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ALMOST SURE EXPONENTIAL STABILITY OF NEUTRAL DIFFERENTIAL DIFFERENCE EQUATIONS WITH DAMPED STOCHASTIC PERTURBATIONS

Xiao Xin Liao and Xuerong Mao

Department of Statistics and Modelling Science, University of Strathclyde, Glasgow G1 1XH, U.K.

E-mail: xuerong@stams.strath.ac.uk

Abstract: In this paper we shall discuss the almost sure exponential stability for a neutral differential difference equation with damped stochastic perturbations of the form $d[x(t) - G(x(t-\tau))] = f(t, x(t), x(t-\tau))dt + \sigma(t)dw(t)$. Several interesting examples are also given for illustration. It should be pointed out that our results are even new in the case when $\sigma(t) \equiv 0$, i.e. for deterministic neutral differential difference equations.

Keywords: neutral equations, stochastic perturbation, exponential martingale inequality, Borel-Cantelli's lemma, Lyapunov exponent.

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Almost Sure Exponential Stability of Neutral Differential Difference Equations with Damped Stochastic Perturbations [1]

X. X. Liao[2] and X. Mao[3]
Department of Statistics and Modelling Science
University of Strathclyde
Glasgow G1 1XH, Scotland, U.K.

Abstract: In this paper we shall discuss the almost sure exponential stability for a neutral differential difference equation with damped stochastic perturbations of the form

$$d[x(t)-G(x(t-\tau))]=f(t,x(t),x(t-\tau))dt+\sigma(t)dw(t).$$

Several interesting examples are also given for illustration. It should be pointed out that our results are even new in the case when $\sigma(t)\equiv 0$, i.e. for deterministic neutral differential difference equations.

Key Words: neutral equations, stochastic perturbation, exponential martingale inequality, Borel-Cantelli's lemma, Lyapunov exponent.

1. Introduction

Deterministic neutral differential difference equations and their stability have been studied by many authors e.g. Hale [4], Hale & Meyer [5]. Such equations arise in various situations. For example, the reactors in chemical engineering systems can sometimes be described by a linear neutral differential difference equation

$$\dot{x}(t) - C\dot{x}(t-\tau) = Ax(t) + Bx(t-\tau), \tag{1.1}$$

and in the theory of aeroelasticity, one often meets non-linear neutral differential difference equations of the form

$$\frac{d}{dt}[x(t) - G(x(t-\tau))] = f(t, x(t), x(t-\tau)) \tag{1.2}$$

(for the details please see Kolmanovskii & Nosov [8]). If there exist damped stochastic perturbations to equations (1.1) and (1.2) we arrive at a linear neutral stochastic differential difference equation

$$d[x(t) - Cx(t-\tau)] = [Ax(t) + Bx(t-\tau)]dt + \sigma(t)dw(t), \tag{1.3}$$

or a non-linear neutral stochastic differential difference equation

$$d[x(t) - G(x(t-\tau))] = f(t, x(t), x(t-\tau))dt + \sigma(t)dw(t). \tag{1.4}$$

[1] Supported by the Royal Society.

[2] Permanent Address: Department of Mathematics, Huazhong University of Science and Technology, Wuhan, P.R.China.

[3] For any correspondence regarding this paper please address to this author.

Such neutral stochastic differential difference equations were introduced by Kolmanovskii & Nosov [7], and the stability and asymptotic stability of the equations have also been studied by Kolmanovskii [6], Kolmanovskii & Nosov [7]. However, so far little is known about the almost sure exponential stability of such neutral stochastic differential difference equations and the aim of this paper is to close this gap.

In this paper we are only concerned with the damped stochastic perturbations. That is, we require that the diffusion coefficient $\sigma(t)$ tends to zero sufficiently rapidly as $t \to \infty$ (more precisely, please see condition (2.2) below). Such damped stochastic perturbations appear frequently in many branches of science e.g. stochastic mechanics (cf. Albeverio et al. [1], Durran & Truman [2]). For example, Durran & Truman [2] showed that under the damped stochastic perturbation, the orbit of the planetesimal is in an L^2-neighbourhood of the Keplerian circular orbit after a sufficiently long time. Damped stochastic perturbations also appear in a natural way in hierarchically controlled systems (cf. Mao [9, 10]). For example, consider a stochastic hierarchical system

$$d[y(t) - G_1(y(t-\tau))] = f_1(t, y(t), y(t-\tau)) + g_1(t, y(t), y(t-\tau))dw(t), \quad (1.5a)$$

$$\begin{aligned} d[x(t) - G_2(x(t-\tau))] &= f_2(t, x(t), x(t-\tau), y(t), y(t-\tau))dt \\ &\quad + g_2(t, y(t), y(t-\tau))dw(t). \end{aligned} \quad (1.5b)$$

Note that the 'lower' subsystem (1.5a) will not depend on the 'higher' subsystem (1.5b) but will feed to the 'higher' one. Assume that subsystem (1.5a) is exponentially stable, that is $y(t)$ tends to zero exponentially fast. Assume also that $||g_2(t,x,y)|| \le K(|x|+|y|)$ for some $K > 0$. We then see that the diffusion coefficient $g_2(t, y(t), y(t-\tau))$ in (1.5b) tends to zero exponentially rapidly. In other words, we have the damped stochastic perturbation in subsystem (1.5b). The stability problem of subsystem (1.5b) is then reduced to the study of equation (1.4).

