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A Stochastic Fixed Point Equation Related to Weighted Branching with Deterministic Weights

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For real numbers $C, T_1, T_2, \ldots$ we find all solutions μ to the stochastic fixed point equation $W \stackrel{d}{=} \sum_{j\geq 1} T_j W_j + C$, where $W, W_1, W_2, \ldots$ are independent real-valued random variables with distribution μ and $\stackrel{d}{=}$ means equality in distribution. All solutions are infinitely divisible. The set of solutions depends on the closed multiplicative subgroup of $\mathbb{R}_* = \mathbb{R}\backslash\{0\}$ generated by the T_j. If this group is continuous, i.e. $\mathbb{R}_*$ itself or the positive halfline $\mathbb{R}_+$, then all nontrivial fixed points are stable laws. In the remaining (discrete) cases further periodic solutions arise. A key observation is that the Lévy measure of any fixed point is harmonic with respect to $\Lambda = \sum_{j\geq 1} \delta_{T_j}$, i.e. $\Gamma = \Gamma \star \Lambda$, where $\star$ means multiplicative convolution. This will enable us to apply the powerful Choquet-Deny theorem.

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1. Introduction

Given a sequence $C, T_1, T_2, ...$ of real-valued random variables with known joint distribution, consider the stochastic fixed point equation

$$W \stackrel{d}{=} \sum_{j\geq 1} T_j W_j + C, \tag{1.1}$$

for i.i.d. real-valued random variables $W, W_1, W_2, ...$ which are independent of (C, T), $T \stackrel{\text{def}}{=} (T_j)_{j\geq 1}$. The general goal is to determine all possible distributions of W such that (1.1) holds true. Every such solution is called a (distributional) fixed point.

Fixed points of (1.1) turn up in a natural way as limits of recursive equations of the form

$$W^{(n+1)} \stackrel{d}{=} \sum_{j\geq 1} T_j^{(n)} W_j^{(n)} + C^{(n)}, \quad n \geq 0. \tag{1.2}$$

Here $(T^{(n)}, C^{(n)})$, $W_1^{(n)}, W_2^{(n)}, ...$ are independent random variables for each $n \geq 0$. The joint distribution of $C^{(n)}$ and $T^{(n)} = (T_1^{(n)}, T_2^{(n)}, ...)$ is given and converges to that of (C, T). The $W_j^{(n)}$'s are copies of $W^{(n)}$. The first study of a recursive system of type (1.2) appears in [26] for the sorting algorithm Quicksort which might still be the most prominent example. The method provided in [26] which applies to general divide and conquer algorithms is based on the contraction of the map K associated with (1.2) defined on the set of $\mathfrak{L}^p$ probability measures ($p \geq 1$) endowed with the Mallows metric (see Section 2); [24],[27] and [30] may be consulted for good descriptions of the method, generalizations, more examples and accounts of relevant literature. [27] contains a systematic study of (1.1) as a contraction limit of (1.2). The results focus on conditions for the convergence of $X^{(n)}$ in the Mallows metric, the existence of solutions of (1.1), their moments and also their tail behavior. They all require at least second moments.

A very special case of equation (1.1), namely

$$W \stackrel{d}{=} T_1 W_1 + C,$$

has been studied in insurance mathematics under the keyword "perpetuity", see [10], [12], and for related results in higher dimensions [19] (stochastic matrices) and [4] (random affine mappings in Hilbert space). (1.1) and (1.2) are intimately connected with weighted branching processes. For the simplest case, the Galton-Watson branching process $(Z_n)_{n\geq 0}$ with offspring distribution $(p_j)_{j\geq 0}$ and offspring mean m, (1.2) holds true upon setting $W^{(n)} = Z_n$, $C^{(n)} = 0$ and $T^{(n)} = (1, ..., 1, 0, ...)$ with $\mathbb{P}(\sum_{j\geq 1} T_j^{(n)} = k) = p_k$ for all $k, n \in \mathbb{N}_0$. Replacing the 1's in $T^{(n)}$ with i.i.d. nonnegative random variables, we get a branching random walk which has been studied by many authors, see e.g. [2] and [3]. In a general weighted branching process the weight of an individual is given as the weight of the mother times an independent random factor, see e.g. [28] for a further discussion including general $C^{(n)}$.

Let us also mention a number of special examples, all having $C = 0$. Mandelbrot [23] introduced fractals and a class of models he called "canonique" which lead to equations of type

(1.1). In this context, random measures and Hausdorff dimension attracted some attention, see [15] and [1] for results using contraction arguments. From a different perspective, Kahane and Peyriére [18] and Guivarc'h [16] considered positive solutions of (1.1) under the restriction of a fixed number of i.i.d. nonzero components T_j of T. Motivated by questions related to infinite particle systems and again only for finitely many nonzero T_j, Durrett and Liggett [11] provided a description of all nontrivial nonnegative fixed points of (1.1). Their approach relies on monotonicity properties of the Laplace transforms which cannot be used in case of general real-valued factors T_j. Partial extensions of their results appear in [21] and [22] under the condition that the number of factors is random but still a.s. finite.

The purpose of this article is to study (1.1) for the case where (C, T) is a vector of real-valued *deterministic* components $C, T_1, T_2, ...$, that is when $C, T_1, T_2, ...$ are just constants. We will determine all fixed points. The major part will focus on the homogeneous case $C = 0$ and builds on an unpublished report by the second author [29]. The case where $C \neq 0$ can then be treated rather shortly by drawing to a large extent on the findings in the homogeneous situation. In terms of the characteristic function φ of W equation (1.1) with $C = 0$ may be rewritten in the equivalent form

$$\varphi(t) \;=\; \prod_{j\geq 1} \varphi(T_j t), \quad t \in \mathbb{R}. \tag{1.3}$$

It will be shown that, trivial cases excluded, any fixed point is infinitely divisible (Prop. 4.1). Hence we may take logarithms to obtain

$$\log \varphi(t) \;=\; \sum_{j\geq 1} \log \varphi(T_j t) \;=\; \int \log \varphi(xt)\; \Lambda(dx) \tag{1.4}$$

with $\Lambda \stackrel{\text{def}}{=} \sum_{j\geq 1} \mathbf{1}_{\{T_j \neq 0\}} \delta_{T_j}$ and δ_x denoting the Dirac measure at x. In a series of papers, Davies, Shimizu, Ramachandran, Rao, and others considered (1.4) for quite general Λ under various assumptions, see [25] for an account. They combine the integrated Cauchy functional equation with additional arguments. Our approach is similar, but uses a stronger form of the Choquet-Deny theorem involving characters and disintegration of measures (see the Appendix to this paper). This simplifies the structural arguments as well as the assumptions.

Here are some explicit examples. The normal distributions with mean 0 and variance $\sigma^2 > 0$ are the unique nonzero solutions to the equation

$$W \;\stackrel{d}{=}\; \frac{W_1}{\sqrt{2}} + \frac{W_2}{\sqrt{2}}, \tag{1.5}$$

see [9], [31] and [25]. The distribution of a random variable W is called *α-stable* for $\alpha \in (0, 2]$ if, for all $a, b > 0$, there exists $d \in \mathbb{R}$ such that

$$cW + d \;\stackrel{d}{=}\; aW_1 + bW_2 \tag{1.6}$$

where W_1, W_2 are independent copies of W and $c > 0$ is determined through $a^\alpha + b^\alpha = c^\alpha$. A symmetric α-stable distribution satisfies (1.6) for $d = 0$. Notice that (1.6) with $d = 0$ is of the form (1.1) with $T_1 = a/c$ and $T_2 = b/c$. Symmetric α-stable distributions have characteristic function

$$\varphi(t) = \exp(-c|t|^\alpha)$$

for some $c > 0$. They solve equation (1.6) as well as (1.1) with constant $T_1, T_2, ...$ whenever $\sum_{j\geq 1} |T_j|^\alpha = 1$. There is also a partial converse. If (1.6) holds for some a, b, c with $|a/c|^\alpha + |b/c|^\alpha = 1$ and $d = 0$, and if the smallest closed multiplicative group generated by a, b equals $\mathbb{R}_+ \stackrel{\text{def}}{=} (0, \infty)$ or $\mathbb{R}_* \stackrel{\text{def}}{=} \mathbb{R}\backslash\{0\}$, the solution is a stable distribution [9], [31], [25].

The condition on the generated group is necessary. Lévy (see [13, p.567]) gave an example of a symmetric non-stable distribution solving (1.6) in the special case

$$W \stackrel{d}{=} \frac{W_1}{2} + \frac{W_2}{2}, \tag{1.7}$$

i.e. $a = b = \frac{1}{2}$. In this case the smallest closed multiplicative group generated by a, b is obviously discrete. Corollary 7.5 will provide all symmetric solutions to (1.1) (and thus (1.6) with $d = 0$). In our notation used there Lévy's example corresponds to the case $\alpha = 1$, $\beta = \gamma = 0$, $c = 1$, $r = 2$, $s = 1$, and $\mathbb{G}_\Lambda = r^\mathbb{Z} \cup -r^\mathbb{Z} \stackrel{\text{def}}{=} \{-r^z, r^z : z \in \mathbb{Z}\}$ where $\mathbb{G}_\Lambda$ denotes the closed multiplicative group generated by $\Lambda = \sum_{j\geq 1} \mathbf{1}_{\{T_j \neq 0\}} \delta_{T_j}$.

We proceed with an outline of the main necessary steps that will furnish our main results to be presented in Section 7 ($C = 0$) and Section 9 ($C \neq 0$). Let $C = 0$ unless stated otherwise and let $\mathfrak{F}$ denote the set of all distributional fixed points of (1.1). Section 2 provides some basic information on the map K associated with (1.1). K is defined on the set of probability measures on $\mathbb{R}$ (see (2.1)), and the elements of $\mathfrak{F}$ are the fixed points of K. Trivial cases where all T_j have modulus 0 or 1 are discussed in Section 3 and thus excluded from the subsequent analysis. Lemma 3.2 collects the then necessary conditions on T for the existence of nontrivial fixed points which are thus standing assumptions throughout the rest of the article. That all elements of $\mathfrak{F}$ are infinitely divisible was already proved in [31] but will be reproved here for completeness by using the weighted branching representation of a fixed point (Section 4).

The next step is to show that $\mathfrak{F}$ contains nontrivial elements iff there exists an $\alpha \in (0, 2]$, in fact uniquely determined, such that

$$\sum_{j\geq 1} |T_j|^\alpha = 1, \tag{1.8}$$

see Proposition 5.1. We call α the *characteristic exponent of* T. One may directly check that all symmetric α-stable distributions then belong to $\mathfrak{F}$ as mentioned above already. However, there may be more fixed points. To determine $\mathfrak{F}$ completely requires to account for the closed multiplicative subgroup of $\mathbb{R}_*$ generated by Λ, that is $\mathbb{G}_\Lambda$. A key fact, first obtained for symmetric fixed points in the proof of Proposition 5.1 and then in Lemma 8.1 for any element of $\mathfrak{F}$, is that the Lévy measure Γ of such a fixed point satisfies

$$\Gamma = \Gamma \star \Lambda \tag{1.9}$$

where $\star$ means multiplicative convolution. So Γ is Λ-*harmonic* which brings the powerful Choquet-Deny theorem [6] into play. This result tells us that any Λ-harmonic ν is a mixture over $\mathbb{R}_*/\mathbb{G}_\Lambda$ of extremal measures of the form $e(s^{-1}x)\lambda\!\!\lambda_{s\mathbb{G}_\Lambda}(dx)$, where $\lambda\!\!\lambda_{s\mathbb{G}_\Lambda}$ denotes the Haar measure on the coset $s\mathbb{G}_\Lambda \stackrel{\text{def}}{=} \{sx : x \in \mathbb{G}_\Lambda\}$, $s \in \mathbb{R}_*/\mathbb{G}_\Lambda$, and e is a character of $\mathbb{G}_\Lambda$ satisfying

$$\int e(x)\ \Lambda(dx)\ =\ 1. \tag{1.10}$$

Though a rather abstract result in the general setting of locally compact Abelian groups (an Appendix collects some necessary general facts) it leads to a very explicit conclusion about Γ in the given situation because the set of characters satisfying (1.10) contains only one element, namely $e(x) = |x|^{-\alpha}$ with α the characteristic exponent of T. We arrive at the conclusion that all nontrivial elements of $\mathfrak{F}$ are stable laws or mixtures of certain periodic variants described in Section 6. Given this information the fixed point equation (1.1), more precisely its characteristic function version (1.4), boils down to an equation for the parameters of these distributions.

