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STRICT CONCAVITY OF THE HALF PLANE INTERSECTION EXPONENT FOR PLANAR BROWNIAN MOTION

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Abstact The intersection exponents for planar Brownian motion measure the exponential decay of probabilities of nonintersection of paths. We study the intersection exponent $\xi(\lambda_1, \lambda_2)$ for Brownian motion restricted to a half plane which by conformal invariance is the same as Brownian motion restricted to an infinite strip. We show that ξ is a strictly concave function. This result is used in [11] to establish a universality result for conformally invariant intersection exponents.

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1 Introduction

The intersection exponents for planar Brownian motion give the exponential rate of decay of probabilities of certain nonintersection events. The importance of the exponents can be seen both in their relevance to fine properties of Brownian paths [6, 7, 8] and in their apparent relation to critical exponents for self-avoiding walks and percolation [5, 11]. Most of the work (see [9] and references therein) has focused on the exponents for Brownian motion in the whole plane. However, recent work using conformal invariance [10] shows that the exponents for Brownian motions restricted to a half plane are fundamental in understanding all exponents.

The purpose of this paper is to study the half plane exponents $\xi = \xi(\lambda_1, \lambda_2)$. These exponents, which we define below, are denoted by $\tilde{\xi}(\lambda_1, 1, \lambda_2)$ in [10]; however, since we will only be considering these exponents in this paper we choose the simpler notation. The main result of this paper is that $\xi(\lambda_1, \lambda_2)$ is a strictly concave function. The corresponding result for the whole space exponent was proved in [9]. While the basic framework of the argument in this paper is similar to that in [9], there are two differences which make the arguments in this paper somewhat nicer. First, while [9] discussed both two and three dimensional Brownian motions, this paper only considers planar Brownian motions. Hence, conformal invariance can be exploited extensively. Second, a coupling argument is used which improves the rate of convergence to an invariant measure; in particular, the stretched exponential rate, $O(e^{-\beta\sqrt{n}})$, in [9] is improved to an exponential rate, $O(e^{-\beta n})$, here. The coupling argument is similar to the argument used in [4] (see [4] for other references for coupling arguments). The main theorem in this paper is used in [11] to show universality among conformal invariance exponents. This latter paper gives the first rigorous result that indicates why self-avoiding walk and percolation exponents in two dimensions should be related to Brownian exponents.

We will now give the definition of $\xi(\lambda_1, \lambda_2)$. Rather than considering Brownian motions restricted to the upper half plane

$$\mathcal{H} = \{x + iy : y > 0\},$$

we will study Brownian motions restricted to the infinite strip

$$\mathcal{J} = \{x + iy : -\frac{\pi}{2} < y < \frac{\pi}{2}\}.$$

Note that the map $z \mapsto \exp\{z + i(\pi/2)\}$ takes $\mathcal{J}$ conformally onto $\mathcal{H}$, and hence, there is an immediate relation between paths in $\mathcal{J}$ and paths in $\mathcal{H}$. Let B_t be a complex valued Brownian motion defined on the probability space $(\Omega, \mathbf{P})$, and for $n \in \mathbb{R}$, let T_n be the stopping time

$$T_n = \inf\{t : \Re[B_t] = n\}.$$

Assume for now that B_0 has a uniform distribution on $[-i\pi/2, i\pi/2]$. Let J_n be the event

$$J_n = \{B[0, T_n] \subset \mathcal{J}\}.$$

It is well known that

$$\mathbf{P}(J_n) \asymp e^{-n}, \quad n \to \infty, \tag{1}$$

where we write $\asymp$ to denote that each side is bounded above by a positive constant times the other side. (If we use the map $z \mapsto \exp\{z + i(\pi/2)\}$ to take paths to the upper half plane, this estimate can be deduced from the "gambler's ruin" estimate for one dimensional Brownian motion.)

Let B^1_t be another complex valued Brownian motion defined on a different probability space $(\Omega_1, \mathbf{P}_1)$ with stopping times

$$T^1_n = \inf\{t : \Re[B^1_t] = n\}.$$

Assume for now that B^1_0 has a uniform distribution on $[-i\pi/2, i\pi/2]$. If $w, z \in \mathbb{C}$, we write $w \succ z$ if $\Im(w) > \Im(z)$. Define the $(\Omega, \mathbf{P})$ random variables

$$Z^+_n = \mathbf{P}_1\{B^1[0, T^1_n] \subset \mathcal{J} \setminus B[0, T_n];\ B^1(T^1_n) \succ B(T_n)\}\ 1_{J_n},$$

$$Z^-_n = \mathbf{P}_1\{B^1[0, T^1_n] \subset \mathcal{J} \setminus B[0, T_n];\ B(T_n) \succ B(T^1_n)\}\ 1_{J_n}.$$

If $\lambda_1, \lambda_2 \geq 0$, the half plane exponent $\xi = \xi(\lambda_1, \lambda_2)$ is defined by

$$\mathbf{E}[(Z^+_n)^{\lambda_1}(Z^-_n)^{\lambda_2}] \approx e^{-\xi n}, \quad n \to \infty.$$

Here we write $\approx$ for logarithmic asymptotics, i.e., the logarithms of both sides are asymptotic. If $\lambda_1 = 0$ or $\lambda_2 = 0$ we use the convention $0^0 = 0$, i.e.,

$$(Z^+_n)^0 = (Z^-_n)^0 = 1_{J_n}.$$

The existence of such a ξ was established in [10]; we will reprove this in this paper and show, in fact, that

$$\mathbf{E}[(Z^+_n)^{\lambda_1}(Z^-_n)^{\lambda_2}] \asymp e^{-\xi n}, \quad n \to \infty. \tag{2}$$

Moreover, for each $M < \infty$, the implicit multiplicative constants in (2) can be chosen uniformly for $0 \leq \lambda_1, \lambda_2 \leq M$. The estimate (1) shows that

$$\xi(0,0) = 1.$$

Also by considering Brownian motions restricted to rectangles of height $\pi/3$, we can see that (1) implies

$$\xi(\lambda_1, \lambda_2) \leq 3(1 + \lambda_1 + \lambda_2) \leq 3(2M+1), \quad 0 \leq \lambda_1, \lambda_2 \leq M.$$

The main result of this paper that is used in [11] is the following.

Theorem 1 *Let $\lambda_2 \geq 0$ and let $\xi(\lambda) = \xi(\lambda, \lambda_2)$. Then ξ is C^2 for $\lambda > 0$ with*

$$\xi''(\lambda) < 0, \quad \lambda > 0.$$

It is conjectured [10], in fact, that

$$\xi(\lambda_1, \lambda_2) = \frac{(\sqrt{24\lambda_1 + 1} + \sqrt{24\lambda_2 + 1} + 3)^2 - 1}{24}.$$

Of course, if this conjecture is true, Theorem 1 would follow immediately. However, this conjecture is still open, and it is possible that Theorem 1 will be useful in proving the conjecture. In [10] a whole family of half plane intersection exponents

$$\tilde{\xi}(a_1, \ldots, a_p)$$

were defined for any nonnegative $a_1, \ldots, a_p$; however, it was shown that all of these values can be determined from the values of $\xi(\lambda_1, \lambda_2)$. This is why we restrict our attention to $\xi(\lambda_1, \lambda_2)$ in this paper.

Studying exponential decay rates (i.e., large deviation rates) generally leads to studying the behavior of processes conditioned on this exceptional behavior, and this is the case here. Fix λ_2 and let

$$\Psi_n = -\log Z_n^+,$$

$$\xi_n(\lambda) = -\frac{1}{n} \log \mathbf{E}[(Z_n^+)^{\lambda}(Z_n^-)^{\lambda_2}] = -\frac{1}{n} \log \mathbf{E}[e^{-\lambda \Psi_n}(Z_n^-)^{\lambda_2}].$$

Note that (2) implies as $n \to \infty$,

$$\xi_n(\lambda) = \xi(\lambda) + O(\frac{1}{n}).$$

Direct differentiation give

$$\xi_n'(\lambda) = \frac{1}{n} \tilde{\mathbf{E}}_n(\Psi_n),$$

$$\xi_n''(\lambda) = -\frac{1}{n} \mathbf{var}_n(\Psi_n),$$

where $\tilde{\mathbf{E}}_n$ and $\mathbf{var}_n$ denote expectation and variance with respect to the measure

$$\frac{(Z_n^+)^{\lambda}(Z_n^-)^{\lambda_2}}{\mathbf{E}[(Z_n^+)^{\lambda}(Z_n^-)^{\lambda_2}]}\, d\mathbf{P}. \tag{3}$$

What we prove is that there is an $a = a(\lambda, \lambda_2)$ and a $v = v(\lambda, \lambda_2)$ such that

$$\tilde{\mathbf{E}}_n(\Psi_n) = an + O(1),$$

$$\mathbf{var}_n(\Psi_n) = vn + O(1).$$

Moreover, we show that $|\xi'''(\lambda)|$ is bounded so that ξ is C^2 with

$$\xi'(\lambda) = a, \quad \xi''(\lambda) = -v.$$

Part of the work is showing that the measures on paths (3) approach an invariant measure and that the random variables

$$\Psi_1 - \Psi_0, \ldots, \Psi_n - \Psi_{n-1}$$

are approximately a stationary sequence with exponentially small correlations. A separate argument is given to show that $v > 0$ which then gives Theorem 1.

We now outline the paper. Section 2 derives a number of results using conformal invariance. We start by reviewing facts about Brownian motion in a rectangle with the intent of using conformal invariance to relate these results to regions that are conformally equivalent to a rectangle. We assume that the reader is familiar with the conformal invariance of Brownian motion (see, e.g., [2, V]). An important conformal invariant is extremal distance. This quantity was first studied in complex variables (see, e.g., [1]), but we give a self-contained treatment of the facts that we need. Section 3 discusses the intersection exponent. The main goals of this section are to derive the separation lemma, Lemma 9, and to use this to derive (2). Since the results in this section are very similar to results in corresponding sections of [6, 7, 8], we are somewhat brief. The last section constructs the invariant measure on paths and uses this to justify the differentiation of $\xi(\lambda)$. The proof here is easier than that of the corresponding parts of [6, 7, 8]; we use a coupling argument derived from that in [4] to show the exponential decay of correlations.

We make some assumptions about constants in this paper. We fix $M < \infty$ and we consider only $0 \leq \lambda_1, \lambda_2 \leq M$. Constants $c, c_1, c_2, \ldots$ and $\beta, \beta_1, \beta_2, \ldots$ are positive constants that may depend on M but do not depend on anything else. (In fact, the constants in Section 2 do not depend on M.) In particular, constants do not depend on the particular λ_1, λ_2. Constants c, c_1, c_2 and β may change from line to line, but $c_3, c_4, \ldots$ and $\beta_1, \beta_2, \ldots$ will not vary. If $\delta_n \downarrow 0$, we write

$$f(n) = g(n) + O(\delta_n)$$

if there is a constant c such that

$$|f(n) - g(n)| \leq c\delta_n.$$

Similarly, if f, g are positive, we write

$$f(n) = g(n)[1 + O(\delta_n)]$$

if

$$\log f(n) = \log g(n) + O(\delta_n).$$

All implicit constants in the notations $O(\cdot)$ and $\asymp$ will depend only on M and not on λ_1, λ_2. We will write $\mathbf{P}^z, \mathbf{E}^z$ to denote probabilities and expectations assuming $B_0 = z$. If the z is omitted the assumption is that B_0 has a uniform distribution on $[-i\pi/2, i\pi/2]$. The same assumptions will be made about $\mathbf{P}_1^z, \mathbf{P}_1$.