In this paper we shall first study the almost sure exponential stability of equation (1.3) in Section 2 and then equation (1.4) in Section 3. Also several interesting examples will be given to illustrate our theory in Section 4. It should be pointed out that our results are even new in the case when $\sigma(t) \equiv 0$, i.e. for deterministic neutral differential difference equations.

2. Almost Sure Exponential Stability of Linear Neutral Stochastic Differential Difference Equations

Throughout this paper, unless otherwise specified, let $w(t) = (w_1(t), \cdots, w_m(t))^T$ be an m-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$ with a natural filtration $\{\mathcal{F}_t\}_{t\ge 0}$ (i.e. $\mathcal{F}_t = \sigma\{w(s) : 0 \le s \le t\}$). If A is a vector or matrix, denote by A^T its transpose. Denote by $|\cdot|$ the Euclidean norm, i.e. $|x| = \sqrt{x^T x}$ if $x \in R^n$. If A is a matrix, denote by $||A||$ the operator norm of A, i.e. $||A|| = sup\{|Ax| : |x| = 1\}$. Furthermore, let $\tau > 0$ and denote by $\mathcal{C}^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$ the family of all continuous bounded R^n-valued stochastic processes $\xi(s), -\tau \le s \le 0$ such that $\xi(s)$ is $\mathcal{F}_0$-measurable for every s.

In this section we shall consider an n-dimensional linear neutral differential difference equation with damped stochastic perturbations of the form

$$d[x(t) - Cx(t-\tau)] = [Ax(t) + Bx(t-\tau)]dt + \sigma(t)dw(t) \quad (2.1)$$

on $t \geq 0$ with initial data $x(t) = \xi(t)$ for $-\tau \leq t \leq 0$, where A, B, C are all $n \times n$ matrices, $\sigma(t), t \geq 0$ is an $\mathcal{F}_t$-adapted bounded $R^{n\times m}$-valued stochastic process, and $\xi := \{\xi(s) : -\tau \leq s \leq 0\} \in \mathcal{C}^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$. An $\mathcal{F}_t$-adapted process $x(t), -\tau \leq t < \infty$ (let $\mathcal{F}_t = \mathcal{F}_0$ for $-\tau \leq t \leq 0$) is said to be the solution of equation (2.1) if it satisfies the initial condition, and moreover for every $t \geq 0$,

$$x(t) - Cx(t-\tau) = \xi(0) - C\xi(-\tau)$$
$$+ \int_0^t [Ax(s) + Bx(s-\tau)]ds + \int_0^t \sigma(s)dw(s). \tag{2.1$'$}$$

To see that equation (2.1) has a unique solution, we first restrict t on the interval $[0, \tau]$. In this case, equation (2.1)$'$ becomes

$$x(t) = \xi(0) + C[\xi(t-\tau) - \xi(-\tau)]$$
$$+ \int_0^t [Ax(s) + B\xi(s-\tau)]ds + \int_0^t \sigma(s)dw(s).$$

By the theory of stochastic differential equations (cf. [3], [9] or [12]) one can find a unique solution $x(t)$ on $[0, \tau]$. Similarly, a unique solution exists on $[\tau, 2\tau], [2\tau, 3\tau]$ and so on, hence on the whole $t \geq 0$. Denote by $x(t;\xi)$ the unique solution. It is also easy to see that the solution is square integrable.

Theorem 2.1 *Assume that there exist two symmetric $n \times n$ matrices Q and D with Q positive definite and D non-negative definite such that the symmetric matrix*

$$H = \begin{pmatrix} QA + A^TQ + D & QB - A^TQC \\ B^TQ - C^TQA & -D - C^TQB - B^TQC \end{pmatrix}$$

is negative definite, and denote by $-\lambda$ the biggest eigenvalue of H. Assume $||C|| < 1$. Assume also that there exists a pair of positive constants γ and δ such that

$$trace[\sigma^T(t)\sigma(t)] \leq \delta e^{-\gamma t} \quad \text{for all } t \geq 0. \tag{2.2}$$

Then equation (2.1) is almost surely exponentially stable. Moreover, the top Lyapunov exponent of the solution should not be greater than $-(\gamma \wedge \alpha \wedge \beta)/2$, that is

$$\limsup_{t\to\infty} \frac{1}{t} \log(|x(t;\xi)|) \leq -\frac{1}{2}(\gamma \wedge \alpha \wedge \beta) \qquad a.s. \tag{2.3}$$

where $\alpha \in (0, \lambda)$ is the unique root of

$$2\alpha||Q|| + \alpha\tau e^{\alpha\tau}||D|| = \lambda, \tag{2.4}$$

and

$$\beta = -\frac{2}{\tau}\log(||C||) > 0.$$

In order to prove this theorem let us prepare a lemma which is very useful in the study of exponential stability for neutral stochastic differential difference equations.

