

Lower estimates for random walks on a class of amenable p -adic groups

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Abstract

We give central lower estimates for the transition kernels corresponding to symmetric random walks on certain amenable p -adic groups.

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1 Introduction

Let G be a locally compact amenable group. We shall denote by $d^r g$ (resp. $d^l g$) the right (resp. left) Haar measure on G and by $\delta(g) = d^r g / d^l g$ the modular function on G normalized by $\delta(e) = 1$ where e denotes the identity of G . Let $d\mu(g) = \varphi(g)d^r g \in \mathbf{P}(G)$ be a probability measure on G where $\varphi(g) \in L^\infty(G)$ is assumed to have a compact support or a fast decay at infinity. Let us assume that μ is symmetric (i.e. the involution $g \rightarrow g^{-1}$ stabilizes μ) and consider the random walk on G induced by μ , i.e. the G -valued process that evolves as follows: if $X_n = g$ is the position at time n then $X_{n+1} = gh$ where h is chosen according to μ . We shall denote by

$$d\mu^{*n}(g) = \varphi_n(g)d^r g$$

the n^{th} convolution power of μ and examine the behaviour of the decay of $\varphi_n(e)$ as $n \rightarrow \infty$.

If we restrict ourselves to unimodular real Lie groups then the answer lies in the behaviour of the volume growth of G . Let us recall that if G is a locally compact group that is generated by some symmetric compact neighbourhood $\Omega \subset G$ of the identity in G , the volume growth function is (cf. [9])

$$\gamma(t) = \text{Vol}(B_t(e)), \quad t = 1, 2, \dots$$

where the volume is taken with respect to $d^r g$ (or $d^l g$) and where $B_t(e)$ is the ball of radius t centred on e defined by

$$B_t(e) = \Omega \dots \Omega, \quad t \text{ times.}$$

For $x \in G$ the distance from e is defined by $|x| = \inf\{t, x \in B_t(e)\}$ and a left invariant distance can be defined on G by setting $d(x, y) = |y^{-1}x|$, $x, y \in G$. If Ω_1, Ω_2 are two neighbourhoods of e as above it is not difficult to check that there exists $C > 0$ such that $C^{-1} \leq |\cdot|_2 / |\cdot|_1 \leq C$ and that the corresponding growth functions satisfy the obvious equivalence $\gamma_1 \approx \gamma_2$, i.e.

$$\gamma_1(t) \leq C\gamma_2(Ct) + C \leq C'\gamma_1(C't) + C', \quad t \geq 1.$$

For real Lie groups we have the following dichotomy (cf. [9], [13]): either

$$\gamma(t) \approx t^D$$

where $D = D(G) = 1, 2, \dots$, or

$$\gamma(t) \approx e^t.$$

In the first case we say that G is of polynomial growth and in the second case we say that G is of exponential growth and the answer to our problem in the case of unimodular amenable real Lie groups was given by Varopoulos and is the following

$$\varphi_n(e) \approx n^{-D/2} \iff \gamma(t) \approx t^D$$

$$\varphi_n(e) \approx e^{-n^{1/3}} \iff \gamma(t) \approx e^t$$

(cf. [1], [6], [7], [10], [28]).

Varopoulos showed that the $e^{-n^{1/3}}$ versus polynomial behaviour extends to the non-unimodular amenable case depending on whether the Lie group G is (C) or (NC). This classification introduced

in [24] can be expressed in terms of the roots of the ad -action of the radical of the Lie algebra of the group on its nilradical.

The discret case is more complicated (cf. [1], [11], [16], [17], [22], [23]). First there is no dichotomy in the volume growth (cf. [8]). On the other hand if we suppose that the group G is of exponential growth then one can claim only the upper bound (cf. [11])

$$\varphi_n(e) \leq C \exp(-cn^{1/3}), \quad n \geq 1.$$

In general, the matching lower bound fails. Ch. Pittet and L. Saloff-Coste showed (cf. [16]) that there are soluble groups with exponential volume growth for which the heat kernel decays as $\exp(-cn^\alpha)$ with $\alpha \in (0, 1)$ which can be taken arbitrarily close to 1. This, as mentioned above, can not happen in the case of real Lie groups.

