^{-1}(x)\right| \le c$. So the first part of the statement follows.
Denote $|\eta|_0 = \sup_x (1+|x|)^{-1}|\eta(x)|$. It follows from (5) applied for $y = 0$

$$c|x| - |\eta(0)| \le |\eta(x)| \le c_1|x| + |\eta(0)|$$

for all $\eta \in D$. Since $|\eta|_0 \ge |\eta(0)|$ we have for all $\eta \in D$

$$\begin{aligned} &|\eta|_0 \ge (1/2)c,\ 1 + |\eta(x)| \le \bar{C}(1+|x|), \text{ and} \\ &c|x| + |\eta|_0 \le |\eta(x)| + 2|\eta|_0. \end{aligned} \tag{7}$$

where $\bar{C} = \max\{c_1,\, 1+c_0\}$. Thus for all $\eta \in D$

$$\bar{c}(1+|x|) \le 1 + |\eta(x)|,$$

where $\bar{c} = (1/2)c\max\{1,\, 2c_0\}^{-1}$.

Using this Lemma and Remark 1 we derive easily the following two statements.

**Corollary 2** Let $D = \{\eta \in \mathbf{B}_{l+\alpha}^{l+1,\alpha} : \sup_x |\eta(x)| (1+|x|)^{-1} \leq c_0,\ \sup_x |\nabla\eta(x)| \leq c_1,\ \inf_x |\det \nabla\eta(x)| \geq c_2 > 0\}$, $D^{-1} = \{\sigma = \eta^{-1} : \eta \in D\}$. Then there exist some constants $k_i = k_i(c_0, c_1, c_2) > 0$, $i = 0, 1, 2$ such that for all $\sigma \in D^{-1}$

$$\sup_x |\sigma(x)| (1+|x|)^{-1} \leq k_0, \sup_x |\nabla\sigma(x)| \leq k_1,\ \inf_x |\det \nabla\sigma(x)| \geq k_2.$$

**Corollary 3** Let $B_{r_1} = \{v \in C_{\delta+l}^{l+1,\alpha} : |v|_{C_{\delta+l}^{l+1,\alpha}} \leq r_1\}$, $D_{r_2,r} = \{\eta \in C_{l+\alpha}^{l+1,\alpha} : |\eta|_{C_{l+\alpha}^{l+1,\alpha}} \leq r_2,\ \inf_x |\det \nabla\eta(x)| \geq r > 0\}$. Then there exists $R = R(r_1, r_2, r)$ such that $B_{r_1} \circ D_{r_2,\, r} = \{v \circ \eta : v \in B_{r_1},\ \eta \in D_{r_2,r}\} \subset B_R$.

## 1.2 Derivation of a new equation

Assume $u(t,x)$ satisfies (1) and $\eta(t,x)$ is its flow, i.e. (2) holds. Then

$$\begin{cases} \partial_t \eta_{x_l}^k(t) = u_{x_m}^k(t, \eta(t))\, \eta_{x_l}^m(t), & k, l = 1, \dots, d \\ \eta_{x_l}^k(0, x) = \delta_{kl}.. \end{cases} \tag{8}$$

We now derive an equation for $\eta(t) = \eta(t,x)$ equivalent to (1). We have by chain rule and (8)

$$\frac{d}{dt}\left((u^k(t,\eta(t)) - \epsilon W_t^k)\eta_{x_l}^k(t)\right) = (\partial_t u^k(t,\eta(t)) + u_{x_m}^k(t,\eta(t))\, \partial_t \eta^m(t) -$$

$$-\epsilon \dot{W}_t^k \eta_{x_l}^k(t)) + \left(u^k(t,\eta(t)) - \epsilon W_t^k\right) \partial_t \eta_{x_l}^k(t) = -p_{x_k}(t,\eta(t))\, \eta_{x_l}^k(t) +$$

$$+(u^k(t,\eta(t)) - \epsilon W_t^k) u_{x_m}^k(t,\eta(t))\, \eta_{x_l}^m(t) = \frac{\partial}{\partial x_l}(-p(t,\eta(t)) + \tfrac{1}{2}|u(t,\eta(t)) - \epsilon W_t|^2).$$

Taking solenoidal projection of both sides of this equality we have

$$\frac{d}{dt}\mathcal{P}\left((u^k(t,\eta(t)) - \epsilon W_t^k)\eta_{x_l}^k(t)\right) = 0.$$

Since $\mathcal{P}\left(u^k(0,\eta(0))\, \eta_{x_l}^k(0)\right) = \mathcal{P}h = h$, it follows that for some scalar field $G(t,x)$

$$(u^k(t,\eta(t,x)) - \epsilon W_t^k)\eta_{x_l}^k(t,x) = h^l(x) + G_{x_l}(t,x).$$

Denote $\sigma(t) = \sigma(t,x) = \eta^{-1}(t) = \eta^{-1}(t,x)$. Then

$$(u^k(t,x) - \epsilon W_t^k)\eta_{x_l}^k(t,\sigma(t)) = h^l(\sigma(t)) + G_{x_l}(t,\sigma(t)),$$

and therefore

$$u^l(t,x) = h^k(\sigma(t))\, \sigma_{x_l}^k(t) + G_{x_k}(t,\sigma(t))\, \sigma_{x_l}^k(t) + \epsilon W_t =$$

$$= h^k(\sigma(t))\, \sigma_{x_l}^k(t) + (G(t,\sigma(t)))_{x_l} + \epsilon W_t.$$

Since $\operatorname{div} u = 0$, we have

$$u(t,x) = \mathcal{P}\left(\left(h^k(\sigma(t))\, \sigma_{x_l}^k(t)\right)_l\right) + \epsilon W_t. \tag{9}$$

Given $h \in C_{l+\delta}^{l+1,\alpha}$, consider the following function $\omega : \mathbf{B}_{l+\alpha}^{l+1,\alpha} \longrightarrow C_{l+\delta}^{l,\alpha}$ defined by

$$\eta \longmapsto \omega(\eta) = (\nabla \eta(y)^{-1})^{*} \nabla h(y) \nabla \eta(y)^{-1} J\eta(y),$$

where $J\eta(y)$ is the Jacobian determinant of $\eta(y)$. So, according to (2), (9) and (4) (see Remark 5), the following equation for $\eta(t)$ holds:

$$\begin{aligned} &\partial_t \eta(t) = F(\eta(t)) + \epsilon W_t, \\ &\eta(0,x) = x, \end{aligned} \tag{10}$$

where $F = \left(F^l\right)_{1 \leq l \leq d}$,

$$F^l(\eta) = \sum_i \int \Gamma_{x_i}(\eta(x) - \eta(y)) \left[\omega(\eta)(y)_{li} - \omega(\eta)(y)_{il}\right] dy. \tag{11}$$