Lemma 2.2 *Assume that* $G : R^n \to R^n$ *is a Borel measurable function such that for some* $\kappa \in (0,1)$

$$|G(x)| \le \kappa |x| \qquad \text{for all } x \in R^n. \tag{2.5}$$

Let $\varphi(t), -\tau \le t < \infty$ *be a Borel measurable* R^n*-valued function. Let* $\alpha > 0$, $K > 0$ *and* k_o *be a positive integer. Assume*

$$|\varphi(t) - G(\varphi(t-\tau))|^2 \le K(1+\log k)e^{-\alpha(k-1)\tau} \tag{2.6}$$

for $(k-1)\tau \le t \le k\tau$, $k \ge k_o$. *Then*

$$\limsup_{t\to\infty} \frac{1}{t}\log(|\varphi(t)|) \le -\frac{1}{2}(\alpha \wedge \beta), \tag{2.7}$$

where

$$\beta = -\frac{2}{\tau}\log\kappa > 0.$$

Proof. Let $\theta \in (\kappa, 1)$ be arbitrary. Note

$$\begin{aligned}
&|\varphi(t) - G(\varphi(t-\tau))|^2 \\
&\ge |\varphi(t)|^2 - 2|\varphi(t)|\,|G(\varphi(t-\tau))| + |G(\varphi(t-\tau))|^2 \\
&\ge |\varphi(t)|^2 - \theta|\varphi(t)|^2 - \theta^{-1}|G(\varphi(t-\tau))|^2 + |G(\varphi(t-\tau))|^2 \\
&\ge (1-\theta)|\varphi(t)|^2 - \kappa^2(\theta^{-1}-1)|\varphi(t-\tau)|^2.
\end{aligned}$$

Thus

$$|\varphi(t)|^2 \le \frac{1}{1-\theta}|\varphi(t) - G(\varphi(t-\tau))|^2 + \frac{\kappa^2}{\theta}|\varphi(t-\tau)|^2. \tag{2.8}$$

Now define, for $k = k_o - 1,\ k_o,\ k_o+1, \cdots,$

$$\phi_k = \sup_{(k-1)\tau \le t \le k\tau} |\varphi(t)|^2.$$

One can see from (2.8) and (2.6) that

$$\begin{aligned}
\phi_k &\le \frac{1}{1-\theta}\sup_{(k-1)\tau \le t \le k\tau}|\varphi(t) - G(\varphi(t-\tau))|^2 + \frac{\kappa^2}{\theta}\phi_{k-1} \\
&\le \frac{K}{1-\theta}(1+\log k)e^{-\alpha(k-1)\tau} + \frac{\kappa^2}{\theta}\phi_{k-1} \qquad \text{for all } k \ge k_o.
\end{aligned} \tag{2.9}$$

Set

$$\beta_\theta = \frac{1}{\tau}\log\left(\frac{\theta}{\kappa^2}\right)$$

and let $\varepsilon \in (0, \alpha \wedge \beta_\theta)$ be arbitrary. It then follows from (2.9) that

$$\begin{aligned}
\max_{k_o \le i \le k}(\phi_i e^{\varepsilon i \tau}) &\le \frac{K}{1-\theta}\,e^{\alpha\tau}\max_{k_o \le i \le k}\Big((1+\log i)e^{-(\alpha-\varepsilon)i\tau}\Big) + \frac{\kappa^2}{\theta}\max_{k_o \le i \le k}(\phi_{i-1}e^{\varepsilon i\tau}) \\
&\le \frac{Kc_1}{1-\theta}\,e^{\alpha\tau} + \frac{\kappa^2}{\theta}e^{\varepsilon\tau}\max_{k_o \le i \le k}(\phi_{i-1}e^{\varepsilon(i-1)\tau}) \\
&\le \frac{Kc_1}{1-\theta}\,e^{\alpha\tau} + \frac{\kappa^2}{\theta}e^{\varepsilon\tau}\Big(\phi_{k_o-1}e^{\varepsilon(k_o-1)\tau} + \max_{k_o \le i \le k}(\phi_i e^{\varepsilon i\tau})\Big),
\end{aligned} \tag{2.10}$$

where

$$c_1 = \sup_{k_o \le i < \infty} \Big((1+\log i)e^{-(\alpha-\varepsilon)i\tau})\Big) < \infty.$$

Note

$$\frac{\kappa^2}{\theta} e^{\varepsilon\tau} < 1.$$

One sees from (2.10) that

$$\max_{k_o \le i \le k} (\phi_i e^{\varepsilon i \tau}) \le c_2,$$

that is

$$\phi_k \le c_2 e^{-\varepsilon k \tau} \qquad \text{for all } k \ge k_o,$$

where

$$c_2 = \Big(\frac{Kc_1}{1-\theta}\, e^{\alpha\tau} + \frac{\kappa^2}{\theta}\phi_{k_o-1}e^{\varepsilon k_o \tau}\Big)\Big(1-\frac{\kappa^2}{\theta}e^{\varepsilon\tau}\Big)^{-1},$$

By the definition of ϕ_k it is therefore easy to derive that

$$\limsup_{t\to\infty} \frac{1}{t}\log(|\varphi(t)|) \le -\frac{\varepsilon}{2}.$$

Finally, letting $\varepsilon \to \alpha \wedge \beta_\theta$ we get

$$\limsup_{t\to\infty} \frac{1}{t}\log(|\varphi(t)|) \le -\frac{1}{2}(\alpha \wedge \beta_\theta),$$

and then letting $\theta \to 1$ we obtain the required (2.7). The proof is complete.