We will consider two norms on probability measures. The first is the standard variation measure

$$\|P_1 - P_2\| = \sup_E |P_1(E) - P_2(E)|.$$

The second norm will be

$$\|P_1 - P_2\|_1 = \sup_E |\log P_1(E) - \log P_2(E)|. \tag{4}$$

Here the supremum is over all E for which $P_1(E) + P_2(E) > 0$. Equivalently, we could define

$$\|P_1 - P_2\|_1 = \|\log \frac{dP_1}{dP_2}\|_\infty,$$

where the norm on the right hand side is the standard L^∞ norm. This norm could be infinite. Note that

$$\|P_1 - P_2\|_1 \geq \|P_1 - P_2\|.$$

If $\delta_n \downarrow 0$, then

$$\|P_n - Q_n\| = O(\delta_n)$$

will also be written

$$P_n = Q_n + O(\delta_n),$$

and

$$\|P_n - Q_n\|_1 = O(\delta_n)$$

will also be written

$$P_n = Q_n[1 + O(\delta_n)].$$

I would like to thank Wendelin Werner for many useful conversations on intersection exponents.

2 Conformal Invariance

2.1 Rectangle estimates

Let $\mathcal{J}$ be the infinite strip as in the introduction, and

$$\mathcal{J}_+ = \{z \in \mathcal{J} : \Re(z) > 0\},$$

$$\mathcal{J}_L = \{z \in \mathcal{J}_+ : \Re(z) < L\}.$$

Let ∂_1, ∂_2 be the vertical boundaries of $\mathcal{J}_L$,

$$\partial_1 = \{iy : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\},$$

$$\partial_2 = \partial_{2,L} = \{L + iy : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\}.$$

We will need some standard facts about Brownian motion in rectangles and half infinite strips. What we need can be derived from the exact form of the solution of the Dirichlet problem in a rectangle (see, e.g., [3, Section 11.3]); we will just state the results that we will need. If U is any open region, let

$$\tau_U = \inf\{t : B_t \in \partial U\}.$$

For this subsection, let $\tau = \tau_{\mathcal{J}_+}, \tau_L = \tau_{\mathcal{J}_L}$. First, for real $n > 0$

$$\mathbf{P}^n\{B(\tau) \in \partial_1\} \asymp e^{-n}.$$

Also,

$$\mathbf{P}^n\left\{B(\tau) \in [-\frac{i\pi}{4}, \frac{i\pi}{4}];\ B[0,\tau] \cap \{\Re(z) \geq n+1\} = \emptyset\right\} \asymp e^{-n}. \tag{5}$$

Consider an "excursion" W_t from $\{\Re(z) = L\}$ to $\{\Re(z) = 0\}$. There are a number of ways to get such an excursion. One way is to let $z \in \mathbb{C}$ with $\Re(z) \geq L$, setting

$$T_0 = \inf\{t : \Re(B_t) = 0\},$$

$$S_L = \sup\{t \leq T_0 : \Re(B_t) = L\},$$

and defining

$$W_t = B(t + S_L), \quad 0 \leq t \leq T_0 - S_L. \tag{6}$$

By vertical translation, we can allow the excursion to start at $L + iy$ for any $y \in \mathbb{R}$. An excursion starts on $\{\Re(z) = L\}$, immediately enters $\{\Re(z) < L\}$, and then has the distribution of a Brownian motion conditioned to leave $\{0 < \Re(z) < L\}$ at $\{\Re(z) = 0\}$. Such a process can also be given by $(L - X_t) + iY_t$ where X_t is a Bessel-3 process stopped when $X_t = L$, and Y_t is an independent Brownian motion. Suppose W_t is such an excursion with $W_0 \in \mathcal{J}_+$ and let

$$\hat{T} = \hat{T}_0 = \inf\{t : \Re[W_t] = 0\}.$$

Consider the event

$$G_L = \{W[0, \hat{T}) \subset \mathcal{J}_+\} = \{W(0, \hat{T}) \subset \mathcal{J}_L\}.$$

For each $z \in \mathcal{J}_+$ with $\Re(z) = L$, there is a probability measure $Q_{z,L}$ on ∂_1 given by

$$Q_{z,L}(E) = \mathbf{P}^z\{W(\hat{T}) \in E \mid G_L\}.$$

The following lemma whose proof we omit can be proved either by direct examination of the solution of the Dirichlet problem or by a coupling of h-processes.

Lemma 1 *Let*

$$g(L) = \sup \|Q_{L+iy_1,L} - Q_{L+iy_2,L}\|_1,$$

where the supremum is over all $-\pi/2 < y_1, y_2 < \pi/2$, and $\|\cdot\|_1$ is the norm as defined in (4). Then $g(L) < \infty$ for every $L > 0$. Moreover, there exist c_3, β_1 such that for all $L \geq 1$,

$$g(L) \leq c_3 e^{-L\beta_1}. \tag{7}$$

Suppose $U \subset \mathcal{J}_+$ is open and connected with

$$\mathcal{J}_+ \cap \{\Re(z) \leq L\} \subset U.$$

Let $z \in U$ with $\Re(z) \geq L$. Let $\tilde{Q}_{z,U}$ be the measure on ∂_1,

$$\tilde{Q}_{z,U}(E) = \mathbf{P}^z\{B(\tau_U) \in E \mid B(\tau_U) \in \partial_1\}.$$

By splitting the path $B[0, T_0]$ as in (6), we see that

$$\|\tilde{Q}_{z,U} - Q_{L+iy,L}\|_1 \leq g(L), \quad -\frac{\pi}{2} < y < \frac{\pi}{2}. \tag{8}$$

If we let $L \to \infty$, we see that there is a density

$$H(y), \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2},$$

such that if Q denotes the measure $H(y)\, dy$, then

$$\|Q_{L+iy,L} - Q\|_1 \leq g(L). \tag{9}$$

Similarly, (8) holds with Q replacing $Q_{L,L}$,

$$\|\tilde{Q}_{z,U} - Q\|_1 \leq g(L). \tag{10}$$

2.2 Extremal distance

Let U be a bounded simply connected domain in $\mathbb{C}$ whose boundary is a Jordan curve. Let A_1, A_2 be disjoint closed connected subsets of ∂U, each larger than a single point. We denote the other arcs by A_3, A_4 so that $\partial U = A_1 \cup A_2 \cup A_3 \cup A_4$; the arcs A_1, A_2, A_3, A_4 are closed and the intersection of two arcs is empty or a single point; and going counterclockwise the order of the arcs is A_1, A_3, A_2, A_4. There is a unique $L = L(A_1, A_2; U)$ such that there is a conformal transformation

$$F : U \to \mathcal{J}_L$$

which can be extended continuously to the boundary so that

$$F(A_1) = \partial_1,$$

$$F(A_2) = \partial_2,$$

$$F(A_3) = \partial_3 = \partial_{3,L} \doteq \{x - \frac{i\pi}{2} : 0 \le x \le L\},$$

$$F(A_4) = \partial_4 = \partial_{4,L} \doteq \{x + \frac{i\pi}{2} : 0 \le x \le L\}.$$

We call L the *extremal distance* (this is actually π^{-1} times the standard extremal length or extremal distance as in [1], but this definition will be more convenient for us). As defined, extremal distance is clearly a conformal invariant.

Let us define a similar quantity which is sometimes easier to estimate. Let B_t be a complex valued Brownian motion and for $j = 1, 2, 3, 4$, let

$$f_j(z) = f_j(z; A_1, A_2, U) = \mathbf{P}^z\{B(\tau_U) \in A_j\}.$$

In other words, f_j is the solution of the Dirichlet problem with boundary value 1 on A_j and 0 on $\partial U \setminus A_j$. Let $f^*_{j,L}(z)$ be the corresponding functions on $\mathcal{J}_L$,

$$f^*_{j,L}(z) = f_j(z; \partial_1, \partial_2, \mathcal{J}_L).$$

Let

$$\phi(A_1, A_2; U) = \sup_{z \in U} \min\{f_1(z), f_2(z)\},$$

and

$$\phi_L = \phi(\partial_1, \partial_2; \mathcal{J}_L).$$

By conformal invariance,

$$\phi(A_1, A_2; U) = \phi_{L(A_1, A_2; U)}.$$

It is easy to check that $L \mapsto \phi_L$ is a continuous, strictly decreasing function with

$$\phi_L \to \frac{1}{2}, \quad L \to 0+.$$

Estimates for rectangles tell us that $\phi_L \asymp e^{-L/2}$ as $L \to \infty$, i.e., there is a c_4 such that for all $L \ge 1$,

$$|\log \phi(L) + \frac{L}{2}| \le c_4. \tag{11}$$

Using conformal equivalence of rectangles and symmetry,

$$\phi(\partial_3, \partial_4; \mathcal{J}_L) = \sup_{z \in \mathcal{J}_L} \min\{f^*_{3,L}(z), f^*_{4,L}(z)\} = \frac{1}{2} - \phi_L,$$

and the supremum is obtained at the center. By conformal invariance, we then get for all such domains,

$$\phi(A_1, A_2; U) + \phi(A_3, A_4; U) = \frac{1}{2}.$$

Lemma 2 *If $z \in \mathcal{J}_L$ and $\Re(z) = L/2$,*

$$f^*_{3,L}(z) + f^*_{4,L}(z) \geq f^*_{3,L}(\frac{L}{2}) + f^*_{4,L}(\frac{L}{2}) = 1 - 2\phi_L.$$

Proof. We only need to prove the first inequality. Let $z = (L/2) + iy$ with $|y| < \pi/2$. Let

$$S' = S'_L = \inf\{t : \Re[B_t] \in \{0, L\}\},$$

$$S = \inf\{t : |\Im[B_t]| = \frac{\pi}{2}\}.$$

Note that S' and S are independent, the distribution of S' does not depend on y, and

$$f^*_{3,L}(z) + f^*_{4,L}(z) = \mathbf{P}^y\{S < S'\}.$$

Hence it suffices to show for all $|y| < \pi/2$ and all $t > 0$,

$$\mathbf{P}^0\{S < t\} < \mathbf{P}^y\{S < t\}.$$

Let

$$S^y = \inf\{t : |\Im[B_t]| = y\}.$$

Then, if $|\Im[B_0]| \leq y$,

$$S = S^y + (S - S^y).$$

But $S - S^y$ has the distribution of S given $\Im[B_0] = y$. $\square$

Lemma 3 *Suppose U is a domain as above such that for some $n > 0$,*

$$A_1 \subset \{\Re(z) \leq 0\},$$

$$A_2 \subset \{\Re(z) \geq n\},$$

and

$$U \cap \{0 < \Re(z) < n\} \subset \mathcal{J}.$$

Then

$$L(A_1, A_2; U) \geq n.$$

Proof. By comparison with $\mathcal{J}_n$, if $z \in U$ with $\Re(z) = n/2$,

$$f_3(z) + f_4(z) \geq f^*_{3,n}(z) + f^*_{4,n}(z) \geq 1 - 2\phi_n.$$

By continuity, we can find a z_0 with $\Re(z_0) = n/2$ and $f_3(z_0) = f_4(z_0)$. Hence,

$$\frac{1}{2} - \phi(A_1, A_2; U) \geq \min\{f_3(z_0), f_4(z_0)\} \geq \frac{1}{2} - \phi_n.$$

Hence $\phi_n \geq \phi(A_1, A_2; U)$ and $L(A_1, A_2; U) \geq n$. $\square$

Suppose $\eta : [0, 1] \to \mathbb{C}$ is a simple, continuous path with

$$\Im[\eta(0)] = -\frac{\pi}{2}, \quad \Im[\eta(1)] = \frac{\pi}{2},$$

$$\eta(0,1) \subset \mathcal{J}.$$

We will call such a path a *crossing path.* If η_1, η_2 are two crossing paths we write $\eta_1 \prec \eta_2$ if $\eta_1[0,1] \cap \eta_2[0,1] = \emptyset$ and η_1 is to the "left" of η_2. If $\eta_1 \prec \eta_2$, we write $\mathcal{U} = \mathcal{U}(\eta_1, \eta_2)$ for the bounded domain with $A_1 = \eta_1[0,1], A_2 = \eta_2[0,1]$,

$$A_3 = \{x - \frac{i\pi}{2} : \Re[\eta_1(0)] \leq x \leq \Re[\eta_2(0)]\},$$

$$A_4 = \{x + \frac{i\pi}{2} : \Re[\eta_1(1)] \leq x \leq \Re[\eta_2(1)]\}.$$

We call $\mathcal{U}$ (more precisely, $\mathcal{U}, A_1, A_2$) the *generalized rectangle* generated by η_1, η_2. Note that the rectangles $\mathcal{J}_L$ are generalized rectangles.

