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EXPONENTIAL ASYMPTOTIC STABILITY OF LINEAR ITÔ-VOLTERRA EQUATIONS WITH DAMPED STOCHASTIC PERTURBATIONS

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Abstract: This paper studies the convergence rate of solutions of the linear Itô - Volterra equation

$$dX(t) = \left(AX(t) + \int_0^t K(t-s)X(s)\,ds \right) dt + \Sigma(t)\,dW(t) \tag{0.1}$$

where K and Σ are continuous matrix–valued functions defined on $\mathbb{R}^+$, and $(W(t))_{t\geq 0}$ is a finite-dimensional standard Brownian motion. It is shown that when the entries of K are all of one sign on $\mathbb{R}^+$, that (i) the almost sure exponential convergence of the solution to zero, (ii) the p-th mean exponential convergence of the solution to zero (for all $p > 0$), and (iii) the exponential integrability of the entries of the kernel K, the exponential square integrability of the entries of noise term Σ, and the uniform asymptotic stability of the solutions of the deterministic version of (0.1) are equivalent. The paper extends a result of Murakami which relates to the deterministic version of this problem.

Keywords and phrases: Exponential asymptotic stability; Itô-Volterra equation; Volterra equation; Liapunov exponent; p-th mean exponential asymptotic stability; almost sure exponential asymptotic stability.

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1 Introduction

In recent years, several researchers have investigated the almost sure and p-th mean exponential asymptotic stability of solutions of stochastic differential equations and stochastic delay differential equations with bounded delay. Among the extensive literature on this topic, we cite [11, 13, 14, 15, 20, 10], and [16], though this list is by no means exhaustive.

However, comparatively little research has been carried out examining the exponential asymptotic stability of solutions of Itô-Volterra equations, in which the delay is unbounded. In particular, if solutions are asymptotically stable in an almost sure or p-th mean sense, it would be interesting to see whether the convergence still occurs at an exponential rate, or whether the extra "inertia" introduced into the system via an unbounded delay retards the convergence rate.

This question has partial, but satisfactory answers in the deterministic theory. The question as to whether slower than exponential convergence of solutions of linear, time homogeneous Volterra equations with unbounded delay was possible was raised by Lakshmikantham and Corduneannu [6]. More specifically, they asked whether the uniform asymptotic stability of the zero solution implied its exponential asymptotic stability, as is the case for deterministic equations with bounded delay. The question was answered in the negative in two seminal papers of Murakami; the scalar case is covered in [25]; the general finite dimensional case in [24]. He proved, for the Volterra equation

$$x'(t) = Ax(t) + \int_0^t K(t-s)x(s)\,ds, \tag{1.1}$$

that when the zero solution is uniformly asymptotically stable, it is also exponentially asymptotically stable if and only if

$$\int_0^\infty \|K(s)\|e^{\gamma s}\,ds < \infty, \quad \text{for some } \gamma > 0 \tag{1.2}$$

whenever the entries of the kernel do not change sign on $\mathbb{R}^+$.

The existence of such sharp results for deterministic Volterra equations indicates that it should be possible for Itô-Volterra equations to exhibit non-exponential asymptotic stability. Results towards this end are established in [3], in which an almost sure non-exponential lower bound on the solution of a linear scalar Itô-Volterra equation with *multiplicative noise*

$$dX(t) = (-aX(t) + \int_0^t k(t-s)X(s)\,ds)\,ds + \sigma X(t)\,dB(t)$$

is established. Using results developed in the deterministic theory in [2], the first two authors of this paper will present corresponding non-exponential upper bounds elsewhere. In each case, the kernel k violates (1.2) above.

In this paper, we study the exponential asymptotic stability of the linear Itô-Volterra equation with *additive noise*:

$$dX(t) = (AX(t) + \int_0^t K(t-s)X(s)\,ds)\,dt + \Sigma(t)\,dW(t), \tag{1.3}$$

where A, K are $d \times d$ matrices Σ is a $d \times r$ matrix and

$$W(t) = (W^1(t), W^2(t), \dots, W^r(t))$$

is an r-dimensional Brownian motion where each of the component Brownian motions $W^j(t)$ are independent. The moment asymptotic stability of the solution of this equation has been studied by Mizel and Trutzer [22]. The Gaussian character of the solution, and stationarity of the solution of (1.3) in the bounded delay case is studied in [9], [23]. Almost sure asymptotic stability is studied in [1]; exponential asymptotic stability is studied in [19], and [18] for nonlinear bounded delay problems. Sufficient conditions under which nonlinear Itô-Volterra equations are exponentially asymptotically stable are established in Mao [17] (we contrast the results obtained in this latter paper with results obtained here presently). Some properties of infinite dimensional versions of this equation are studied in [5], in the context of heat conduction in materials with memory. The solutions of (1.3) are Gaussian processes, if the initial conditions are deterministic or Gaussian and independent of the driving Brownian motion W. Furthermore, the solutions can be represented in terms of the fundamental matrix solution (denoted by $Z(t)$) of the deterministic version of (1.3), viz.,

$$x'(t) = Ax(t) + \int_0^t K(t-s)x(s)\,ds. \tag{1.4}$$

This representation means that the mean vector and variance-covariance matrix can be written in terms of Z and Σ.

The first result of this note (Theorem 3.3), makes use of this representation. We show that if

$$\int_0^\infty \|K(s)\| e^{\gamma_1 s}\,ds < \infty, \quad \text{for some } \gamma_1 > 0, \tag{1.5}$$

and

$$\int_0^\infty \|\Sigma(s)\|^2 e^{2\gamma_2 s}\,ds < \infty, \quad \text{for some } \gamma_2 > 0, \tag{1.6}$$

then the uniform asymptotic stability of the zero solution of (1.4) implies that there exists constants $M_p(X_0) \geq 0$, $\beta_p > 0$ for each $p > 0$ such that

$$\mathbb{E}[\|X(t)\|^p] \leq M_p(X_0)e^{-\beta_p t}, \quad t \geq 0. \tag{1.7}$$

This result is used in Theorem 3.4 to establish that the uniform asymptotic stability of (1.4) in conjunction with (1.6), (1.5) imply the existence of an a.s. negative top Liapunov exponent for all solutions, in that there exists $\beta_0 > 0$, such that

$$\limsup_{t\to\infty} \frac{1}{t} \log \|X(t)\| \leq -\beta_0, \quad \text{a.s.} \tag{1.8}$$

Thus all solutions of (1.3) are exponentially convergent to zero, in both a p-th mean and an almost sure sense.

The result uses the idea of Theorem 4.3.1 in [13] (for stochastic differential equations) that $\mathbb{E}[\|X(t)\|^2] \leq C_2(X_0)e^{-\beta_2 t}$ for some $\beta_2 > 0$ implies a.s. exponential asymptotic stability. In contrast to Mao [17] (for unbounded delay integrodifferential equations), and Liao and Mao [19] and Mao [18] (for bounded delay equations) it is unnecessary for Σ to be bounded above in a pointwise sense— indeed, (1.6) shows, to the contrary, that Σ need only be exponentially bounded "on average". Moreover, we do not require a pointwise exponential decay on the kernel K required in Mao [17] in order to guarantee exponential stability. In fact, as discussed below, it is our purpose to show here that the conditions (1.6) and (1.5) are, under the same circumstances which apply to the deterministic theory, the optimal ones required to ensure exponential stability.