Indeed, by (4) (see Remark 5),

$$\begin{aligned}
&\mathcal{P}\left(\left(h^k(\sigma(t))\,\sigma_{x_l}^k(t)\right)_l\right) \\
= \;& (\sum_{i \neq l} \int \Gamma_{x_i}(x-y)\left((h^k(\sigma(t,y))\,\sigma_{x_l}^k(t,y))_{x_i} - (h^k(\sigma(t,y))\,\sigma_{x_i}^k(t,y))_{x_l}\right) dy) \\
= \;& (\sum_{i \neq l} \int \Gamma_{x_i}(x-y)\,(h_{x_m}^k(\sigma(t,y))\,\sigma_{x_i}^m(t,y)\,\sigma_{x_l}^k(t,y) \\
& - h_{x_m}^k(\sigma(t,y))\,\sigma_{x_l}^m(t,y)\,\sigma_{x_i}^k(t,y))\,dy) \\
= \;& (\int \{[(\nabla\sigma(t))^{*}\nabla h(\sigma(t))\,\nabla\sigma(t)]_{li} \\
& -[(\nabla\sigma(t))^{*}\nabla h(\sigma(t))\,\nabla\sigma(t)]_{il}\}\Gamma_{x_i}(x-y)\;\; dy) \\
= \;& (\int \{[(\nabla\sigma(t,\eta(t,y)))^{*}\nabla h(y)\,\nabla\sigma(t.\eta(t,y))]_{li} \\
& -[(\nabla\sigma(t,\eta(t,y)))^{*}\nabla h(y)\,\nabla\sigma(t.\eta(t,y))]_{il}\}\Gamma_{x_i}(x-\eta(t,y))\,J\eta(t,y)\;dy.
\end{aligned}$$

So, (10) follows from (9) and (2).

The above derivation shows that if $u(t,x)$ satisfies (1) with $u(0,x) = h(x)$, and if $\eta$ is the flow of $u$, then $\eta$ satisfies (10) with $\eta(0)$, the identity. Reading the same derivation backwards, we find that $\eta$ satisfies (10) implies that in order for $u$ to verify (1) it must be defined by

$$u(t,x) = H(t,x) + \epsilon W_t, \tag{12}$$

where

$$H^l(t,x) = \sum_i \int \Gamma_{x_i}(x - \eta(t,y)) \left[\omega(\eta(t))(y)_{li} - \omega(\eta(t))(y)_{il}\right] dy. \tag{13}$$

Thus solving (1) is equivalent to solving (10).

**Remark 7** According to Proposition 7 $F(\eta)$ is $C^{l+1,\alpha}_{l+\delta}$-valued function on $\mathbf{B}^{l+1,\alpha}_{l+\alpha}$. Obviously it is continuous and locally bounded.

**Remark 8** If $\varepsilon = 0$, then the equations are deterministic. Denote $u_0$, $\eta_0$ the corresponding solutions of (1) and (10). The formulas (10) and (11) show that

$$\eta(t) = \eta_0(t) + \epsilon \int_0^t W_s \, ds. \tag{14}$$

Notice the trajectories of particles in this case are nice functions ($\eta(t)$ is differentiable in $t$), $\nabla\eta(t) = \nabla\eta_0(t)$ is deterministic, $\sigma(t,x) = \sigma_0\left(t, x - \epsilon \int_0^t W_s \, ds\right)$. Also,

$$u(t,x) = u_0\left(t, x - \epsilon \int_0^t W_s \, ds\right) + \epsilon W_t. \tag{15}$$

So, it is not true that $u(t,x) = u_0(t,x) + \epsilon W_t$.

Indeed, $F(\eta + a) = F(\eta)$ for each $a \in \mathbf{R}^d$. So, $\eta(t) - \varepsilon \int_0^t W_s \, ds$ is a solution of (10) with $\varepsilon = 0$, and (14) holds. Let $H_0(t,x)$ be a function defined by (13) with $\epsilon = 0$. By (14)

$$H(t,x) = H_0\left(t, x - \epsilon \int_0^t W_s \, ds\right), \; u(t,x) = u_0\left(t, x - \epsilon \int_0^t W_s \, ds\right) + \epsilon W_t.$$

### 1.3 Local result: solving (10)

The last Remark and trivial formulas (14), (15) determine that we start with deterministic equation (10), i.e. $\epsilon = 0$.

**Proposition 3** Let $l \geq 0$, $\delta \in (2+\alpha, 3+\alpha)$, $h \in C^{l+1,\alpha}_{l+\delta}, \operatorname{div} h = 0$. Then there exists a time interval $(-\lambda, \kappa)$ such that $(10), \epsilon = 0$, has a unique solution $\eta_0(t) \in \mathbf{B}^{l+1,\alpha}_{l+\alpha}$, $t \in (-\lambda, \kappa)$ such that $\eta_0(0)$ is the identity. Moreover, it depends smoothly on $h$.

**Proof 3** We consider $(10), \epsilon = 0$, as deterministic ODE in Banach space $C^{l+1,\alpha}_{l+\alpha}$. We mentioned already (see Remark 7) that $F(\eta)$ is continuous $C^{l+1,\alpha}_{l+\delta}$-valued function on $C^{l+1,\alpha}_{l+\alpha}$ (notice $C^{l+1,\alpha}_{l+\delta} \subseteq C^{l+1,\alpha}_{l+\alpha}$. The local existence and uniqueness follow from the smoothness of $F(\eta)$. Indeed, given $h \in C^{l+1,\alpha}_{l+\delta}$ the following function $\omega : \mathbf{B}^{l+1,\alpha}_{l+\alpha} \longrightarrow C^{l,\alpha}_{l+\delta}$ defined by

$$\eta \longmapsto \omega(\eta) = (\nabla\eta(y)^{-1})^* \nabla h(y) \, \nabla\eta(y)^{-1} \, J\eta(y),$$

is smooth. It is smooth (linear) in $h$ as well. The function $G(\eta, w) = (G^l(\eta, w))_{1 \leq l \leq d}$ defined by

$$w \longmapsto G^l(\eta, w) = \int \Gamma_{x_i}(\eta(x) - \eta(y)) \, w^{li}(y) \, dy$$

is linear in $w = (w^{li})_{1 \leq l, i \leq d}$ and bounded from $C^{l,\alpha}_{l+\delta}$ to $C^{l+1,\alpha}_{l+\delta}$ (by Proposition 7, Appendix 1). Now we show that $G(\eta, w)$ depends smoothly on $\eta \in \mathbf{B}^{l+1,\alpha}_{l+\alpha}$. Differentiating $G(\eta, w)$ with respect to $\eta$ in direction $v \in C_{l+\alpha}^{l+1,\alpha}$ we get:

$$\begin{aligned}
\partial_\eta G(\eta, w) \cdot v &= \\
&= \int \Gamma_{x_i x_j} \left(\eta\left(x\right) - \eta\left(y\right)\right) \left(v^j\left(y\right) - v^j\left(x\right)\right) w^{li}\left(y\right)\, dy \\
&= \int \Gamma_{x_i x_j} \left(\eta\left(x\right) - y\right) \left(v^j(\sigma(y)) - v^j(x)\right) w^{li}\left(\sigma(y)\right) J\eta(y)\, dy = H^l(\eta(x)),
\end{aligned}$$

where

$$H^l(x) = \int \Gamma_{x_i x_j} \left(x - y\right) \left(v^j(\sigma(y)) - v^j(\sigma(x)\right) w^{li}\left(\sigma(y)\right) J\eta(y)\, dy.$$