Lemma 4 *There exists a c_5 such that the following holds. Suppose $\eta_1 \prec \eta_2$ are crossing paths with*

$$\sup\{\Re[\eta_1(t)] : 0 \leq t \leq 1\} = 0,$$

and

$$\inf\{\Re[\eta_2(t)] : 0 \leq t \leq 1\} = n > 0.$$

Let $\mathcal{U} = \mathcal{U}(\eta_1, \eta_2)$ be the generalized rectangle generated by η_1, η_2. Then

$$n \leq L(A_1, A_2; U) \leq n + c_5.$$

Proof. The first inequality follows immediately from Lemma 3. For the other direction let $z = n/2$. Then by (5),

$$\mathbf{P}^z\{B(\tau_{\mathcal{J}_n}) \in [-\frac{i\pi}{4}, \frac{i\pi}{4}]\} \geq ce^{-n/2}.$$

But geometric considerations give

$$\mathbf{P}^z\{B(\tau_U) \in A_1 \mid B(\tau_{\mathcal{J}_n}) \in [-\frac{i\pi}{4}, \frac{i\pi}{4}]\} \geq c.$$

Hence

$$\mathbf{P}^z\{B(\tau_U) \in A_1\} \geq ce^{-n/2}.$$

A similar argument holds for A_2 giving

$$\phi(A_1, A_2; U) \geq \min\{f_1(z), f_2(z)\} \geq ce^{-n/2}.$$

The lemma then follows from (11). □

Lemma 5 *Let $\psi(d)$ be the maximum of*

$$L(\eta_1[0,1], \eta_2[0,1]; \mathcal{U}(\eta_1, \eta_2)).$$

over all crossing paths $\eta_1 \prec \eta_2$ with

$$\text{dist}(\eta_1[0,1], \eta_2[0,1]) \leq d.$$

Then

$$\lim_{d \to 0+} \psi(d) = 0.$$

Proof. Let
$$d = \mathrm{dist}(\eta_1[0,1], \eta_2[0,1]),$$
and choose s, t with
$$|\eta_1(s) - \eta_2(t)| = d.$$
Without loss of generality assume $d < 1/10$ and $\Im[\eta_1(s)] \leq 0$. Let
$$w = \frac{1}{2}[\eta_1(s) + \eta_2(t)] + \frac{d}{4}i,$$
and consider the curve consisting of the straight line from $\eta_1(s)$ to w followed by the line from w to $\eta_2(t)$. It is easy to see that there is a c such that for all z in this line
$$f_1(z) + f_2(z) \geq c.$$
By continuity we can find a z_0 on this line so that
$$f_1(z_0), f_2(z_0) \geq c/2. \tag{12}$$
But the Beurling projection theorem (see, e.g., [2, V.4]) gives
$$f_4(z_0) \leq c_2 d^{1/2}. \tag{13}$$
Consideration of (12) and (13) on the rectangle $\mathcal{J}_L$ shows that $L \to 0$ as $d \to 0+$. □

2.3 Path domains

Let $\mathcal{X}$ denote the set of continuous functions
$$\gamma : [0, \infty) \longrightarrow \mathcal{J}$$
with
$$\Re[\gamma(0)] = 0,$$
$$\Re[\gamma(t)] > 0, \quad t > 0,$$
$$\Re[\gamma(t)] \to \infty, \quad t \to \infty.$$
Given $\gamma \in \mathcal{X}$, let $D = D(\gamma)$ be the connected component of $\mathcal{J}_+ \setminus \gamma(0, \infty)$ whose boundary includes
$$\{x + \frac{i\pi}{2} : 0 \leq x < \infty\}$$
and
$$\partial_\gamma \doteq \{iy : \Im[\gamma(0)] \leq y \leq \frac{\pi}{2}\}.$$
We will call such domains *path domains*; in particular, $D(\gamma)$ is the path domain associated to γ. We also consider $\mathcal{J}_+$ as the path domain associated to the function
$$\gamma^*(t) = t - \frac{i\pi}{2}, \quad 0 \leq t < \infty,$$
even though this function is not actually in $\mathcal{X}$.

Let Δ denote the open unit disk and let A be the arc $[e^{i\pi/2}, e^{3i\pi/2}]$ on $\partial\Delta$. There is a unique conformal transformation $f = f_\gamma$,

$$f : D(\gamma) \to \Delta,$$

with

$$f(\partial_\gamma) = A,$$

$$f(\frac{i\pi}{2}) = i, \quad f(\gamma(0)) = -i, \quad f(\infty) = 1.$$

We let $F_\gamma = f_{\gamma^*}^{-1} \circ f_\gamma$, which is a conformal transformation taking $D(\gamma)$ to $\mathcal{J}_+$ and ∂_γ to ∂_1. There is a well-defined probability measure on Brownian excursions in the disk Δ starting at 1 and conditioned to leave Δ at A. (One way to obtain the measure is to define the measure for Brownian paths starting at $z \in \Delta$ conditioned to leave the disk at A and then taking a limit as the initial point z approaches 1.) By reversing time, we can consider these paths as starting at A and leaving the disk at 1. The lifetimes of these paths are random and finite. However, if these paths are conformally mapped to $D(\gamma)$ by f_γ^{-1} we get paths with infinite lifetime. This measure could also be defined by taking appropriate limits of paths in $\mathcal{J}_+$, and clearly the limit is conformally invariant. We denote this measure by $\nu(\gamma)$.

Let $\gamma \in \mathcal{X}$, and let $D = D(\gamma)$. If $n > 0$, let $y_n = y_n(\gamma)$ be the largest y such that

$$n + iy \in \gamma(0, \infty),$$

and let

$$V_n = V_n(\gamma) = \{n + iy : y_n \le y \le \frac{\pi}{2}\},$$

$$V_n^o = V_n^o(\gamma) = \{n + iy : y_n < y < \frac{\pi}{2}\}.$$

Note that $D \setminus V_n^o$ consists of two connected components, a bounded component that we denote by V_n^- and an unbounded component that we denote by V_n^+. While we think of V_n^- as being to the left of $\{\Re(z) = n\}$ and V_n^+ as being to the right, note that both

$$V_n^- \cap \{\Re(z) > n\}$$

and

$$V_n^+ \cap \{\Re(z) < n\}$$

can be nonempty. Let

$$\sigma_n = \sigma_n(\gamma) = \sup\{t : \Re[\gamma(t)] = n\},$$

$$\kappa_n = \kappa_n(\gamma) = \inf\{t : \Re[\gamma(t)] = n\}.$$

Note that if $0 < m < n$, and $\sigma_m < \kappa_n$, then

$$V_m^- \cap \{\Re(z) \ge n\} = \emptyset.$$

If $0 < m < n$, let

$$V_{m,n} = V_m^+ \cap V_n^-.$$

Note that by Lemma 3,

$$L(V_m, V_n; V_{m,n}) \ge n - m. \tag{14}$$

2.4 Truncated paths

Fix $\gamma \in \mathcal{X}$ and let $F = F_\gamma$ be the conformal transformation taking $D = D(\gamma)$ to $\mathcal{J}_+$ as above. For $z \in D$, let

$$K(z) = K_\gamma(z) = \Re[F(z)],$$

and for $n > 0$, let

$$K_n = \sup\{K(z) : z \in D, \Re(z) = n\}.$$

We let $\nu = \nu(\gamma)$ be the measure on paths as in Section 2.3, and let ν^* be the corresponding measure on paths in $\mathcal{J}_+$. Note that ν and ν^* are related by the conformal map F. We write informally $\nu^* = F(\nu)$ although this notation ignores the time change involved in the conformal transformation.

If $\eta \in \mathcal{X}$, we define $\sigma_n(\eta), \kappa_n(\eta)$ as in Section 2.3. If $n > 0$, we use $\Phi_n\eta$ to represent the bounded path obtained from η by truncating the domain at $\sigma_n(\eta)$,

$$\Phi_n\eta(t) = \eta(t), \quad 0 \leq t \leq \sigma_n(\eta).$$

Let ν_n, ν_n^* denote the measures on truncated paths obtained from ν, ν^* by performing this truncation.

There is another way to get ν_n (or similarly ν_n^*) that we will describe now. For any $z \in D$ with $\Re(z) \geq n$, start a Brownian motion B_t at z. As before, let

$$T_n = \inf\{t : \Re(B_t) = n\}.$$

On the event

$$\{B(\tau_D) \in \partial_\gamma\},$$

consider the time reversal of the path $B[T_n, \tau_D]$. More precisely, we let

$$\eta(t) = B(\tau_D - t), \quad 0 \leq t \leq \tau_D - T_n.$$

The conditional measure of these paths given $B(0) = z$ and $B(\tau_D) \in \partial_\gamma$ gives a measure that we denote $\nu_{n,z}$. Let $Q_{n,z}$ be the measure on $\{w : \Re(w) = n\}$ obtained from the distribution of $B(T_n)$ given $B(0) = z$ and $T_n \leq \tau_D$. Then we can also describe $\nu_{n,z}$ by starting a Brownian motion on $\{w : \Re(w) = n\}$ using the initial distribution $Q_{n,z}$; conditioning on the event

$$\{B(\tau_D) \in \partial_\gamma\};$$

and considering the paths

$$\eta(t) = B(\tau_D - t), \quad 0 \leq t \leq \tau_D.$$

(Note that we do not fix a w and then do the conditioning, but rather we condition just once. In particular, the measure on $\{\Re(w) = n\}$ given by terminal points under $\nu_{n,z}$ is not the same as $Q_{n,z}$.) We can do the same construction on paths in $\mathcal{J}_+$ giving the measures $\nu_{n,z}^*, Q_{n,z}^*$.

By (7), if $s \geq 1$, $\Re(z), \Re(w) \geq n + s$, then

$$\|Q_{n,z}^* - Q_{n,w}^*\|_1 \leq c_3 e^{-\beta_1 s}.$$

By conformal transformation, we see that a similar result holds for $Q_{n,z}$. More precisely, if $s \geq 1$, $K(z), K(w) \geq K_n + s$, then

$$\|Q_{n,z} - Q_{n,w}\|_1 \leq c_3 e^{-\beta_1 s}.$$

Letting z tend to infinity, we therefore get measures Q_n, Q_n^* such that if $s \geq 1$ and $K(z) \geq K_n + s$,

$$\|Q_{n,z} - Q_n\|_1 \leq c_3 e^{-\beta_1 s}.$$

The measure ν_n as above can be obtained in the same was as the $\nu_{n,z}$, using Q_n as the initial measure. Note that Q_0 is the same as the Q of Section 2.1, and Q_n is just a translation of this measure.