The second type of result of the paper (Theorems 4.1, 4.3) give conditions under which the exponential convergence of solutions of (1.3) imply results about the kernel or noise terms. Theorem 4.1 shows that (1.5) together with (1.8) imply (1.6). Thus, under (1.5) and the uniform asymptotic stability of the zero solution of (1.4), we get exponential asymptotic stability if and only if the noise declines exponentially. This phenomenon is alluded to in Liu and Mao [10] for stochastic differential equations, and, in part, forms the subject of Mao's paper on polynomial stability of stochastic differential equations [12]. The proof of Theorem 4.3 is closely modelled on that of Theorem 4.1, and, as in that result, the Gaussian properties of the process $\int_0^t \Sigma(s)\, dW(s)$ are crucial. We show that (1.5) together with (1.7) implies (1.6).

Theorem 5.1 constitutes the equivalences that can be drawn between (1.5), (1.6), and (1.7) and (1.8), and is the main result of the paper. In the first part of this proof, we show that the a.s. exponential convergence of all solutions (1.8) is equivalent to (1.6), (1.5) when the zero solution of (1.3) is uniformly asymptotically stable, provided that the entries of K do not change sign. In conjunction with Theorem 3.4, this follows by proving that a.s. convergence of solutions of (1.3) at an exponential rate implies the exponential convergence of solutions of (1.4). Theorem 2 in [24] then establishes (1.5). Therefore, Theorem 4.1, together with (1.5) and the exponential convergence of solutions yields (1.6).

Next we establish that p-th mean exponential convergence of all solutions (1.7) is equivalent to (1.6), (1.5) when the zero solution of (1.3) is uniformly asymptotically stable, providing that the entries of K do not change sign on $\mathbb{R}^+$. Theorem 3.4 proves the reverse implication; Theorem 4.3 the forward implication. Consequently, we have proven when the entries of K do not change sign, that (1.6), (1.5), and the uniform asymptotic stability of the zero solution of (1.1), is equivalent to (1.7) and (1.8) for the problem (1.3).

The organisation of the note is as follows: the problem to be studied is formally stated in Section 2, alongside background theory and definitions. Theorems 3.3 and 3.4 are proven in Section 3. Section 4 covers the proof of Theorems 4.1 and 4.3. The equivalences captured by Theorems 5.1 are established in Section 5.

2 Background Material

We first fix some standard notation. As usual, let $x \vee y$ denote the maximum of x, $y \in \mathbb{R}$, and $x \wedge y$ their minimum.

Denote by $C(I;J)$ the space of continuous functions taking the finite dimensional Banach space I onto the finite dimensional Banach space J. Let d be a positive integer. Let $(\mathbb{R}^d)^n$ be the n-fold Cartesian product of $\mathbb{R}^d$ with itself. Let $M_{d,d}(\mathbb{R})$ denote the space of all $d \times d$ matrices with real entries. We say that the function $f : \mathbb{R}^+ \to M_{n,m}(\mathbb{R})$ is in $L^1(\mathbb{R}^+)$ if each of its entries is a scalar Lebesgue integrable function, and in $L^2(\mathbb{R}^+)$ if each of its entries is a scalar square integrable function. The convolution of the function f with g is denoted by $f * g$. The transpose of any matrix A is denoted A^T; the trace of a square matrix is denoted by $\text{tr}(A)$. Further denote by I_d the identity matrix in $M_{d,d}(\mathbb{R})$. Let $\|x\|$ stand for the Euclidean norm of $x \in \mathbb{R}^d$, and $\|x\|_1$ be the sum of the absolute values of the components of x. If $A = (A_{ij}) \in M_{d,r}(\mathbb{R})$, A has operator norm denoted by $\|A\|$, and given by

$$\|A\| = \sup\{\|Ax\| \,:\, x \in \mathbb{R}^r,\ \|x\| = 1\}.$$

It further has Frobenius norm, denoted by $\|A\|_F$, and defined as follows: if $A = (a_{i,j})$, is an $d \times r$ matrix, then

$$\|A\|_F = \left(\sum_{i=1}^{d} \sum_{j=1}^{r} |a_{i,j}|^2 \right)^{1/2} = \text{tr}(AA^T)^{\frac{1}{2}}.$$

Since $M_{d,r}(\mathbb{R})$ is a finite dimensional Banach space, $\|\cdot\|$, $\|\cdot\|_F$ are equivalent, so there exist positive universal constants $c_1(d,r) \le c_2(d,r)$ such that

$$c_1(d,r)\|A\| \le \|A\|_F \le c_2(d,r)\|A\|, \quad A \in M_{d,r}(\mathbb{R}). \tag{2.1}$$

We first turn our attention to the deterministic Volterra equation (1.1), where A and K are in $C(\mathbb{R}^+; M_{d,d}(\mathbb{R}) \cap L^1(\mathbb{R}^+)$. For any $t_0 \ge 0$ and $\phi \in C([0,t_0], \mathbb{R}^d)$, there is a unique $\mathbb{R}^d$-valued function $x(t)$, which satisfies (1.1) on $[t_0, \infty)$ and for which $x(t) = \phi(t)$ for $t \in [0, t_0]$. We denote such a solution by $x(t; t_0, \phi)$. The function $x(t) \equiv 0$ is a solution of (1.1) and is called the zero solution of (1.1).

Consider now the matricial equation

$$Z'(t) = AZ(t) + \int_0^t K(t-s)Z(s)\,ds, \quad t \ge 0 \tag{2.2}$$

with $Z(0) = I_d$. The unique $Z \in C(\mathbb{R}^+; M_{d,d}(\mathbb{R}))$ which satisfies (2.2) is called the resolvent, or principal matrix solution for (1.1).

Existence and uniqueness results for deterministic Volterra equations are covered in [7].

In this paper, we will consider d-dimensional linear stochastic integro-differential equations with stochastic perturbations of the form

$$dX(t) = \left(AX(t) + \int_0^t K(t-s)X(s)\,ds \right) dt + \Sigma(t)\,dW(t) \tag{2.3}$$

on $t \geq 0$ where $(W(t))_{t\geq 0}$ is an r-dimensional Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where the filtration is the natural one $\mathcal{F}_t = \sigma\{W(s) : 0 \leq s \leq t\}$. X has deterministic initial condition $X_0 = x$, $A \in M_{d,d}(\mathbb{R})$, $\Sigma \in C(\mathbb{R}^+; M_{d,r}(\mathbb{R}))$ and $K \in C(\mathbb{R}^+; M_{d,d}(\mathbb{R})) \cap L^1(\mathbb{R}^+)$.

The existence and uniqueness of a continuous solution of (2.3) is covered in Berger and Mizel [4], for instance. We can also represent the solution of (2.3) in terms of the resolvent of (1.1).

Lemma 2.1. *Under the above conditions on K and Σ, if $X(0) = X_0$ is deterministic, there exists a unique a.s. continuous solution to (2.3) on every interval of $\mathbb{R}^+$. Moreover, if Z satisfies (2.2), with $Z(0) = I_d$, then*

$$X(t) = Z(t)X_0 + \int_0^t Z(t-s)\Sigma(s)\,dW(s) \tag{2.4}$$

for all $t \geq 0$.

Proof. The proof follows the line of reasoning in Küchler and Mensch [9], or Mohammed [23]. □

Remark 2.2.