By Appendix 2 Proposition 8 $|H|_{C_{l+\delta}^{l+1,\alpha}} \leq C|v \circ \sigma|_{C_{l+\alpha}^{l+1,\alpha}} |w \circ \sigma J\eta|_{C_{l+\delta}^{l,\alpha}}$, if $d > 2$, and

$$|H|_{C_{l+\alpha}^{l+1,\alpha}} \leq C|v \circ \sigma|_{C_{l+\alpha}^{l+1,\alpha}} |w \circ \sigma J\eta|_{C_{l+\delta}^{l,\alpha}}, \text{ if } d = 2.$$

Since

$$F^l\left(\eta\right) = \sum_i \int \Gamma_{x_i}\left(\eta\left(x\right) - \eta\left(y\right)\right) \left[\omega\left(\eta\right)\left(y\right)_{li} - \omega\left(\eta\right)\left(y\right)_{il}\right]\, dy,$$

the function $F\left(\eta\right)$ from $\mathbf{B}_{l+\alpha}^{l+1,\alpha}$ into $C_{l+\delta}^{l,\alpha}$ (or $C_{l+\alpha}^{l,\alpha}$, if $d = 2$) is smooth, and the statement follows.

**Remark 9** Notice that the function

$$K : \mathbf{B}_{l+\alpha}^{l+1,\alpha} \to \mathbf{B}_{l+\alpha}^{l+1,\alpha};\ \eta \longmapsto \sigma = \eta^{-1}$$

is smooth as well.

Indeed, $\left(\eta + v\right)^{-1}\left(\eta\right) - \eta^{-1}\left(\eta\right) = \left(\eta + v\right)^{-1}\left(\eta\right) - \left(\eta + v\right)^{-1}\left(\eta + v\right) = -D(\eta + v)^{-1}\left(\eta\right) v \approx -\left(\nabla\eta\left(\eta\right)\right)^{-1} v$. So,

$$\left(\eta + \mathbf{v}\right)^{-\mathbf{1}} - \eta^{-\mathbf{1}} \approx -(\nabla\eta)^{-\mathbf{1}}\mathbf{v}\left(\eta^{-\mathbf{1}}\right),$$

and the first derivative $DK\left(\eta\right) \cdot v = -(\nabla\eta)^{-\mathbf{1}} v\left(\eta^{-\mathbf{1}}\right)$. Similarly the higher derivatives formulas can be obtained.

Having in mind Remark 8 and formula (14), we derive easily the following obvious statement.

**Corollary 4** Let $l \geq 0$, $\delta \in (2 + \alpha, 3 + \alpha)$, $h \in C_{l+\delta}^{l+1,\alpha}$, $\operatorname{div} h = 0$. Let $(-\lambda, \kappa)$ be a time interval specified in Proposition 3.

Then for each $\epsilon$ (10) has a unique solution $\eta\left(t\right) \in \mathbf{B}_{l+\alpha}^{l+1,\alpha}$, $t \in (-\lambda, \kappa)$ such that $\eta\left(0\right)$ is the identity. Moreover, it depends smoothly on $h$, and (14) holds.

Now, form the equivalence of (1) and (10) we have obviously

**Proposition 4** Let $l \geq 0$, $\delta \in (2 + \alpha, 3 + \alpha)$, $h \in C_{l+\delta}^{l+1,\alpha}$, $\operatorname{div} h = 0$. Then there exists a time interval $(-\lambda, \kappa)$ such that (10), $\epsilon = 0$, has a unique solution $C_{l+\delta}^{l+1,\alpha}$-solution $u_0\left(t, x\right)$, $t \in (-\lambda, \kappa)$.

**Corollary 5** Let $l \geq 0$, $\delta \in (2 + \alpha, 3 + \alpha)$, $h \in C_{l+\delta}^{l+1,\alpha}$, $\operatorname{div} h = 0$. Let $(-\lambda, \kappa)$ be the time interval specified in Proposition 4.

Then for each $\epsilon$ (1) has a unique solution $C_{l+\delta}^{l+1,\alpha}$-solution $u\left(t, x\right)$, $t \in (-\lambda, \kappa)$.

### 1.3.1 The case $d = 2$. Global result

Now we prove that for $d = 2$, $\lambda = \kappa = \infty$.

**Proposition 5** Let $d = 2$, $l \geq 0$, $\delta \in (2+\alpha, 3+\alpha)$, $h \in C_{l+\delta}^{l+1,\alpha}$, $\operatorname{div} h = 0$. Then (10), $\epsilon = 0$, has a unique solution $\eta_0(t) \in \mathbf{B}_{l+\alpha}^{l+1,\alpha}$ defined for all $t \in (-\infty, \infty)$.

**Proof 4** Since (10) is a deterministic ODE on $\mathbf{B}_{l+\alpha}^{l+1,\alpha}$, to show that a solution exists for all $t$, it is enough to show that the solution $\eta(t)$ remains bounded in $C_{l+\alpha}^{l+1,\alpha}$-norm on any finite time interval. To do this we will need again some estimates of $F(\eta)$. First of all we notice that the Jacobian determinant $J\eta(t) = 1$ for all $t$. It simply satisfies a linear ODE

$$\frac{d}{dt} J\eta(t) = \operatorname{div} u(t, \eta(t))\ J\eta(t)$$

with the initial condition $J\eta(0) = 1$. Then $J\sigma(t) = 1$ as well. So,

$$\begin{aligned} F^1(\eta) &= \int \Gamma_{x_2}(\eta - y)\, h_{x_2}^1(\sigma(t,y)) - h_{x_1}^2(\sigma(t,y))\, dy \\ &= \int \Gamma_{x_2}(\eta(t,x) - \eta(t,y))\,(h_{x_2}^1(y) - h_{x_1}^2(y))\ dy, \end{aligned} \tag{16}$$

and similarly

$$F^2(\eta) = \int \Gamma_{x_1}(\eta(t,x) - \eta(t,y))\,(h_{x_1}^2(y) - h_{x_2}^1(y)))\ dy. \tag{17}$$