Proof of Theorem 2.1. Fix the initial data ξ arbitrarily and write $x(t;\xi) = x(t)$ simply. Define a Lyapunov function

$$V(z,t) = z^T Q z + \int_{-\tau}^{0} x^T(t+s) D x(t+s) ds \tag{2.11}$$

for $(z,t) \in R^n \times [0,\infty)$. Applying the Itô formula to $V(x(t)-Cx(t-\tau),t)$ and using the assumptions we derive that

$$\begin{aligned}
& dV(x(t)-Cx(t-\tau),t) \\
& = 2(x(t)-Cx(t-\tau))^T Q\Big([Ax(t)+Bx(t-\tau)]dt + \sigma(t)dw(t)\Big) \\
& + trace[\sigma^T(t)Q\sigma(t)]dt + \big(x^T(t)Dx(t) - x^T(t-\tau)Dx(t-\tau)\big)dt \\
& \le (x^T(t), x^T(t-\tau))\, H \begin{pmatrix} x(t) \\ x(t-\tau) \end{pmatrix} dt + ||Q||trace[\sigma^T(t)\sigma(t)]dt \\
& + 2(x(t)-Cx(t-\tau))^T Q\sigma(t)dw(t) \\
& \le -\lambda(|x(t)|^2 + |x(t-\tau)|^2)dt + \delta||Q||e^{-\gamma t}dt \\
& + 2(x(t)-Cx(t-\tau))^T Q\sigma(t)dw(t).
\end{aligned} \tag{2.12}$$

Let $\theta \in (0, \alpha \wedge \gamma)$ be arbitrary. Applying Itô's formula once again we derive that

$$\begin{aligned}
&d\big(e^{\theta t}V(x(t) - Cx(t-\tau), t)\big) \\
&= \theta e^{\theta t}V(x(t) - Cx(t-\tau), t)dt + e^{\theta t}dV(x(t) - Cx(t-\tau), t) \\
&\leq \theta e^{\theta t}\Big(2||Q||\big(|x(t)|^2 + ||C||^2|x(t-\tau)|^2\big) + ||D||\int_{-\tau}^{0}|x(t+s)|^2ds\Big)dt \\
&+ e^{\theta t}\Big(-\lambda(|x(t)|^2 + |x(t-\tau)|^2)dt + \delta||Q||e^{-\gamma t}dt \\
&\qquad + 2(x(t) - Cx(t-\tau))^T Q\sigma(t)dw(t)\Big) \\
&\leq e^{\theta t}\Big(-(\lambda - 2\theta||Q||)(|x(t)|^2 + |x(t-\tau)|^2) + \theta||D||\int_{-\tau}^{0}|x(t+s)|^2ds\Big)dt \\
&+ \delta||Q||e^{-(\gamma-\theta)t}dt + 2e^{\theta t}(x(t) - Cx(t-\tau))^T Q\sigma(t)dw(t). \qquad (2.13)
\end{aligned}$$

That is

$$\begin{aligned}
&e^{\theta t}V(x(t) - Cx(t-\tau), t) \\
&\leq c_3 - (\lambda - 2\theta||Q||)\int_0^t e^{\theta s}(|x(s)|^2 + |x(s-\tau)|^2)ds \\
&+ \theta||D||\int_0^t e^{\theta s}\int_{-\tau}^{0}|x(s+r)|^2drds + M(t), \qquad (2.14)
\end{aligned}$$

where

$$c_3 = V(\xi(0) - C\xi(-\tau), 0) + \frac{\delta||Q||}{\gamma - \theta}$$

and

$$M(t) = 2\int_0^t e^{\theta s}(x(s) - Cx(s-\tau))^T Q\sigma(s)dw(s).$$

But

$$\begin{aligned}
&\int_0^t e^{\theta s}\int_{-\tau}^{0}|x(s+r)|^2drds = \int_0^t e^{\theta s}\int_{s-\tau}^{s}|x(r)|^2drds \\
&= \int_{-\tau}^{t}\Big(\int_{r\vee 0}^{(r+\tau)\wedge t} e^{\theta s}ds\Big)|x(r)|^2dr \leq \int_{-\tau}^{t}\tau e^{\theta(r+\tau)}|x(r)|^2dr \\
&\leq \tau e^{\theta\tau}\int_{-\tau}^{0}|\xi(s)|^2ds + \tau e^{\theta\tau}\int_0^t e^{\theta s}|x(s)|^2ds. \qquad (2.15)
\end{aligned}$$

Substituting (2.15) into (2.14) yields

$$e^{\theta t}V(x(t) - Cx(t-\tau), t) \leq c_4 - \bar{\theta}\int_0^t e^{\theta s}(|x(s)|^2 + |x(s-\tau)|^2)ds + M(t), \qquad (2.16)$$

where

$$\bar{\theta} = \lambda - 2\theta||Q|| - \theta\tau e^{\theta\tau}||D||,$$

$$c_4 = c_3 + \theta\tau e^{\theta\tau}||D||\int_{-\tau}^{0}|\xi(s)|^2ds.$$

Recalling $0 < \theta < \gamma \wedge \alpha$ as well as (2.4) one sees that $\bar{\theta} > 0$. On the other hand, note that $M(t)$ is a continuous martingale vanishing at $t = 0$ and its quadratic variation

$$\langle M(t) \rangle = 4 \int_0^t e^{2\theta s} |x(s) - Cx(s-\tau))^T Q \sigma(s)|^2 ds$$
$$\leq 8\delta ||Q||^2 \int_0^t e^{\theta s} (|x(s)|^2 + |x(s-\tau))|^2) ds. \tag{2.17}$$

Let

$$\varepsilon = \frac{\bar{\theta}}{4\delta ||Q||^2}.$$

By the well-known exponential martingale inequality (cf. Mao [9] or Métivier [11]) we have that for $k = 1, 2, \cdots$,

$$P\Big(\omega : \sup_{0 \leq t \leq k\tau} \big[M(t) - \frac{\varepsilon}{2} \langle M(t) \rangle\big] > \frac{2}{\varepsilon} \log k\Big) \leq \frac{1}{k^2}.$$