In a recent paper (cf. [17]) Ch. Pittet and L. Saloff-Coste established the lower bound $\varphi_{2n}(e) \geq c \exp(-Cn^{1/3})$, $n = 1, 2, \dots$ for the large times asymptotic behaviours of the probabilities of return to the origin at even times $2n$, for random walks associated with finite symmetric generating sets of solvable groups of finite Prüfer rank. They asked in this paper (cf. [17], §8) if a similiar lower bound is available in the case of analytic p -adic groups. An answer to this problem was given in [15] (cf. also [14]). The aim of this paper is to show that the $e^{-n^{1/3}}$ lower bound obtained in [14] can be substantially improved for a large class of amenable p -adic groups.

2 Amenable p -adic groups

In this section G will denote an algebraic connected amenable group over \mathbb{Q}_p the field of p -adic numbers; $U \subset Q \subset G$ will denote the radical and the unipotent radical (cf. [3], [5]). Amenability of G is equivalent to the fact that the semi-simple group G/Q is compact (cf. [19]). Let S denote a fixed levi subgroup S of G (cf. [3]). The group G can then be written as a semi-direct product:

$$(1) \quad G = Q \rtimes S = (U \rtimes A) \rtimes S \cong U \rtimes (A \times S)$$

where A is abelian and can be identified to the direct product of a finite group and a d -torus $T \cong (\mathbb{Q}_p^*)^d$ (cf. [3], [5]). Here \mathbb{Q}_p^* denotes the multiplicative group of the field \mathbb{Q}_p . Since $(\mathbb{Q}_p^*)^d \cong \mathbb{Z}^d \times K$ (where K is compact, cf. [2], [4]), by considering the projection

$$(2) \quad \pi : A \longrightarrow \mathbb{Z}^d,$$

we can fix $\pi_1, \dots, \pi_d \in A$ so that each $z \in A$ admits a unique decomposition

$$(3) \quad z = \pi_1^{n_1} \dots \pi_d^{n_d} \tau, \quad n_1, \dots, n_d \in \mathbb{Z}, \quad \tau \in \tilde{K},$$

where \tilde{K} denotes a compact subgroup of A . Let $\mathcal{U} = Lie(U)$ denote the Lie algebra of U . Let $\overline{\mathbb{Q}_p}$ denotes a finite extension of \mathbb{Q}_p which contains all the eigenvalues defined by

$$\det (Ad(\pi_j) - \lambda I) = 0, \quad j = 1, \dots, d.$$

The Ad -action of A on \mathcal{U} extends to $\mathcal{U} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}$ and it follows from the proof of the Zassenhaus lemma (cf. [12]) that we have a decomposition of $\mathcal{U} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}$ into a direct sum

$$(4) \quad \mathcal{U} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p} = W_1 \oplus \dots \oplus W_r$$

where the subspaces W_j ($1 \leq j \leq r$) are invariant by $Ad(\pi_i)$, $i = 1, \dots, d$, and such that the restriction of $Ad(\pi_i)$ to W_j is the sum of the scalar $\lambda_j(\pi_i)$ and a nilpotent endomorphism.

Let $\chi_j : A \rightarrow \overline{\mathbb{Q}}_p^*$ defined by

$$\chi_j(z) = \lambda_j(\pi_1)^{n_1} \dots \lambda_j(\pi_d)^{n_d}, \quad z = \pi_1^{n_1} \dots \pi_d^{n_d} \tau, \quad j = 1, \dots, r.$$

Let $|\cdot|_p$ denote the standard p -adic norm. We shall denote by $|\cdot|'_p$ its extension to $\overline{\mathbb{Q}}_p$. We have $|x|'_p \in \{\bar{p}^n, n \in \mathbb{Z}\} \cup \{0\}$, $x \in \overline{\mathbb{Q}}_p$, where \bar{p} denotes a rational power of the prime p (cf. [2]). Let $\alpha_1, \alpha_2, \dots, \alpha_s$ denote the different norms of the χ_j 's, i.e. the different homomorphisms obtained by considering $A \rightarrow \overline{\mathbb{Q}}_p^*$, $z \rightarrow |\chi_j(z)|'_p$, $j = 1, \dots, r$. Let $\mathcal{L} = \{\alpha_1, \alpha_2, \dots, \alpha_s\}$. We shall assume that \mathcal{L} is nonempty and that

$$1 \notin \mathcal{L}.$$

Here 1 denotes the homomorphism $A \rightarrow \overline{\mathbb{Q}}_p^*$ identically equal to 1. This assumption guarantees the fact that the group G is compactly generated (cf. [3]). Observe that the group G is then of exponential volume growth (cf. [18]).