*Basic estimates:*

1) Denoting $H^{ij}(y) = h_{x_j}^i(y) - h_{x_i}^j(y)$, we find that

$$\begin{aligned} &\left|F^2(\eta)\right| \leq \left|\int \Gamma_{x_1}(\eta(x) - \eta(y))\, H^{21}(y)\ dy\right| \leq \\ &\leq C \int |\eta(x) - \eta(y)|^{-1} (1+|y|)^{\alpha-\delta}\ dy \leq \\ &\leq \int_{|\eta(x)-\eta(y)|\leq 1} |\eta(x) - \eta(y)|^{-1}\ dy + \int_{|\eta(x)-\eta(y)|>1} (1+|y|)^{\alpha-\delta}\ dy \leq \\ &\leq C \int_{|z|\leq 1} |z|^{-1}\ dz + \int (1+|y|)^{\alpha-\delta}\ dy < \infty. \end{aligned}$$

Thus for each $T$ there is $C$ such that for all $t \leq T$

$$\begin{aligned} &|\eta(t)|_0 = \sup_x |\eta(t,x)|\,(1+|x|)^{-1} \leq C, \\ &\sup_x |\eta(t,x) - x| \leq C. \end{aligned} \tag{18}$$

Then for the same constant $C$

$$\begin{aligned} &|\eta(t)|_0 = \sup_x |x|\,(1+|\sigma(t,x)|)^{-1} \leq C, \\ &\sup_x |\sigma(t,x) - x| \leq C. \end{aligned} \tag{19}$$

2) Estimate of $\nabla F(\eta)$.

Since by chain rule $\nabla (F(\eta)) = \nabla F(\eta) \nabla \eta$ and $F$ is defined by (16), (17), we need a "good" estimate of the second derivative of $w(x) = \int \Gamma(x-y) g(\sigma(y)) \, dy$, where $g \in C_{l+\delta}^{l,\alpha}$. By Lemma 4.2 in [6] we have

$$\begin{aligned}
\partial_{ij}^2 w(x) &= \int_{E(x)} \partial_{ij}^2 \Gamma(x-y) \left[ (1+|\sigma(y)|)^{\delta} \left( g(\sigma(y)) - g(\sigma(x)) \right) \right] (1+ \\
&\quad + |\sigma(y)|)^{-\delta} \, dy + \int_{E(x)^c} \partial_{ij}^2 \Gamma(x-y) (1+ \\
&\quad + |\sigma(y)|^{\delta-\alpha}) g(\sigma(y)) (1+|\sigma(y)|)^{-\delta+\alpha} \, dy \\
&\quad - g(\sigma(x)) \int_{\partial E(x)} \Gamma_{x_j}(x-y) n_i(y) \, S(dy),
\end{aligned}$$

where $E(x) = \{y : |x-y| \le \max\{1, \min\{|x|, |y|\}\}\}$.
Notice, $|x-y| \le \max\{1, \min\{|x|, |y|\}\}$ implies that

$$(1+|y|)/2 \le 1+|x| \le 2(1+|y|). \tag{20}$$

¿From the opposite inequality $|x-y| > \max\{1, \min\{|x|, |y|\}\}$ it follows

$$|x-y| \ge (|x|+|y|)/8 + 1/2. \tag{21}$$

By (18), (19), (20), (21)

$$\begin{aligned}
&\left| \partial_{ij}^2 w \right| \le C \textstyle\int_{|x-y| \le \max\{1, \min\{|x|,|y|\}\}} |x-y|^{-2} |\sigma(y) - \sigma(x)|^{\alpha} (1+|y|)^{-\delta} \, dy + C_1, \\
&\textstyle\int_{|x-y| \le \max\{1, \min\{|x|,|y|\}\}} |x-y|^{-2} |\sigma(y) - \sigma(x)|^{\alpha} (1+|y|)^{-\delta} \, dy \le \\
&\le |\nabla \eta|_{\infty} \textstyle\int_{|x-y| \le |\nabla \eta|_{\infty}^{-1}} |x-y|^{-2+\alpha} \, dy + \\
&+ \textstyle\int_{|\nabla \eta|_{\infty}^{-1} \le |x-y| \le \max\{1, \min\{|x|,|y|\}\}} |x-y|^{-2} (C + |x-y|^{\alpha}) (1+|y|)^{-\delta} \, dy \le \\
&\le \textstyle\int_{|\nabla \eta|_{\infty}^{-1} \le |x-y| \le 1} |x-y|^{-2} \, dy + \int_{1 \le |x-y| \le \min\{|x|,|y|\}} \dots \le C(1 + \ln(1 + |\nabla \eta|_{\infty})),
\end{aligned} \tag{22}$$

where $|\nabla \eta(t)|_{\infty} = \sup_x |\nabla \eta(t,x)|$. This estimate is "good" enough (see [3]) in order to obtain that for each $T$ there is $C$ such that for all $t$ $|\nabla \eta(t)|_{\infty} \le C$. Since $J\eta(t) = 1$, $|\nabla \sigma(t)|_{\infty} \le C$ as well. So,

$$|\eta(t,x) - \eta(t,y)| \le C|x-y|,$$

$$|\sigma(t,x) - \sigma(t,y)| \le C|x-y|,$$

and

$$\begin{aligned}
C^{-1}|x-y| &\le |\eta(t,x) - \eta(t,y)| \le C|x-y|, \\
C^{-1}|x-y| &\le |\sigma(t,x) - \sigma(t,y)| \le C|x-y|.
\end{aligned} \tag{23}$$

3) Using (18), (19), (23), the estimate of $|\eta(t)|_{C_{l+\alpha}^{l+1,\alpha}}$ is now straightforward (by induction).

## 2 Stochastic Euler equation

Let $u, p$ be a solution of (1) and $\eta$ be a solution of (10) in some deterministic interval $[0, \kappa)$. Define

$$\bar{\eta}(t, x) = \eta(t, x) + \mu B_t, \tag{1}$$

where $B$ is a standard Wiener process independent of $W$. It means that besides the random forces acting by the second Newton law there are some other reasons deflecting individual trajectories away. Instead of $u(t,x) = H(t, \eta(t, \sigma(t,x)) + \epsilon W_t = u(t, \eta(t, \sigma(t,x)))$, we go with $\bar{u}(t,x) = u(t, \eta(t, \bar{\sigma}(t,x)))$, where $\bar{\sigma}(t,x) = \bar{\eta}^{-1}(t,x)$.