Hence the well-known Borel-Cantelli lemma yields that for almost all $\omega \in \Omega$ there exists an integer $k_o = k_o(\omega)$ such that

$$\sup_{0 \leq t \leq k\tau} \big[M(t) - \frac{\varepsilon}{2} \langle M(t) \rangle\big] \leq \frac{2}{\varepsilon} \log k \qquad \text{whenever } k \geq k_o.$$

This, together with (2.17) and the definition of ε, implies that for almost all $\omega \in \Omega$

$$M(t) \leq \frac{2}{\varepsilon} \log k + \bar{\theta} \int_0^t e^{\theta s} (|x(s)|^2 + |x(s-\tau))|^2) ds \tag{2.18}$$

whenever $0 \leq t \leq k\tau$, $k \geq k_o$. Substituting (2.18) into (2.16) we obtain that for almost all $\omega \in \Omega$

$$e^{\theta t} V(x(t) - Cx(t-\tau), t) \leq c_4 + \frac{2}{\varepsilon} \log k$$

whenever $0 \leq t \leq k\tau$, $k \geq k_o$. Consequently, for almost all $\omega \in \Omega$

$$V(x(t) - Cx(t-\tau), t) \leq \big(c_4 + \frac{2}{\varepsilon}\big)(1 + \log k) e^{-\theta(k-1)\tau} \tag{2.19}$$

whenever $(k-1)\tau \leq t \leq k\tau$, $k \geq k_o$. However

$$V(x(t) - Cx(t-\tau), t) \geq (x(t) - Cx(t-\tau))^T Q (x(t) - Cx(t-\tau))$$
$$\geq \lambda_{min} |x(t) - Cx(t-\tau)|^2,$$

where $\lambda_{min} > 0$ is the smallest eigen-value of Q. Therefore, for almost all $\omega \in \Omega$

$$|x(t) - Cx(t-\tau)|^2 \leq \frac{1}{\lambda_{min}} \big(c_4 + \frac{2}{\varepsilon}\big)(1 + \log k) e^{-\theta(k-1)\tau} \tag{2.20}$$

whenever $(k-1)\tau \leq t \leq k\tau$, $k \geq k_o$. Now applying Lemma 2.2 one gets that

$$\limsup_{t \to \infty} \frac{1}{t} \log(|x(t)|) \leq -\frac{1}{2}(\theta \wedge \beta) \qquad \text{a.s.}$$

Finally letting $\theta \to \gamma \wedge \alpha$ we obtain the required (2.3). The proof is complete.

We now apply Theorem 2.1 to establish one useful corollary.

Corollary 2.3 *Assume that the symmetric matrix $A + A^T$ is negative definite and denote by $-\eta$ its biggest eigenvalue. Assume $||C|| < 1$ and*

$$\frac{\eta}{2} > ||C^T B|| + ||B - A^T C||.$$

Assume furthermore that there exists a pair of positive constants γ and δ such that

$$trace[\sigma^T(t)\sigma(t)] \le \delta e^{-\gamma t} \quad \text{for all } t \ge 0.$$

Then equation (2.1) is almost surely exponentially stable. Moreover, the top Lyapunov exponent of the solution should not be greater than $-(\gamma \wedge \alpha \wedge \beta)/2$, where β is the same as defined in Theorem 2.1, $\alpha > 0$ is the unique root to the equation

$$2\alpha + \alpha\tau e^{\alpha\tau}\Big(\frac{\eta}{2} + ||C^T B||\Big) = \frac{\eta}{2} - ||C^T B|| - ||B - A^T C||.$$

Proof. Let $Q = I$ and $D = \theta I$, where I is the $n \times n$ identity matrix and

$$\theta = \frac{\eta}{2} + ||C^T B||.$$

Let the matrix H be the same as defined in Theorem 2.1. We claim that H is negative definite and its biggest eigenvalue is not greater than

$$-\lambda := -\big(\frac{\eta}{2} - ||C^T B|| - ||B - A^T C||\big).$$

In fact, for any $x, y \in R^n$,

$$\begin{aligned}
&(x^T, y^T)\, H \begin{pmatrix} x \\ y \end{pmatrix} \\
&= x^T(A + A^T)x + \theta|x|^2 + 2x^T(B - A^T C)y - \theta|y|^2 - 2y^T C^T B y \\
&\le -(\eta - \theta)|x|^2 + 2||B - A^T C||\,|x|\,|y| - (\theta - 2||C^T B||)|y|^2 \\
&\le -(\eta - \theta - ||B - A^T C||)|x|^2 - (\theta - 2||C^T B|| - ||B - A^T C||)|y|^2 \\
&= -\lambda(|x|^2 + |y|^2).
\end{aligned}$$

Therefore the conclusion of the corollary follows from Theorem 2.1. The proof is complete.

3. Almost Sure Exponential Stability of Non-linear Neutral Stochastic Differential Difference Equations

In this section we shall consider an n-dimensional non-linear neutral differential difference equation with damped stochastic perturbations of the form

$$d[x(t) - G(x(t-\tau))] = f(t, x(t), x(t-\tau))dt + \sigma(t)dw(t) \tag{3.1}$$

on $t \geq 0$ with initial data $x(t) = \xi(t)$ for $-\tau \leq t \leq 0$, where $w(t)$, $\sigma(t)$, ξ are the same as in the previous section; $f : R_+ \times R^n \times R^n \to R^n$ is a locally Lipschitz continuous function satisfying the linear growth condition; moreover G is a continuous function from R^n to itself such that for some $\kappa \in (0,1)$

$$|G(x)| \leq \kappa |x| \qquad \text{for all } x \in R^n. \tag{3.2}$$

It is easy to see that the equation (3.1) has a unique solution which is denoted by $x(t;\xi)$ again.