Our main results, which provide a complete description of $\mathfrak{F}$, are presented in Section 7. They must distinguish between five different cases because there are essentially five different closed multiplicative subgroups of $\mathbb{R}_*$ that T can generate (listed in Section 5). If $\mathbb{G}_\Lambda$ equals $\mathbb{R}_*$ or $\mathbb{R}_+$, which is the most pleasant situation, all nontrivial elements of $\mathfrak{F}$ are α-stable laws, thus normal distributions if $\alpha = 2$. However, further fixed points of periodic type may occur if $\mathbb{G}_\Lambda$ is discrete. Proofs are presented in Section 8. The case $C \neq 0$ is treated in Section 9. This can be done by a straightforward reduction to the homogeneous case (Theorem 9.2) whenever $\sum_{j\geq 1} T_j \in \mathbb{R}\backslash\{1\}$. If $\sum_{j\geq 1} T_j$ equals 1 or does not exist in $\mathbb{R}$ there may be no fixed points at all (Theorem 9.3). The final Section 10 contains a brief discussion of the associated multiplicative random walk associated with T. The latter was used in [11] for the determination of all solutions to (1.1) for the case where $C = 0$ and T consists of a finite fixed number of positive random variables.

2. The fixed point equation and some properties

Let T_j, $j \geq 1$, be given real numbers and define the map K on the set of probability measures on $\mathbb{R}$ by

$$K(\mu)\ \stackrel{\text{def}}{=}\ \mathcal{L}\left(\sum_{j\geq 1} T_j W_j\right) \tag{2.1}$$

where the random variables $W_1, W_2, ...$ are independent with distribution μ and where $\mathcal{L}(X)$ means the distribution of a random variable X. The infinite sum need not exist but if it does it is understood in the sense of convergence in distribution of the finite sums $\sum_{j=1}^n T_j X_j$ as $n \to \infty$. Since these are sums of independent random variables the convergence actually holds true a.s., see e.g. [7, p. 292]. Denote by $\mathfrak{D}(K)$ the domain of K, i.e. the set of all probability distributions μ for which $K(\mu)$ is well defined.

As usual, denote by $*$ the convolution of measures on the additive group $\mathbb{R}$. The domain $\mathfrak{D}(K)$ is closed under $*$ because one can easily see that

$$K(\mu * \nu) \;=\; K(\mu) * K(\nu) \tag{2.2}$$

for all $\mu, \nu \in \mathfrak{D}(K)$. K further commutes with the reflection operator R defined as $R(\mu) = \mathcal{L}(-X)$ where X has distribution μ. So we have

$$KR \;=\; RK, \tag{2.3}$$

and the domain $\mathfrak{D}(K)$ is closed under the operator R.

Our purpose is to completely describe the set of fixed points of K, that is of

$$\mathfrak{F} \;\stackrel{\text{def}}{=}\; \{\mu \in \mathfrak{D}(K) : \mu = K(\mu)\}$$

or, equivalently, the set $\hat{\mathfrak{F}}$ of associated characteristic functions. The point measure at 0 is a trivial fixed point.

Lemma 2.1. *The set $\mathfrak{F}$ is closed under convolution and reflection.*

Proof. Given two fixed points μ, ν, we infer with the help of (2.2) and (2.3) that $K(\mu * \nu) = K(\mu) * K(\nu) = \mu * \nu$ and $KR(\mu) = RK(\mu) = \mu$, respectively. ◇

As an immediate consequence we note:

Corollary 2.2. *If μ is a fixed point then the same holds true for its symmetrization $\mu^s \stackrel{\text{def}}{=} \mu * R(\mu)$.*

In the following we will write the fixed point equation $K(\mu) = \mu$ in terms of random variables, that is in the form (1.1) with $C = 0$:

$$W \;\stackrel{d}{=}\; \sum_{j \geq 1} T_j W_j, \tag{2.4}$$

where $W, W_1, W_2, \ldots$ have distribution μ. If

$$\varphi(t) \;=\; \varphi_W(t) \;\stackrel{\text{def}}{=}\; \mathbb{E}(e^{itW}) \;=\; \int e^{itx}\;\mu(dx)$$

denotes the characteristic function of W (or μ) then (2.4) is equivalent to

$$\varphi(t) \;=\; \prod_{j \geq 1} \varphi(T_j t), \quad t \in \mathbb{R}. \tag{2.5}$$

The right hand side is necessarily well defined as the limit of $\prod_{j=1}^{n}\varphi(T_jt)$ by Lévy's continuity theorem. We note that the order of summation or multiplication in (2.4), respectively (2.5) is fixed and can in fact be crucial in cases where $\sum_{j=1}^{n}T_j$ converges to a finite limit while $\sum_j |T_j| = \infty$.

3. Trivial cases

We already noted after the definition of $\mathfrak{F}$ that $\mu = \delta_0$ is always a fixed point of K. However, there may be more such trivial solutions under special assumptions on $T = (T_1, T_2, ...)$. We call $\mu \in \mathfrak{F}$ trivial if $\mu = \delta_c$ for some c in which case the characteristic function equals $\varphi(t) = e^{ict}$. For example, μ is trivial if the smallest closed additive group generated by $\{t \in \mathbb{R} : |\varphi(t)| = 1\}$ is $\mathbb{R}$ itself [20, Cor. 2 to Thm. 2.1.4]. We will next briefly show that the trivially cases are those where the T_j take values only in $\{-1, 0, 1\}$.

Proposition 3.1. *Suppose that all T_j are in $\{-1, 0, 1\}$.*
(a) If $|\{j : T_j \neq 0\}| \neq 1$ then there exist only trivial solutions.
(b) If $|\{j : |T_j| = 1\}| = |\{j : T_j = 1\}| = 1$ then every distribution is a fixed point.
(c) If $|\{j : |T_j| = 1\}| = |\{j : T_j = -1\}| = 1$ then $\mathfrak{F}$ consists of all symmetric distributions.

Proof. If all T_j are 0 there is nothing to prove ($\mathfrak{F} = \{\delta_0\}$). Given the assumption of (a), the estimate

$$|\varphi(t)| \;\le\; \prod_{j\ge 1}|\varphi(T_jt)| \;=\; \prod_{j:|T_j|=1}|\varphi(T_jt)| \;\le\; |\varphi(t)|^2, \quad t \in \mathbb{R}$$

implies that $|\varphi(t)|$ is either 0 or 1 for every t, hence $|\varphi| \equiv 1$ because φ is continuous. This shows $\mu = \delta_c$ for some c. The proofs of (b) and (c) are easy and thus omitted. ◇

Assuming that not all T_j are in $\{-1, 0, 1\}$, we finally prove that the existence of a nontrivial fixed point not only implies that there must be nonzero T_j with modulus $\neq 1$ but that in fact all nonzero T_j must have modulus less than 1 and that there are at least two of them.

Lemma 3.2. *Suppose that not all T_j are in $\{-1, 0, 1\}$. If there exists a nontrivial fixed point then $\sup_{j\ge 1}|T_j| < 1$, $\lim_{j\to\infty} T_j = 0$ and $\sum_{j\ge 1}\mathbf{1}_{\{T_j\neq 0\}} \ge 2$.*

Proof. Let φ be the characteristic function of a nontrivial fixed point μ. We first prove that $\sup_{j\ge 1}|T_j| \le 1$. Indeed, if $|T_j| > 1$ for some j, then, by (2.5), $|\varphi(t)| \le |\varphi(T_jt)|$ for all t which inductively implies $|\varphi(t)| \ge |\varphi(t/T_j^n)| \to \varphi(0) = 1$ as $n \to \infty$. Hence $|\varphi| \equiv 1$ and μ must be trivial.

Now suppose that $\sup_{j\ge 1}|T_j| = 1$. If $|T_j| = 1$ for some $j \in \mathbb{N}$, w.l.o.g. $j = 1$, then (2.5) implies $|\psi(t)| = 1$ for all $t \in \{\varphi \neq 0\}$, where $\psi(t) \overset{\text{def}}{=} \prod_{j\ge 2}\varphi(T_jt)$. Since $\{\varphi \neq 0\} \supset [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$ we infer $|\psi| \equiv 1$ on $[-\varepsilon, \varepsilon]$ and then everywhere because ψ is a ch.f. (use Cor.

2 on p. 298 in [7]). This yields the contradiction $|\varphi| \equiv 1$. If all nonzero T_j have modulus less than 1 choose an infinite subsequence $(T_{j(k)})_{k\geq 1}$ with $|T_{j(k)}| \to 1$. Then, by continuity, $\lim_{k\to\infty} |\varphi(T_{j(k)}t)| = |\varphi(t)|$ which in turn implies $|\varphi(T_{j(k)}t)| \leq |\varphi(t)|+\varepsilon < 1$ for infinitely many k and some suitable $\varepsilon > 0$, whenever $|\varphi(t)| < 1$. By another appeal to (2.5) we thus conclude

$$|\varphi(t)| \;\leq\; \prod_{k\geq 1} |\varphi(T_{j(k)}t)| \;=\; 0 \tag{3.1}$$

for all t with $|\varphi(t)| < 1$, i.e. $|\varphi|$ is 0-1-valued. But this leads once again to the contradiction $|\varphi| \equiv 1$ because φ is continuous. So we have proved $\sup_{j\geq 1} |T_j| < 1$.

If $c \overset{\text{def}}{=} \limsup_{j\to\infty} |T_j| > 0$ let $(T_{j(k)})_{k\geq 1}$ be a subsequence with $|T_{j(k)}| \to c$ and infer $\lim_{k\to\infty} |\varphi(T_{j(k)}t)| = |\varphi(ct)|$ for all t. Again (3.1) follows, now for all t with $|\varphi(ct)| < 1$, and leads once more to the contradicting conclusion $|\varphi| \equiv 1$. So $\lim_{j\to\infty} T_j = 0$.

Finally, it is immediately seen that $\sum_{j\geq 1} \mathbf{1}_{\{T_j\neq 0\}} = 1$ in combination with $\sup_{j\geq 1} |T_j| < 1$ implies $\mathfrak{F} = \{\delta_0\}$ and thus that there can be no nontrivial solution. ♢

4. The weighted branching representation and infinite divisibility

In this section we will show, under the assumption $\sup_{j\geq 1} |T_j| < 1$, that any fixed point of K is infinitely divisible. For this and later purposes we next give a brief description of the weighted branching process associated with equation (2.1).

Let $\mathbb{V}$ be the infinite tree with vertex set $\cup_{n\geq 0}\mathbb{N}^n$ where $\mathbb{N}^0 \overset{\text{def}}{=} \{\emptyset\}$. Each vertex $v = (v_1, ..., v_n)$, which we also write as $v_1v_2...v_n$, is uniquely connected to the root $\emptyset$ by the path $\emptyset \to v_1 \to v_1v_2 \to ... \to v$ of length $|v| = n$. Put $L(\emptyset) \overset{\text{def}}{=} 1$ and define

$$L(v) \;=\; T_{v_1}T_{v_2} \cdot ... \cdot T_{v_n}.$$

for any $v = v_1...v_n \in \mathbb{V}$ of length $n \geq 1$. Then $L(v)$ gives the total weight of the unique path from the root to v under multiplication. Now let $X(v)$, $v \in \mathbb{V}$, be i.i.d. random variables with common distribution μ and define

$$W_n \;=\; \sum_{|v|=n} L(v)X(v) \tag{4.1}$$

for $n \geq 0$. The notation $\sum_{|v|=n}$ suggests that the order of summation does not matter. However, this may not be true because our assumptions will not guarantee that the $L(v)X(v)$ are absolutely summable. In such cases we stipulate that the summation is to be understood in lexicographic order: $\sum_{|v|=n} = \sum_{v_1\geq 1} ... \sum_{v_n\geq 1}$. Indeed, this is the right summation when iterating equation (2.1), see (4.3) below.

$(W_n)_{n\geq 0}$ forms a stochastic sequence called *weighted branching process*. It satisfies the *backward equation*

$$W_{n+1} \;=\; \sum_{j\geq 1} T_j W_{n,j} \tag{4.2}$$

where the $W_{n,j} = \sum_{|v|=n} \frac{L(jv)}{T_j} X(jv)$, $j \geq 1$, are i.i.d. copies of W_n. This shows that $K^n(\mu) = \mathcal{L}(W_n)$ for each $n \geq 0$. In particular, all W_n have distribution μ if μ is a fixed point of K. Under this assumption it is now easily seen that an n-fold iteration of equation (2.1) takes the form

$$W \stackrel{d}{=} \sum_{|v|=n} L(v) W(v) \tag{4.3}$$

for every $n \in \mathbb{N}$ where the $W(v)$, $|v| = n$, are i.i.d. copies of W. This representation will be used in the proof of Proposition 4.1 below.