**Proposition 6** Let $l \geq 1, \delta \in (2+\alpha,\, 3+\alpha)$, $h \in C^{l+1,\,\alpha}_{l+\delta}$, $\operatorname{div} h = 0$. Then in the interval $[0, \kappa)$ $\bar{u}(t, x)$ is a solution of the following SPDE:

$$\begin{cases} \partial_t \bar{u}(t, x) = \\ = -\bar{u}^k(t, x)\partial_k \bar{u}(t, x) - \mu \partial_k \bar{u}(t, x) \dot{B}^k_t - \nabla \bar{p}(t, x) + (1/2)\mu^2 \Delta \bar{u}(t, x) + \epsilon \dot{W}_t, \\ \\ \bar{u}(0, x) = h(x),\ \operatorname{div} \bar{u} = 0. \end{cases} \tag{2}$$

**Proof 5** For the inverse $\bar{\sigma}(t, x)$ of $\bar{\eta}(t, \cdot)$ we have

$$x = \eta(t, \bar{\sigma}(t, x)) + \mu B_t,$$

i.e. $\eta(t, \bar{\sigma}(t, x)) = x - \mu B_t$. Thus $\bar{u}(t, x) = u(t, x - \mu B_t)$, and by Ito formula we have

$$\partial_t \bar{u}(t, x) + \bar{u}^k(t, x)\partial_k \bar{u}(t, x) + \mu \partial_k \bar{u}(t, x)\dot{B}^k_t - (1/2)\mu^2 \Delta \bar{u}(t, x) + \nabla \bar{p}(t, x) = \epsilon W_t,$$

where $\bar{p}(t, x) = p(t, x - \epsilon W_t)$. Also, obviously $\operatorname{div} \bar{u} = 0$, $\bar{u}(0, \cdot) = h(x)$.

**Corollary 6** Let $l \geq 1$, $\delta \in (2+\alpha,\, 3+\alpha)$, $h \in C^{l+1,\,\alpha}_{l+\delta}$, $\operatorname{div} h = 0$, and $\bar{u} = \bar{u}^{\epsilon,\mu}$ be a solution of (2) constructed in Proposition 6. Then $\mathbf{P}$-a.s. for each $t \in [0, \kappa)$

$$\lim_{\mu \to 0} \sup_{s \leq t,\, x} |\bar{u}^{\epsilon,\mu}(s, x) - u(s, x)| = 0,$$

where $u$ is a solution in $[0, \kappa)$ of (1).

**Proof 6** Indeed, as we noticed $\bar{u}^{\epsilon}(t, x) = u(t, x - \epsilon W_t)$, and the statement follows immediately.

**Remark 10** Denote $u_0$, $\eta_0$ the corresponding solutions of (1) and (10) with $\epsilon = 0$. By Remark 8 and definition of $\bar{\eta}$ we have

$$\bar{\eta}(t) = \eta_0(t) + \epsilon \int_0^t W_s\, ds + \mu B_t. \tag{3}$$

Note that the trajectories of particles in this case are not as regular as in (14) ($\bar{\eta}(t)$ is not differentiable in $t$), $\nabla \bar{\eta}(t) = \nabla \eta_0(t)$ is deterministic. Also,

$$\bar{u}(t,x) = u_0\left(t, x - \epsilon \int_0^t W_s\, ds - \mu B_t\right) + \epsilon W_t. \tag{4}$$

# 3 Appendix 1: Laplace equation in $\mathbf{R}^d$

Consider the following equation in the whole space $\mathbf{R}^d$, $d \geq 2$,

$$\Delta u(x) = f(x),\ x \in \mathbf{R}^d, \tag{1}$$

where $f \in C_0^\infty(\mathbf{R}^d)$. It is well known that

$$w(x) = w^d\,(x) = \int \Gamma^d(x-y) f(y)\,dy$$

is a solution of (1), where

$$\Gamma\,(x-y) = \Gamma^d(x-y) = \Gamma^d(|x-y|) = \begin{cases} |x-y|^{2-d}/d(2-d)\omega_d, & d > 2 \\ \frac{1}{2\pi}\ln|x-y|\,, & d = 2. \end{cases}.$$

It is rather straightforward to show that

$$\partial_i w(x) = \int \partial_i \Gamma(x-y) f(y)\,dy.$$

We notice here also that

$$\begin{gathered} (1+|x|)^{\beta-1}\,|\partial_i \Gamma(x-y)|\,(1+|y|)^{-\beta} \leq \\ \leq C(1+|x|)^{\beta-1}\,|x-y|^{1-d}\,(1+|y|)^{-\beta},\ d \geq 2; \\ (1+|x|)^{\beta-2}\,|\Gamma(x-y)|\,(1+|y|)^{-\beta} \leq \\ \leq C(1+|x|)^{\beta-2}\,|x-y|^{2-d}\,(1+|y|)^{-\beta},\ d > 2. \end{gathered} \tag{2}$$

Introduce the space $C_\delta^{l,\,\alpha}(\mathbf{R}^d)$ (see [5]), $l = 0,\,1,\,\ldots\,,\ \alpha \in (0,\,1)$, $\delta \in \mathbf{R}$, of $l$ times continuously differentiable functions $u$ on $\mathbf{R}^d$ with finite norm

$$|u|_{C_\delta^{l,\,\alpha}} = \sup_{x,y} |x-y|^{-\alpha} \textstyle\sum_{k=0}^l \left|(1+|x|)^{\delta-l+k}\partial^k u(x) - (1+|y|)^{\delta-l+k}\partial^k u(y)\right| +$$

$$+ \sup_x \textstyle\sum_{k=0}^l (1+|x|)^{\delta-l+k-\alpha}\left|\partial^k u(x)\right|.$$

**Remark 11** It is easy to see (cf. Proposition 2.3.16 in [5]) that the norm $|u|_{C_\delta^{l,\,\alpha}}$ is equivalent to the norm

$$|u|_{l,\,\alpha;\,\delta} = \sup_{x,y} \min\{1+|x|\,,\,1+|y|\}^\delta \frac{\left|\partial^l u(x) - \partial^l u(y)\right|}{|x-y|^\alpha} +$$

$$+ \sup_x \textstyle\sum_{k=0}^l (1+|x|)^{\delta-l+k-\alpha}\left|\partial^k u(x)\right|.$$

Notice that obviously

$$\begin{gathered} |u|_{l,\alpha;\delta} = \sup_{R>0}\sup_{|x|\leq R} \textstyle\sum_{k=0}^l (1+|x|)^{\delta-l+k-\alpha}\left|\partial^k u(x)\right| + \\ + \sup_{R>0}\sup_{|x|\leq R,|y|\leq R} \min\{1+|x|\,,\,1+|y|\}^\delta \frac{\left|\partial^l u(x) - \partial^l u(y)\right|}{|x-y|^\alpha}. \end{gathered} \tag{3}$$

Let $\mathcal{D} = \left\{ \phi \in C_0^\infty \left( \mathbf{R}^2 \right) \text{ and } \phi = 1 \text{ in some neighbourhood of } 0 \right\}$. Fix $\delta \in (2+\alpha,\, 3+\alpha)$, $\alpha \in (0,1)$. For $d = 2$ and $\phi \in \mathcal{D}$ we introduce the space

$$\tilde{C}_\delta^{l,\,\alpha}(\mathbf{R}^2) = \left\{ \tilde{u} = u + \lambda \left(1 - \phi\left(x\right)\right) \ln |x| \,,\, u \in C_\delta^{l,\,\alpha}(\mathbf{R}^2),\, \lambda \in \mathbf{R} \right\},$$

with a norm

$$|\tilde{u}|_{\tilde{C}_\delta^{l,\alpha}} = |u|_{C_\delta^{l,\alpha}} + |\lambda|.$$

Notice that $\tilde{C}_\delta^{l,\,\alpha}(\mathbf{R}^2)$ does not depend on a particular $\phi \in \mathcal{D}$ and all norms are equivalent.