Estimates for the rectangle tell us that if $n, s > 0$,

$$\nu^*\{\eta : \sigma_n(\eta) > \kappa_{n+s}(\eta)\} \leq$$

$$\nu^*\{\eta : \eta[\kappa_{n+s}(\eta), \infty) \cap \{\Re(z) \leq n\} = \emptyset\} \leq ce^{-2s}.$$

By conformal invariance, we get a similar result for ν,

$$\nu\{\eta : \exists\, z \in \eta[0, \sigma_n(\eta)] \text{ with } K(z) \geq K_n + s\} \leq ce^{-2s}. \tag{15}$$

Let

$$k_n = \inf\{K(z) : z \in D, \Re(z) = n\}.$$

If $\nu_n(m)$ denotes the conditional measure derived from ν_n by conditioning on the event

$$\{\kappa_m(\eta) > \sigma_n(\eta)\},$$

then (15) implies

$$\|\nu_n(s) - \nu_n\| \leq ce^{-2(k_m - K_n)},$$

where $\|\cdot\|$ denotes variation measure as in the introduction. Similarly, if $Q_{n,z}(m)$ denotes the measure on $\{\Re(w) = n\}$,

$$Q_{n,z}(m)[E] = \mathbf{P}^z\{B(T_n) \in E \mid T_n \leq \min\{\tau_D, T_m\}\},$$

and if z satisfies

$$K(z) \leq k_m - s,$$

then

$$\|Q_{n,z}(m) - Q_{n,z}\| \leq ce^{-2s}.$$

Let $\nu_{n,z}(m)$ be the measure defined similarly to $\nu_{n,z}$ except that the conditioning is on the event

$$\{B(\tau_D) \in \partial'; \tau_D < T_m\}.$$

We have derived the following lemma.

Lemma 6 *There exists a c such that the following holds. Suppose $0 < n < m < \infty$ and $z \in D$ with*

$$K_n + s \leq K(z) \leq k_m - s.$$

Then

$$\|\nu_{n,z}(m) - \nu_n\| \leq ce^{-2s}.$$

Let $n \geq 1$ and
$$S_{n-1} = \inf\{t : B_t \in V^o_{n-1}\}.$$
Let $U = V^-_{n-(1/4)}$. Then by (14) and comparison to the rectangle, we can see that if $z \in U \setminus V^-_{n-(1/2)}$,
$$\mathbf{P}^z\{B(\tau_U) \in \partial_\gamma \mid S_{n-1} < \tau_U\} \asymp \mathbf{P}^z\{B(\tau_D) \in \partial_\gamma \mid S_{n-1} < \tau_D\}. \tag{16}$$
Let $\tilde{\mathcal{X}}_n$ be the set of $\gamma \in \mathcal{X}$ such that
$$\sigma_{jn}(\gamma) < \kappa_{(j+1)n}(\gamma), \quad j = 1, 2, 3, 4,$$
or, equivalently,
$$\gamma[\kappa_{(j+1)n}(\gamma), \infty) \cap \{\Re(z) \leq jn\} = \emptyset, \quad j = 1, 2, 3, 4.$$
Note that if $\gamma \in \tilde{\mathcal{X}}_n$, then
$$V^-_{jn}(\gamma) \cap \{\Re(z) \geq (j+1)n\} = \emptyset, \quad j = 1, 2, 3, 4,$$
$$V^+_{(j+1)n}(\gamma) \cap \{\Re(z) \leq jn\} = \emptyset, \quad j = 1, 2, 3, 4.$$
It follows from Lemmas 3 and 4 that there is a c such that if $j = 1, 2$ and $z, w \in D(\gamma)$ with
$$\Re(z) \geq (j+2)n, \qquad \Re(w) \leq jn,$$
then
$$K(z) - K(w) \geq n - c.$$
In particular, if $2n \leq \Re(z) \leq 3n$,
$$\|\nu_{n,z}(5n) - \nu_n\| \leq ce^{-2n}. \tag{17}$$
Note that the measure $\nu_{n,z}(5n)$ depends only on
$$D \cap \{\Re(z) \leq 5n\},$$
and not on all of D.

2.5 An important lemma

In this subsection we will consider two paths $\gamma^1, \gamma^2 \in \mathcal{X}$. We use the notation of the previous subsection except that we use superscripts to indicate which path we are considering. For example, the measure ν in the previous section corresponding to γ^1 and γ^2 will be denoted ν^1 and ν^2 respectively. We will write
$$\gamma^1 =_n \gamma^2$$
if
$$\Phi_n \gamma^1 = \Phi_n \gamma^2.$$
Note that if
$$\gamma^1 =_n \gamma^2,$$

then

$$D(\gamma^1) \cap \{\Re(z) < n\} = D(\gamma^2) \cap \{\Re(z) < n\}.$$

Also, if

$$\gamma^1 =_{5n} \gamma^2,$$

then $\gamma^1 \in \tilde{\mathcal{X}}_n$ if and only if $\gamma^2 \in \tilde{\mathcal{X}}_n$. The following lemma is then a corollary of (17).

Lemma 7 *There is a constant c_8 such that the following holds. Suppose $\gamma^1, \gamma^2 \in \tilde{\mathcal{X}}_n$ and*

$$\gamma^1 =_{5n} \gamma^2.$$

Then

$$\|\nu_n^1 - \nu_n^2\| \leq ce^{-2n}.$$

3 Intersection exponent

In this section we define the intersection exponent and derive some important properties. We use the notation in the introduction. On the event J_n, the random variables Z_n^+, Z_n^- can be defined by

$$Z_n^+ = \mathbf{P}_1\{B^1[0,T_n^1] \cap B[0,T_n] = \emptyset; B^1[0,T_n^1] \subset \mathcal{J}; B^1(T_n^1) \succ B(T_n)\},$$

$$Z_n^- = \mathbf{P}_1\{B^1[0,T_n^1] \cap B[0,T_n] = \emptyset; B^1[0,T_n^1] \subset \mathcal{J}; B(T_n) \succ B^1(T_n^1)\}.$$

On the event J_n^c, we have $Z_n^+ = Z_n^- = 0$. Let

$$\Theta_n = \Theta_n(\lambda_1, \lambda_2) = (Z_n^+)^{\lambda_1}(Z_n^-)^{\lambda_2}.$$

Recall that we use the convention that $0^0 = 0$, i.e., $(Z_n^+)^0$ is the indicator function of the event $\{Z_n^+ > 0\}$. Let

$$q_n = q_n(\lambda_1, \lambda_2) = \mathbf{E}[\Theta_n].$$

The Harnack inequality applied separately to B and B^1 can be used to show that there is c_7 such that

$$q_{n+m} \leq c_7 q_n q_{m-1}.$$

In particular, $\log(c_7 q_{n-1})$ is a subadditive function, and hence by standard arguments there is a $\xi = \xi(\lambda_1, \lambda_2)$, which we call the intersection exponent, such that

$$q_n \approx e^{-\xi n}, \quad n \to \infty.$$

Moreover, there is a c_8 such that

$$q_n \geq c_8 e^{-\xi n}.$$

3.1 Separation lemma

Let $\mathcal{F}_n$ denote the σ-algebra generated by

$$B_t, \quad 0 \le t \le T_n.$$

Note that the random variables Z_n^+, Z_n^-, Θ_n are functions of $B[0, T_n]$ and hence are $\mathcal{F}_n$-measurable. Let δ_n be the $\mathcal{F}_n$-measurable random variable

$$\delta_n = \mathrm{dist}\Big\{ \{n + \frac{i\pi}{2}, n - \frac{i\pi}{2}\} \ , \ B[0, T_n] \Big\},$$

and let U_n be the $\mathcal{F}_n$-measurable event

$$U_n = \Big\{ B[0, T_n] \cap \{\Re(z) \ge n - 1\} \subset \{|\Im(z)| \le \frac{\pi}{6}\} \Big\}.$$

The following lemma can be proved easily using conformal invariance and estimates for Brownian motion in a wedge. We omit the proof.

Lemma 8 *There exist c, β such that the following is true. Suppose $n \ge 0$, $3/2 \le r \le 3$, and $\epsilon > 0$. Then,*

$$\mathbf{E}[\Theta_{n+r}\ J_{n+r}\ U_{n+r}\ 1_{\{\delta_n \ge \epsilon\}} \mid \mathcal{F}_n] \ge c \Theta_n J_n 1_{\{\delta_n \ge \epsilon\}} \epsilon^\beta.$$

The next lemma is a key lemma to show quick convergence to equilibrium. We call it the separation lemma because it states roughly that paths conditioned not to intersect actually get a reasonable distance apart with positive probability.

Lemma 9 *There exists a c such that the following is true. If $n \ge 0$,*

$$\mathbf{E}[\Theta_{n+2}\ J_{n+2}\ U_{n+2} \mid \mathcal{F}_n] \ge c \mathbf{E}[\Theta_{n+2}\ J_{n+2} \mid \mathcal{F}_n].$$

Proof. Let N be the smallest positive integer so that

$$\sum_{j=N}^{\infty} j^2 2^{-j} \le \frac{1}{4}.$$

Let $m_k = 3/2$ for $k \le N$, and for $k > N$,

$$m_k = \frac{3}{2} + \sum_{j=N}^{k-1} j^2 2^{-j} < 2.$$

For $k \ge N$, let $h_k = h_k(\lambda_1, \lambda_2)$ be the infimum of

$$\frac{\mathbf{E}[\Theta_{n+r}\ J_{n+r}\ U_{n+r} \mid \mathcal{F}_n]}{\mathbf{E}[\Theta_{n+r}\ J_{n+r} \mid \mathcal{F}_n]},$$

where the infimum is over all $n \ge 0$, $m_k \le r \le 2$, and on the event

$$\delta_n \ge 2^{-k}.$$

It follows from Lemma 8 that $h_k \geq c2^{-\beta k} > 0$. We need to show that

$$\inf_k h_k > 0,$$

and to show this it suffices to show that

$$h_{k+1} \geq h_k(1 - o(k^{-2})).$$

Assume that $k \geq N$, $n \geq 0$, $m_{k+1} \leq r \leq 2$, and that

$$2^{-(k+1)} \leq \delta_n < 2^{-k}.$$

Let $\rho = \rho(n,k)$ be the smallest positive integer l such that

$$\delta_{n+l2^{-k+1}} \geq 2^{-k}.$$

If $B(n+(l-1)2^{-k+1}) \in \mathcal{J}$ then there is a positive probability (independent of the exact position of $B(n+(l-1)2^{-k+1})$) that $\delta_{n+l2^{-k+1}} \geq 2^{-k}$. Hence, for some β_2

$$\mathbf{P}\{\rho > \frac{k^2}{4}; J_{n+k^2 2^{-k-1}}\} \leq 2^{-\beta_2 k^2}. \tag{18}$$

If $j \leq k^2/4$, the strong Markov property and the definition of h_k imply that

$$\mathbf{E}[\Theta_{n+r}; \rho = j; U_{n+r} \mid \mathcal{F}_{n+j2^{-k+1}}] \geq$$
$$h_k \mathbf{E}[\Theta_{n+r}; \rho = j \mid \mathcal{F}_{n+j2^{-k+1}}].$$

This implies

$$\mathbf{E}[\Theta_{n+r};\ \rho = j;\ U_{n+r} \mid \mathcal{F}_n] \geq h_k \mathbf{E}[\Theta_{n+r}; \rho = j \mid \mathcal{F}_n].$$

Summing over j, this gives

$$\mathbf{E}[\Theta_{n+r}; \rho \leq \frac{k^2}{4}\ U_{n+r} \mid \mathcal{F}_n] \geq h_k \mathbf{E}[\Theta_{n+r}; \rho \leq \frac{k^2}{4} \mid \mathcal{F}_n].$$