Let $\gamma_{j,1}, \dots, \gamma_{j,d} \in \mathbb{Z}$ the integers defined by

$$(5) \quad \alpha_j(z) = \bar{p}^{\gamma_{j,1}n_1 + \gamma_{j,2}n_2 + \dots + \gamma_{j,d}n_d}, \quad z = \pi_1^{n_1} \dots \pi_d^{n_d} \tau, \quad j = 1, \dots, s.$$

We shall denote by L_j , $j = 1, \dots, s$, the linear forms on \mathbb{Z}^d defined by

$$(6) \quad L_j(n_1, \dots, n_d) = \gamma_{j,1}n_1 + \gamma_{j,2}n_2 + \dots + \gamma_{j,d}n_d,$$

$(n_1, \dots, n_d) \in \mathbb{Z}^d$ and by \check{L}_j the linear forms on \mathbb{R}^d induced by the L_j 's.

Let now $d\mu(g) = \varphi(g)d^r g \in \mathbf{P}(G)$ denote a symmetric probability measure on G . The density $\varphi(g)$ is assumed to be a continuous compactly supported function on G . To avoid unnecessary complications we shall assume that there exists $e \in \Omega = \Omega^{-1} \subset G$ such that

$$(7) \quad \inf \{ \varphi(g), g \in \Omega \} > 0, \quad G = \bigcup_{n \geq 0} \Omega^n;$$

the last condition in (7) implies that $\text{supp}(\mu)$ generates the group G .

Let

$$p : G \rightarrow G/U \rightarrow A$$

denote the projection that we obtain from the identifications (1) and let

$$(8) \quad \check{\mu} = (\pi \circ p)(\mu) \in \mathbf{P}(\mathbb{Z}^d) \subset \mathbf{P}(\mathbb{R}^d),$$

where π denotes the canonical projection (2). It follows from (7) that there exists a choice of coordinates on \mathbb{R}^d for which the covariance matrix of $\check{\mu}$ satisfies

$$(9) \quad \int_{\mathbb{R}^d} x_i x_j d\check{\mu}(x) = \delta_{i,j}, \quad 1 \leq i, j \leq d.$$

We shall assume that \mathbb{R}^d is equipped with the Euclidean structure associated to these coordinates.

From now on we shall assume that the L_j 's induce a "Weyl chamber". More precisely we suppose that

$$\Pi_{\mathcal{L}} = \{x \in \mathbb{R}^d, \tilde{L}_j(x) > 0, j = 1, 2, \dots, s\} \subset \mathbb{R}^d$$

define a nonempty convex cone in \mathbb{R}^d . This condition is the analogue of the (NC)-condition introduced by Varopoulos in [24] in the setting of real amenable Lie groups. We shall prove that, under this condition, we have a lower bound of the form $\varphi_n(e) \geq cn^{-\nu}$. The argument follows the approach introduced by Varopoulos in [24].

The exact value of ν is defined as in the real case and is expressed in terms a parameter $\lambda = \lambda(d, \mathcal{L})$ that is defined as follows. In the the rank one case (i.e. $d = 1$) we shall set

$$(10^a) \quad \lambda = 0.$$

In the case $d \geq 2$ let us denote by $\Sigma = \{x \in \mathbb{R}^d, |x| = 1\}$ the unit sphere in \mathbb{R}^d . Let Δ_{Σ} be the corresponding spherical Laplacian. Let $\Pi_{\Sigma} = \Sigma \cap \Pi_{\mathcal{L}} \subset \Sigma$. We then set

$$(10^b) \quad \lambda = \inf\{-(\Delta_{\Sigma}f, f), \|f\|_2 = 1, f \in C_0^{\infty}(\Pi_{\Sigma})\};$$

i.e. λ is the first Dirichlet eigenvalue of the region Π_{Σ} . The scalar product and the L^2 -norm in (10^b) are taken with respect to the Euclidean volume element on Σ .