**Proposition 7** Let $\delta \in (2+\alpha,\, 3+\alpha)$, $\alpha \in (0,1)$. Then for each $l = 0,\, 1,\, \ldots$

a) the Laplace operator $\Delta$ is a continuous bijection from $C_{l+\delta}^{2+l,\,\alpha}$ onto $C_{\delta+l}^{l,\,\alpha}$, if $d > 2$;

b) the Laplace operator $\Delta$ is a continuous bijection from $\tilde{C}_{l+\delta}^{2+l,\,\alpha}$ onto $C_{\delta+l}^{l,\,\alpha}$, if $d = 2$.

**Proof 7** Let

$$\tilde{w}\left(x\right) = \tilde{w}^d\left(x\right) = \begin{cases} w^d\left(x\right), & \text{if } d > 2, \\ w^2\left(x\right) - \Gamma^2\left(x\right)\left(1 - \phi\left(x\right)\right) \int f\left(y\right)\, dy, & \text{if } d = 2. \end{cases},$$

and

$$\tilde{f}\left(x\right) = \tilde{f}^d\left(x\right) = \begin{cases} f\left(x\right), & \text{if } d > 2, \\ f\left(x\right) - \Delta(\Gamma^2\left(x\right)\left(1 - \phi\left(x\right)\right) \int f\left(y\right)\, dy, & \text{if } d = 2. \end{cases}$$

Notice (for large $x$)

$$\begin{aligned} |\tilde{w}^2\left(x\right)| &\leq \int |\ln|x-y| - (1 - \phi\left(x\right) \ln|x|)|\, |f\left(y\right)| \, dy = \\ &= |x|^2 \int |\ln|x|\, \phi\left(x\right) + \ln|z-y||\, |f\left(|x|\, y\right)|\, dy \leq \\ &\leq \left[ |x|^2 \ln|x|\, \phi\left(x\right) \int \tfrac{1}{(1+|x||y|)^{\delta-\alpha}}\, dy + \int \tfrac{|\ln|z-y| - \ln|z||}{(1+|x||y|)^{\delta-\alpha}}\, dy \right] |f|_{C_{\delta+l}^{0,\alpha}}, \end{aligned} \tag{4}$$

where $z = x/\left|x\right|$. So,

$$\sup_x \left(1+|x|\right)^{\delta-2-\alpha} \left|\tilde{w}^2\left(x\right)\right| \leq C \sup_x \left(1+|x|\right)^{\delta-\alpha} \left|\tilde{f}\left(x\right)\right|. \tag{5}$$

Let

$$\begin{aligned} U_0 &= \{x : |x| < 2\},\, V_0 = \{x : |x| < 4\}, \\ U &= \{x : 1/2 < |x| < 2\},\, V = \{x : 1/4 < |x| < 4\}. \end{aligned}$$

By inner Hölder estimates for each $l = 0,\, 1,\, \ldots$ there exist some constants $C$ such that

$$\begin{aligned} |\tilde{w}|_{2+l,\,\alpha;\,U} &\leq C(\left|\tilde{f}\right|_{l+\alpha,\,V} + |\tilde{w}|_{0,\,V}), \\ |\tilde{w}|_{2+l,\,\alpha;\,U_0} &\leq C(\left|\tilde{f}\right|_{l+\alpha,\,V_0} + |\tilde{w}|_{0,\,V_0}) \end{aligned} \tag{6}$$

where

$$|g|_{k,\alpha;\,B} = |g|_{0;\,B} + \left\langle \partial^k g \right\rangle_{\alpha;\,B},$$

$$|g|_{0;\,B} = \sup_{x\in B}|g(x)|\,,\ \langle g\rangle_{\alpha;\,B} = \sup_{x,\,y\in B}\frac{|g(x)-g(y)|}{|x-y|^{\alpha}}.$$

Let $U_j = 2^j U$, $V_j = 2^j V$, $j = 1, \dots$, $\tilde{w}_j(x) = \tilde{w}(2^j x)$, $\tilde{f}_j(x) = \tilde{f}(2^j x)$, $x \in V$. Then

$$\Delta \tilde{w}_j(x) = 2^{2j}\tilde{f}_j(x) \text{ in } V.$$

Also $\partial^k \tilde{w}_j(x) = 2^{kj}\tilde{w}(2^j x)$, $x \in V$, $k = 0, 1, \dots$, and $\left\langle \partial^k \tilde{w}_j \right\rangle_{\alpha,\,U} = 2^{(k+\alpha)j}\left\langle \partial^k \tilde{w} \right\rangle_{\alpha;\,U_j}$. By (6) we obtain

$$2^{(2+l+\alpha)j}\left\langle \partial^{2+l}\tilde{w} \right\rangle_{0,\,\alpha;\,U_j} + |\tilde{w}|_{0,\,U_j} \le C(2^{(2+l+\alpha)j}\left\langle \partial^{l}\tilde{f} \right\rangle_{\alpha,\,V_j} + 2^{2j}\left|\tilde{f}\right|_{0,\,V_j} + |\tilde{w}|_{0,\,V_j})$$

Multiplying both sides by $2^{(\delta-2-\alpha)j}$ we have

$$\begin{aligned} &2^{(\delta+l)j}\left\langle \partial^{2+l}\tilde{w} \right\rangle_{0,\,\alpha;\,U_j} + 2^{(\delta-2-\alpha)j}|\tilde{w}|_{0,\,U_j} \le C(2^{(\delta+l)j}\left\langle \partial^{l}\tilde{f} \right\rangle_{0,\,\alpha;\,V_j} + \\ &+2^{(\delta-\alpha)j}\left|\tilde{f}\right|_{0,\,V_j} + 2^{(\delta-2-\alpha)j}|\tilde{w}|_{0,\,V_j}). \end{aligned} \tag{7}$$