¿From Lemma 2.1, it is immediate that X is a Gaussian process with expectation vector

$$\mathbb{E}X(t) = Z(t)X_0 \tag{2.5}$$

and covariance matrix

$$\begin{aligned}\rho(s,t) &= \mathbb{E}\left[(X(s) - \mathbb{E}X(s))(X(t) - \mathbb{E}X(t))^T\right] \\ &= \int_0^{s\wedge t} Z(s-u)\Sigma(u)\Sigma(u)^T Z(t-u)^T\,du.\end{aligned} \tag{2.6}$$

This can be seen by fixing t and adapting the line of argument of Problem 6.2 p.355 in [8].

We recall the various standard notions of stability of the zero solution required for our analysis; the reader may refer further to Miller [21]. Here, and subsequently, if $t_0 \in \mathbb{R}^+$ and $\phi \in C([0,t_0],\mathbb{R}^n)$, we define $|\phi|_{t_0} = \max_{0\leq s\leq t_0} \|\phi(s)\|_1$.

The zero solution of (1.1) is said to be *uniformly stable* (US), if, for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $t_0 \in \mathbb{R}^+$ and $\phi \in C([0,t_0],\mathbb{R}^n)$ with $|\phi|_{t_0} < \delta(\varepsilon)$ implies $\|x(t;t_0,\phi)\|_1 < \varepsilon$ for all $t \geq t_0$. The zero solution of (1.1) is said to be *uniformly asymptotically stable* (UAS) if is US and there exists $\delta > 0$ with the following property: for each $\varepsilon > 0$ there exists a $T(\varepsilon) > 0$ such that $t_0 \in \mathbb{R}^+$ and $\phi \in C([0,t_0],\mathbb{R}^n)$ with $|\phi|_{t_0} < \delta$ implies $\|x(t,t_0,\phi)\|_1 < \varepsilon$ for all $t \geq t_0 + T(\varepsilon)$. The zero solution of (1.1) is said to *exponentially asymptotically stable* (ExAS) if there exists $C > 0$ and $\lambda > 0$ independent of $t_0 \in \mathbb{R}^+$ and $\phi \in C([0,t_0],\mathbb{R}^n)$, such that $\|x(t,t_0,\phi)\|_1 \leq C|\phi|_{t_0} e^{-\lambda(t-t_0)}$ for all $t \geq t_0$.

The properties of the resolvent Z are deeply linked to the stability of the zero solution of (1.1). It is shown in [21] that the zero solution of (1.1) is UAS if and only if $Z \in L^1(\mathbb{R}^+)$.

We now make precise the notion of p-th mean and a.s. exponential convergence of solutions of (1.3).

Definition 2.3. The $\mathbb{R}^d$ valued stochastic process $(X(t))_{t\geq 0}$ is *p-th mean exponentially convergent*, for $p > 0$, if there exists $\beta_p > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E}[\|X(t)\|^p]) \leq -\beta_p.$$

The definition of a.s. exponential convergence has a similar form:

Definition 2.4. The $\mathbb{R}^d$ valued stochastic process $(X(t))_{t\geq 0}$ is *almost surely exponentially convergent*, if there exists $\beta_0 > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log \|X(t)\| \leq -\beta_0, \quad \text{a.s.}$$

This definition is, in turn, equivalent to the following: for every $\varepsilon > 0$, there exists $\Omega_\varepsilon \subset \Omega$, with $\mathbb{P}[\Omega_\varepsilon] = 1$, and an a.s. finite random variable $C(\varepsilon) > 0$ such that for all $\omega \in \Omega_\varepsilon$ we have

$$\|X(t)(\omega)\| \leq C(\varepsilon)(\omega)e^{-(\beta_0-\varepsilon)t}, \quad t \geq 0.$$

The definitions are, of course, closely based on those of p-th mean and almost sure exponential asymptotic *stability*. It is not appropriate, however, to talk about the "stability of the zero solution" of (2.3), as the process $X(t) = 0$ for all $t \geq 0$ is not a solution of (2.3). If however, we view the random contribution in (2.3) as a perturbation, we may ask whether the equilibrium solution of the unperturbed problem (1.1) is asymptotically stable in the presence of this perturbation, and determine the conditions under which the solutions are exponentially convergent to the equilibrium solution of the unperturbed problem.

As evidenced by Lemma 2.1, the solution of the Itô-Volterra equation (2.3) relies strongly on that of the deterministic Volterra equation (1.1). The exponential asymptotic stability of this problem has been studied by Murakami [24]. We summarise his results here. Firstly, the uniform asymptotic stability of the zero solution, together with the exponential integrability of the kernel imply exponential asymptotic stability of solutions of (1.1).

Lemma 2.5. *If the zero solution of (1.1) is uniformly asymptotically stable, and (1.5) holds, then the zero solution of (1.1) is exponentially asymptotically stable.*

Lemma 2.6. *Suppose that all of the entries of $K \in L^1(\mathbb{R}^+)$ do not change sign on $\mathbb{R}^+$. If there exists $C > 0$ and $\lambda > 0$ such that the fundamental matrix solution of (2.2) satisfies*

$$\|Z(t)\| \leq Ce^{-\lambda t}, \quad t \geq 0, \tag{2.7}$$

then there exists $\gamma_1 > 0$ such that (1.5) holds.

Therefore, if the entries of K do not change sign, and the zero solution of (1.1) is uniformly asymptotically stable, the exponential convergence of all solutions of (1.1) is equivalent to (2.7), which is in turn equivalent to (1.5).

3 Sufficient conditions for exponential convergence of solutions

The following estimate is used throughout this section.

Lemma 3.1. *Suppose that $f \in C(\mathbb{R}^+; \mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ satisfies*

$$\int_0^\infty f(t)e^{\gamma t}\,dt < \infty, \quad \text{for some } \gamma > 0.$$

If $\lambda > 0$, and $\lambda' = \lambda \wedge \gamma$, then

$$\int_0^t e^{-\lambda(t-s)} f(s)\,ds \le e^{-\lambda' t} \int_0^\infty e^{\gamma s} f(s)\,ds.$$

Proof. We consider the cases $0 < \lambda < \gamma$, $0 < \gamma \le \lambda$ separately. If $0 < \lambda < \gamma$, then $\lambda' = \lambda$, and we have

$$\int_0^t e^{-\lambda(t-s)} f(s)\,ds \le e^{-\lambda t} \int_0^t e^{\gamma s} f(s)\,ds \le e^{-\lambda' t} \int_0^\infty e^{\gamma s} f(s)\,ds,$$

proving the result in this case. For $0 < \gamma \le \lambda$, we have $\lambda' = \gamma$, so

$$\begin{aligned}\int_0^t e^{-\lambda(t-s)} f(s)\,ds &= e^{-\gamma t} \int_0^t e^{-(\lambda-\gamma)(t-s)} e^{\gamma s} f(s)\,ds \\ &\le e^{-\gamma t} \int_0^t e^{\gamma s} f(s)\,ds \le e^{-\lambda' t} \int_0^\infty e^{\gamma s} f(s)\,ds,\end{aligned}$$

which gives the result. □

Proposition 3.2. *Suppose that the zero solution of (1.1) is uniformly asymptotically stable, and that (1.5), (1.6) hold. Then there exists $\lambda' > 0$ and $M = M(X_0) > 0$ such that the solution of (2.3) satisfies*

$$\mathbb{E}[\|X(t)\|^2] \le M(X_0)e^{-2\lambda' t}, \quad t \ge 0. \tag{3.1}$$

Proof. With $Z(t)$ defined by (2.2), and $Z(0) = I_d$, the uniform asymptotic stability of the zero solution of (1.1), together with (1.5) ensures the existence of $C > 0$ and $\lambda > 0$ such that (2.7) holds, by Lemma 2.5 above. Hence by (2.1) and (2.7)

$$\|Z(t-u)\Sigma(u)\| \le \|Z(t-u)\|\|\Sigma(u)\| \le Ce^{-\lambda(t-u)} \frac{1}{c_1(d,r)} \|\Sigma(u)\|_F,$$

for all $0 \le u \le t$, so

$$\|Z(t-u)\Sigma(u)\|_F^2 \le C_1' e^{-2\lambda(t-u)} \|\Sigma(u)\|_F^2, \tag{3.2}$$

where $C_1' = \frac{C^2 c_2(d,r)^2}{c_1(d,r)^2}$. By Lemma 2.1 and Remark 2.2 (specifically, equations (2.5) and (2.6)), we have

$$\mathbb{E}[X(t)X(t)^T] = Z(t)X_0(Z(t)X_0)^T + \int_0^t Z(t-u)\Sigma(u)(Z(t-u)\Sigma(u))^T\,du. \tag{3.3}$$