A probability measure μ as well as its characteristic function φ (or a random variable X with $\mathcal{L}(X) = \mu$) is called *infinitely divisible* if for each $n \in \mathbb{N}$ there exists a characteristic function φ_n such that $\varphi = \varphi_n^n$. Equivalently, μ can be decomposed as the n-fold convolution of a probability measure μ_n having characteristic function φ_n for each n.

A triangular scheme is an array of random variables $Y_{j,k}$, $j \in \mathbb{N}$ and $1 \leq k \leq k_j$ for some $k_j \in \mathbb{N}$. It is called *independent* if the rows consist of independent random variables, and it is called *infinitesimal* or *asymptotically negligible* if

$$\lim_{j\to\infty} \sup_{1\leq k\leq k_j} \mathbb{P}(|Y_{j,k}| \geq \varepsilon) \ = \ 0$$

for any $\varepsilon > 0$. The connection with infinite divisibility is the following: The distributional limit of the row sums of any independent infinitesimal triangular scheme is necessarily inifinitely divisible (if the limit exists), see e.g. [14]. This is the crucial fact to be used in the proof of the following result concerning the fixed points of K given by (2.1).

PROPOSITION 4.1. *Suppose* $\sup_{j\geq 1} |T_j| < 1$. *Then every solution to the fixed point equation (2.4) is infinitely divisible.*

PROOF. Since trivial solutions are cleary infinitely divisible assume there is a nontrivial solution μ and suppose $W \stackrel{d}{=} \mu$. Then W satisfies (4.3) which implies that for each $n \geq 1$ we find a finite set $\mathbb{V}_n \subset \{v \in \mathbb{V} : |v| = n\}$ such that

$$\sum_{v\in\mathbb{V}_n} L(v)W(v) \ \stackrel{d}{\to} \ \mu \quad (n \to \infty)$$

where $\stackrel{d}{\to}$ means convergence in distribution. Observe that the $L(v)W(v)$, $v \in \mathbb{V}_n$ and $n \in \mathbb{N}$, form an independent infinitesimal triangular scheme with row sums $\sum_{v\in\mathbb{V}_n} L(v)W(v)$. It is asymptotically negligible because $|L(v)| \leq (\sup_{j\geq 1} |T_j|)^n$ if $|v| = n$. Hence we conclude the asserted infinite divisibility of μ from the result stated above. $\diamondsuit$

The Lévy representation for an infinitely divisible distribution μ states that the logarithm of its characteristic function φ has a unique integral representation of the form

$$\log\varphi(t) \ = \ i\gamma t - \frac{\sigma^2 t^2}{2} + \int \left(e^{itu} - 1 - it\chi(u)\right) \Gamma(du). \tag{4.4}$$

Here $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$, $\chi(u) \stackrel{\text{def}}{=} u\mathbf{1}_{[-1,1]}(u) + \mathbf{1}_{(1,\infty)}(u) - \mathbf{1}_{(-\infty,-1)}(u)$ is the truncated (at ± 1) identity function and Γ a not necessarily finite measure on $\mathbb{R}_*$, called *Lévy measure* associated with μ (or φ). Γ satisfies

$$\int (u^2 \wedge 1)\ \Gamma(du) \ < \ \infty. \tag{4.5}$$

See [14] for different representations and [17] for an approach via Choquet representation theory. In the situation of Proposition 4.1 where μ is a fixed point we further have for all $u > 0$

$$\Gamma([-u,u]^c) \ = \ \lim_{n\to\infty} \sum_{|v|=n} \mathbb{P}(|W| > u/|L(v)|), \quad W \stackrel{d}{=} \mu, \tag{4.6}$$

see [5, Lemma 6]. Note that, if μ is symmetric, then $\gamma = 0$ and Γ is also symmetric whence (4.4) simplifies to

$$\log\varphi(t) \ = \ -\frac{\sigma^2 t^2}{2} - \int (1-\cos(tu))\ \Gamma(du). \tag{4.7}$$

5. The characteristic exponent of $T = (T_1, T_2, ...)$

Throughout the remainder of this article we always assume that $T = (T_1, T_2, ...)$ satisfies

$$0 \ < \ \sup_{j\geq 1} |T_j| \ < \ 1, \quad \lim_{j\to\infty} T_j \ = \ 0 \quad \text{and} \quad \sum_{j\geq 1} \mathbf{1}_{\{T_j \neq 0\}} \ \geq \ 2. \tag{A}$$

This is justified because, by Lemma 3.2, nontrivial fixed points of K exist only under this condition when excluding the trivial cases discussed in Proposition 3.1.

Define the function $m : [0,\infty) \to (0,\infty]$ by

$$m(\beta) \ \stackrel{\text{def}}{=} \ \sum_{j\geq 1} |T_j|^\beta \mathbf{1}_{\{T_j\neq 0\}} \tag{5.1}$$

and note that $m(0) = \sum_{j\geq 1} \mathbf{1}_{\{T_j\neq 0\}} \geq 2$. It follows from assumption (A) that m is strictly decreasing on $\{\beta \geq 0 : m(\beta) < \infty\}$. Consequently, there exists at most one α with $m(\alpha) = 1$. It is positive because $m(0) \geq 2$. We call this α the *characteristic exponent of* T hereafter and will next show that it always exists in $(0,2]$ whenever K has nontrivial fixed points.

Proposition 5.1. *Suppose* (A) *and that* $\mathfrak{F}$ *contains a nontrivial element. Then the characteristic exponent* α *of* T *exists and is an element of* $(0,2]$.

Remark. If (A) holds but $\mathfrak{F} = \{\delta_c : c \in \mathbb{R}\}$ then the characteristic exponent α of T may be > 2 or not even exist. As an example for the first situation take $T_1 = -T_2 = \frac{2}{3}$, $T_3 = T_4 = \frac{1}{2}$ and $T_i = 0$, otherwise. Then $\sum_{j\geq 1} T_j = 1$ ensures $\delta_c \in \mathfrak{F}$ for $c \in \mathbb{R}$, while $m(3) < 1 < m(2)$ implies $2 < \alpha < 3$. An example where α does not exist is given by $T_j = (-1)^{j+1}\frac{1}{k}$ for $2^{k-1} \leq j < 2^k$ and $k \geq 1$. Indeed, $\sum_{j\geq 1} T_j = 1$ again holds true, but

$$m(\beta) \ = \ \sum_{k\geq 1} 2^{k-1} k^{-\beta} \ = \ \infty$$

for every $\beta > 0$.

Before we can proceed with the proof some important facts must be collected. Recall that $\Lambda = \sum_{j\ge1} \mathbf{1}_{\{T_j\neq0\}}\delta_{T_j}$ and $\mathbb{G}_\Lambda$ denotes the closed multiplicative subgroup of $\mathbb{R}_*$ generated by Λ. There are five possible cases:

(C1) $\mathbb{G}_\Lambda = \mathbb{R}_*$.
(C2) $\mathbb{G}_\Lambda = \mathbb{R}_+$.
(D1) $\mathbb{G}_\Lambda = r^{\mathbb{Z}} \cup -r^{\mathbb{Z}}$ for some $r > 1$, where $r^{\mathbb{Z}} \stackrel{\text{def}}{=} \{r^z : z \in \mathbb{Z}\}$.
(D2) $\mathbb{G}_\Lambda = r^{\mathbb{Z}}$ for some $r > 1$.
(D3) $\mathbb{G}_\Lambda = (-r)^{\mathbb{Z}}$ for some $r > 1$.

Notice that condition (A) excludes the trivial subgroups $\{-1, 1\}$ and $\{1\}$. The Haar measure (unique up to multiplicative constants), denoted as $\lambda\!\!\lambda_{\mathbb{G}_\Lambda}$ hereafter, equals $|u|^{-1}du$ in the continuous cases (C1) and (C2), and counting measure in the discrete ones. Let $E(\mathbb{G}_\Lambda)$ be the set of characters of $\mathbb{G}_\Lambda$, that is the set of all continuous positive functions $e : \mathbb{G}_\Lambda \to \mathbb{R}_+$ satisfying

$$e(xy) \;=\; e(x)e(y)$$

for all $x, y \in \mathbb{G}_\Lambda$. Of particular interest for our purposes is the subset

$$E_1(\Lambda) \;\stackrel{\text{def}}{=}\; \left\{ e \in E(\mathbb{G}_\Lambda) : \int e(x^{-1})\,\Lambda(dx) = 1 \right\}.$$

It is not difficult to check that in all five cases the characters are given by the functions $e_l(x) \stackrel{\text{def}}{=} |x|^{-l}$, $l \in \mathbb{R}$, so $E(\mathbb{G}_\Lambda) = E$ is independent of Λ. Moreover, we infer upon noting $\int e_l(x^{-1})\,\Lambda(dx) = \sum_{j\ge1} |T_j|^l = m(l)$ that, under (A), $E_1(\Lambda)$ is either void or consists of the single element e_α, α the characteristic exponent of T.

Now consider a Radon measure μ on $\mathbb{R}_*$ and suppose that μ is Λ-*harmonic*, defined by $\mu = \mu \star \Lambda$. Here $\star$ means multiplicative convolution, that is

$$\int f(x)\ \mu \star \Lambda(dx) \;\stackrel{\text{def}}{=}\; \int\!\!\int f(xy)\ \mu(dx)\ \Lambda(dy)$$

for any measurable $f : \mathbb{G}_\Lambda \to [0, \infty)$. The set of all Λ-harmonic measures is a convex cone. By the Choquet-Deny theorem which we state in an Appendix at the end of the paper we infer that any nonzero Λ-harmonic μ has a unique integral representation

$$\mu \;=\; \int \mu_e(y^{-1}\cdot)\ \overline{\mu}(de, dy)$$

where $\mu_e(dx) \stackrel{\text{def}}{=} e(x)\lambda\!\!\lambda_{\mathbb{G}_\Lambda}(dx)$ for $e \in E$ and $\overline{\mu}$ is a finite measure on $E_1(\Lambda) \times \mathbb{R}_*/\mathbb{G}_\Lambda$ endowed with the Baire σ-field. If $E_1(\Lambda) = \emptyset$ there is no Λ-harmonic measure. Otherwise, $E_1(\Lambda) = \{e_\alpha\}$ so that $\overline{\mu}$ must equal $c(\delta_{e_\alpha} \otimes \tilde{\mu})$ for some probability measure $\tilde{\mu}$ on the factor group $\mathbb{R}_*/\mathbb{G}_\Lambda$

and a $c > 0$. This means that

$$\mu(dx) \;=\; \int c\left|\frac{x}{y}\right|^{-\alpha} \lambda\!\!\lambda_{y\mathbb{G}_\Lambda}(dx)\;\tilde\mu(dy). \tag{5.2}$$

Of course, if $\mathbb{G}_\Lambda = \mathbb{R}_*$ then $\mathbb{R}_*/\mathbb{G}_\Lambda = \{1\}$, $\tilde\mu = \delta_1$ and thus

$$\mu(dx) \;=\; c|x|^{-\alpha}\lambda\!\!\lambda_{\mathbb{R}_*}(dx) \;=\; c|x|^{-\alpha-1}dx \tag{5.3}$$

for some $c > 0$. If $\mathbb{G}_\Lambda = \mathbb{R}_+$ then $\mathbb{R}_*/\mathbb{G}_\Lambda = \{-1, 1\}$, $\tilde\mu = b\delta_{-1} + (1-b)\delta_1$ for some $b \in [0,1]$ and thus

$$\mu(dx) \;=\; (c_1\mathbf{1}_{(-\infty,0)}(x) + c_2\mathbf{1}_{(0,\infty)}(x))|x|^{-\alpha-1}dx \tag{5.4}$$

for some $c_1, c_2 \geq 0$ with $c_1 + c_2 > 0$.