According to Remark 2.3.17 in [5] it follows from (7) that

$$|\tilde{w}|_{C^{l+2,\,\alpha}_{\delta+l}} \le C(\left|\tilde{f}\right|_{C^{l,\alpha}_{\delta+l}} + \sup_x (1+|x|)^{\delta-2-\alpha}|\tilde{w}(x)|). \tag{8}$$

According to (5) and Theorem 1.3.5 (generalization of Hardy-Littlewood inequality) in [5] (for $\beta = \delta - \alpha$, $\delta \in (2+\alpha,\, 3+\alpha)$ and $d > 2$)

$$\begin{aligned} &\sup(1+|x|)^{\delta-2-\alpha}|\tilde{w}(x)| \le C\sup_x(1+|x|)^{\delta-\alpha}\left|\tilde{f}(x)\right|, \\ &\sup_x(1+|x|)^{\delta-1-\alpha}|\partial_i\tilde{w}(x)| \le C\sup_x(1+|x|)^{\delta-\alpha}\left|\tilde{f}(x)\right|. \end{aligned} \tag{9}$$

Thus by (8) and (9) for each $l = 0, 1, \dots$ there exists a constant $C$ independent of $f \in C_0^\infty(\mathbf{R}^d)$ such that

$$|\tilde{w}|_{C^{l+2,\,\alpha}_{\delta+l}} \le C\left|\tilde{f}\right|_{C^{l,\alpha}_{\delta+l}} \le C\,|f|_{C^{l,\alpha}_{\delta+l}}.$$

By Theorem 2.3.19 in [5] we can extend this inequality to the whole $C^{l,\,\alpha}_{\delta+l}$.

# 4 Appendix 2: Hölder norm estimate

The Hölder norm estimate of the following function $H = (H^k(x))_{1\le k\le d}$ is important. Let $v = (v^j)_{1\le j\le d} \in C^{l+1,\alpha}_{l+\alpha}$, $w = (w^{ki})_{1\le i,k\le d} \in C^{l,\alpha}_{l+\delta}$, $\delta \in (2+\alpha,\, 3+\alpha)$. We consider the function $H = (H^k(x))_{1\le k\le d}$ defined by

$$\begin{aligned} H^k(x) &= \int \Gamma_{x_ix_j}(x-y)(v^j(x)-v^j(y))w^{ki}(y)\,dy \\ &= \int \Gamma_{x_ix_j}(z)(v^j(x-z)-v^j(x))w^{ki}(x-z)\,dz. \end{aligned}$$

We assume $\int w^{ki}\,dx = 0$ for all $k$, $i$, if $d = 2$.

**Proposition 8** There is a constant $C$ independent of $v$, $w$ such that

$$\begin{aligned}
|H|_{C^{l+1,\alpha}_{l+\delta}} &\leq C|v|_{C^{l+1,\alpha}_{l+\alpha}}|w|_{C^{l,\alpha}_{l+\delta}}, \text{ if } d > 2; \\
|H|_{C^{l+1,\alpha}_{l+\alpha}} &\leq C|v|_{C^{l+1,\alpha}_{l+\alpha}}|w|_{C^{l,\alpha}_{l+\delta}}, \text{ if } d = 2.
\end{aligned}$$

**Proof 8** For each multiindex $\beta$ of length $l$

$$\begin{aligned}
\partial^\beta H^k(x) &= \sum_{\gamma+\mu=\beta} \int \Gamma_{x_i x_j}(z)(\partial^\gamma v^j(x) - \partial^\gamma v^j(x-z))\partial^\mu w^{ki}(x-z)\,dz \\
&= \sum_{\gamma+\mu=\beta} \int \Gamma_{x_i x_j}(x-y)(\partial^\gamma v^j(x) - \partial^\gamma v^j(y))\partial^\mu w^{ki}(y)\,dz \\
&= \sum_{\gamma+\mu=\beta} [\int \frac{\partial}{\partial y_i}(\Gamma_{x_j}(x-y)(\partial^\gamma v^j(y) - \partial^\gamma v^j(x)))\partial^\mu w^{ki}(y)\,dz \\
&\quad - \int \Gamma_{x_j}(x-y)\partial^\gamma v^j_{x_i}(y)\partial^\mu w^{ki}(y)\,dz] \\
&= I^1(x) + I^2(x).
\end{aligned}$$

Notice

$$\begin{aligned}
I^2 &= -\int \Gamma_{x_j}(x-y)\partial^\beta(v^j_{x_i}(y)w^{ki}(y)\,dy = \\
&= \partial^\beta g_{x_j}(x),
\end{aligned}$$

where $\Delta g = -v^j_{x_i}(y)w^{ki}(y) \in C^{l,\alpha}_{l+\delta}$. If $d > 2$, by Proposition 7

$$|I^2|_{C^{1,\alpha}_{l+\delta}} \leq |g|_{C^{l+2,\alpha}_{l+\delta}} \leq C|v^j w^{ki}|_{C^{l.\alpha}_{l+\delta}} \leq C|v^j_{x_i}|_{C^{l,\alpha}_{l+\alpha}}|w^{ki}|_{C^{l.\alpha}_{l+\delta}}. \tag{1}$$

If $d = 2$, we have a decomposition $g = \tilde{g} + F$, where $\tilde{g} \in C^{l+2,\alpha}_{l+\delta}$, $F(x) = \lambda \ln|x|(1-\phi(x))$, and

$$|I^2|_{C^{1,\alpha}_{l+\alpha}} \leq |\tilde{g}|_{C^{l+2,\alpha}_{l+\delta}} + |F|_{C^{l+2,\alpha}_{l+\alpha}} \leq C|v^j|_{C^{l+1,\alpha}_{l+\alpha}}|w^{ki}|_{C^{l.\alpha}_{l+\delta}}. \tag{2}$$

Choose $\bar{x}$ and $x$ so that $\varepsilon = |x-\bar{x}| \leq \frac{1}{32}[1 \vee (|x| \wedge |\bar{x}|)]$. Let $\xi = \frac{1}{2}(x+\bar{x})$,

$$E(\xi) = \{y \in \mathbf{R}^d : |\xi - y| \leq 1 \vee (|\xi| \wedge |y|)\}.$$

In a standard way (see Lemma 4.2 in [6]), we have the following representation

$$\begin{aligned} I^1_{x_p}(x) &= \sum_{\gamma+\mu=\beta} \int_{E(\xi)} \frac{\partial^2}{\partial y_i \partial x_p} (\Gamma_{x_j}(x-y)(\partial^\gamma v^j(y) - \partial^\gamma v^j(x)))(\partial^\mu w^{ki}(y) \\ &\quad -\partial^\mu w^{ki}(x))\, dy \\ &\quad + \sum_{\gamma+\mu=\beta} \int_{E(\xi)^c} \frac{\partial^2}{\partial y_i \partial x_p} (\Gamma_{x_j}(x-y)(\partial^\gamma v^j(y) - \partial^\gamma v^j(x)))\partial^\mu w^{ki}(y)\, dy \\ &\quad + \sum_{\gamma+\mu=\beta} \partial^\mu w^{ki}(x) \int_{\partial E(\xi)} \frac{\partial}{\partial x_p} (\Gamma_{x_j}(y-x)(\partial^\gamma v^j(y) - \partial^\gamma v^j(x)))n_i(y)\, S(dy) \\ &= A^1(x) + A^2(x) + A^3(x), \end{aligned}$$