Since $\|x\|^2 = \text{tr}(xx^T)$ for any $x \in \mathbb{R}^d$, (3.2) together with (3.3) yields

$$\begin{aligned}
&\mathbb{E}[\|X(t)\|^2] \\
&= \|Z(t)X_0\|^2 + \int_0^t \text{tr}\left(Z(t-u)\Sigma(u)(Z(t-u)\Sigma(u))^T\right) du \\
&\leq \|Z(t)\|^2\|X_0\|^2 + \int_0^t \|Z(t-u)\Sigma(u)\|_F^2 \, du \\
&\leq C^2 e^{-2\lambda t}\|X_0\|^2 + C_1' \int_0^t e^{-2\lambda(t-u)}\|\Sigma(u)\|_F^2 \, du.
\end{aligned} \tag{3.4}$$

Let

$$\lambda' = \lambda \wedge \gamma_2. \tag{3.5}$$

¿From Lemma 3.1, (1.5), and (3.5), we obtain

$$\int_0^t e^{-2\lambda(t-u)}\|\Sigma(u)\|_F^2 \, du \leq e^{-2\lambda' t} \int_0^\infty e^{2\gamma_2 u}\|\Sigma(u)\|_F^2 \, du. \tag{3.6}$$

Applying this to (3.4) gives

$$\begin{aligned}
\mathbb{E}[\|X(t)\|^2] &\leq C^2 e^{-2\lambda t}\|X_0\|^2 + C_1' e^{-2\lambda' t} \int_0^\infty e^{2\gamma_2 u}\|\Sigma(u)\|_F^2 \, du \\
&\leq M(X_0)e^{-2\lambda' t},
\end{aligned}$$

where $M(X_0) = C^2\|X_0\|^2 + C_1' \int_0^\infty e^{2\gamma_2 u}\|\Sigma(u)\|_F^2 \, du$. This establishes the result. □

This calculation enables us to obtain an exponential upper bound on $\mathbb{E}[\|X(t)\|^p]$ for every $p > 0$.

Theorem 3.3. *Suppose that the zero solution of (1.1) is uniformly asymptotically stable, and that (1.5), (1.6) hold. Then there exists $\lambda' > 0$ and $M_p = M_p(X_0) > 0$ such that for each $p > 0$, the solution of (2.3) satisfies*

$$\mathbb{E}[\|X(t)\|^p] \leq M_p(X_0)e^{-\lambda' p t}, \quad t \geq 0. \tag{3.7}$$

Proof. We consider the cases $p > 2$, and $0 < p \leq 2$ separately. The case $0 < p \leq 2$ follows directly from Liapunov's inequality, and (3.1), to wit:

$$\mathbb{E}[\|X(t)\|^p] \leq \mathbb{E}[\|X(t)\|^2]^{p/2} \leq M_p(X_0)e^{-\lambda' p t},$$

where we define $M_p(X_0) = M(X_0)^{p/2}$.

The case $p \geq 2$ is less trivial. For $p \geq 2$, there exists $m \in \mathbb{N}$, $m \geq 2$, such that $2(m-1) \leq p < 2m$. We now seek an upper bound on $\mathbb{E}[\|X(t)\|^{2m}]$, which in turn gives us an upper bound on $\mathbb{E}[\|X(t)\|^p]$.

Using the inequality $(x+y)^{2m} \le 2^{2m-1}(x^{2m}+y^{2m})$, $x, y \ge 0$, together with Lemma 2.1 and (2.5), we obtain

$$\mathbb{E}[\|X(t)\|^{2m}] \le 2^{2m-1}(\|\mathbb{E}[X(t)]\|^{2m} + \mathbb{E}[\|Y(t)\|^{2m}]), \tag{3.8}$$

where $Y(t) = \int_0^t Z(t-s)\Sigma(s)\,dW(s)$. Define $\sigma \in C(\mathbb{R}^+ \times \mathbb{R}^+; M_{d,r}(\mathbb{R}))$ by $\sigma(s,t) = Z(t-s)\Sigma(s)$ for $0 \le s \le t$. Then

$$\|Y(t)\|^2 = \sum_{i=1}^d \left(\sum_{j=1}^r \int_0^t \sigma_{i,j}(s,t)\,dW^j(s) \right)^2.$$

For $n \in \mathbb{N}$, $k \in \mathbb{N}$, we may use the inequality

$$\left(\sum_{i=1}^n |x_i| \right)^k \le n^k \sum_{i=1}^n |x_i|^k,$$

twice (first with $n = d$, $k = m$, and then with $n = r$, $k = 2m$) to get

$$\begin{aligned}
\|Y(t)\|^{2m} &\le d^m \sum_{i=1}^d \left(\sum_{j=1}^r \int_0^t \sigma(s,t)\,dW^j(s) \right)^{2m} \\
&\le d^m \sum_{i=1}^d r^{2m} \sum_{j=1}^r \left(\int_0^t \sigma_{i,j}(s,t)\,dW^j(s) \right)^{2m}.
\end{aligned}$$

Since

$$\int_0^t \sigma_{i,j}(s,t)\,dW^j(s) \sim \mathcal{N}\left(0, \int_0^t \sigma_{i,j}(s,t)^2\,ds\right),$$

we must have

$$\mathbb{E}[\|Y(t)\|^{2m}] \le d^m \cdot r^{2m} \cdot \frac{(2m)!}{m!2^m} \sum_{i=1}^d \sum_{j=1}^r \left(\int_0^t \sigma_{i,j}(s,t)^2\,ds \right)^m. \tag{3.9}$$

To obtain an upper bound on this summation, note that the definition of the Frobenius norm, together with (3.2) gives

$$\sigma_{i,j}(s,t)^2 = (Z(t-s)\Sigma(s))_{i,j}^2 \le \|Z(t-s)\Sigma(s)\|_F^2 \le C_1' e^{-2\lambda(t-s)} \|\Sigma(s)\|_F^2,$$

for all $0 \le s \le t$. Therefore, from (3.9), we have

$$\mathbb{E}[\|Y(t)\|^{2m}] \le d^m \cdot r^{2m} \cdot \frac{(2m)!}{m!2^m} \cdot (C_1')^m \cdot dr \cdot \left(\int_0^t e^{-2\lambda(t-s)} \|\Sigma(s)\|_F^2\,ds \right)^m,$$

which, employing (3.6), and defining $L_m(d,r)$ by

$$L_m(d,r) = d^m \cdot r^{2m} \cdot \frac{(2m)!}{m!2^m} \cdot (C_1')^m \cdot dr \left(\int_0^\infty e^{2\gamma_2 u} \|\Sigma(u)\|_F^2\,du \right)^m.$$

yields

$$\mathbb{E}[\|Y(t)\|^{2m}] \le L_m(d,r)\cdot e^{-2\lambda' mt}, \tag{3.10}$$

where λ' is given by (3.5). Using (2.7), we have

$$\|\mathbb{E}[X(t)]\|^{2m} \le \|Z(t)\|^{2m}\|X_0\|^{2m} \le C^{2m}e^{-2\lambda mt}\|X(0)\|^{2m}. \tag{3.11}$$