Proof of Proposition 5.1. Let μ be a nontrivial fixed point with characteristic function φ. Since $\mu \in \mathfrak{F}$ implies $\mu^s = \mu * R(\mu) \in \mathfrak{F}$ (Corollary 2.2) it is no loss of generality to assume that μ is symmetric with nonnegative φ. Using (2.5) and the Lévy representation (4.7) we infer the equation

$$\begin{aligned}\frac{\sigma^2t^2}{2} + \int(1-\cos(tu))\;\Gamma(du) \;&=\; -\log\varphi(t) = \;-\sum_{j\geq 1}\log\varphi(T_jt)\\ &=\; \frac{\sigma^2t^2\sum_{j\geq1}T_j^2}{2} + \int\sum_{j\geq1}(1-\cos(T_jtu))\;\Gamma(du)\end{aligned} \tag{5.5}$$

for all $t \in \mathbb{R}$ with unique $\sigma^2 \geq 0$ and Lévy measure Γ. Note that the last expression exists as the limit of the nondecreasing sequence $\frac{1}{2}\sigma^2t^2\sum_{j=1}^n T_j^2 + \int\sum_{j=1}^n(1-\cos(T_jtu))\,\Gamma(du)$, $n \in \mathbb{N}$. We conclude that either $\sigma^2 > 0$ in which case $m(2) = \sum_{j\geq1}T_j^2 = 1$ (thus $\alpha = 2$), or $\sigma^2 = 0$ and (5.5) reduces to

$$\int(1-\cos(tu))\;\Gamma(du) \;=\; \int\sum_{j\geq1}(1-\cos(T_jtu))\;\Gamma(du) \;=\; \int(1-\cos(tu))\;\Gamma\star\Lambda(du) \tag{5.6}$$

for all $t \in \mathbb{R}$, in particular

$$\int_{[-\varepsilon,\varepsilon]} u^2\;\Gamma\star\Lambda(du) \;\leq\; 4\int(1-\cos(u))\;\Gamma\star\Lambda(du) \;<\; \infty \tag{5.7}$$

for sufficiently small $\varepsilon > 0$. With the help of (4.6), we also have

$$\begin{aligned}\Gamma\star\Lambda([-\varepsilon,\varepsilon]^c) \;&=\; \sum_{j\geq1}\Gamma\left(\left[\frac{-\varepsilon}{|T_j|},\frac{\varepsilon}{|T_j|}\right]\right)\\ &=\; \lim_{n\to\infty}\sum_{j\geq1}\sum_{|v|=n}\mathbb{P}\left(|W| > \frac{\varepsilon}{|T_jL(v)|}\right), \quad W \stackrel{d}{=} \mu,\\ &=\; \lim_{n\to\infty}\sum_{|v|=n+1}\mathbb{P}\left(|W| > \frac{\varepsilon}{|L(v)|}\right)\end{aligned}$$

$$= \Gamma([-\varepsilon,\varepsilon]^c) \; < \; \infty.$$

By combining this with (5.7) we see that $\Gamma \star \Lambda$ is also a Lévy measure, and the uniqueness of Γ entails

$$\Gamma = \Gamma \star \Lambda \tag{5.8}$$

and therefore, by (5.2), that

$$\Gamma(du) \;=\; \int c\left|\frac{y}{u}\right|^{\alpha} \lambda\!\!\lambda_{y\mathbb{G}_\Lambda}(du)\ \tilde{\Gamma}(dy) \tag{5.9}$$

for some $c > 0$ and a probability measure $\tilde{\Gamma}$ on $\mathbb{R}_*/\mathbb{G}_\Lambda$. The number $\alpha > 0$ is the unique characteristic exponent of T, and it satisfies $\alpha < 2$ because

$$\infty \;>\; \int_{[-1,1]} u^2\ \Gamma(du) \;=\; c\int_{\mathbb{R}_*/\mathbb{G}_\Lambda}\int_{[-1,1]} |u|^{2-\alpha}\ \lambda\!\!\lambda_{y\mathbb{G}_\Lambda}(du)\ |y|^{\alpha}\tilde{\Gamma}(dy)$$

in any of the five cases for $\mathbb{G}_\Lambda$ mentioned above. ◊

6. Stable and $s\mathbb{G}$-stable distributions

An infinitely divisible distribution μ is called *stable* if the set

$$\{\mu(a^{-1}\cdot) * \delta_b : a > 0, b \in \mathbb{R}\} \tag{6.1}$$

is closed with respect to additive convolution $*$. This is just an equivalent formulation of property (1.6) stated in the Introduction. The characteristic function φ of a stable distribution has the representation

$$\log\varphi(t) \;=\; i\gamma t - c|t|^{\alpha}(1 + i\beta\,\mathrm{sgn}(t)\omega(t,\alpha)), \tag{6.2}$$

where $\alpha \in (0,2]$ is the so-called *index of* μ, $\beta \in [-1,1]$, $\gamma \in \mathbb{R}$, $c \geq 0$ are further parameters, $\mathrm{sgn}(t)$ denotes the sign of t ($\mathrm{sgn}(0) \stackrel{\text{def}}{=} 0$), and

$$\omega(t,\alpha) \;\stackrel{\text{def}}{=}\; \begin{cases} \tan(\alpha\pi/2), & \text{if } \alpha \neq 1,\\ \dfrac{2}{\pi}\log|t|, & \text{if } \alpha = 1.\end{cases}$$

The Lévy measure Γ of μ is the null measure in case $\alpha = 2$ and equals

$$\Gamma(dx) \;=\; \big(c_1\mathbf{1}_{(-\infty,0)}(x) + c_2\mathbf{1}_{(0,\infty)}(x)\big)\alpha|x|^{-\alpha-1}dx \tag{6.3}$$

otherwise. Here c_1, c_2 are nonnegative number satisfying $c_1 + c_2 > 0$ and $\beta = \frac{c_1-c_2}{c_1+c_2}$. Moreover,

$$c \;=\; \begin{cases} -(c_1+c_2)\cos(\alpha\pi/2)\int_0^\infty (e^{-x}-1)x^{-\alpha-1}\,dx, & \text{if } \alpha \in (0,1),\\ (c_1+c_2)\pi/2, & \text{if } \alpha = 1,\\ -(c_1+c_2)\cos(\alpha\pi/2)\int_0^\infty (e^{-x}-1+x)x^{-\alpha-1}\,dx, & \text{if } \alpha \in (1,2),\\ \sigma^2/2, & \text{if } \alpha = 2.\end{cases}$$

The representation (6.2) is unique unless $\alpha = 2$ or $c = 0$ in which case β is arbitrary.

For the continuous cases (C1) and (C2) where the subgroup $\mathbb{G}_\Lambda$ generated by Λ is uncountable we will see in the next section that all nontrivial fixed points of K are stable distributions. However, fixed points of a more general type occur when $\mathbb{G}_\Lambda$ is discrete (cases (D1-3)). The following notion of a $s\mathbb{G}$-*stable* distribution provides an appropriate class for these additional solutions. Here $\mathbb{G}$ is an infinite closed multiplicative subgroup of $\mathbb{R}_*$, s an element of the factor group $\mathbb{R}_*/\mathbb{G}$ and $s\mathbb{G}$ the usual coset $\{sx : x \in \mathbb{G}\}$. Recall that $\lambda\!\!\lambda_{s\mathbb{G}}$ equals $|u|^{-1}\mathbf{1}_{s\mathbb{G}}(u)du$ if $\mathbb{G}$ is continuous and counting measure on $s\mathbb{G}$ if $\mathbb{G}$ is discrete.

Definition 6.1. Given $\mathbb{G}$ and s as just stated, an infinitely divisible distribution μ with characteristic function φ is called $s\mathbb{G}$-*stable with index* $\alpha \in (0, 2)$ if

$$\log \varphi(t) \ = \ i\gamma t + c \int \left(e^{istu} - 1 - it\chi(su)\right)|u|^{-\alpha} \lambda\!\!\lambda_{\mathbb{G}}(du) \tag{6.4}$$

for some $\gamma \in \mathbb{R}$ and $c > 0$.

One can immediately check that if $\mathbb{G}$ equals $\mathbb{R}_*$ itself and thus $s = 1$ then the $\mathbb{G}$-stable distributions are just the ordinary symmetric stable distributions with index $\alpha \in (0, 2)$. If $\mathbb{G} = \mathbb{R}_+$ then $s \in \{-1, 1\}$ and the $s\mathbb{G}$-stable distributions are the one-sided stable distributions concentrated either on $\mathbb{R}_+$ or $\mathbb{R}_-$. However, if $\mathbb{G}$ is one of the discrete subgroups listed in (D1-3) the $s\mathbb{G}$-stable distributions are no longer stable. On the other hand, the set defined in (6.1), but with $a \in r^{\mathbb{N}}$, is again closed under additive convolution for any $s\mathbb{G}$-stable distribution μ. Furthermore, if $X \stackrel{d}{=} \mu$ then rX is $rs\mathbb{G}$-stable with rs computed modulo $\mathbb{G}$. Note finally that an $s\mathbb{G}$-stable distribution is symmetric iff $\gamma = 0$ and $\mathbb{G} = -\mathbb{G}$, thus $\mathbb{G} = \mathbb{R}_*$ or $\mathbb{G} = r^{\mathbb{Z}} \cup -r^{\mathbb{Z}}$ for some $r > 1$. In this case (6.4) simplifies to

$$\log \varphi(t) \ = \ c \int (\cos(su) - 1)|u|^{-\alpha} \lambda\!\!\lambda_{\mathbb{G}}(du) \tag{6.5}$$

for some $c > 0$.

In the discrete cases (D1-3) mixtures of $s\mathbb{G}_\Lambda$-stable distributions will arise as additional solutions of the fixed point equation $K(\mu) = \mu$.

7. Main Results

We are now in the position to present our main results. The following theorems provide a full description of all nontrivial solutions to $K(\mu) = \mu$ in the possible cases (C1-2) and (D1-3). They are given in terms of their charactersistic function which amounts to a description of $\hat{\mathfrak{F}}$. Condition (A) will be in force throughout and α always denotes the characteristic exponent of T. Let us stipulate hereafter that $\sum_{j\ge1} T_j \neq 1$ can mean $\lim_{n\to\infty} \sum_{j=1}^n T_j \neq 1$ or that this limit does not exist at all.

THEOREM 7.1. ($\mathbb{G}_\Lambda = \mathbb{R}_*$). *Let $\mathbb{R}_*$ be the smallest closed multiplicative subgroup generated by the T_j.*

(a) If $\alpha \in (0,1]$, or if $\alpha \in (1,2]$ and $\sum_{j\geq 1} T_j \neq 1$, then

$$\hat{\mathfrak{F}} \;=\; \{\exp(-c|t|^\alpha) : c \geq 0\},$$

i.e. the nontrivial fixed points of K are exactly the symmetric stable laws of index α.

(b) If $\alpha \in (1,2]$ and $\sum_{j\geq 1} T_j = 1$ then

$$\hat{\mathfrak{F}} \;=\; \{\exp(i\gamma t - c|t|^\alpha) : c \geq 0, \gamma \in \mathbb{R}\},$$

i.e. the nontrivial fixed points of K are exactly the symmetric stable laws of index α plus a shift γ.

THEOREM 7.2. ($\mathbb{G}_\Lambda = \mathbb{R}_+$). *Let $\mathbb{R}_+$ be the smallest closed multiplicative subgroup generated by the T_j.*

(a) If $\alpha = 1$ then

$$\hat{\mathfrak{F}} \;=\; \{\exp(i\gamma t - c|t|) : c \geq 0, \gamma \in \mathbb{R}\},$$

i.e. the nontrivial fixed points are the symmetric stable laws with index 1 plus a shift.

(b) If $\alpha \in (0,1) \cup (1,2)$ then

$$\hat{\mathfrak{F}} \;=\; \{\exp(-c|t|^\alpha(1 + i\beta\,\mathrm{sgn}(t)\omega(t,\alpha))) : c \geq 0, \beta \in [-1,1]\},$$

i.e. the nontrivial fixed points of K are exactly the stable laws of index α with $\gamma = 0$.

(c) If $\alpha = 2$ then

$$\hat{\mathfrak{F}} \;=\; \{\exp(-\sigma^2 t^2/2) : \sigma^2 \geq 0\},$$

i.e. the nontrivial fixed points of K are exactly the normal distributions with mean 0.

The next theorem provides a complete description of the fixed points of K (in terms of their characteristic functions) for discrete $\mathbb{G}_\Lambda$, but without a distinction of the three cases (D1-3). A specialization to these follows in a subsequent corollary. Let us note that we have

- $\mathbb{R}_*/(r^{\mathbb{Z}} \cup -r^{\mathbb{Z}}) \simeq [1,r)$ if $r > 1$ in Case (D1).
- $\mathbb{R}_*/r^{\mathbb{Z}} \simeq (-r,-1] \cup [1,r)$ for all $r > 1$ in Case (D2).
- $\mathbb{R}_*/(-r)^{\mathbb{Z}} \simeq (-r,-1] \cup [1,r)$ for all $r > 1$ in Case (D3).

Here $\simeq$ means isomorphic equality.