where $\partial E(\xi) = \{y : |\xi - y| = 1 \vee (|\xi| \wedge |y|)$ and $n = (n_i(y))$ is exterior unit normal at $y \in \partial E(\xi)$. Similar formula holds for $I_{x_p}(\bar{x})$ as well. Now,

$$\begin{aligned} A^2(x) &= -\int_{E(\xi)^c} \Gamma_{x_i x_p x_j}(x-y)\partial^\beta(v^j(y)w^{ki}(y))\, dy \\ &\quad + \int_{E(\xi)^c} \Gamma_{x_p x_j}(x-y)\partial^\beta(v^j_{yi}(y)w^{ki}(y))\, dy \\ &\quad + \partial^\gamma v^j(x) \int_{E(\xi)^c} \Gamma_{x_i x_p x_j}(x-y)\partial^\mu w^{ki}(y)\, dy \\ &\quad + \partial^\gamma v^j_{x_p}(x) \int_{E(\xi)^c} \Gamma_{x_i x_j}(x-y)\partial^\mu w^{ki}(y)\, dy. \end{aligned} \tag{3}$$

We have the estimate

$$(1+|x|)^{l+\delta}|A^2(x) - A^2(\bar{x})| \le C|x-\bar{x}|^\alpha |v|_{C^{l+1,\alpha}_{l+\alpha}} |w|_{C^{l,\alpha}_{l+\delta}}, \text{ if } d > 2; \tag{4}$$

$$(1+|x|)^{l+\alpha}|A^2(x) - A^2(\bar{x})| \le C|x-\bar{x}|^\alpha |v|_{C^{l+1,\alpha}_{l+\alpha}} |w|_{C^{l,\alpha}_{l+\delta}}, \text{ if } d = 2$$

For example, integrating by parts (applying divergence theorem) the first term of RHS of (3)

$$\begin{aligned} &\int_{E(\xi)^c} \Gamma_{x_i x_p x_j}(x-y)\partial^\beta(v^j(y)w^{ki}(y))\, dy \\ &= \int_{E(\xi)^c} \partial^\beta_x \Gamma_{x_i x_p x_j}(x-y) v^j(y) w^{ki}(y)\, dy \\ &\quad + \sum_{\hat{m}+\tau+\mu=\beta} \int_{\partial E(\xi)} \partial^\mu_y \Gamma_{x_i x_p x_j}(x-y)\partial^\tau(v^j(y)w^{ki}(y))n_m(y)\, S(dy) \\ &= B^1(x) + B^2(x), \end{aligned}$$

where $\hat{m}$ is a unit vector whose $m$-th component is 1. By Remark 2

$$\begin{aligned} & (1+|x|)^{l+\delta}|B^1(x)-B^1(\bar{x})| \\ \leq\ & C(1+|\xi|)^{l+\delta}|A^2(x)-A^2(\bar{x})| \\ \leq\ & C(1+|\xi|)^{l+\delta}|x-\bar{x}|^\alpha \int_{E(\xi)^c} \frac{dy}{|\xi-y|^{l+d+1+\alpha}(1+|y|)^{\delta-1-\alpha}} \\ \leq\ & C|x-\bar{x}|^\alpha(1+\int_{G(\xi)} \frac{dy}{|\tilde{\xi}-y|^{l+d+1+\alpha}(|\xi|^{-1}+|y|)^{\delta-1-\alpha}}), \end{aligned}$$

where $\tilde{\xi}=\xi/|\xi|$, $G(\xi)=\{y:|\tilde{\xi}-y|>|\xi|^{-1}\vee(1\wedge|y|),\ |y|\leq 1\}$ and the integral is uniformly bounded in $\xi$. Similarly we estimate $B^2$ and the remaining terms of $A^2$. Let $B(\xi)=B_\varepsilon(\xi)=\{y:|y-\xi|<\varepsilon\}$, $\varepsilon=|x-\bar{x}|$. Then

$$\begin{aligned} A^1(x) &= \sum_{\gamma+\mu=\beta}\int_{E(\xi)\cap B(\xi)} \frac{\partial^2}{\partial y_i\partial x_p}(\Gamma_{x_j}(x-y)(\partial^\gamma v^j(y) \\ &\quad -\partial^\gamma v^j(x)))(\partial^\mu w^{ki}(y)-\partial^\mu w^{ki}(x))\,dy \\ &\quad +\sum_{\gamma+\mu=\beta}\int_{E(\xi)\cap B(\xi)^c} \frac{\partial^2}{\partial y_i\partial x_p}(\Gamma_{x_j}(x-y)(\partial^\gamma v^j(y) \\ &\quad -\partial^\gamma v^j(x)))(\partial^\mu w^{ki}(y)-\partial^\mu w^{ki}(x))\,dy \\ &= C^1(x)+C^2(x). \end{aligned}$$

Since on $E(\xi)$ the distances $(1+|x|), (1+|\bar{x}|)$, $(1+|\xi|)$ are equivalent (see Remark 2), we can simply follow the proof in [6]. It is straightforward that

$$\begin{aligned} (1+|x|)^{l+\delta}|C^1(x)-C^1(\bar{x})| &\leq (1+|x|)^{l+\delta}(|C^1(x)|+|C^1(\bar{x})|) \\ &\leq C\varepsilon^\alpha |v|_{C^{l+1,\alpha}_{l+\alpha}}|w|_{C^{l,\alpha}_{l+\delta}}. \end{aligned}$$

Now,

$$\begin{aligned} C^2(\bar{x})-C^2(x) &= \sum_{\gamma+\mu=\beta}[(\partial^\mu w(x)-\partial^\mu w(\bar{x}))\int\frac{\partial^2}{\partial y_i\partial x_p}K^\gamma(x,y)\,dy \\ &\quad +\int(\frac{\partial^2}{\partial y_i\partial x_p}K^\gamma(\bar{x},y)- \\ &\quad -\frac{\partial^2}{\partial y_i\partial x_p}K^\gamma(x,y))(\partial^\mu w^{ki}(y)-\partial^\mu w^{ki}(\bar{x}))\,dy], \end{aligned}$$

where $K^\gamma(x,y)=\Gamma_{x_j}(x-y)(\partial^\gamma v^j(y)-\partial^\gamma v^j(x))$. Then we have the estimate by repeating the proof of Lemma 4.4 in [6].

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