Taking (3.8), (3.10), and (3.11) together, and using (3.5), we find that

$$\mathbb{E}[\|X(t)\|^{2m}] \le L'_m(d,r)e^{-2\lambda' mt},$$

where $L'_m(d,r,X_0) = 2^{2m-1}(C^{2m}\|X(0)\|^{2m} + L_m(d,r))$. Recalling that $m = m(p)$ is defined to be the integer satisfying $2(m-1) \le p < 2m$, using Liapunov's inequality, we get

$$\mathbb{E}[\|X(t)\|^p] \le \mathbb{E}[\|X(t)\|^{2m}]^{p/2m} \le L'_m(d,r,X_0)^{p/2m}e^{-\lambda' pt}.$$

Hence, we have (3.7) with $M_p(X_0) = L'_{m(p)}(d,r,X_0)^{p/2m(p)}$, where $2m(p)$ is the minimal even integer greater than p. $\square$

We can use the exponential estimates obtained above in the following adaptation of a result of Mao. Under the conditions which assure exponential convergence in p-th mean, we can also guarantee that the solution converges almost surely to zero exponentially fast.

Theorem 3.4. *Suppose that (1.5), (1.6) hold, and that the zero solution of (1.1) is uniformly asymptotically stable. Then there exists $\beta_0 > 0$ such that*

$$\limsup_{t\to\infty} \frac{1}{t}\log\|X(t)\| \le -\beta_0, \quad a.s. \tag{3.12}$$

Proof. By the hypotheses of the theorem, Proposition 3.2 ensures that there exists $M = M(X_0) > 0$ and $\lambda > 0$ (where λ satisfies (2.7)) such that

$$\mathbb{E}[\|X(t)\|^2] \le M(X_0)e^{-2\lambda' t}$$

where $\lambda' > 0$ satisfies (3.5); i.e., the process $(X(t))_{t\ge 0}$ satisfies (3.1). To prove (3.12), we first show that there exists $M' = M'(X_0) > 0$, $\lambda_3 > 0$ such that

$$\mathbb{E}\left[\sup_{n-1\le t\le n}\|X(t)\|^2\right] \le M'e^{-\lambda_3(n-1)}. \tag{3.13}$$

For each $t > 0$, there exists $n \in \mathbb{N}$ such that $n-1 \le t < n$. Since

$$X(t) = X(n-1) + \int_{n-1}^t AX(s) + (K*X)(s)\,ds + \int_{n-1}^t \Sigma(s)\,dW(s),$$

taking the triangle inequality, squaring, using the inequality $(x+y+z)^2 \le 3^2(x^2+y^2+z^2)$, taking suprema, using Jensen's inequality on the Riemann integral term, and finally taking expectations yields

$$\begin{aligned}
&\mathbb{E}\left[\sup_{n-1\le t\le n}\|X(t)\|^2\right] \\
&\quad \le 3^2\Bigg(\mathbb{E}[\|X(n-1)\|^2] + \mathbb{E}\int_{n-1}^{n}\left(\|A\|\|X(s)\| + \int_0^s \|K(s-u)\|\|X(u)\|\,du\right)^2 ds \\
&\qquad\qquad + \mathbb{E}\left[\sup_{n-1\le t\le n}\left\|\int_{n-1}^{t}\Sigma(s)\,dW(s)\right\|^2\right]\Bigg).
\end{aligned} \tag{3.14}$$

To obtain an exponentially decaying upper bound on the second term on the right hand side of (3.14), we may use the Cauchy-Schwarz inequality, together with the inequality $(x+y)^2 \le 2(x^2+y^2)$, to obtain

$$\begin{aligned}
&\int_{n-1}^{n}\left(\|A\|\|X(s)\| + \int_0^s \|K(s-u)\|\|X(u)\|\,du\right)^2 ds \\
&\le\ 2\int_{n-1}^{n}\|A\|^2\|X(s)\|^2 \\
&\qquad + \left(\int_0^s \|K(s-u)\|^{\frac{1}{2}}e^{\frac{\gamma_1}{2}(s-u)}\|K(s-u)\|^{\frac{1}{2}}e^{-\frac{\gamma_1}{2}(s-u)}\|X(u)\|\,du\right)^2 ds \\
&\le\ 2\int_{n-1}^{n}\Bigg(\|A\|^2\|X(s)\|^2 \\
&\qquad + \int_0^s \|K(u)\|e^{\gamma_1 u}\,du \int_0^s \|K(s-u)\|e^{-\gamma_1(s-u)}\|X(u)\|^2\,du\Bigg)\,ds.
\end{aligned}$$

Taking expectations across the above inequality, and using (1.5), (3.1) yields

$$\begin{aligned}
&\mathbb{E}\int_{n-1}^{n}\left(\|A\|\|X(s)\| + \int_0^s \|K(s-u)\|\|X(u)\|\,du\right)^2 ds \\
&\le\ 2M\int_{n-1}^{n}\left(\|A\|^2 e^{-2\lambda' s} + \overline{K}\int_0^s \|K(u)\|e^{-\gamma_1 u}e^{-2\lambda'(s-u)}\,du\right) ds,
\end{aligned}$$

where $\overline{K} = \int_0^\infty \|K(u)\|e^{\gamma_1 u}\,du$. Next, define $\lambda_2 = \lambda' \wedge \gamma_1$. Using the same kind of argument as in Lemma 3.1, we obtain

$$\int_0^s \|K(s-u)\|e^{-\gamma_1 u}e^{-2\lambda'(s-u)}\,du \le e^{-2\lambda_2 s}\int_0^\infty \|K(u)\|e^{\gamma_1 u}\,du.$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\int_{n-1}^{n}\left(\|A\|\|X(s)\|+\int_{0}^{s}\|K(s-u)\|\|X(u)\|\,du\right)^{2}ds\\
&\leq\ 2M\left(\|A\|^{2}+\left(\int_{0}^{\infty}\|K(u)\|e^{\gamma_1 u}\,du\right)^{2}\right)\int_{n-1}^{n}e^{-2\lambda_2 s}\,ds\\
&\leq\ 2M\left(\|A\|^{2}+\left(\int_{0}^{\infty}\|K(u)\|e^{\gamma_1 u}\,du\right)^{2}\right)\frac{1}{2\lambda_2}e^{-2\lambda_2(n-1)}. \qquad (3.15)
\end{aligned}$$

Next, we obtain an exponential upper bound on the third term on the right hand side of (3.14). By the Burkholder-Davis-Gundy inequality, there exists $C_4>0$ such that

$$\mathbb{E}\left[\sup_{n-1\leq t\leq n}\left\|\int_{n-1}^{t}\Sigma(s)\,dW(s)\right\|^{2}\right]\leq C_4\int_{n-1}^{n}\|\Sigma(s)\|_F^2\,ds\leq C_4\int_{n-1}^{\infty}\|\Sigma(s)\|_F^2\,ds.$$