THEOREM 7.3. ($\mathbb{G}_\Lambda$ *discrete*). *Let the smallest closed multiplicative subgroup generated by the T_j be one of the discrete subgroups listed in (D1-3).*

(a) If $\alpha \in (0,2)$ then $\hat{\mathfrak{F}}$ consists of all φ of the form

$$\varphi(t) \;=\; \exp\Big(it\gamma + c \iint (e^{itsu} - 1 - it\chi(su))\,|u|^{-\alpha}\, \lambda\!\!\lambda_{\mathbb{G}_\Lambda}(du)\,\nu(ds)\Big), \quad t \in \mathbb{R}, \qquad (7.1)$$

where the probability measure ν on $\mathbb{R}_*/\mathbb{G}_\Lambda$ and the constants $c \geq 0$, $\gamma \in \mathbb{R}$ are subject to the following contraint: If

$$F_{s,c}(x) \stackrel{\text{def}}{=} c \int (x\chi(su) - \chi(xsu))\,|u|^{-\alpha}\,\lambda\!\!\lambda_{\mathbb{G}_\Lambda}(du) \tag{7.2}$$

for $x \in [-1, 1]$, then

$$\gamma = \lim_{n\to\infty} \sum_{j=1}^{n} \big(\gamma T_j + \textstyle\int F_{s,c}(T_j)\,\nu(ds)\big). \tag{7.3}$$

(b) If $\alpha = 2$, then

$$\hat{\mathfrak{F}} = \{\exp(it\gamma - \sigma^2 t^2/2) : \sigma^2 \geq 0, \gamma \in \mathbb{R}\},$$

in case $\sum_{j\geq 1} T_j = 1$, and

$$\hat{\mathfrak{F}} = \{\exp(\sigma^2 t^2/2) : \sigma^2 \geq 0\},$$

in case $\sum_{j\geq 1} T_j \neq 1$. So the nontrivial fixed points of K are the normal distributions if $\sum_{j\geq 1} T_j = 1$, and the centered normal distributions otherwise.

Let $\mathbb{F}$ be the set of all triples (γ, c, ν) for which (7.3) holds true. Then, for the discrete case, Theorem 7.3 provides us in principle with a complete description of $\mathfrak{F}$ (or $\hat{\mathfrak{F}}$) in terms of $\mathbb{F}$ (which is one to one unless $c = 0$ in which case the triple $(\gamma, 0, \nu)$ pertains to $\varphi(t) = e^{i\gamma t}$ regardless of ν). On the other hand, the appearing condition (7.3) naturally demands for further examination. Doing so while considering the cases (D1-3) separately, one is led to a more explicit description of $\mathbb{F}$ stated as Theorem 7.4 below.

Let $\mathfrak{M}_\Lambda$ be the set of probability measures ν on $\mathbb{R}_*/\mathbb{G}_\Lambda$ (when identified with the subsets of $\mathbb{R}_*$ given before Theorem 7.3) and $\mathfrak{M}_\Lambda^s$ the subset of symmetric ν. Note that $\mathfrak{M}_\Lambda^s$ is empty in Case (D1) because then $\mathbb{R}_*/\mathbb{G}_\Lambda \subset \mathbb{R}_+$. For the Cases (D2) and (D3) we further need the class $\mathfrak{M}_\Lambda^0$ consisting of all centered $\nu \in \mathfrak{M}_\Lambda$, if $\alpha = 1$, and otherwise of those $\nu \in \mathfrak{M}_\Lambda$ satisfying, respectively, $\int \operatorname{sgn}(s)|s|^\alpha B(s)\,\nu(ds) = 0$ and $\int \operatorname{sgn}(s)|s|^\alpha \Delta(s)\,\nu(ds) = 0$, where the bounded and even functions B and Δ are defined in (8.5) and (8.8) of the next section. Notice that $\mathfrak{M}_\Lambda^s \subset \mathfrak{M}_\Lambda^0$ holds in any case.

THEOREM 7.4. *Let $\alpha \in (0, 2)$ and the smallest closed multiplicative subgroup generated by the T_j be one of the discrete subgroups listed in (D1-3).*

(a) Suppose (D1) holds. If

(i) $\alpha > 1$ and $\sum_j T_j = 1$, then $\mathbb{F} = \mathbb{R} \times [0, \infty) \times \mathfrak{M}_\Lambda$, that is, $\mathfrak{F}$ consists of all mixtures of $s\mathbb{G}_\Lambda$-stable distributions.

(ii) $\alpha \leq 1$ or $\sum_j T_j \neq 1$, then $\mathbb{F} = \{0\} \times [0, \infty) \times \mathfrak{M}_\Lambda$, that is, $\mathfrak{F}$ consists of all mixtures of symmetric $s\mathbb{G}_\Lambda$-stable distributions (including trivial solutions where $c = 0$).

(b) Suppose (D2) holds in which case all T_j are nonnegative. If

(i) $\alpha = 1$, thus $\sum_j T_j = 1$, then $\mathbb{F} = \mathbb{R} \times [0, \infty) \times \mathfrak{M}_\Lambda^0$.

(ii) $\alpha \neq 1$, *then for each pair* $(c,\nu) \in [0,\infty) \times \mathfrak{M}_\Lambda$ *there exists exactly one* γ *such that* $(\gamma, c, \nu) \in \mathbb{F}$.

(c) Suppose (D3) holds. If

(i) $\alpha = 1$, *or* $\alpha \neq 1$ *and* $\sum_j T_j = \sum_j \operatorname{sgn}(T_j)|T_j|^\alpha$, *then* $\mathbb{F} = \{0\} \times [0,\infty) \times \mathfrak{M}_\Lambda$.

(ii) $\alpha \neq 1$ *and* $\sum_j T_j = 1$, *then* $\mathbb{F} = \mathbb{R} \times [0,\infty) \times \mathfrak{M}_\Lambda^0$.

(iii) $\alpha \neq 1$ *and* $\sum_j T_j$ *exists in* $\mathbb{R} \cup \{\pm\infty\}$ *but does not equal* $\sum_j \operatorname{sgn}(T_j)|T_j|^\alpha$ *or* 1, *then for each pair* $(c,\nu) \in (0,\infty) \times \mathfrak{M}_\Lambda$ *there exists exactly one* γ *such that* $(\gamma, c, \nu) \in \mathbb{F}$.

(iv) $\alpha > 1$ *and* $\sum_j T_j$ *does not exist, then* $\mathbb{F} = \{0\} \times [0,\infty) \times \mathfrak{M}_\Lambda^0$.

Remark. It should be clear that the fixed points provided in Theorem 7.1 for the case $\mathbb{G}_\Lambda = \mathbb{R}_*$ (under respective conditions on α and T) remain to be fixed points for any discrete subgroup $\mathbb{G}_\Lambda$. They are obtained when choosing ν as the uniform distribution on $\mathbb{R}_*/\mathbb{G}_\Lambda$. If $\mathbb{G}_\Lambda$ is also a subgroup of $\mathbb{R}_+$ (Case (D2)) the same holds true for the fixed points given in Theorem 7.2. These are obtained by choosing ν as a mixture of uniform distributions on the two congruent connected components of $\mathbb{R}_*/\mathbb{G}_\Lambda$.

A description of the set $\mathfrak{F}^s$ of symmetric fixed points of K or, equivalently, the associated set $\hat{\mathfrak{F}}^s$ of characteristic functions is easily derived from the previous results and thus summarized without proof in the subsequent corollary. As for the discrete cases, we only note that φ of the form (7.1) belongs to $\hat{\mathfrak{F}}^s$ iff $\gamma = 0$ and at least one of $\lambda\!\!\lambda_{\mathbb{G}_\Lambda}$ and ν is symmetric. Plainly, $\lambda\!\!\lambda_{\mathbb{G}_\Lambda}$ is symmetric in the case (D2) where $\mathbb{G}_\Lambda = r^{\mathbb{Z}} \cup -r^{\mathbb{Z}}$ for some $r > 1$.

Corollary 7.5. *(Symmetric fixed points).*

(a) If $\alpha = 2$ *then, regardless of* $\mathbb{G}_\Lambda$, $\hat{\mathfrak{F}}^s = \{\exp(\sigma^2 t^2/2) : \sigma^2 \geq 0\}$, *i.e. the nontrivial symmetric fixed points are the centered normal distributions.*

(b) If $\alpha \in (0,2)$ *and* $\mathbb{G}_\Lambda$ *equals* $\mathbb{R}_*$ *or* $\mathbb{R}_+$, *then* $\hat{\mathfrak{F}}^s = \{\exp(-c|t|^\alpha) : c \geq 0\}$, *i.e. the nontrivial symmetric fixed points are the symmetric stable laws of index* α.

(c) If $\alpha \in (0,2)$ *and* $\mathbb{G}_\Lambda = r^{\mathbb{Z}} \cup -r^{\mathbb{Z}}$ *for some* $r > 1$, *then*

$$\hat{\mathfrak{F}}^s \;=\; \{\exp(-c \textstyle\iint (1-\cos(stu))|u|^{-\alpha} \lambda\!\!\lambda_{\mathbb{G}_\Lambda}(du)\, \nu(ds)) : c \geq 0, \nu \in \mathfrak{M}_\Lambda\},$$

i.e. the nontrivial symmetric fixed points are the mixtures of symmetric $s\mathbb{G}_\Lambda$*-stable distributions.*

(d) If $\alpha \in (0,2)$ *and* $\mathbb{G}_\Lambda$ *equals* $r^{\mathbb{Z}}$ *or* $(-r)^{\mathbb{Z}}$ *for some* $r > 1$, *then*

$$\hat{\mathfrak{F}}^s \;=\; \{\exp(-c \textstyle\iint (1-\cos(stu))|u|^{-\alpha} \lambda\!\!\lambda_{\mathbb{G}_\Lambda}(du)\, \nu(ds)) : c \geq 0, \nu \in \mathfrak{M}_\Lambda^s\}$$

i.e. the nontrivial symmetric fixed points are the symmetric mixtures of $s\mathbb{G}_\Lambda$*-stable distributions.*

Remark. Let us point out that in cases where $\sum_j T_j$ exists in $\mathbb{R}$ while $\sum_j |T_j| = \infty$, the set $\mathfrak{F}$ may depend on the order of summation of the T_j. It is indeed a well known fact in such a situation that for each $x \in \mathbb{R}$ we can find a rearrangement $T_{\pi(1)}, T_{\pi(2)}, \ldots$ satisfying $\sum_j T_{\pi(j)} = x$. But our results show that the set of fixed points belonging to a rearrangement π with $\sum_j T_{\pi(j)} = 1$ generally differs from the corresponding set when $\sum_j T_{\pi(j)} = x \neq 1$.

8. Proofs

In the proof of Proposition 5.1 we were led to the conclusion that the Lévy measure Γ of any nontrivial *symmetric* fixed point μ satisfies $\Gamma = \Gamma \star \Lambda$ where $\star$ denotes multiplicative convolution, see (5.8). Lemma 8.1(c) shows this be true for any nontrivial fixed point and provides us with the key to determine Γ by an application of the powerful Choquet-Deny theorem (see (5.9) for symmetric μ).

Lemma 8.1. *Let φ be the characteristic function of any infinitely divisible distribution μ with Lévy-Khinchine representation (4.4). Put $\Lambda_n \stackrel{\text{def}}{=} \sum_{j=1}^n \mathbf{1}_{\{T_j \neq 0\}} \delta_{T_j}$ for $n \geq 1$. Then μ is a fixed point of K iff the following conditions hold true:*

(a) As $n \to \infty$,

$$\gamma_n \stackrel{\text{def}}{=} \gamma \int x \, \Lambda_n(dx) + \iint (x\chi(u) - \chi(xu)) \, \Gamma(du) \, \Lambda_n(dx) \ \to \ \gamma. \tag{8.1}$$

(b) Either $\sigma^2 = 0$, or $\sigma^2 > 0$ and $m(2) = \sum_{j \geq 1} T_j^2 = 1$ (thus $\alpha = 2$).

(c) The Lévy measure is Λ-harmonic, i.e. $\Gamma = \Gamma \star \Lambda$.

Proof. The "only if-part" is easily obtained by checking that any φ that meets the conditions in (a-c) satisfies the fixed point equation (1.4). So we can immediately proceed with the proof of the "if-part".