Theorem 3.1 *Let (3.2) and (2.2) hold with $\kappa \in (0,1)$ and δ, $\gamma > 0$. Assume that there exist two symmetric $n \times n$ matrices Q and D with Q positive definite and D non-negative definite as well as two constants λ_1, $\lambda_2 > 0$ such that*

$$2(x - G(y))^T Q f(t,x,y) + x^T D x - y^T D y \leq -\lambda_1 |x|^2 - \lambda_2 |y|^2 \tag{3.3}$$

for all $(t,x,y) \in R_+ \times R^n \times R^n$. Then equation (3.1) is almost surely exponentially stable. Moreover, the top Lyapunov exponent of the solution should not be greater than $-(\gamma \wedge \alpha_1 \wedge \alpha_2 \wedge \beta)/2$, that is

$$\limsup_{t \to \infty} \frac{1}{t} \log(|x(t;\xi)|) \leq -\frac{1}{2}(\gamma \wedge \alpha_1 \wedge \alpha_2 \wedge \beta) \qquad a.s. \tag{3.4}$$

where

$$\beta = -\frac{2}{\tau} \log \kappa > 0, \qquad \alpha_2 = \frac{\lambda_2}{2\kappa^2 ||Q||} \tag{3.5}$$

and $\alpha_1 \in (0, \lambda_1)$ is the unique root of

$$2\alpha_1 ||Q|| + \alpha_1 \tau e^{\alpha_1 \tau} ||D|| = \lambda_1. \tag{3.6}$$

Proof. Again fix any initial data ξ and write $x(t;\xi) = x(t)$ simply. Define the Lyapunov function $V(z,t)$ by (2.11). By Itô's formula and condition (3.3) one can show that

$$\begin{aligned} dV(x(t) - G(x(t-\tau)), t) &\leq \big(-\lambda_1 |x(t)|^2 - \lambda_2 |x(t-\tau)|^2\big) dt \\ &+ \delta ||Q|| e^{-\gamma t} + 2(x(t) - G(x(t-\tau)))^T Q \sigma(t) dw(t). \end{aligned}$$

Let $\theta \in (0,\, \gamma \wedge \alpha_1 \wedge \alpha_2)$ be arbitrary. Applying Itô's formula once again we can derive that

$$\begin{aligned} & d\big(e^{\theta t} V(x(t) - G(x(t-\tau)), t)\big) \\ & \leq e^{\theta t} \Big(-(\lambda_1 - 2\theta ||Q||) |x(t)|^2 - (\lambda_2 - 2\theta \kappa^2 ||Q||) |x(t-\tau)|^2 \\ & \quad + \theta ||D|| \int_{-\tau}^{0} |x(t+s)|^2 ds \Big) dt \\ & + \delta ||Q|| e^{-(\gamma - \theta) t} dt + 2 e^{\theta t} (x(t) - G(x(t-\tau)))^T Q \sigma(t) dw(t). \end{aligned}$$

That is

$$
\begin{aligned}
& e^{\theta t} V(x(t)-G(x(t-\tau)),t) \\
& \le c_5 - (\lambda_1 - 2\theta||Q||)\int_0^t e^{\theta s}|x(s)|^2 ds - (\lambda_2 - 2\theta\kappa^2||Q||)\int_0^t e^{\theta s}|x(s-\tau)|^2)ds \\
& + \theta||D||\int_0^t e^{\theta s}\int_{-\tau}^0 |x(s+r)|^2 dr ds + N(t), \qquad (3.7)
\end{aligned}
$$

where

$$
c_5 = V(\xi(0)-G(\xi(-\tau)),0) + \frac{\delta||Q||}{\gamma-\theta}
$$

and

$$
N(t) = 2\int_0^t e^{\theta s}(x(s)-G(x(s-\tau)))^T Q\sigma(s)dw(s).
$$

Substituting (2.15) into (3.7) gives

$$
\begin{aligned}
e^{\theta t} V(x(t)-G(x(t-\tau)),t) \le c_6 - \theta_1 \int_0^t e^{\theta s}|x(s)|^2 ds \\
- \theta_2 \int_0^t e^{\theta s}|x(s-\tau)|^2 ds + N(t), \qquad (3.8)
\end{aligned}
$$

where

$$
\theta_1 = \lambda_1 - 2\theta||Q|| - \theta\tau e^{\theta\tau}||D||, \qquad \theta_2 = \lambda_2 - 2\theta\kappa^2||Q||,
$$
$$
c_6 = c_5 + \theta\tau e^{\theta\tau}||D||\int_{-\tau}^0 |\xi(s)|^2 ds.
$$