But, for any $t\geq 0$, (1.6) implies

$$\int_{t}^{\infty}\|\Sigma(s)\|_F^2\,ds\leq\int_{t}^{\infty}e^{2\gamma_2(s-t)}\|\Sigma(s)\|_F^2\,ds\leq e^{-2\gamma_2 t}\int_{0}^{\infty}e^{2\gamma_2 s}\|\Sigma(s)\|_F^2\,ds.$$

Therefore,

$$\mathbb{E}\left[\sup_{n-1\leq t\leq n}\left\|\int_{n-1}^{t}\Sigma(s)\,dW(s)\right\|^{2}\right]\leq C_4 e^{-2\gamma_2(n-1)}\int_{0}^{\infty}e^{2\gamma_2 s}\|\Sigma(s)\|_F^2\,ds. \qquad (3.16)$$

Setting $\lambda_3=\lambda_2\wedge(2\gamma_2)$, and inserting the estimates (3.1), (3.15), and (3.16) into (3.14) gives (3.13), with $M'=M'(X_0)$ given by

$$\begin{aligned}
M'(X_0)=3^{2}\Bigg(&M(X_0)+2M(X_0)\left(\|A\|^{2}+\left(\int_{0}^{\infty}\|K(u)\|e^{\gamma_1 u}\,du\right)^{2}\right)\frac{1}{2\lambda_2}\\
&+C_4\int_{0}^{\infty}e^{2\gamma_2 s}\|\Sigma(s)\|_F^2\,ds\Bigg).
\end{aligned}$$

The proof now follows directly by applying the line of reasoning in Mao, Theorem 4.3.1, [13], and sequels in [16]: indeed we obtain

$$\limsup_{t\to\infty}\frac{1}{t}\log\|X(t)\|\leq-\frac{\lambda_3}{2},\quad \text{a.s.}$$

Setting $\beta_0=\lambda_3/2=(\lambda_2/2)\wedge\gamma_2=\lambda'\wedge\gamma_1\wedge\gamma_2=\lambda\wedge\gamma_1\wedge\gamma_2>0$, we are done. □

The above result establishes an upper bound on the almost sure exponential rate of decay of the solution $(-\beta_0)$; the estimate is $\beta_0=\lambda\wedge\gamma_1\wedge\gamma_2$, where λ is the exact top Liapunov exponent of resolvent of the deterministic problem (1.1). Based on the asymptotic behaviour of linear stochastic differential equations with damped noise, we conjecture that a better— perhaps optimal— estimate for β_0 is $\lambda \wedge \gamma_2$; however, to date we have been unable to prove this result. Another reasonable conjecture is that $\beta_0 = \gamma_1 \wedge \gamma_2$. However, the following counterexample shows that this conjecture cannot be true, in general.

Let $a, \mu, \alpha \in \mathbb{R}$, and consider the scalar version of equation (1.1) with $A = -a$, $K(t) = \alpha e^{-\mu t}$. Let $a > 0$, $\mu > 0$, and $\alpha < 0$. Suppose moroever that

$$-\alpha > \left(\frac{\mu - a}{2}\right)^2 > 0, \quad \mu > a.$$

Then the equation $\beta^2 + \beta(a + \mu) + (a\mu - \alpha) = 0$ has two complex-valued solutions with

$$\Re e(\beta) = -\frac{\mu + a}{2},$$

where $\Re e(\beta)$ is the real part of the complex number β. Next, it is easy to reformulate the resolvent equation (2.2) with $Z(0) = 1$ as the following second-order initial value problem:

$$Z''(t) + (\mu + a)Z'(t) + (a\mu - \alpha)Z(t) = 0, \quad t > 0,$$
$$Z(0) = 1, \quad Z'(0) = -a.$$

Therefore $\lambda = (\mu + a)/2$.

Next, notice that we can choose γ_1 to be any number less than μ. In particular, we may choose $\varepsilon \in (0, \mu - a)$ and set $\gamma_1 = \mu - \varepsilon/2$. Therefore, $\lambda < \gamma_1$.

If $\Sigma(t) \equiv 0$, we can interpret $\gamma_2 = \infty$; therefore, for the equation (2.3), the conjecture that $\beta_0 = \gamma_1 \wedge \gamma_2$ is equivalent to $\beta_0 = \gamma_1$. On the other hand, as Σ is identically zero, $\beta_0 = \lambda$. As $\lambda < \gamma_1$, we have $\beta_0 = \gamma_1 > \lambda = \beta_0$, a contradiction.

In the case where $\Sigma(t) \not\equiv 0$, it is still possible to show (using the same underlying deterministic equation) that the conjecture $\beta_0 = \gamma_1 \wedge \gamma_2$ does not hold in general; the proof requires an amount of explicit calculation, and will not be given here. However, the plan of the of the proof is simple: the scalar Itô-Volterra equation (2.3) can be reformulated as a two-dimensional linear stochastic differential equation, with solution $Y(t) = (Y_1(t), Y_2(t))$, where $Y_1(t) = X(t)$, $Y_2(t) = \int_0^t e^{-\mu(t-s)} X(s)\, ds$. The solution of this SDE can be written explicitly in terms of Φ, the fundamental matrix solution of

$$\Phi'(t) = \begin{pmatrix} -a & \alpha \\ 1 & -\mu \end{pmatrix} \Phi(t), \quad t > 0, \qquad \Phi(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Indeed, we have

$$\Phi(t)^{-1} Y(t) = \begin{pmatrix} x \\ 0 \end{pmatrix} + \int_0^t \Phi(s)^{-1} \begin{pmatrix} \Sigma(t) \\ 0 \end{pmatrix} dB(s), \quad t \geq 0.$$

The size of the stochastic perturbation on the righthand side can be estimated using the martingale time change theorem. Once this is done, the rate of decay to zero of $Y_1(t) = X(t)$ can be readily determined.

In the case when $\Sigma(t)$ obeys (1.6) with $\gamma_2 > \gamma_1$, the above calculation yields the estimate $\beta_0 = \lambda$, since $\gamma_1 > \lambda$. However, this is once again inconsistent with $\beta_0 = \gamma_1 \wedge \gamma_2 = \gamma_1$, because $\gamma_1 > \lambda$.

4 Necessary conditions for exponential convergence of solutions

We now turn our attention to conditions which must hold if the solution of (2.3) is exponentially convergent in an almost sure sense, or a p-th mean sense. We start with an almost sure result, and then modify the analysis to tackle the p-th mean case.