Obviously, trivial fixed points must satisfy the asserted conditions so that we may focus on nontrivial ones. Given the characteristic function φ of any nontrivial fixed point with Lévy-Khinchine representation (4.4), let Γ^s be the Lévy measure of its symmetrization μ^s. We have $\Gamma^s(du) = \Gamma(du) + \Gamma(-du)$ and $\Gamma^s = \Gamma^s \star \Lambda$, the latter being true by (5.8) because μ^s is symmetric. It follows that

$$\int (u^2 \wedge 1) \, \Gamma \star \Lambda_n(du) \ \leq \ \int (u^2 \wedge 1) \, \Gamma \star \Lambda(du) \ \leq \ \int (u^2 \wedge 1) \, \Gamma^s \star \Lambda(du) \ < \ \infty. \tag{8.2}$$

for all $n \geq 1$. By combining (2.5) with the Lévy-Khinchine representation of φ we get

$$\begin{aligned}
\log \varphi(t) \ &= \ it\gamma - \frac{\sigma^2 t^2}{2} + \int (e^{itu} - 1 + it\chi(u)) \, \Gamma(du) \\
&= \ \lim_{n\to\infty} \sum_{j=1}^n \log \varphi(T_j t) \ = \ \lim_{n\to\infty} \int \log \varphi(xt) \, \Lambda_n(dx) \\
&= \ \lim_{n\to\infty} \left(it\gamma \int x \, \Lambda_n(dx) - \frac{\sigma^2 t^2}{2} \int x^2 \, \Lambda_n(dx) \right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \iint (e^{itxu} - 1 + itx\chi(u))\,\Gamma(du)\,\Lambda_n(dx)\Big) \\
&= \lim_{n\to\infty} \bigg(it\Big(\gamma \int x\,\Lambda_n(dx) + \iint (x\chi(u) - \chi(ux))\,\Gamma(du)\,\Lambda_n(dx)\Big) \\
&\qquad - \frac{\sigma^2 t^2}{2} \int x^2\,\Lambda_n(dx) + \iint (e^{itxu} - 1 + it\chi(xu))\,\Gamma(du)\,\Lambda_n(dx)\bigg) \\
&= \lim_{n\to\infty} \bigg(it\gamma_n - \frac{\sigma^2 t^2}{2} \int x^2\,\Lambda_n(dx) + \int (e^{itu} - 1 + it\chi(u))\,\Gamma \star \Lambda_n(du)\bigg) \qquad (8.3)
\end{aligned}
$$

for all $t \in \mathbb{R}$. Since $\Gamma \star \Lambda_n$ converges weakly to $\Gamma \star \Lambda$, $|e^{itu} - 1 + it\chi(u)| \le C_t t(u^2 \wedge 1)$ for all $u, t \in \mathbb{R}$ and suitable constants $C_t > 0$, and by (8.2) we infer

$$
\lim_{n\to\infty} \int (e^{itu} - 1 + it\chi(u))\,\Gamma \star \Lambda_n(du) \;=\; \int (e^{itu} - 1 + it\chi(u))\,\Gamma \star \Lambda(du)
$$

Moreover, $m(2) = \sum_{j\ge1} T_j^2 \le 1$ ensures

$$
\lim_{n\to\infty} \frac{\sigma^2 t^2}{2} \int x^2\,\Lambda_n(dx) \;=\; \frac{\sigma^2 t^2}{2}\, m(2).
$$

Consequently, γ_n also converges to some $\gamma' \in \mathbb{R}$ as $n \to \infty$. Returning to (8.3) this yields

$$
\begin{aligned}
&it\gamma - \frac{\sigma^2 t^2}{2} + \int (e^{itu} - 1 + it\chi(u))\,\Gamma(du) \\
&\qquad = \; it\gamma' - \frac{\sigma^2 t^2}{2} m(2) + \int (e^{itu} - 1 + it\chi(u))\,\Gamma \star \Lambda(du).
\end{aligned}
$$

for all $t \in \mathbb{R}$. The uniqueness of the Lévy-Khinchine representation finally implies that $\gamma = \gamma'$, $\sigma^2 = 0$ or $m(2) = 1$, and $\Gamma = \Gamma \star \Lambda$. $\diamondsuit$

Proof of Theorem 7.1. It is easily checked that all elements of $\hat{\mathfrak{F}}$ as asserted pertain to fixed points of K for the respective cases. So we must conversely show that there are no other ones. To that end let μ be any fixed point of K with characteristic function φ having Lévy-Khinchine representation (4.4). Since $\mathbb{G}_\Lambda = \mathbb{R}_*$ and Γ is Λ-harmonic, we have by (5.3) that $\Gamma(du) = c|u|^{-\alpha-1}du$ for some $c \ge 0$.

Suppose first $\alpha = 2$. Then $\infty > \int_{-1}^{1} u^2\,\Gamma(du) = c\int_{-1}^{1} |u|^{-1}\,du$ entails $c = 0$, thus $\Gamma \equiv 0$. Since μ is nontrivial we must further have $\sigma^2 > 0$. With this at hand, condition (8.1) in Lemma 8.1 simplifies to

$$
0 \;=\; \lim_{n\to\infty} \gamma\Big(\int x\,\Lambda_n(dx) - 1\Big) \;=\; \lim_{n\to\infty} \gamma\Big(\sum_{j=1}^{n} T_j - 1\Big).
$$

So either $\gamma = 0$ and thus $\varphi(t) = \exp(-\sigma^2 t^2/2)$, or $\gamma \ne 0$, $\sum_{j\ge1} T_j = 1$ and thus $\varphi(t) = \exp(i\gamma t - \sigma^2 t^2/2)$ must hold.

If $\alpha < 2$ then $\sigma^2 = 0$ by Lemma 8.1(b) and $\Gamma(du) = c|u|^{-\alpha-1}du$ for some *positive* c because μ is nontrivial. We infer that μ is a stable distribution with index α and characteristic function

$$\varphi(t) \;=\; \exp(it\gamma - c|t|^\alpha),$$

see (6.2). Again, using (2.5) and the Lévy-Khinchine representation we have

$$it\gamma - c|t|^\alpha \;=\; it\gamma \sum_j T_j - c|t|^\alpha \sum_j |T_j|^\alpha$$

and therefore

$$t\gamma \;=\; t\gamma \sum_j T_j$$

for all $t \in \mathbb{R}$. The uniqueness of the representation implies that

$$\gamma(\sum_j T_j - 1) = 0$$

and thus $\gamma = 0$ unless $\sum_{j\geq 1} T_j = 1$. Note that $\sum_{j\geq 1} T_j < \sum_{j\geq 1} |T_j| \leq 1$ for $\alpha \leq 1$. Now one can easily check that μ is of the asserted type. ♢

Proof of Theorem 7.2. The proof in case $\mathbb{G}_\Lambda = \mathbb{R}_+$ is very similar to the previous one and we therefore restrict ourselves to a few comments. Again we must only verify that a nontrivial solution μ is of the type asserted in the theorem for the respective cases. By (5.4), its Lévy measure Γ this time has the form $\Gamma(du) = (c_1\mathbf{1}_{(-\infty,0)}(u) + c_2\mathbf{1}_{(0,\infty)}(u))|u|^{-\alpha-1}du$ with $c_1, c_2 \geq 0$. If $\alpha = 2$, we conclude $\Gamma \equiv 0$ by the same argument as above, while $c_1 + c_2 > 0$ must hold if $\alpha \in (0,2)$. The nonnegativity of the T_j together with the uniqueness of α as a solution to $\sum_{j\geq 1} T_j^\alpha = 1$ implies $\sum_{j\geq 1} T_j \neq 1$ whenever $\alpha \neq 1$. For the case $\alpha = 2$ this entails that only the centered normal distributions can be fixed points ($\gamma = 0$). In all other cases a fixed point must be a stable law, and the uniqueness of the Lévy-Khinchine representation may once again be employed to arrive at the asserted constraints of the parameters. Further details are omitted. ♢

Proof of Theorem 7.3. We only consider the case $\alpha \in (0,2)$ because for $\alpha = 2$ the Lévy measure of any nontrivial fixed point equals again 0. After this observation the remaining arguments are the same as in the previous two theorems.

If $\alpha \in (0,2)$ then, by (5.9), the Lévy measure Γ of any nontrivial fixed point equals

$$\Gamma(du) \;=\; \int c\left|\frac{s}{u}\right|^\alpha \lambda\!\!\lambda_{s\mathbb{G}_\Lambda}(du)\,\nu(ds)$$

for some probability measure ν on $\mathbb{R}_*/\mathbb{G}_\Lambda$ and some $c > 0$. We have $\sigma^2 = 0$ by Lemma 8.1(b). Hence

$$\begin{aligned}\log\varphi(t) \;&=\; it\gamma + c\iint (e^{itu} - 1 + it\chi(u))\left|\frac{s}{u}\right|^\alpha \lambda\!\!\lambda_{s\mathbb{G}_\Lambda}(du)\,\nu(ds)\\ &=\; it\gamma + c\iint (e^{itsu} - 1 + it\chi(su))|u|^{-\alpha}\,\lambda\!\!\lambda_{\mathbb{G}_\Lambda}(du)\,\nu(ds).\end{aligned}$$

This shows that a nontrivial fixed point is necessarily a mixture of $s\mathbb{G}_\Lambda$-stable distributions for which we must still check the contraints resulting from the fixed point equation or, equivalently, from condition (8.1) in Lemma 8.1 which here takes the form

$$\begin{aligned}\gamma &= \lim_{n\to\infty}\left(\gamma\sum_{j=1}^n T_j + \sum_{j=1}^n \int \big(T_j\chi(u) - \chi(T_ju)\big)\,\Gamma(du)\right)\\ &= \lim_{n\to\infty}\sum_{j=1}^n\left(\gamma T_j + c\iint (T_j\chi(su) - \chi(suT_j))\,|u|^{-\alpha}\,\lambda\!\!\lambda_{\mathbb{G}_\Lambda}(du)\,\nu(ds)\right)\\ &= \lim_{n\to\infty}\sum_{j=1}^n\left(\gamma T_j + \int F_{s,c}(T_j)\,\nu(ds)\right)\end{aligned}$$

which is the asserted condition (7.3). $\diamondsuit$

In order for a proof of Theorem 7.4 based upon a discussion of condition (7.3) the following lemma will provide us with a tractable expression for $\sum_{j=1}^n F_{s,c}(T_j)$, $F_{s,c}$ as defined in (7.2).

Lemma 8.2. *Define $A(x,u) \stackrel{\text{def}}{=} x\chi(u) - \chi(xu)$ for $x,u \in \mathbb{R}$. Then*

$$\sum_{n\in\mathbb{Z}} r^{-\alpha n}A(x,sr^n) = \begin{cases} B(s)\,\mathrm{sgn}(xs)\,|s|^\alpha\,(|x| - |x|^\alpha), & \text{if } \alpha \neq 1\\ sx\,\log_r|x|, & \text{if } \alpha = 1\end{cases} \tag{8.4}$$

for all $x \in (-1,1)$, $s \neq 0$ and $r > 1$, where

$$B(s) = B(\alpha,r,s) \stackrel{\text{def}}{=} \frac{r^{-\alpha\eta}}{1-r^{-\alpha}} - \frac{r^{(1-\alpha)\eta}}{1-r^{1-\alpha}}. \tag{8.5}$$

for $\alpha \neq 1$ and η denotes the fractional part of $\log_r|s|$. The function $B(s)$ is bounded and further positive if $\alpha < 1$ and negative if $\alpha > 1$.

Proof. The function $A(x,u)$ is odd in each argument and satisfies for $x \in (0,1)$

$$A(x,u) = \begin{cases} 0, & \text{if } 0 \le u \le 1,\\ x(1-u), & \text{if } 1 < u \le 1/x,\\ x-1, & \text{if } u > 1/x.\end{cases}$$

It thus suffices to consider the case where $x \in (0,1)$ and $s > 0$. Suppose $s = r^\zeta$, $x = r^{-m}$, and note that $\zeta = l + \eta$ for some integer $l \in \mathbb{Z}$. If $\alpha \neq 1$ then

$$\begin{aligned} 0 &> \sum_{n\in\mathbb{Z}} r^{-\alpha n}A(x,sr^n)\\ &= \sum_{-\zeta\le n<m-\zeta} r^{-\alpha n}r^{-m}(1-r^{\zeta+n}) + (r^{-m}-1)\sum_{n\ge m-\zeta} r^{-\alpha n}\\ &= r^{-m}\sum_{n=-l}^{m-l-1} r^{-\alpha n} - r^{\zeta-m}\sum_{n=-l}^{m-l-1} r^{(1-\alpha)n} + (r^{-m}-1)\sum_{n\ge m-l} r^{-\alpha n}\end{aligned}$$

$$
\begin{aligned}
&= \frac{r^{-m+\alpha l} - r^{-m-\alpha(m-l)}}{1-r^{-\alpha}} - \frac{r^{\zeta-(1-\alpha)l}(r^{-m} - r^{-\alpha m})}{1-r^{1-\alpha}} + \frac{(r^{-m}-1)r^{-\alpha(m-l)}}{1-r^{-\alpha}} \\
&= r^{\alpha l}\left(\frac{r^{-m}-r^{-\alpha m}}{1-r^{-\alpha}} - \frac{r^{\eta}(r^{-m}-r^{-\alpha m})}{1-r^{1-\alpha}}\right) \\
&= r^{\alpha(\zeta-\eta)}(r^{-m}-r^{-\alpha m})\left(\frac{1}{1-r^{-\alpha}} - \frac{r^{\eta}}{1-r^{1-\alpha}}\right) \\
&= s^{\alpha}(x-x^{\alpha})\left(\frac{r^{-\alpha\eta}}{1-r^{-\alpha}} - \frac{r^{(1-\alpha)\eta}}{1-r^{1-\alpha}}\right)
\end{aligned}
$$

which proves the assertion including $B(s) > 0$, if $\alpha < 1$, and $B(s) < 0$ if $\alpha > 1$. If $\alpha = 1$ it suffices to note that the middle term of the third line in the above computation simplifies to

$$
-r^{\zeta-m}\sum_{n=-l}^{m-l-1} r^{(1-\alpha)n} \;=\; -r^{\zeta-m}m \;=\; sx\log_r x,
$$

while the first and last one cancel each other. ♢

Given one of the cases (D2) or (D3), the previous lemma does now easily lead to an expression for $\sum_{j=1}^{n} F_{s,c}(T_j)$ that will enable us to prove Theorem 7.4. We obtain:

Case (D2). If $\mathbb{G}_\Lambda = r^{\mathbb{Z}}$ for some $r > 1$, then all T_j are nonnegative and

$$
\begin{aligned}
\sum_{j=1}^{n} F_{s,c}(T_j) &= c\sum_{j=1}^{n}\sum_{k\in\mathbb{Z}} r^{-\alpha k} A(T_j, sr^k) \\
&= \begin{cases} cB(s)\,\mathrm{sgn}(s)|s|^{\alpha}\sum_{j=1}^{n}(T_j - T_j^{\alpha}), & \text{if } \alpha \neq 1, \\ cs\sum_{j=1}^{n} T_j \log_r T_j, & \text{if } \alpha = 1 \end{cases}
\end{aligned} \tag{8.6}
$$

for each $s \in \mathbb{R}_*/r^{\mathbb{Z}}$

Case (D3). Suppose $\mathbb{G}_\Lambda = (-r)^{\mathbb{Z}}$ for some $r > 1$. It is easily checked with the help of (8.4) (though not directly seen upon inspection) that $F_{s,c} \equiv 0$ in case $\alpha = 1$. If $\alpha \neq 1$, let $s \in \mathbb{R}_*/(-r)^{\mathbb{Z}}$ and note that $\eta \overset{\text{def}}{=} \log_{r^2}|s| \in [0, \frac{1}{2})$. Since $F_{s,c}$ is odd in s the following computation is only done for the case $s > 0$. We infer with (8.4)

$$
\begin{aligned}
\sum_{j=1}^{n} F_{s,c}(T_j) &= c\sum_{j=1}^{n}\sum_{k\in\mathbb{Z}}\Big(r^{-2k\alpha}A(T_j, sr^{2k}) - r^{-(2k+1)\alpha}A(T_j,(rs)r^{2k})\Big) \\
&= c\Big(B(\alpha, r^2, s)s^{\alpha} - r^{-\alpha}B(\alpha, r^2, rs)(rs)^{\alpha}\Big)\sum_{j=1}^{n}(T_j - \mathrm{sgn}(T_j)|T_j|^{\alpha}) \qquad (8.7) \\
&= c\,\Delta(s)\,s^{\alpha}\sum_{j=1}^{n}(T_j - \mathrm{sgn}(T_j)|T_j|^{\alpha})
\end{aligned}
$$

for each positive $s \in \mathbb{R}_*/(-r)^{\mathbb{Z}}$, where $\Delta(s) = \Delta(\alpha, r, s) \stackrel{\text{def}}{=} B(\alpha, r^2, s) - B(\alpha, r^2, rs)$ and $\Delta(-s) \stackrel{\text{def}}{=} \Delta(s)$. Since $\log_{r^2}(sr) = \eta + \frac{1}{2}$, a straightforward calculation using (8.5) gives

$$\Delta(s) \;=\; \frac{r^{-2\alpha\eta}}{1 + r^{-\alpha}} - \frac{r^{2\eta(1-\alpha)}}{1 + r^{1-\alpha}}. \tag{8.8}$$

We note that $\Delta(\alpha, r, s)$ can be zero or nonzero and that in the case $\alpha = 2\eta = \frac{1}{2}$ we have $\Delta(\alpha, r, s) = 0$ for all $r > 1$.

Proof of Theorem 7.4. Suppose $\alpha \in (0, 2)$.

Case (D1) If $\mathbb{G}_\Lambda = r^{\mathbb{Z}} \cup -r^{\mathbb{Z}}$ for some $r > 1$, then $F_{s,c} \equiv 0$ for all s, c and (7.3) simplifies to

$$\lim_{n\to\infty} \gamma \left(\sum_{j=1}^{n} T_j - 1 \right) \;=\; 0, \tag{8.9}$$

i.e. either $\sum_j T_j = 1$ and $\gamma \in \mathbb{R}$, or $\sum_j T_j \neq 1$ and $\gamma = 0$. This yields part (a) of the theorem.

Case (D2). Suppose $\mathbb{G}_\Lambda = r^{\mathbb{Z}}$ for some $r > 1$. Using (8.6) and $\sum_j T_j^\alpha = 1$, condition (7.3) becomes

$$\begin{aligned} \gamma \;&=\; \left(\gamma + \textstyle\int \operatorname{sgn}(s)|s|^\alpha B(s)\,\nu(ds)\right) \textstyle\sum_{j\ge1} T_j - c \textstyle\int \operatorname{sgn}(s)|s|^\alpha B(s)\,\nu(ds) \\ &=\; \gamma \textstyle\sum_{j\ge1} T_j + c \textstyle\int \operatorname{sgn}(s)|s|^\alpha B(s)\,\nu(ds) \big(\textstyle\sum_{j\ge1} T_j - 1 \big) \end{aligned} \tag{8.10}$$

if $\alpha \neq 1$, and

$$0 \;=\; c \textstyle\int s\,\nu(ds) \sum_{j\ge1} T_j \log T_j, \tag{8.11}$$

if $\alpha = 1$. The assertions (i)-(iii) are now easily concluded when recalling in the case $\alpha \neq 1$ that $B(s)$ is either positive or negative on $\mathbb{R}_*/\mathbb{G}_\Lambda$.

Case (D3). Suppose $\mathbb{G}_\Lambda = (-r)^{\mathbb{Z}}$ for some $r > 1$. If $\alpha = 1$ then $F_{s,c} \equiv 0$ implies that (7.3) again simplifies to (8.9). Since here $\sum_j T_j < \sum_j |T_j| = 1$ (not all T_j are nonnegative) we infer $\gamma = 0$ and thus assertion (i). If $\alpha \neq 1$, use (8.7) to rewrite (7.3) in the form

$$\begin{aligned} \gamma \;&=\; \left(\gamma + c\textstyle\int \operatorname{sgn}(s)|s|^\alpha \Delta(s)\,\nu(ds)\right) \textstyle\sum_{j\ge1} T_j - c \textstyle\int \operatorname{sgn}(s)|s|^\alpha \Delta(s)\,\nu(ds) \textstyle\sum_{j\ge1} \operatorname{sgn}(T_j)|T_j|^\alpha \\ &=\; \gamma \textstyle\sum_{j\ge1} T_j + c\textstyle\int \operatorname{sgn}(s)|s|^\alpha \Delta(s)\,\nu(ds) \big(\textstyle\sum_{j\ge1} T_j - \textstyle\sum_{j\ge1} \operatorname{sgn}(T_j)|T_j|^\alpha \big) \end{aligned} \tag{8.12}$$

Note that $\sum_j \operatorname{sgn}(T_j)|T_j|^\alpha$ is always finite and < 1. Note also that

$$\lim_{n\to\infty} \frac{\sum_{j\le n} T_j - \sum_{j\le n} \operatorname{sgn}(T_j)|T_j|^\alpha}{\sum_{j\le n} T_j - 1} \;=\; 1,$$

if $|\sum_j T_j| = \infty$. Assertions (ii)-(iv) are now easily concluded. ♢

9. The inhomogeneous case

In this section we will briefly discuss the fixed point equation (1.1) with a *nonzero* constant C (inhomogeneous case). So the map K is now defined as

$$K(\mu) \stackrel{\text{def}}{=} \mathcal{L}\left(\sum_{j\geq 1} T_j X_j + C\right) \tag{9.1}$$

with independent $X_1, X_2, ...$ having common distribution μ. The weighted branching representation of any fixed point $W \stackrel{d}{=} \mu$ of K becomes

$$W \stackrel{d}{=} \sum_{|v|=n} L(v)W(v) + C \sum_{|v|<n} L(v) \tag{9.2}$$

for all $n \geq 1$ and is obtained by successive iteration of (1.1). The $W(v)$, $v \in \mathbb{V}$, are independent copies of W. Summation may again be a subtle point. Recall from Section 4 that $\sum_{|v|=n} = \sum_{v_1 \geq 1} ... \sum_{v_n \geq 1}$. If we iterate (1.1) once we get

$$\begin{aligned} W &\stackrel{d}{=} \lim_{n\to\infty} \sum_{i=1}^{n} T_i \left(\sum_{j\geq 1} T_j W(ij) + C \right) + C \\ &= \lim_{n\to\infty} \left(\sum_{i=1}^{n} \sum_{j\geq 1} L(ij) W(ij) + C \sum_{i=1}^{n} T_j \right) + C \end{aligned} \tag{9.3}$$

and thus see that the summands going with C by rearrangement may be separated from those going with the $W(ij)$. On the other hand, we cannot conclude at this point that the limit in (9.3) exists when taken for both terms $\sum_{i=1}^{n} \sum_{j\geq 1} L(ij)W(ij)$ and $C \sum_{i=1}^{n} T_j$ separately. In particular, it is not clear at this point whether or not $\sum_j T_j$ exists in $\mathbb{R}$ when there is a fixed point (see Theorem 9.3 for an answer). These remarks apply, of course, to all higher order iterations of (1.1) as well.

Refraining from a discussion of trivial cases we assume from the beginning that T satisfies condition (A) hereafter. By (9.2) and a similar argument as in the proof of Proposition 4.1, we see that each nontrivial fixed point μ of K can be obtained as the limit of an independent infinitesimal triangular scheme and is thus infinitely divisible. So the logarithm of its characteristic function φ exists everywhere and satisfies (compare (1.4))

$$\log \varphi(t) = iCt + \sum_{j\geq 1} \log \varphi(T_j t). \tag{9.4}$$

Embarking on this observation the following lemma is just the straightforward extension of Lemma 8.1 to general $C \in \mathbb{R}$.

Lemma 9.1. *Let φ be the characteristic function of any infinitely divisible distribution μ with Lévy-Khinchine representation (4.4). Then μ is a fixed point of K iff the conditions* *(a-c) in Lemma 8.1 hold true, where (8.1) is modified to*

$$C+\gamma_n \;=\; C+\gamma\int x\,\Lambda_n(dx)+\iint (x\chi(u)-\chi(xu))\,\Gamma(du)\,\Lambda_n(dx)\;\to\;\gamma, \qquad (9.5)$$

as $n\to\infty$.

Let $\mathfrak{F}_C$ be the set of fixed points of K for general C, hence $\mathfrak{F}=\mathfrak{F}_0$. If $\sum_j T_j$ exists in $\mathbb{R}$ and does not equal 1, then we can provide a simple description of $\mathfrak{F}_C$ in terms of $\mathfrak{F}_0$ for which we may resort to the results in Section 7. Put

$$C(T) \;\stackrel{\text{def}}{=}\; \frac{C}{1-\sum_{j\ge1}T_j}.$$

THEOREM 9.2. *If T satisfies (A) and if $\sum_j T_j\in\mathbb{R}\backslash\{1\}$, then*

$$\mathfrak{F}_C \;=\; \mathfrak{F}*\delta_{C(T)} \;\stackrel{\text{def}}{=}\; \{\mu*\delta_{C(T)}:\mu\in\mathfrak{F}\}$$

for all $C\in\mathbb{R}$.

PROOF. It suffices to note that W solves equation (1.1) with $C\ne0$ iff (under the given assumptions)

$$W-C(T) \;\stackrel{d}{=}\; \sum_{j\ge1}T_j(W_j-C(T))+C(T)\sum_{j\ge1}T_j-C(T)+C \;=\; \sum_{j\ge1}T_j(W_j-C(T)),$$

which means that $W+C(T)$ is a fixed point for the homogeneous equation ($C=0$). ♢

Left with the cases that $\sum_j T_j$ equals 1 or does not exist in $\mathbb{R}$, the following two results provide complete answers. Theorem 9.3 shows in particular that the existence of $\sum_j T_j$ in $\mathbb{R}$ constitutes a necessary condition for $\mathfrak{F}_C\ne\emptyset$ (for any $C\ne0$).