Recalling (3.5), (3.6) and $\theta \in (0,\ \gamma\wedge\alpha_1\wedge\alpha_2)$ one sees that both θ_1 and θ_2 are positive. Let

$$
\varepsilon = \frac{\theta_1\wedge\theta_2}{4\delta||Q||^2}.
$$

Since $N(t)$ is a continuous martingale vanishing at $t=0$, we can show in the same way as the proof of Theorem 2.1 that for almost all $\omega\in\Omega$ there exists an integer $k_o = k_o(\omega)$ such that

$$
N(t) \le \frac{2}{\varepsilon}\log k + (\theta_1\wedge\theta_2)\int_0^t e^{\theta s}(|x(s)|^2 + |x(s-\tau)|^2)ds
$$

whenever $0\le t\le k\tau$, $k\ge k_o$. Substituting this into (3.8) one can easily derive that for almost all $\omega\in\Omega$

$$
|x(t)-G(x(t-\tau))|^2 \le \frac{1}{\lambda_{min}}\big(c_6 + \frac{2}{\varepsilon}\big)(1+\log k)e^{-\theta(k-1)\tau} \qquad (3.9)
$$

whenever $(k-1)\tau\le t\le k\tau$, $k\ge k_o$, where $\lambda_{min}>0$ is the smallest eigenvalue of Q. An application of Lemma 2.2 implies that

$$
\limsup_{t\to\infty}\frac{1}{t}\log(|x(t)|) \le -\frac{1}{2}(\theta\wedge\beta) \qquad \text{a.s.}
$$

Finally letting $\theta \to \gamma \wedge \alpha_1 \wedge \alpha_2$ we obtain the required (3.4). The proof is complete.

The following corollary is sometimes convenient for applications.

Corollary 3.2 *Let (3.2) and (2.2) hold with* $\kappa \in (0,1)$ *and* $\delta, \gamma > 0$. *Assume that there exists a symmetric positive definite* $n \times n$ *matrix* Q *as well as two constants* $\lambda_1 > 0$, $\lambda_2 < \lambda_1$ *such that*

$$2(x - G(y))^T Q f(t,x,y) \le -\lambda_1 |x|^2 + \lambda_2 |y|^2$$

for all $(t,x,y) \in R_+ \times R^n \times R^n$. *Then equation (3.1) is almost surely exponentially stable. Moreover, the top Lyapunov exponent of the solution should not be greater than* $-(\gamma \wedge \alpha_1 \wedge \alpha_2 \wedge \beta)/2$, *where*

$$\beta = -\frac{2}{\tau} \log \kappa > 0, \qquad \alpha_2 = \frac{\lambda_1 - \lambda_2}{4\kappa^2 ||Q||}$$

and $\alpha_1 \in (0, (\lambda_1 - \lambda_2)/2)$ *is the unique root of*

$$2\alpha_1 ||Q|| + \frac{1}{2} \alpha_1 \tau e^{\alpha_1 \tau} (\lambda_1 + \lambda_2) = \frac{1}{2}(\lambda_1 - \lambda_2).$$

Proof. Let

$$D = \frac{1}{2}(\lambda_1 + \lambda_2) I,$$

where I is the $n \times n$ identity matrix. Then

$$2(x - G(y))^T Q f(t,x,y) + x^T D x - y^T D y \le -\frac{1}{2}(\lambda_1 - \lambda_2)(|x|^2 + |y|^2)$$

for all $(t,x,y) \in R_+ \times R^n \times R^n$. By Theorem 3.1, equation (3.1) is almost surely exponentially stable. The proof is complete.

We shall now establish one more useful corollary.

Corollary 3.3 *Let (3.2) and (2.2) hold with* $\kappa \in (0,1)$ *and* $\delta, \gamma > 0$. *Assume that there exists a symmetric positive definite* $n \times n$ *matrix* Q *as well as four positive constants* λ_1–λ_4 *such that*

$$x^T Q f(t,x,0) \le -\lambda_1 |x|^2,$$
$$|f(t,x,y) - f(t,x,0)| \le \lambda_2 |y|,$$
$$|f(t,x,y)| \le \lambda_3 |x| + \lambda_4 |y|$$

for all $t \ge 0$, $x, y \in R^n$. *If*

$$(\lambda_2 + \kappa\lambda_3 + \kappa\lambda_4)||Q|| < \lambda_1,$$

then equation (3.1) is almost surely exponentially stable.

Proof. Compute, for all $t \ge 0$, $x, y \in R^n$,

$$\begin{aligned} &2(x - G(y))^T Qf(t,x,y) = 2x^T Qf(t,x,y) - 2G(y)^T Qf(t,x,y) \\ \le & 2x^T Qf(t,x,0) + 2x^T Q[f(t,x,y) - f(t,x,0)] + 2|G(y)|\,||Q||\,|f(t,x,y)| \\ \le & -2\lambda_1|x|^2 + 2\lambda_2||Q||\,|x|\,|y| + 2\kappa||Q||\,|y|(\lambda_3|x| + \lambda_4|y|) \\ \le & -2\lambda_1|x|^2 + \lambda_2||Q||(|x|^2 + |y|^2) + \kappa\lambda_3||Q||(|x|^2 + |y|^2) + 2\kappa\lambda_4||Q||\,|y|^2 \\ \le & -\big(2\lambda_1 - (\lambda_2 + \kappa\lambda_3)||Q||\big)|x|^2 + (\lambda_2 + \kappa\lambda_3 + 2\kappa\lambda_4)||Q||\,|y|^2 \end{aligned}$$

Therefore, by Corollary 3.2, equation (3.1) is almost surely exponentially stable. The proof is complete.

4. Examples

In this section we shall discuss a number of interesting examples in order to illustrate our theory. In these examples we shall omit mentioning the initial data which are always assumed to be in $\mathcal{C}^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$ anyway.