Theorem 4.1. *Suppose that (1.8) and (1.5) hold for the equation (2.3). Then (1.6) holds.*

Proof. Let β_0 be defined by (1.8), $\gamma_1 > 0$ by (1.5). Let γ_2 be a positive constant satisfying $\gamma_2 < \beta_0 \wedge \gamma_1$, and define the process $Y(t) = e^{\gamma_2 t} X(t)$. Hence (1.8) implies

$$\limsup_{t\to\infty} \frac{1}{t} \log \|Y(t)\| \leq -(\beta_0 - \gamma_2) < 0, \quad \text{a.s.},$$

which means that

$$\lim_{t\to\infty} Y(t) = 0, \quad Y(t) \in L^1(\mathbb{R}^+) \quad \text{a.s.} \tag{4.1}$$

Further define $\tilde{K}(t) = e^{\gamma_2 t} K(t)$. This gives $\tilde{K}(t) \in L^1(\mathbb{R}^+)$, by (1.5). Integration by parts on the process Y, and (2.3) gives, on rearrangement,

$$\begin{aligned}\int_0^t e^{\gamma_2 s} \Sigma(s)\, dW(s) = Y(t) - X_0 \\ - (\gamma_2 I_d + A) \int_0^t Y(s)\, ds - \int_0^t \int_0^s \tilde{K}(s-u) Y(u)\, du\, ds.\end{aligned} \tag{4.2}$$

Consider the limit as $t \to \infty$ on the right hand side of (4.2). By (4.1), the first and third terms have almost sure limits as $t \to \infty$. Since $\tilde{K}(t) \in L^1(\mathbb{R}^+)$, and $Y(t) \in L^1(\mathbb{R}^+)$ (by (4.1)), the fourth term has almost sure limit as $t \to \infty$. Therefore

$$\lim_{t\to\infty} \int_0^t e^{\gamma_2 s} \Sigma(s)\, dW(s)$$

exists almost surely. Thus, for $i = 1, \ldots, d$, the i-th component converges almost surely, i.e.,

$$\lim_{t\to\infty} U_i(t)$$

exists, a.s, where

$$U_i(t) = \sum_{j=1}^{r} \int_0^t e^{\gamma_2 s} \Sigma_{i,j}(s)\, dW^j(s).$$

Further define

$$\tilde{\Sigma}_i(s)^2 = \sum_{j=1}^{r} \Sigma_{i,j}^2(s).$$

Now, suppose that there exists $i \in \{1, \ldots, d\}$ such that

$$\int_0^\infty e^{2\gamma_2 s} \tilde{\Sigma}_i(s)^2\, ds = \infty. \tag{4.3}$$

for all $\gamma_2 > 0$. Then, there exists an increasing sequence of times $t_n \nearrow \infty$ with $t_0 = 0$ and which satisfies

$$\int_{t_n}^{t_{n+1}} e^{2\gamma_2 s} \tilde{\Sigma}_i(s)^2\, ds = 1$$

for $n \geq 0$. Next, set

$$Z_n = \sum_{j=1}^{r} \int_{t_n}^{t_{n+1}} e^{\gamma_2 s} \Sigma_{i,j}(s)\, dW^j(s).$$

By construction, and the independence of the Brownian motions W^j, the sequence of random variables Z_n are identically and independently distributed (indeed, standardised) Gaussian random variables. Hence

$$\sum_{n=1}^{\infty} \mathbb{E}[Z_n^2] = \infty. \tag{4.4}$$

Since $U_i(t)$ has an almost sure limit as $t \to \infty$, we have, by definition of Z_n,

$$\sum_{n=1}^{\infty} Z_n \quad \text{exists a.s.} \tag{4.5}$$

This introduces a contradiction, as (4.4), (4.5) are inconsistent, by Kolmogorov's Three Series theorem. Hence (4.3) must be false for all $i = 1, \ldots, d$, and some $\gamma_2 > 0$. Thus, there exists $\gamma_2 > 0$ such that

$$\sum_{i=1}^{d} \int_0^\infty e^{2\gamma_2 s} \tilde{\Sigma}_i(s)^2\, ds < \infty.$$

Using the definition of $\tilde{\Sigma}_i(s)$, we have (1.6), by the invoking the definition of the Frobenius norm, and appealing to the equivalence of norms on $M_{d,r}(\mathbb{R})$. □

Before we can prove the corresponding result for p-th mean exponentially convergent solutions of (2.3), we need the following technical result.

Lemma 4.2. *Suppose $N_i \sim \mathcal{N}(0, v_i^2)$ for $i = 1, \dots, d$, and*

$$Z = \left(\sum_{i=1}^{d} N_i^2 \right)^{1/2}.$$

Then there exists a v_i-independent constant $D_1 > 0$ such that

$$\mathbb{E}[Z^2] \le D_1 \mathbb{E}[|Z|]^2.$$

Proof. By the equivalence of norms on $\mathbb{R}^d$, there exist d-dependent positive constants $c_1 \le c_2$ such that for $x_1, x_2, \dots, x_d \in \mathbb{R}$,

$$c_1 \sum_{i=1}^{d} |x_i| \le \left(\sum_{i=1}^{d} x_i^2 \right)^{1/2} \le c_2 \sum_{i=1}^{d} |x_i|. \tag{4.6}$$

Applying (4.6) yields

$$\mathbb{E}[|Z|] \ge c_1 \sum_{i=1}^{d} \mathbb{E}|N_i|.$$

Since $N_i \sim \mathcal{N}(0, v_i^2)$, there exists a v_i-independent constant $c_3 > 0$ such that $\mathbb{E}[|N_i|] = c_3 v_i$. Hence, using (4.6), and the fact that $N_i \sim \mathcal{N}(0, v_i^2)$, we obtain

$$\mathbb{E}[|Z|]^2 \ge c_1^2 c_3^2 \left(\sum_{i=1}^{d} v_i \right)^2 \ge \frac{c_1^2 c_3^2}{c_2^2} \sum_{i=1}^{d} v_i^2 = \frac{c_1^2 c_3^2}{c_2^2} \mathbb{E}[Z^2].$$

Putting $D_1 = \frac{c_2^2}{c_1^2 c_3^2}$ gives the result. $\square$

This enables us to establish the corresponding p-th mean result, for $p \ge 1$.

Theorem 4.3. *Suppose for some $p \ge 1$ that there exists $\beta_p > 0$, $M_p > 0$ such that*

$$\mathbb{E}[\|X(t)\|^p] \le M_p e^{-\beta_p t}, \quad t \ge 0, \tag{4.7}$$

and that (1.5) holds for (2.3). Then (1.6) holds.

Proof. By (4.7) and Liapunov's inequality, there exists $M_1 > 0$, $\beta_1 > 0$ such that

$$\mathbb{E}[\|X(t)\|] \le M_1 e^{-\beta_1 t}.$$

Choose $\gamma_2 > 0$ such that $\gamma_2 < \beta_1 \wedge \gamma_1$, where $\gamma_1 > 0$ is defined by (1.5). Define $Y(t)$ as in Theorem 4.1 above. Then

$$\mathbb{E}[\|Y(t)\|] \le M_1 e^{-(\beta_1 - \gamma_2)t}.$$

Hence

$$\lim_{t \to \infty} \mathbb{E}[\|Y(t)\|] = 0, \quad \mathbb{E}[\|Y(t)\|] \in L^1(\mathbb{R}^+).$$

Defining $\tilde{K}$ as in Theorem 4.1 means that $\tilde{K}(t) \in L^1(\mathbb{R}^+)$. Taking the triangle inequality across (4.2), using Hölder's inequality, and then taking expectations yields

$$\mathbb{E}\left[\left\|\int_0^t e^{\gamma_2 s}\Sigma(s)\,dW(s)\right\|\right] \leq \mathbb{E}\|Y(t)\| + \|X_0\| \\ + \|\gamma_2 I_d + A\|\int_0^t \mathbb{E}[\|Y(s)\|]\,ds + \int_0^t\int_0^s \|\tilde{K}(s-u)\|\mathbb{E}[\|Y(u)\|]\,du\,ds. \quad (4.8)$$