THEOREM 9.3. *The set $\mathfrak{F}_C$ is empty for each $C\ne0$ whenever $\sum_j T_j$ does not exist in $\mathbb{R}$, or when $\sum_j T_j=1$ in one of the cases (C1), (D1), or $\alpha=2$ holds true.*

PROOF. Suppose $C\ne0$ and that $\sum_j T_j$ equals 1 or does not exist in $\mathbb{R}$. The Lévy measure Γ of any fixed point must satisfy $\Gamma=\Gamma\star\Lambda$ (Lemma 9.1). In each of the three cases (C1), (D1), and $\alpha=2$ we have seen in Section 8 that this in turn implies the symmetry of Γ, in particular $\int(x\chi(u)-\chi(xu))\,\Gamma(du)=0$. Hence we infer $\mathfrak{F}_C=\emptyset$ because condition (9.5), which simplifies to $C+\gamma\sum_{j=1}^n T_j\to\gamma$, is clearly impossible to satisfy. It is a matter of checking (9.4) or (9.5) to arrive at the same conclusion whenever $\sum_j T_j$ does not exist in $\mathbb{R}$ (and no further condition on α or $\mathbb{G}_\Lambda$). We omit the details. ♢

It remains to look at the cases (C2), (D2) and (D3) for any T additionally satisfying $\sum_j T_j=1$.

THEOREM 9.4. *Suppose $\alpha \in (0,2)$, $\sum_j T_j = 1$ and $C \neq 0$.*

(a) If $\mathbb{G}_\Lambda = \mathbb{R}_+$ (Case (C2)), then $\alpha = 1$ and

$$\hat{\mathfrak{F}}_C \;=\; \{\exp(i\gamma t - c|t|(1+\beta\,\mathrm{sgn}(t)\omega(t,\alpha))) : \gamma \in \mathbb{R}, (\beta,c) \in \Theta_{\Lambda,C}\},$$

where

$$\Theta_{\Lambda,C} \;\stackrel{\text{def}}{=}\; \{(\beta,c) \in [-1,1]\times\mathbb{R}_+ : \tfrac{2\beta c}{\pi}\textstyle\sum_j T_j \log T_j = C\}.$$

(b) If $\mathbb{G}_\Lambda = r^{\mathbb{Z}}$ for some $r > 1$ (Case (D2)), then $\alpha = 1$ and $\hat{\mathfrak{F}}_C$ consists of all φ of the form (7.1) with $\gamma \in \mathbb{R}$ and $(c,\nu) \in \mathbb{R}_+ \times \mathfrak{M}_\Lambda \backslash \mathfrak{M}_\Lambda^0$ satisfying

$$C + c \textstyle\int s\,\nu(ds) \sum_j T_j \log T_j \;=\; 0.$$

In particular, $\hat{\mathfrak{F}}_C = \emptyset$ whenever $\sum_j T_j|\log T_j| = \infty$.

(c) If $\mathbb{G}_\Lambda = (-r)^{\mathbb{Z}}$ for some $r > 1$ (Case (D3)), then $\alpha \neq 1$ and $\hat{\mathfrak{F}}_C$ consists of all φ of the form (7.1) with $\gamma \in \mathbb{R}$ and $(c,\nu) \in \mathbb{R}_+ \times \mathfrak{M}_\Lambda \backslash \mathfrak{M}_\Lambda^0$ satisfying

$$C + c \textstyle\int \mathrm{sgn}(s)|s|^\alpha \Delta(s)\,\nu(ds)\big(1 - \sum_j \mathrm{sgn}(T_j)|T_j|^\alpha\big) \;=\; 0$$

where $\Delta(s)$ was defined in (8.8).

PROOF. (a) By the same arguments as in the proofs of Theorem 7.1 and 7.2 we infer here that a fixed point μ must be stable of index α with characteristic function $\varphi(t) = \exp(it\gamma - c|t|^\alpha(1 + i\beta\,\mathrm{sgn}(t)\omega(t,\alpha)))$ for suitable $\gamma \in \mathbb{R}$, $\beta \in [-1,1]$ and $c \geq 0$. In case $\alpha = 1$ equation (9.4) enforces that γ, β, c must satisfy

$$\gamma = \gamma \textstyle\sum_j T_j + C - \frac{2\beta c}{\pi}\sum_j T_j \log T_j,$$

so $\gamma \in \mathbb{R}$ and $(\beta,c) \in B_\Lambda$.

The assertions of (b) and (c) are immediately obtained when observing that equation (9.5) is equivalent to $\gamma = C + \sum_j \big(\gamma T_j + \int F_{s,c}(T_j)\,\nu(ds)\big)$ (compare (7.3)) which in turn leads to (8.11), respectively (8.12) with C added on the right hand side. Further details can be omitted. ♢

10 The associated random walk

For a given vector T which satisfies (A) and has characteristic exponent $\alpha \in (0,2]$ we next define the probability measure Λ_α on $\mathbb{R}_*$ by

$$\Lambda_\alpha \;\stackrel{\text{def}}{=}\; \sum_{j\geq 1} |T_j|^\alpha \mathbf{1}_{\{T_j\neq 0\}}\delta_{T_j} \tag{10.1}$$

and note that Λ_α generates the same closed multiplicative subgroup of $\mathbb{R}_*$ as the measure Λ. Let $(M_n)_{n\geq 0}$ be a multiplicative random walk with $M_0 = 1$ and $M_1 \stackrel{d}{=} \Lambda_\alpha$. If all nonzero T_j

are positive one may as well consider the additive counterpart of $(M_n)_{n\ge 0}$ obtained by taking logarithms. The latter was used in [11] in the given context, see Section 2 there. Recalling from Section 4 the weighted branching model associated with T, we further introduce the point measures

$$\Lambda_{\alpha,1:n} \ \stackrel{\text{def}}{=} \ \sum_{|v|=n} |L(v)|^{\alpha}\mathbf{1}_{\{L(v)\neq 0\}} \bigotimes_{j=1}^{n} \delta_{L(v|j)} \tag{10.2}$$

and

$$\Lambda_{\alpha,n} \ \stackrel{\text{def}}{=} \ \sum_{|v|=n} |L(v)|^{\alpha}\mathbf{1}_{\{L(v)\neq 0\}}\delta_{L(v|j)} \tag{10.3}$$

for each $n \in \mathbb{N}$, where $v|j \stackrel{\text{def}}{=} (v_1, ..., v_j)$ if $v = (v_1, ..., v_n)$.

LEMMA 10.1. *Under the given assumptions,*

$$(M_1, ..., M_n) \ \stackrel{d}{=} \ \Lambda_{\alpha,1:n} \quad \textit{and} \quad M_n \ \stackrel{d}{=} \ \Lambda_{\alpha,n} \tag{10.4}$$

for each $n \in \mathbb{N}$. Moreover,

$$\mathbb{E}f(M_n) \ = \ \sum_{|v|=n} |L(v)|^{\alpha} f(L(v)) \tag{10.5}$$

for any real-valued f for which the expectation exists.

PROOF. The assertions are easily verified when using the independence structure in the weighted branching model described in Section 4. The result appears also in [3, Lemma 4.1] for the case where T consists of a random number of i.i.d. nonzero random variables (branching random walk case). We therefore omit the details. ♢

Returning to the fixed point equation (2.1), i.e. (1.1) with $C = 0$, we now have the following result.

LEMMA 10.2. *Suppose that T satisfies (A) and has characteristic exponent $\alpha \in (0, 2]$. If μ is any solution to (2.1) with characteristic function φ satisfying*

$$\mathbb{E}\left(\frac{|\log\varphi(M_1)|}{|M_1|^{\alpha}}\right) < \infty, \tag{10.6}$$

then $\psi(t) \stackrel{\text{def}}{=} |t|^{-\alpha}\log\varphi(t)$ is harmonic for $(M_n)_{n\ge 0}$, that is

$$\mathbb{E}\psi(tM_1) \ = \ \psi(t) \tag{10.7}$$

for all $t \in \mathbb{R}_*$.

PROOF. One can check by using the Lévy representation of φ that (10.6) implies the (Lebesgue) integrability of $\psi(tM_1)$ for every $t \in \mathbb{R}_*$. It then follows from (10.5) and (1.4) that

$$\mathbb{E}\psi(tM_1) \;=\; \sum_{j\geq 1} |T_j|^\alpha \left(\frac{\log\varphi(T_j t)}{|T_j t|^\alpha}\right) \;=\; \frac{1}{|t|^\alpha}\sum_{j\geq 1}\log\varphi(T_j t) \;=\; \frac{\log\varphi(t)}{|t|^\alpha} \;=\; \psi(t)$$

for all $t \in \mathbb{R}_*$. ◊

Lemma 10.2 might suggest to the probabilistic reader that the fixed points of (2.1) can also be found via a more appealing itinerary than the one chosen here when studying the harmonic functions of the multiplicative random walk $(M_n)_{n\geq 0}$, for instance by drawing on multiplicative renewal theory in combination with the fact that $(\psi(tM_n))_{n\geq 0}$ is a martingale for any harmonic ψ. Although this approach does indeed lead to solutions to (2.1), it generally fails to provide a complete description of $\mathfrak{F}$. For instance, we saw that in some situations where $\sum_j T_j = 1$ and $\sum_j |T_j| = \infty$ the constants γ are solutions. Their characteristic functions $\varphi(t) = e^{i\gamma t}$, however, do not satisfy the integrability condition (10.6) in the above lemma as is immediately checked. A second problem to be noted is that even in those cases where $\mathfrak{F}$ turns out to be a subset of the set of harmonic functions of $(M_n)_{n\geq 0}$ one must still provide criteria which ensure that a found harmonic ψ corresponds to a solution to (2.1). And finally, there seems to be no alternative to the Choquet-Deny approach in order for finding the quite explicit form of the Lévy measure of a solution. In fact, this is the crucial step to identify a solution as a stable law or a mixture of $s\mathbb{G}$-stable laws.

APPENDIX

Because of their special importance in this article we finally collect very briefly the basic general facts on the Choquet-Deny equation

$$\mu = \mu \star \nu \tag{A.1}$$

where μ, ν are two regular (= Radon) measures on a (multiplicative) Abelian separable locally compact group $\mathbb{G}$. Any regular μ solving (A.1) for a given ν is called *ν-harmonic*. Let $\mathbb{H}_\nu$ denote the set of all such measures. $\mathbb{H}_\nu$ is a convex cone.

A *character* of $\mathbb{G}$ is a positive continuous function $e : \mathbb{G} \to \mathbb{R}_+$ preserving the multiplicative group structure, thus $e(xy) = e(x)e(y)$ for all $x, y \in \mathbb{G}$. We denote by $E = E(\mathbb{G})$ the set of all characters endowed with the topology of uniform convergence on compact sets. Let $\mathbb{G}_\nu$ be the smallest closed subgroup of $\mathbb{G}$ generated by the support of ν. If $\mathbb{G}_\nu = \mathbb{G}$ we say that ν generates the group $\mathbb{G}$. Let $\lambda\!\!\!\lambda_{\mathbb{G}_\nu}$ be the Haar measure on $\mathbb{G}_\nu$, unique up to positive scalars, and put $\mu_e(dx) \stackrel{\text{def}}{=} e(x)\lambda\!\!\!\lambda_{\mathbb{G}_\nu}(dx)$ (thus suppressing the dependence of μ_e on $\mathbb{G}_\nu$ in the notation).

That ν generates $\mathbb{G}$ is the basic assumption in the following theorem by Deny [6]:

THEOREM A.1. (Deny) *Given a regular ν generating $\mathbb{G}$, any solution to (A.1) by a regular measure $\mu \neq 0$ has a unique integral representation*

$$\mu \;=\; \int \mu_e \; \overline{\mu}(de) \tag{A.2}$$

where $\overline{\mu}$ is a finite regular measure on $\{e \in E : \int e(x^{-1})\, \nu(dx) = 1\}$.

Turning to general measures ν, let $\mathbb{G}/\mathbb{G}_\nu$ be the factor group and note that $\mathbb{G}$ can be identified (as a measurable space) with $\mathbb{G}_\nu \otimes (\mathbb{G}/\mathbb{G}_\nu)$ endowed with the naturally induced topology and Borel-σ-field.

THEOREM A.2. (Choquet-Deny) *Given a regular measure ν on $\mathbb{G}$, any solution to (A.1) by a regular measure $\mu \neq 0$ has an integral representation*

$$\mu(\cdot) \;=\; \int \mu_e(s^{-1}\cdot) \; \overline{\mu}(de \times ds), \tag{A.3}$$

where $\overline{\mu}$ is a regular measure on $\{e \in E(\mathbb{G}_\nu) : \int e(x^{-1})\, \nu(dx) = 1\} \times (\mathbb{G}/\mathbb{G}_\nu)$ endowed with the Baire σ-field.

Compared to Deny's theorem the more general Choquet-Deny result provides a disintegration of measures whenever the factor group $\mathbb{G}/\mathbb{G}_\nu$ is nondegenerate.

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