But Lemma 8 implies if $2^{-(k+1)} \leq \delta_n$,

$$\mathbf{E}[\Theta_{n+r}; \rho \leq \frac{k^2}{4} \mid \mathcal{F}_n] \geq ce^{-\beta k}$$

Combining this with (18) we get

$$\mathbf{E}[\Theta_{n+r}; \rho \leq \frac{k^2}{4} \mid \mathcal{F}_n] \geq \mathbf{E}[\Theta_{n+r} \mid \mathcal{F}_n](1 + o(k^{-2})),$$

which finishes the lemma. □

It follows from the lemma that for every $n \geq 2$,

$$\mathbf{E}[\Theta_n; U_n] = \mathbf{E}[\mathbf{E}[\Theta_n; U_n \mid \mathcal{F}_{n-2}]] \geq c\mathbf{E}[\mathbf{E}[\Theta_n \mid \mathcal{F}_{n-2}]] = c\mathbf{E}[\Theta_n].$$

It is easy to check that

$$\mathbf{E}[\Theta_{n+1} \mid \mathcal{F}_n] \geq c\Theta_n 1_{U_n}.$$

From this we conclude that there is a $c_9 > 0$ such that for all $n \geq 0$,

$$q_{n+1} \geq c_9 q_n \tag{19}$$

Let

$$\bar{Z}_n^+ = \sup \mathbf{P}_1^z\{B^1[0, T_n^1] \cap B[0, T_n] = \emptyset; B^1[0, T_n^1] \subset \mathcal{J}; B^1(T_n^1) \succ B(T_n)\},$$

where the supremum is over all z with $\Re(z) = 0$, and define $\bar{Z}_n^-$ similarly. Let

$$\bar{\Theta}_n = (\bar{Z}_n^+)^{\lambda_1}(\bar{Z}_n^-)^{\lambda_2}.$$

Define $\tilde{Z}_n^+$ to be the same as Z_{n-1}^+ for the path

$$\tilde{B}_t = B_{t+T_1} - 1, \quad 0 \leq t \leq T_n - T_1.$$

Let $\tilde{Z}_n^-$ be defined similarly and

$$\tilde{\Theta}_n = (\tilde{Z}_n^+)^{\lambda_1}(\tilde{Z}_n^-)^{\lambda_2}.$$

By the Harnack inequality applied to B^1, there is a constant c such that

$$\bar{\Theta}_n \leq c\tilde{\Theta}_n.$$

But the Harnack inequality applied to B shows that for any z,

$$\mathbf{E}^z[\tilde{\Theta}_n; J_n] \leq c\mathbf{E}[\tilde{\Theta}_n] \leq q_{n-1}.$$

The following then follows from (19).

Lemma 10 *There exists a c_{10} such that for all $y \in \mathbb{R}$ and all $n \geq 0$,*

$$\mathbf{E}^{iy}[\bar{\Theta}_n; J_n] \leq c_{10} q_n.$$

3.2 Other lemmas

In this section we derive a number of lemmas. The main goal is to get the estimate $q_n \leq ce^{-\xi n}$ which can be derived from the estimate $q_{n+m} \geq cq_n q_m$. The separation lemma tells us that a good proportion of the paths under the measure given by Θ_n are separated. To such separated paths we can attach other Brownian paths.

Lemma 11 *There exists a c_{11} such that the following is true. Let $\Lambda_n = \Lambda_n(c_{11})$ be the event*

$$\Lambda_n = \left\{B[0, T_n] \cap \{\Re(z) \leq 1\} \subset \{|\Im(z)| \leq \frac{\pi}{2} - c_{11}\}\right\}.$$

Then

$$\mathbf{E}[\Theta_n; \Lambda_n] \geq \frac{1}{2} q_n.$$

Proof. For any $\epsilon > 0$, let $\kappa = \kappa(\epsilon)$ be the first time t such that

$$B_t \in \{z : \Re(z) \leq 1, |\Im(z)| \geq \frac{\pi}{2} - \epsilon\},$$

and let

$$\rho = \rho(\epsilon) = \inf\{t \geq \kappa : \Re[B_t] = 2\}.$$

By the standard "gambler's ruin" estimate,

$$\mathbf{P}\{B[\kappa, \rho] \subset \mathcal{J}\} \leq c\epsilon.$$

Hence by the strong Markov property, the Harnack inequality, and (19),

$$\mathbf{E}[\Theta_n; \kappa < T_n] \leq c\epsilon q_{n-2} \leq c\epsilon q_n.$$

In particular, by taking ϵ sufficiently small we can make the right hand side smaller than $q_n/2$.
□

We let $\mathcal{B}(z, \epsilon)$ denote the closed disk of radius ϵ about z. We also fix a c_{11} so that the previous lemma holds and let $\Lambda_n = \Lambda_n(c_{11})$.

Lemma 12 *There is a c_{12} such that the following is true. Let Γ_n be the event*

$$\Gamma_n = \left\{ B[0, T_n] \cap \{\Re(z) < 0\} \subset \mathcal{B}(B_0, \frac{c_{11}}{10}) \right\}.$$

Then,

$$\sup_y \mathbf{E}^{iy}[\Theta_n; \Lambda_n \cap \Gamma_n] \geq c_{12} q_n.$$

Proof. Let r_n be the supremum over y of

$$\mathbf{E}^{iy}[\Theta_n; \Lambda_n],$$

and let $y = y_n$ be a number that obtains the supremum. By the previous lemma, $r_n \geq q_n/2$. Let

$$\kappa = \inf\left\{ t : B_t \in \{\Re(z) < 0\} \setminus \mathcal{B}(B_0, \frac{c_{11}}{10}) \right\},$$

$$\rho = \inf\{t \geq \kappa : \Re[B_t] = 0\}.$$

Note that

$$\mathbf{P}\{B[\kappa, \rho] \subset \mathcal{J} \mid \kappa < T_n\} < 1 - \delta,$$

for some $\delta > 0$, independent of n. Hence by the strong Markov property,

$$\mathbf{E}^{iy}[\Theta_n; \Lambda_n \cap \{\kappa < T_n\}] \leq (1 - \delta) r_n,$$

and hence

$$\mathbf{E}^{iy}[\Theta_n; \Lambda_n \cap \Gamma_n] \geq \delta r_n.$$

□

We now let

$$z_+ = (\frac{\pi}{2} - \frac{c_{11}}{8})i, \qquad z_- = -z_+,$$

and let

$$\begin{aligned}
\tilde{Z}_n^+ &= \mathbf{P}_1^{z_+}\{B^1[0,T_n^1] \cap B[0,T_n] = \emptyset; B^1[0,T_n^1] \subset \mathcal{J}; \\
&\qquad\qquad B^1[0,T_n^1] \cap \{\Re(z) < 0\} \subset \mathcal{B}(z_+, \frac{c_{11}}{16})\}, \\
\tilde{Z}_n^- &= \mathbf{P}_1^{z_-}\{B^1[0,T_n^1] \cap B[0,T_n] = \emptyset; B^1[0,T_n^1] \subset \mathcal{J}; \\
&\qquad\qquad B^1[0,T_n^1] \cap \{\Re(z) < 0\} \subset \mathcal{B}(z_-, \frac{c_{11}}{16})\}, \\
\tilde{\Theta}_n &= (\tilde{Z}_n^+)^{\lambda_1} (\tilde{Z}_n^-)^{\lambda_2}.
\end{aligned}$$

Using (16), we can see that there is a c_{13} such that on the event $\Lambda_n \cap \Gamma_n$,

$$\tilde{\Theta}_n \geq c_{13} \Theta_n,$$

and hence

$$\sup \mathbf{E}^{iy}[\tilde{\Theta}_n; \Lambda_n \cap \Gamma_n] \geq c_{13} c_{12} q_n.$$

Another simple use of the Harnack inequality shows that there is a c such that for all $y \in [-\pi/4, \pi/4]$,

$$\mathbf{E}^{iy}[\tilde{\Theta}_n; \Lambda_n \cap \Gamma_n] \geq c q_n.$$

From this and the work of the previous section, we can conclude

$$q_{n+m} \geq c q_n q_m.$$

Hence from standard subadditivity arguments we get the following.

Lemma 13 *There exist c_8, c_{14} such that for all n,*

$$c_8 e^{-n\xi} \leq q_n \leq c_{14} e^{-n\xi}.$$

4 Invariant measures and strict concavity

4.1 Coupling

Let $\mathcal{Y}$ be the set of continuous $\gamma : (-\infty, 0] \to \mathcal{J}$ with

$$\Re[\gamma(0)] = 0,$$

$$\Re[\gamma(t)] < 0, \qquad -\infty < t < 0,$$

$$\Re[\gamma(t)] \to -\infty, \qquad t \to -\infty.$$

In other words $\gamma \in \mathcal{Y}$ if and only if $G\gamma \in \mathcal{X}$ where

$$\Re[G\gamma(t)] = -\Re[\gamma(-t)], \qquad 0 \leq t < \infty,$$

$$\Im[G\gamma(t)] = \Im[\gamma(-t)], \quad 0 \le t < \infty.$$

We will apply results about $\mathcal{X}$ from Section 2 to $\mathcal{Y}$ using the natural identification $\gamma \leftrightarrow G\gamma$.

Let B_t be a complex valued Brownian motion as before with

$$T_n = \inf\{t : \Re[B_t] = n\}.$$

We will start the Brownian motion with "initial condition" $\gamma \in \mathcal{Y}$. More precisely, set $B_0 = \gamma(0)$ and if $n \ge 0$ let

$$\gamma_n(t) = \begin{cases} B_{t+T_n} - n, & -T_n \le t \le 0 \\ \gamma(t + T_n) - n, & -\infty < t \le -T_n. \end{cases}$$

Note that $\gamma_n \in \mathcal{Y}$ if and only if the event $J_n = \{B[0, T_n] \subset \mathcal{J}\}$ holds.

For any $\gamma \in \mathcal{Y}$, $n \ge 0$, let

$$\sigma_{-n} = \sigma_{-n}(\gamma) = -\sigma_n(G\gamma) = \inf\{t : \Re[\gamma(t)] = -n\},$$

$$\kappa_{-n} = \kappa_{-n}(\gamma) = -\kappa_n(G\gamma) = \sup\{t : \Re[\gamma(t)] = -n\}.$$

If $\gamma \in \mathcal{Y}$, $-\infty < r < s \le 0$, define the functionals $Y^+_{r,s}, Y^-_{r,s}$ formally by

$$Y^+_{r,s} = \mathbf{P}_1\{B^1(-\infty, T^1_s] \cap \gamma(-\infty, \sigma_s] = \emptyset \mid$$
$$B^1(-\infty, T^1_r] \cap \gamma(-\infty, \sigma_r] = \emptyset, B^1(-\infty, T^1_r] \subset \mathcal{J}, B^1(T^1_r) \succ \gamma(\sigma_r)\},$$
$$Y^-_{r,s} = \mathbf{P}_1\{B^1(-\infty, T^1_s] \cap \gamma(-\infty, \sigma_s] = \emptyset \mid$$
$$B^1(-\infty, T^1_r] \cap \gamma(-\infty, \sigma_r] = \emptyset, B^1(-\infty, T^1_r] \subset \mathcal{J}, \gamma(\sigma_r) \succ B^1(T^1_r)\}.$$

The conditioning is with respect to the measure discussed in Section 2.3 (where it is done for $\mathcal{X}$), and we recall that $w \succ z$ means $\Im(w) > \Im(z)$. Note that if $-\infty < r < s < u \le 0$, then

$$Y^+_{r,u}(\gamma) = Y^+_{r,s}(\gamma) Y^+_{s,u}(\gamma),$$

$$Y^-_{r,u}(\gamma) = Y^-_{r,s}(\gamma) Y^-_{s,u}(\gamma).$$

In other words, $-\log Y^+_{r,s}$ and $-\log Y^-_{r,s}$ are positive additive functionals.