Every term on the right hand side of (4.8) is uniformly bounded on $\mathbb{R}^+$; the first term is bounded as $t \mapsto \mathbb{E}[\|Y(t)\|]$ is continuous on $\mathbb{R}^+$ and has zero limit at infinity; the second term is constant; the third is bounded as $\mathbb{E}[\|Y(t)\|] \in L^1(\mathbb{R}^+)$, while the fourth is bounded as $(\|K\| * \mathbb{E}[\|Y\|])(t) \in L^1(\mathbb{R}^+)$. Thus, there exists $D_2 > 0$ such that

$$\mathbb{E}\left[\left\|\int_0^t e^{\gamma_2 s}\Sigma(s)\,dW(s)\right\|\right] \leq D_2. \quad (4.9)$$

Next, observe that

$$\left\|\int_0^t e^{\gamma_2 s}\Sigma(s)\,dW(s)\right\|^2 = \sum_{i=1}^d N_i(t)^2,$$

where each of the random variables

$$N_i(t) = \int_0^t \sum_{j=1}^r e^{\gamma_2 s}\Sigma_{i,j}(s)\,dW^j(s)$$

is normally distributed with zero mean, and variance

$$v_i(t)^2 = \int_0^t \sum_{j=1}^r e^{2\gamma_2 s}\Sigma_{i,j}(s)^2\,ds.$$

Therefore, we may apply Lemma 4.2 and (4.8) to get

$$\begin{aligned}\int_0^t e^{2\gamma_2 s}\|\Sigma(s)\|_F^2\,ds &= \sum_{i=1}^d\sum_{j=1}^r \int_0^t e^{2\gamma_2 s}\Sigma_{i,j}(s)^2\,ds \\ &= \mathbb{E}\left[\left\|\int_0^t e^{\gamma_2 s}\Sigma(s)\,dW(s)\right\|^2\right] \\ &\leq D_1\mathbb{E}\left[\left\|\int_0^t e^{\gamma_2 s}\Sigma(s)\,dW(s)\right\|\right]^2 \\ &\leq D_1 D_2^2.\end{aligned}$$

Taking limits as $t \to \infty$ both sides gives the desired result (1.6). $\square$

5 Necessary and sufficient conditions for exponential convergence of solutions

We now use Theorems 3.3-4.3 above to draw equivalences between the statements

(i) (1.5), (1.6), the uniform asymptotic stability of the zero solution of (1.4);

(ii) (1.7) for all $p > 0$;

(iii) (1.8).

for the problem (2.3), under an additional condition on the kernel K. The condition we use is that the each entry of K does not change sign on $\mathbb{R}^+$. This ensures that we can employ Lemma 2.6 above.

We prove the equivalence between (i) and (iii) first, before turning to the equivalence between (i) and (ii).

Theorem 5.1. *Suppose that the entries of K do not change sign on $\mathbb{R}^+$. Then the following are equivalent for the problem (2.3):*

(i) There exists $\beta_0 > 0$, such that

$$\limsup_{t\to\infty} \frac{1}{t} \log \|X(t)\| \leq -\beta_0, \quad a.s.$$

for all solutions of (2.3).

(ii) There exists $\lambda' > 0$, such that for every $p > 0$ there exists $M_p(X_0) > 0$ for which

$$\mathbb{E}[\|X(t)\|^p] \leq M_p(X_0)e^{-\lambda' pt}, \quad t \geq 0,$$

for all solutions of (2.3).

(iii) There exist $\gamma_1 > 0$, $\gamma_2 > 0$ such that

$$\int_0^\infty \|K(s)\|e^{\gamma_1 s}\, ds < \infty, \quad \int_0^\infty \|\Sigma(s)\|^2 e^{2\gamma_2 s}\, ds < \infty,$$

and the zero solution of (1.1) is uniformly asymptotically stable.

Proof. We first show that (iii) and (i) are equivalent.

The implication (iii) implies (i) is the subject of Theorem 3.4, which leaves the implication (i) implies (iii).

To establish this, consider $d+1$ solutions of (2.3), $X^j(t)$ for $j = 0, \ldots, d$ which are associated with the initial conditions $X^0(0) = 0$, $X^j(0) = \mathbf{e}_j$, $j = 1, \ldots, d$. Then, by (2.4), we have,

$$Z(t)\mathbf{e}_j = Z(t)X^j(0) = X^j(t) - X^0(t)$$

for $j = 1, \dots, d$. By hypothesis, there exists a.s. finite random variables $C_j(\varepsilon) > 0$ for every $\varepsilon \in (0, \beta_0/2)$ such that

$$\|X^j(t)\| \le C_j e^{-(\beta_0 - \varepsilon)t}, \quad t \ge 0, \quad \text{a.s.}$$

for $j = 0, \dots, d$. Thus, for $j = 1, \dots, d$, we get

$$\|Z(t)\mathbf{e}_j\| \le \|X^j(t)\| + \|X^0(t)\| \le \tilde{C}_j(\varepsilon) e^{-(\beta_0 - \varepsilon)t},$$

where $\tilde{C}_j(\varepsilon) = C_j(\varepsilon) + C_0(\varepsilon)$. Hence, there exists an almost surely finite random variable $\tilde{C}(\varepsilon) > 0$ such that

$$\|Z(t)\| \le \tilde{C}(\varepsilon) e^{-(\beta_0 - \varepsilon)t}, \quad t \ge 0, \quad \text{a.s.}$$

Since Z is deterministic, we have

$$\limsup_{t \to \infty} \frac{1}{t} \log \|Z(t)\| \le -(\beta_0 - \varepsilon),$$

Thus for every $\varepsilon \in (0, \beta_0/2)$, there exists a deterministic constant $C(\varepsilon)$ such that

$$\|Z(t)\| \le C(\varepsilon) e^{-(\beta_0 - 2\varepsilon)t}, \quad t \ge 0.$$

In particular, with $\varepsilon = \beta_0/4$, and $C = C(\beta_0/4)$, we have

$$\|Z(t)\| \le C e^{-\frac{\beta_0}{4}t}, \quad t \ge 0. \tag{5.1}$$

This immediately ensures the uniform asymptotic stability of the zero solution of (1.1). Since the entries of K do not change sign on $\mathbb{R}^+$, (5.1) taken together with Lemma 2.6 implies (1.5). The hypothesis (i) in conjunction with (1.5) force (1.6), by Theorem 4.1. All the parts of (iii) are satisfied, proving the forward implication.

We now turn to the proof of the equivalence of (i) and (ii); since (i) and (iii) have already been shown to be equivalent, the proof that (i) and (ii) are equivalent now suffices to prove the theorem.

The implication (ii) implies (i) is the subject of Theorem 3.3. We prove the implication (i) implies (ii) as follows: since (i) is true, we have

$$\mathbb{E}[\|X(t)\|] \le M_1(X_0) e^{-\lambda' t}.$$

Thus, by choosing $X_0^j = \mathbf{e}_j$ to be the initial condition associated with the solution $X^j(t)$, $j = 1, \dots, d$, we get

$$\|Z(t)\mathbf{e}_j\| = \|\mathbb{E}[X^j(t)]\| \le \mathbb{E}[\|X^j(t)\|] \le M_1(\mathbf{e}_j) e^{-\lambda' t}.$$

Therefore, there exists $\lambda' > 0$, $C > 0$ such that

$$\|Z(t)\| \le C e^{-\lambda' t}, \quad t \ge 0. \tag{5.2}$$

Therefore, the zero solution of (1.1) is uniformly asymptotically stable. Since the entries of K do not change sign on $\mathbb{R}^+$, and (5.2) holds, Lemma 2.6 implies that (1.5) holds. Since (i), (1.5) are true, Theorem 4.3 now implies that (1.6) is true for some $\gamma_2 > 0$. All the conditions of (ii) are therefore satisfied, and the result proven. □

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