Example 4.1 Let us start with a linear one-dimensional neutral differential difference equation with damped stochastic perturbations of the form

$$d[x(t) - cx(t-\tau)] = [-ax(t) + bx(t-\tau)]dt + \sigma(t)dw(t), \tag{4.1}$$

on $t \ge 0$. We assume that a, b, c are all real numbers such that

$$|c| < 1 \qquad \text{and} \qquad a > |b|.$$

Assume also that $\sigma(t)$ is an m-row-vector-valued function defined on $t \ge 0$ such that

$$|\sigma(t)|^2 \le \delta\varepsilon^{-\gamma t} \qquad \text{for all } t \ge 0$$

with both δ and γ positive constants. To apply Theorem 2.1, let $Q = 1$ and $D = a - bc$ and hence

$$H = \begin{pmatrix} -a - bc & b + ac \\ b + ac & -a - bc \end{pmatrix}.$$

It is easy to verify that H is negative definite and its biggest eigenvalue

$$-\lambda = -a - bc + |b + ac| < 0.$$

Therefore, equation (4.1) is almost surely exponentially stable. Moreover, the top Lyapunov exponent of the solution should not be greater than $-(\gamma \wedge \alpha \wedge \beta)/2$, where $\beta = -(2/\tau)\log|c|$ and $\alpha > 0$ is the unique root to the equation

$$2\alpha + \alpha\tau e^{\alpha\tau}(a - bc) = a + bc - |b + ac|. \tag{4.2}$$

For instance, let

$$a = 3, \quad b = 2, \quad c = 0.5, \quad \tau = 0.1, \quad \gamma = 0.5.$$

Then equation (4.2) becomes

$$2\alpha + 0.2\alpha e^{0.1\alpha} = 0.5$$

whose solution is $\alpha = 0.2268$. Also $\beta = -20 \log 0.5 = 13.8629$. So in this case the top Lyapunov exponent should not greater than -0.1134.

Example 4.2 Let us now consider a two-dimensional neutral stochastic differential difference equation

$$d[x(t) - Cx(t-\tau)] = [Ax(t) + Bx(t-\tau)]dt + \sigma(t)dw(t), \tag{4.3}$$

where

$$A = \begin{pmatrix} -3 & 1 \\ -1 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 0.2 & -0.3 \\ 0.1 & -0.2 \end{pmatrix},$$

and $\sigma : R_+ \to R^{2\times m}$ satisfies

$$trace[\sigma^T(t)\sigma(t)] \le \delta e^{-0.5t} \qquad \text{for all } t \ge 0$$

in which $\delta > 0$ is a constant. Choose $Q =$ the identity matrix and

$$D = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}.$$

Then the symmetric matrix H defined in Theorem 2.1 has the form

$$H = \begin{pmatrix} -3.0 & 0 & -0.3 & -2.1 \\ 0 & -3.0 & 1.2 & 1.5 \\ -0.3 & 1.2 & -2.8 & -0.1 \\ -2.1 & 1.5 & -0.1 & -4.8 \end{pmatrix}.$$

It can be checked that H is negative definite and its biggest eigenvalue is -0.84888. It is also not difficult to compute $||C|| = 0.423607$. Hence, by Theorem 2.1, equation (4.3) is almost surely exponentially stable. Moreover, the top Lyapunov exponent should not be greater than $-(0.5 \wedge \alpha \wedge \beta)/2$, where $\beta = 1.7178/\tau$ and $\alpha > 0$ is the root of

$$2\alpha + 5\alpha\tau e^{\alpha\tau} = 0.84888. \tag{4.4}$$

For instance, if $\tau = 0.1$, then equation (4.4) becomes

$$2\alpha + 0.5\alpha e^{0.1\alpha} = 0.84888$$

which has the solution $\alpha = 0.3372$. Also $\beta = 17.178$. Hence, in this case the top Lyapunov exponent should not be greater than -0.1686.

Example 4.3 Consider an n-dimensional nonlinear neutral differential difference equation with damped stochastic perturbations

$$d[x(t) - G(x(t-\tau))] = [f_1(x(t)) + f_2(x(t-\tau))]dt + \sigma(t)dw(t), \tag{4.5}$$

where G and $\sigma(t)$ are the same as defined in Section 3, f_1, f_2 are both locally Lipschitz continuous functions from R^n to itself. Assume (3.2) and (2.2) hold with $\kappa \in (0,1)$ and δ, $\gamma > 0$. Assume also that there exist positive constants $\lambda_i, 1 \le i \le 4$ such that

$$x^T f_1(x) \le -\lambda_1 |x|^2, \qquad -G^T(y) f_2(y) \le -\lambda_2 |y|^2,$$
$$|f_1(x)| \le \lambda_3 |x|, \qquad |f_2(y)| \le \lambda_4 |y|$$

for all $t \ge 0$ and $x, y \in R^n$. Compute

$$\begin{aligned} &2(x - G(y))^T (f_1(x) + f_2(y)) \\ &\le -2\lambda_1 |x|^2 - 2\lambda_2 |y|^2 + 2\kappa\lambda_3 |x||y| + 2\lambda_4 |x||y| \\ &\le -(2\lambda_1 - \kappa\lambda_3 - \lambda_4)|x|^2 + (-2\lambda_2 + \kappa\lambda_3 + \lambda_4)|y|^2. \end{aligned} \tag{4.6}$$

Hence, by Corollary 3.2, if

$$2\lambda_1 > \kappa\lambda_3 + \lambda_4 \tag{4.7}$$

and

$$2\lambda_1 + 2\lambda_2 > 2\kappa\lambda_3 + 2\lambda_4 \tag{4.8}$$

then equation (4.5) is almost surely exponentially stable.

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