For $0 \le m < n < \infty$, we define the random variables

$$Z^+_{m,n} = Y^+_{m-n,0}(\gamma_n) 1_{\{\gamma_n \in \mathcal{Y}\}},$$

$$Z^-_{m,n} = Y^-_{m-n,0}(\gamma_n) 1_{\{\gamma_n \in \mathcal{Y}\}}.$$

Then

$$\Psi^+_{m,n} \doteq -\log Z^+_{m,n}, \quad \Psi^-_{m,n} \doteq -\log Z^-_{m,n}$$

are positive additive functionals (that can take on the value $+\infty$). We let

$$\Theta_{m,n} = \Theta_{m,n}(\lambda_1, \lambda_2) = (Z^+_{m,n})^{\lambda_1} (Z^-_{m,n})^{\lambda_2},$$

where we again use the convention $0^0 = 0$ if either λ_1 or λ_2 is 0. We also write

$$\Theta_n = \Theta_n(\lambda_1, \lambda_2) = \Theta_{0,n}.$$

We write $\mathbf{P}^\gamma, \mathbf{E}^\gamma$ to denote probabilities and expectations with initial condition γ. For $n \geq 0$, we let $\mathcal{F}_n$ be the σ-algebra generated by γ (in case γ is chosen randomly) and

$$B_t, \quad 0 \leq t \leq T_n.$$

In other words, $\mathcal{F}_n$ is the σ-algebra generated by the random function γ_n. If $n \geq 0$, $\gamma \in \mathcal{Y}$, let

$$R_n(\gamma) = R_n(\gamma; \lambda_1, \lambda_2) = e^{\xi n} \mathbf{E}^\gamma[\Theta_n].$$

We collect some of the results of the previous sections using the notation of this section.

Lemma 14 *There exists a c such that for all $\gamma \in \mathcal{Y}$, $n \geq 0$,*

$$R_n(\gamma) \leq c.$$

Moreover, if $n \geq 2$,

$$|\log R_n(\gamma) - \log R_2(\gamma)| \leq c.$$

Proof. The first inequality is an immediate corollary of Lemma 13. The second follows from Lemma 9 and Lemma 12. $\square$

Lemma 15 *There exists a $\delta : (0, \infty) \to (0, \infty)$ such that if $\gamma \in \mathcal{Y}$ and*

$$\mathcal{B}(\frac{i\pi}{2}, \epsilon) \cap \gamma(-\infty, 0] = \emptyset,$$

$$\mathcal{B}(-\frac{i\pi}{2}, \epsilon) \cap \gamma(-\infty, 0] = \emptyset,$$

then

$$R_3(\gamma) \geq \delta(\epsilon).$$

Moreover, δ may be chosen so that there exist c_{15}, c_{16}, β_2 with

$$\delta(c_{16} e^{-\beta_2 n}) \geq \min\{c_{15}, e^{-n/8}\}.$$

Proof. See Lemma 8. $\square$

If $n \geq 0$, let $\hat{\mathcal{Y}}_n$ be the collection of $\gamma \in \mathcal{Y}$ such that

$$\operatorname{dist}[\gamma(-\infty, 0], \{-\frac{i\pi}{2}, \frac{i\pi}{2}\}] \geq \min\{\frac{1}{100}, c_{16} e^{-\beta_2 n}\}, \tag{20}$$

and if $-n \leq s \leq r - (n/5) \leq -n/5$,

$$\kappa_s(\gamma) \leq \sigma_r(\gamma). \tag{21}$$

The c_{16}, β_2 are the constants from Lemma 15 (which we have now fixed) and $1/100$ is an arbitrarily chosen small number. The condition (21) can also be written

$$\gamma[\sigma_r(\gamma), 0] \cap \{\Re(z) \leq s\} = \emptyset.$$

Note that if $\gamma \in \hat{\mathcal{Y}}_n$,

$$\mathbf{E}^\gamma[\Theta_3] = e^{-3\xi} R_3(\gamma) \geq c e^{-n/8}.$$

Lemma 16 *There exist c_{17}, β_3 such that if $\gamma \in \hat{\mathcal{Y}}_n$,*

$$\mathbf{E}^\gamma[\Theta_3; \gamma_3 \notin \hat{\mathcal{Y}}_{n+3}] \leq c_{17} e^{-\beta_3 n} \mathbf{E}^\gamma[\Theta_3].$$

Proof. It suffices to prove the lemma for n sufficiently large so we may assume that $c_{16} e^{-\beta_2 n} \leq 1/100$. Note that if $\gamma \in \hat{\mathcal{Y}}_n$ and $\gamma_3 \notin \hat{\mathcal{Y}}_{n+3}$, then either

$$B[0, T_3] \cap \{\Re(z) \leq -\frac{n}{5} + 3\} \neq \emptyset, \tag{22}$$

or

$$\mathrm{dist}[B[0, T_3], \{3 + \frac{i\pi}{2}, 3 - \frac{i\pi}{2}\}] \leq c_{16} e^{-\beta_2 n}. \tag{23}$$

For (22), note that for any γ,

$$\mathbf{E}^\gamma[\Theta_3; B[0, T_3] \cap \{\Re(z) \leq -\frac{n}{5} + 3\} \neq \emptyset] \leq$$

$$\mathbf{P}^\gamma[B[0, T_3] \cap \{\Re(z) \leq -\frac{n}{5} + 3 \neq \emptyset\}; B[0, T_3] \subset \mathcal{J}] \leq c e^{-2n/5}.$$

The last inequality follows from rectangle estimates since the Brownian motion has to start at $\{\Re(z) = 0\}$, reach $\{\Re(z) \leq (-n/5) + 3\}$, and return to $\{\Re(z) = 0\}$ staying in $\mathcal{J}$ for the entire time. But, $\mathbf{E}^\gamma[\Theta_3] \geq c e^{-n/8}$ for $\gamma \in \hat{\mathcal{Y}}_n$. Hence,

$$\mathbf{E}^\gamma[\Theta_3; B[0, T_3] \cap \{\Re(z) \leq -\frac{n}{5} + 3\} \neq \emptyset] \leq c e^{-11n/40} \mathbf{E}^\gamma[\Theta_3].$$

For (23), let

$$p(\epsilon) = \sup \mathbf{P}^{iy}[\mathrm{dist}[B[0, T_1], \{1 + \frac{i\pi}{2}, 1 - \frac{i\pi}{2}\}] \leq \epsilon],$$

where the supremum is over all $y \in \mathbb{R}$. Standard estimates give $p(\epsilon) \leq c\epsilon$. But the strong Markov property implies

$$\mathbf{E}^\gamma\left[\Theta_3; \mathrm{dist}[B[0, T_3], \{3 + \frac{i\pi}{2}, 3 - \frac{i\pi}{2}\}] \leq c_{16} e^{-n\beta_2}\right] \leq \mathbf{E}^\gamma[\Theta_2] p(c_{16} e^{-\beta_2 n}),$$

and Lemma 14 implies $\mathbf{E}^\gamma[\Theta_2] \leq c \mathbf{E}^\gamma[\Theta_3]$. □

If $\gamma^1, \gamma^2 \in \mathcal{Y}$ and $n \geq 0$, we write

$$\gamma^1 =_n \gamma^2$$

if $G\gamma_1 =_n G\gamma_2$ as in Section 2.5, i.e., if $\sigma_{-n}(\gamma^1) = \sigma_{-n}(\gamma^2)$ and

$$\gamma^1(t) = \gamma^2(t), \quad \sigma_{-n}(\gamma^1) \leq t \leq 0.$$

Note that if $n \geq 1$ and $\gamma^1 =_n \gamma^2$, then $\gamma^1 \in \hat{\mathcal{Y}}_n$ if and only if $\gamma^2 \in \hat{\mathcal{Y}}_n$.

Lemma 17 *There exist c_{18}, β_4 such that the following is true. Suppose $n \geq 1$ and $\gamma^1, \gamma^2 \in \hat{\mathcal{Y}}_n$ with*

$$\gamma^1 =_n \gamma^2.$$

Then for all $m \geq 0$,

$$|\log R_m(\gamma^1) - \log R_m(\gamma^2)| \leq c_{18} e^{-\beta_4 n},$$

i.e.,

$$R_m(\gamma^2) = R_m(\gamma^1)[1 + O(e^{-\beta_4 n})].$$

Proof. As in the proof of Lemma 16 we have for any $\gamma \in \mathcal{Y}$,

$$\begin{aligned}\mathbf{E}^{\gamma}[\Theta_m; B[0,T_m] \cap \{\Re(z) \leq -\frac{n}{5}\} \neq \emptyset] &\leq ce^{-2n/5} q_m \\ &\leq ce^{-2n/5} e^{-\xi m}.\end{aligned}$$

But if $\gamma \in \hat{\mathcal{Y}}_n$,

$$\mathbf{E}^{\gamma}[\Theta_m] \geq ce^{-\xi m} \mathbf{E}^{\gamma}[\Theta_3] \geq ce^{-\xi m} e^{-n/8}.$$

Hence, for $j = 1, 2$,

$$\mathbf{E}^{\gamma^j}[\Theta_m; B[0,T_m] \cap \{\Re(z) \leq -\frac{n}{5}\} = \emptyset] = \mathbf{E}^{\gamma^j}[\Theta_m][1 + O(e^{-\beta n})].$$

But on the event

$$\{\ B[0,T_m] \cap \{\Re(z) \leq -\frac{n}{5}\} = \emptyset\ \},$$

it follows from Lemma 6 that

$$\Theta^1_m = \Theta^2_m[1 + O(e^{-\beta n})].$$

Hence

$$\mathbf{E}^{\gamma^1}[\Theta_m] = \mathbf{E}^{\gamma^2}[\Theta_m][1 + O(e^{-\beta n})].$$

□

We now fix $N \geq 3$ and consider the following discrete time, time inhomogeneous, Markov chain, $X_0^{(N)}, X_1^{(N)}, \ldots$, indexed by $j = 0, 3, 6, \ldots, \lfloor N/3 \rfloor$ with state space $\mathcal{Y}$. To describe the chain, suppose $X_0^{(N)} = \gamma$. Start a Brownian motion with initial condition γ, and let it run until T_3 giving γ_3 as described above. The weighting on γ_3 is given by the following density with respect to the Wiener measure of the Brownian motion:

$$g(\gamma, \gamma^3) = \frac{\Theta_3(\gamma_3) R_{N-3}(\gamma_3) e^{3\xi}}{R_N(\gamma)}. \tag{24}$$

For fixed γ this gives a probability density since

$$\begin{aligned}\mathbf{E}^{\gamma}[\Theta_N] &= \mathbf{E}^{\gamma}[\mathbf{E}[\Theta_N \mid \mathcal{F}_3]] \\ &= \mathbf{E}^{\gamma}[\Theta_3 \mathbf{E}[\Theta_N \Theta_3^{-1} \mid \mathcal{F}_3]] \\ &= \mathbf{E}^{\gamma}[\Theta_3 e^{-(N-3)\xi} R_{N-3}(\gamma_3)]\end{aligned}$$

and hence

$$R_N(\gamma) = e^{N\xi} \mathbf{E}^{\gamma}[\Theta_N] = \mathbf{E}^{\gamma}[\Theta_3 e^{3\xi} R_{N-3}(\gamma_3)].$$

The distribution of $X_{3(j+1)}^{(N)}$ given $X_{3j}^{(N)} = \gamma$ is the same as the distribution of $X_3^{(N-3j)}$ given $X_0^{(N-3j)} = \gamma$.

Keeping N fixed and assuming $3k \leq N$, let us write just $\gamma_0, \gamma_3, \ldots, \gamma_{3k}$ for $X_0^{(N)}, \ldots, X_{3k}^{(N)}$. We will call

$$(\gamma_t^1, \gamma_t^2), \quad t = 0, 3, \ldots, 3k,$$

a (k, N)-coupling if for $j = 1, 2$,

$$\gamma_0^j, \ldots, \gamma_{3k}^j \ ,$$

has the distribution of this Markov chain with initial condition γ_0^j. Let us describe one coupling. Suppose $\gamma^1,\gamma^2 \in \mathcal{X}$. Let B^3, B^4 be independent Brownian motions (with stopping times T_r^3, T_r^4) started with initial conditions γ^1,γ^2, respectively. Use these Brownian motions to produce $(\gamma^1_{3/2},\gamma^2_{3/2})$. Let $U^j_{3/2}, j=1,2$, be the event

$$U^j_{3/2} = \left\{ \gamma^j_{3/2}(-\infty,0] \cap \{\Re(z) \geq -\frac{1}{2}\} \subset \{|\Im(z)| \leq \frac{\pi}{6}\} \right\}.$$

By the separation lemma, Lemma 9, we can see that

$$\mathbf{E}^{\gamma^j}[\Theta_{3/2}; U^j_{3/2}] \geq c\mathbf{E}^{\gamma^j}[\Theta_{3/2}], \quad j=1,2.$$

Consider the event

$$U^j_2 = \left\{ \gamma^j_2(-\infty,0] \cap \{\Re(z) \geq -\frac{1}{2}\} \subset \{|\Im(z)| \leq \frac{\pi}{6}\} \right\}.$$

Then, the measure on

$$\{\Re(z) = 0; |\Im(z)| \leq \frac{\pi}{12}\},$$

given by

$$\Theta_2 1_{U^j_2} \mathbf{E}^{\gamma^j}[\Theta_2]^{-1},$$

can be seen to be greater than a constant times Lebesgue measure. Hence we can couple the paths B^3, B^4 to produce a coupling (γ^1_2,γ^2_2) with

$$\mathbf{P}\left\{ \gamma^1_2(0) = \gamma^2_2(0); \gamma^j(-\infty,0] \cap \{\Re(z) \geq -1\} \subset \{|\Im(z)| \leq \frac{\pi}{4}\} \right\} \geq c.$$

We now use the same Brownian motion, say B^3, to extend the paths beyond γ^j_2. It is not difficult to see we can extend these rather arbitrarily and still get sets of positive probability. From this we get the following. The function δ_m in the lemma might go to zero very quickly, but this will not be a problem.

Lemma 18 *There is a c_{19} such that for any $\gamma^1,\gamma^2 \in \mathcal{Y}$ and any $N \geq 3$, there is a $(1,N)$ coupling (γ^1_3,γ^2_3) with*

$$\mathbf{P}\{\ \gamma^1_3 =_1 \gamma^2_3;\ \gamma^1_3,\gamma^2_3 \in \hat{\mathcal{Y}}_1\ \} \geq c_{19}.$$

Moreover, for every m, there is a $\delta_m > 0$ such that if $m \leq N/3$, there is an (m,N) coupling with

$$\mathbf{P}\{\ \gamma^1_{3m} =_{3m-2} \gamma^2_{3m};\ \gamma^1_{3m},\gamma^2_{3m} \in \hat{\mathcal{Y}}_{3m-2}\ \} \geq \delta_m.$$

Now suppose that $n \geq 1$ and $\gamma^1 =_n \gamma^2$. There is a natural coupling that can be defined by taking the same Brownian motion starting at $\gamma^1(0) = \gamma^2(0)$. Suppose that $\gamma^1_3 \in \hat{\mathcal{Y}}_{n+3}$. Then $\gamma^2_3 \in \hat{\mathcal{Y}}_{n+3}$. Note that Lemma 17 implies

$$R_{N-3}(\gamma^1_3) = R_{N-3}(\gamma^2_3)[1 + O(e^{-n\beta})].$$

Also, Lemma 7 gives

$$\Theta_3(\gamma^1_3) = \Theta_3(\gamma^2_3)[1 + O(e^{-n\beta})].$$

Combining this with Lemma 16 we get the following lemma.

Lemma 19 *There exist* c_{20}, c_{21}, β_5 *such that if* $\gamma^1, \gamma^2 \in \hat{\mathcal{Y}}_n$ *with* $\gamma^1 =_n \gamma^2$ *and* $N \geq 3$*, there is a* $(1, N)$ *coupling* (γ_3^1, γ_3^2) *with*

$$\mathbf{P}\{\ \gamma_3^1 =_{n+3} \gamma_3^2;\ \gamma_3^1, \gamma_3^2 \in \hat{\mathcal{Y}}_{n+3}\ \} \geq \max\{c_{20}, 1 - c_{21} e^{-n\beta_5}\}.$$

Lemma 20 *There exist* c_{22}, β_6 *such that if* $\gamma^1, \gamma^2 \in \mathcal{Y}$*, and* n *is a positive integer, there is a* $(n, 3n)$ *coupling with*

$$\mathbf{P}\{\ \gamma_{3n}^1 =_n \gamma_{3n}^2\} \geq 1 - c_{22} e^{-\beta_5 n}.$$

Proof. This uses the ideas in [4]. Define a coupling using the couplings above. More precisely, suppose γ^1, γ^2 are given. Produce (γ_3^1, γ_3^2) using the coupling from Lemma 18. Suppose $(\gamma_{3j}^1, \gamma_{3j}^2)$ have been constructed. Let $K_j = K_{j,n}$ be the largest positive multiple of 3, k, such that

$$\gamma_{3j}^1 =_{k-2} \gamma_{3j}^2,$$

$$\gamma_{3j}^1, \gamma_{3j}^2 \in \hat{\mathcal{Y}}_{k-2}.$$

If no such positive k exists, then $K_j = 0$. If $K_j = 0$, construct $(\gamma_{3(j+1)}^1, \gamma_{3(j+1)}^2)$ using the coupling in Lemma 18. If $K_j > 0$, construct $(\gamma_{3(j+1)}^1, \gamma_{3(j+1)}^2)$ using the coupling from Lemma 19. Let $\epsilon_0 = c_{19} \wedge c_{20}$ and for $l > 0$,

$$\epsilon_l = \max\{c_{20}, 1 - c_{21} e^{-(3l-2)\beta_5}\}.$$

Then if $\mathcal{H}_j$ denotes the σ-algebra in this coupling generated by $(\gamma_{3j}^1, \gamma_{3j}^2)$,

$$\mathbf{P}\{K_{j+1} \geq 3 \mid \mathcal{H}_j\} \geq \epsilon_0.$$

$$\mathbf{P}\{K_{j+1} = l + 3 \mid \mathcal{H}_j\} \geq \epsilon_l I\{K_j = l\}.$$

By comparison with an appropriate Markov chain on the nonnegative integers (see [4, 6.3 and A]), we see that

$$\mathbf{P}\{K_n \leq \frac{n}{2}\} \leq c e^{-\beta n},$$

for some c, β. This proves the lemma. $\square$

In particular, we get that there exist c, β such that for all $n \geq 3$ and all $\gamma^1, \gamma^2 \in \mathcal{Y}$,

$$\left| \frac{R_{n+3}(\gamma^1)}{R_n(\gamma^1)} - \frac{R_{n+3}(\gamma^2)}{R_n(\gamma^2)} \right| \leq c e^{-\beta n}.$$

From this we can easily deduce the following.

Lemma 21 *For every* $\gamma \in \mathcal{Y}$*, the limit*

$$R(\gamma) = \lim_{n \to \infty} R_n(\gamma)$$

exists. Moreover, there exist c_{23}, β_7 *such that for all* $n \geq 2$*,*

$$|\log R_n(\gamma) - \log R(\gamma)| \leq c_{23} e^{-\beta_7 n}.$$

In other words,

$$R_n(\gamma) = R(\gamma)[1 + O(e^{-\beta_7 n})].$$

4.2 Invariant measure on $\mathcal{Y}$

If $n > 0$, we let $\mathcal{Y}_n$ be the set of continuous

$$\gamma : [-b, 0] \longrightarrow \mathcal{J},$$

with

$$\Re[\gamma(-b)] = -n, \quad \Re[\gamma(0)] = 0,$$
$$\Re[\gamma(t)] < 0, \quad -b \leq t < 0.$$

Here, $b = b_\gamma$ can be any positive real number. We let ρ be the Skorohod metric on $\mathcal{Y}_n$ such that $\rho(\gamma^1, \gamma^2) < \epsilon$ if there exist an increasing homeomorphism $\phi : [-b_{\gamma^1}, 0] \to [-b_{\gamma^2}, 0]$ with

$$|\phi(t) - t| < \epsilon, \quad |\gamma^1(t) - \gamma^2(\phi(t))| < \epsilon, \quad -b_{\gamma^1} \leq t \leq 0.$$

Measures on $\mathcal{Y}_n$ will be with respect to the corresponding Borel σ-algebra. Let

$$\Phi_n : \mathcal{Y} \to \mathcal{Y}_n,$$

be the projection

$$b_{\Phi_n \gamma} = -\sigma_{-n}(\gamma),$$
$$\Phi_n \gamma(t) = \gamma(t), \quad \sigma_{-n}(\gamma) \leq t \leq 0.$$

Similarly, if $m < n$ we define

$$\Phi_m : \mathcal{Y}_n \to \mathcal{Y}_m.$$

(It might be more precise to write $\Phi_{m,n}$ rather than Φ_m, but there should be no confusion.)

If $m < n$ and ν_n is a measure on $\mathcal{Y}_n$, then we write $\Phi_m \nu_n$ for the measure on $\mathcal{Y}_m$ induced by Φ_m. A collection of measures $\{\nu_n : n > 0\}$ is called consistent if each ν_n is a measure on $\mathcal{Y}_n$ and

$$\Phi_m \nu_n = \nu_m, \quad 0 < m < n < \infty.$$

A measure ν on $\mathcal{Y}$ is a consistent collection of measures $\{\nu_n\}$ (where $\nu_n = \Phi_n \nu$).

If ν is any probability measure on $\mathcal{Y}$ and $n \geq 3$, let $G_n \nu$ be the probability measure on $\mathcal{Y}$ obtained by the distribution of the Markov chain $X_n^{(n)}$, as defined in Section 4.1, given that $X_0^{(n)}$ has distribution ν. (This definition only works if $n = 3k$ for integer k, but slight modifications work for any n.) It follows from Lemma 20 that there exist c, β such that if ν^1, ν^2 are two probability measures on $\mathcal{Y}$ and $s \geq 3n$,

$$\|\Phi_n(G_s \nu^1) - \Phi_n(G_s \nu^2)\| \leq c e^{-\beta s}.$$

By letting $s \to \infty$, we see that there is a limiting measure μ_n such that if ν is any probability measure on $\mathcal{Y}$ and $s \geq 3n$,

$$\|\Phi_n(G_s \nu) - \mu_n\| \leq c e^{-\beta s}.$$

It is easy to check that the $\{\mu_n\}$ form a consistent collection of measures and hence give a measure μ on $\mathcal{Y}$. Also, if $n \geq 0$.

$$\mathbf{E}^\mu[\Theta_n] = e^{-\xi n},$$
$$G_n \mu = \mu.$$

We summarize this in a proposition.

Proposition 22 *There exists a probability measure $\mu = \mu(\lambda_1, \lambda_2)$ on $\mathcal{Y}$ and c, β such that if ν is any probability measure on $\mathcal{Y}$ and $0 \le n \le s/3 < \infty$,*

$$\|\Phi_n(G_s\nu) - \Phi_n\mu\| \le ce^{-\beta s}.$$

A particular application of the proposition is the following. Suppose $n \ge 1$, $k \in \{1,2,3\}$, and $\Phi^+_{n-1,n}$ is as defined in Section 4.1. Then

$$\mathbf{E}^\gamma[(\Psi^+_{n-1,n})^k \Theta_n R(\gamma_n)] = e^{-n\xi} R(\gamma) \mathbf{E}^\mu[(\Psi^+_{0,1})^k e^\xi \Theta_1 R(\gamma_1)][1 + O(e^{-\beta n})]. \tag{25}$$

(This is actually true for all $k = 1, 2, 3, \ldots$, if we allow the implicit constant to depend on k. Since we will only need the result for $k = 1, 2, 3$, it is easier just to state it for these k and have no k dependence in the constant.)

Let $\bar{\mu}$ be the measure on $\mathcal{Y}$ with

$$d\bar{\mu} = R\, d\mu.$$

Then $\bar{\mu}$ is the invariant measure for the time homogeneous Markov chain $X_n = X_n^\infty$ with state space $\mathcal{Y}$ defined for $n > 0$ by saying that the density of X_n with respect to Wiener measure is

$$g(\gamma, \gamma_n) = \frac{e^{\xi n} \Theta_n R(\gamma_n)}{R(\gamma)}.$$

For integer $k > 0$, we define $a = a(\lambda_1, \lambda_2)$ and $b_k = b_k(\lambda_1, \lambda_2)$ by

$$\begin{aligned} a &= \mathbf{E}^\mu[\Psi^+_{0,1} e^\xi \Theta_1 R(\gamma_1)] \\ &= \mathbf{E}^{\bar{\mu}}[\Psi^+_{0,1} e^\xi \Theta_1 R(\gamma_1) R(\gamma_0)^{-1}], \end{aligned}$$

$$\begin{aligned} b_k &= \mathbf{E}^\mu[\Psi^+_{0,1} \Psi^+_{k-1,k} e^{\xi k} \Theta_k R(\gamma_k)] \\ &= \mathbf{E}^\mu[\Psi_{0,1} + \Psi^+_{k-1,k} e^{\xi k} \Theta_k R(\gamma_k) R(\gamma_0)^{-1}] \end{aligned}$$

4.3 Derivatives

In this subsection we fix $\lambda_2 \in [0, M]$ and let

$$\xi(\lambda_1) = \xi(\lambda_1, \lambda_2), \quad 0 < \lambda_1 \le M.$$

We also fix a particular element $\gamma \in \mathcal{Y}$; for ease we shall take the half line,

$$\gamma(t) = t, \quad -\infty < t \le 0.$$

Let

$$\xi_n(\lambda_1) = -\frac{1}{n} \log \mathbf{E}^\gamma[\Theta_n],$$

where $\Theta_n = \Theta_n(\lambda_1, \lambda_2)$. Note that Lemmas 12 and 13 imply

$$\xi_n(\lambda_1) = \xi(\lambda_1) + O(\frac{1}{n}).$$

Direct differentiation gives

$$\xi_n'(\lambda_1) = \frac{1}{n}\tilde{\mathbf{E}}_n[\Psi^+_{0,n}],$$

$$\xi_n''(\lambda_1) = -\frac{1}{n}\mathbf{var}_n[\Psi^+_{0,n}],$$

where $\tilde{\mathbf{E}}_n$ and $\mathbf{var}_n$ denote expectation and variance with respect to the measure with density

$$\frac{\Theta_n}{\mathbf{E}^\gamma[\Theta_n]}$$

with respect to $\mathbf{P}$. By Lemma 17,

$$\mathbf{E}^\gamma[\Theta_n] = e^{-\xi n}R(\gamma)[1 + O(e^{-\beta n})].$$

Recalling that

$$\Psi^+_{0,n} = \Psi^+_{0,1} + \cdots + \Psi^+_{n-1,n},$$

we get

$$\mathbf{E}^\gamma[\Psi^+_{0,n}\Theta_n] = \sum_{j=1}^{n}\mathbf{E}^\gamma[\Psi^+_{j-1,j}\Theta_n].$$

Note that

$$\begin{aligned} \mathbf{E}^\gamma[\Psi^+_{j-1,j}\Theta_n] &= \mathbf{E}^\gamma[\Psi^+_{j-1,j}\Theta_j\mathbf{E}[\Theta_{j,n} \mid \mathcal{F}_j]] \\ &= \mathbf{E}^\gamma[\Psi^+_{j-1,j}\Theta_j e^{-\xi(n-j)}R(\gamma_j)][1 + O(e^{-\beta(n-j)})]. \end{aligned}$$

Also, (25) and the definition of a give

$$\mathbf{E}^\gamma[\Psi^+_{j-1,j}\Theta_j R(\gamma_j)] = e^{-\xi j}aR(\gamma)[1 + O(e^{-\beta j})].$$

Hence,

$$\mathbf{E}^\gamma[\Psi^+_{j-1,j}\Theta_n] = e^{-\xi n}aR(\gamma)[1 + O(e^{-\beta j}) + O(e^{-\beta(n-j)})].$$

Therefore,

$$\tilde{\mathbf{E}}_n[\Psi^+_{0,n}] = an + O(1).$$

Once we have the asymptotic independence with an exponential rate of convergence, it is straightforward and standard to show that

$$\mathbf{var}_n[\Psi^+_{0,n}] = vn + O(1),$$

where $v = v(\lambda_1, \lambda_2)$ is defined by

$$v = (b_1 - a^2) + 2\sum_{j=2}^{\infty}(b_j - a^2).$$

(It is not difficult to show that

$$b_j = a^2[1 + O(e^{-\beta j})]$$

so the sum converges.) By considering third moments, we can show similarly that $|\xi_n'''(\lambda_1)| \leq c$. This allows us take the limits in the first and second derivatives and conclude that $\xi(\lambda)$ is C^2 with

$$\xi'(\lambda) = a, \qquad \xi''(\lambda) = -v.$$

The formula for v is not very useful for determining that $v > 0$. We establish this in the remaining subsection.

4.4 Strict concavity

Let $\mathcal{Y}_\infty$ be the set of continuous

$$\gamma : (-\infty, \infty) \to \mathcal{J},$$

with

$$\Re[\gamma(t)] \to -\infty, \quad t \to -\infty,$$
$$\Re[\gamma(t)] \to \infty, \quad t \to \infty.$$

As before, if $r \in \mathbb{R}$, we set

$$\sigma_r = \sigma_r(\gamma) = \inf\{t : \Re[\gamma(t)] = r\},$$

and we let γ_r be the element of $\mathcal{Y}$,

$$\gamma_r(t) = \gamma(t + \sigma_r) - r, \quad -\infty < t \leq 0.$$

The measure $\bar{\mu}$ gives a probability measure $\mathcal{P}$ on $\mathcal{Y}_\infty$ in a natural way. To be more precise, let $\mathcal{E}$ denote expectations with respect to $\mathcal{P}$. Then if f is a function on $\mathcal{Y}$, $n \geq 0$,

$$\begin{aligned} \mathcal{E}[f(\gamma_n)] &= \mathbf{E}^{\bar{\mu}}[f(\gamma_n) e^{\xi n} \Theta_n R(\gamma_n) R(\gamma_0)^{-1}] \\ &= \mathbf{E}^{\mu}[f(\gamma_n) e^{\xi n} \Theta_n R(\gamma_n)]. \end{aligned}$$

We write $\mathcal{V}$ for variance with respect to this measure. Let $Y^+_{r,s}, Y^-_{r,s}$ be the functionals as in Section 4.1 which are now defined for $-\infty < r < s < \infty$. For integer n, let

$$\psi_n^+ = -\log Y^+_{n-1,n}.$$

Note that

$$\cdots, \psi^+_{-1}, \psi^+_0, \psi^+_1, \cdots,$$

is a stationary sequence of random variables in $(\mathcal{Y}_\infty, \mathcal{P})$. If n is a positive integer, let

$$\Psi^+_n = -\log Y^+_{0,n},$$

so that

$$\Psi^+_n = \psi^+_1 + \cdots + \psi^+_n.$$

If $n \geq 0$, let $\mathcal{G}_n$ be the σ-algebra of $\mathcal{Y}_\infty$ generated by γ_n. (Note that $\mathcal{G}_n$ is really the same as the $\mathcal{F}_n$ of the previous section, but we use the new notation to emphasize that we are considering the measure $\mathcal{P}$ on $\mathcal{Y}_\infty$.). Note that

$$\mathcal{E}[\Psi^+_{0,n}] = n\mathcal{E}[\psi^+_1] = an.$$

Computations similar to those in the previous section give

$$\mathcal{V}[\Psi^+_{0,n}] = \mathcal{E}[(\Psi^+_{0,n} - an)^2] = vn + O(1).$$

Also, it is straightforward to show that there is a c such that

$$\mathcal{E}[\psi^+_2 + \cdots + \psi^+_n \mid \mathcal{G}_1] \geq an - c.$$

In particular, there is a c_{24} such that if $m < n$ are positive integers,

$$\mathcal{E}[\Psi^+_n \mid \mathcal{G}_m] \geq \Psi^+_m + a(n-m) - c_{24}. \tag{26}$$

We end by proving the strict concavity result, i.e., that $v > 0$.

Lemma 23 *For every* $\lambda_1 > 0, \lambda_2 \geq 0$,

$$v = v(\lambda_1, \lambda_2) > 0.$$

Proof. Since

$$\mathcal{V}[\Psi_n^+] = vn + O(1),$$

it suffices to show that

$$\limsup_{n\to\infty} \mathcal{V}[\Psi_n^+] = \infty.$$

Assume on the contrary that for some $C < \infty$ and all $n \geq 0$.

$$\mathcal{V}[\Psi_n^+] \leq C.$$

Let $M < \infty$. By Chebyshev's inequality, if M is sufficiently large,

$$\mathcal{P}\{|\Psi_n^+ - an| \geq M\} \leq \frac{1}{8}. \tag{27}$$

Let

$$\rho = \rho_M = \inf\{m \in \mathbb{Z}_+ : \Psi_m^+ - am \geq M\}.$$

It is not difficult to see directly that there is a $q = q(M) > 0$ such that if $\Psi_{m-1}^+ \geq a(m-1) - M$ then with probability at least q

$$\Psi_m^+ \geq am + M.$$

Combining this with (27), we can find a $n = n(M)$ such that

$$\mathcal{P}\{\rho \leq n\} \geq \frac{1}{2}.$$

For $k = 1, 2, \ldots, n$, it follows from (26) that

$$\mathbf{E}[\Psi_n^+ \mid \rho = k] \geq an + M - c_{17}.$$

Hence

$$\mathbf{E}[\Psi_n^+ - an \mid \rho \leq n] \geq M - c_{24}.$$

By Schwartz inequality,

$$\mathcal{E}[(\Psi_n^+ - an)^2 \mid \rho \leq n] \geq [M - c_{24}]^2.$$

and hence

$$\mathcal{E}[(\Psi_n^+ - an)^2] \geq \frac{1}{2}[M - c_{24}]^2.$$

But we know that the left hand side is bounded by C. By choosing M sufficiently large we get a contradiction. This completes the proof.

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