Theorem 1. *Let G and $d\mu^{*n}(g) = \varphi_n(g)d^r g$, $n = 1, 2, \dots$ be as above. Then there exists $C > 0$ such that*

$$(11) \quad \varphi_n(e) \geq \frac{1}{Cn^{1+\frac{\sqrt{(d-2)^2+4\lambda}}{2}}}, \quad n = 1, 2, \dots$$

where λ is defined by (10).

The following comments may be helpful in placing the above theorem in its proper perspective.

- (i) It is enough to to prove the estimate (11) when n is even since φ satisfies (7).
- (ii) Observe that the group G is automatically non-unimodular, for otherwise we have (cf. [11]):

$$\varphi_n(e) \leq Ce^{-cn^{1/3}}, \quad n = 1, 2, \dots$$

- (iii) The upper estimate

$$\varphi_n(e) \leq \frac{C}{n^{3/2}}, \quad n = 1, 2, \dots$$

is known to hold for general non-unimodular locally compact groups (cf. [26]). This shows that the index $3/2$ cannot be improved in the rank one case.

- (iv) We will show (cf. §4 below) that in the case of metabelian p -adic groups, the lower bound (11) can be complemented with a similar upper bound.

Throughout the remainder of the paper C denotes a positive constant which is not always the same, even in a given line.

3 Proof of Theorem 1

Let G , $d\mu(g) = \varphi(g)d^r g \in \mathbf{P}(G)$ and $d\mu^{*n}(g) = \varphi_n(g)d^r g$ be as in Theorem 1. Let $\xi_1, \xi_2, \dots \in G$ be a sequence of independent equidistributed random variables of law $d\mu(g)$ and let $X_n = \xi_1 \xi_2 \dots \xi_n$, $n = 1, 2, \dots$ denote the corresponding random walk starting at $X_0 = e$. Let $B \subset G$ a borel subset, we have

$$(12) \quad \mathbb{P}_e [X_n \in B] = \int_B \varphi_n(g) d^r g, \quad n = 1, 2, \dots$$

The symmetry of $d\mu(g)$ implies that

$$d\mu(g) = d\mu(g^{-1}) = \varphi(g)d^r g = \varphi(g^{-1})d^r g = \varphi(g^{-1})\delta(g)^{-1}d^r g,$$

hence

$$\varphi(g^{-1}) = \varphi(g)\delta(g), \quad g \in G.$$

We have also

$$\varphi_n(g^{-1}) = \varphi_n(g)\delta(g), \quad g \in G, \quad n = 1, 2, \dots$$

On the other hand we have

$$\varphi_{2n}(e) = \int_G \varphi_n(g^{-1})\varphi_n(g)d^r g = \int_G \varphi_n(g)\delta(g)\varphi_n(g)d^r g = \int_G \varphi_n(g)^2 d^r g.$$

Schwarz inequality applied to (12) gives then

$$(13) \quad \varphi_{2n}(e) \geq \frac{\left(\mathbb{P}_e [X_n \in B] \right)^2}{|B|}, \quad B \subset G, \quad n = 1, 2, \dots$$

where $|B|$ denotes the right Haar measure of B .

Let us further observe that the group G (resp. G/U) decomposes as a semi-direct (resp. direct) product

$$\begin{aligned} G &= Q \times S \cong U \times (\mathbb{Z}^d \times \tilde{S}) \\ G/U &\cong A \times S \cong \mathbb{Z}^d \times \tilde{S} \end{aligned}$$

where \tilde{S} is compact. This follows from (1).

Let us write

$$(14) \quad X_n = \xi_1 \xi_2 \dots \xi_n = u_1 z_1 u_2 z_2 \dots u_n z_n, \quad n = 1, 2, \dots$$

where $\xi_j = u_j z_j$ with $u_j \in U$ and $z_j \in A \times S$, $j = 1, 2, \dots$. Using the interior automorphisms $x^y = yxy^{-1}$, $x, y \in G$, we rewrite (14)

$$(15) \quad \begin{aligned} X_n &= u_1 u_2^{z_1} u_3^{z_1 z_2} \dots u_n^{z_1 \dots z_{n-1}} z_1 z_2 \dots z_n = \Gamma_n Z_n, \\ \Gamma_n &\in U, \quad Z_n \in A \times S, \quad n = 1, 2, \dots \end{aligned}$$

We shall use the exponential map and identify U to its Lie algebra \mathcal{U} (cf. [5], [20]) and write each u_j in the above expression

$$(16) \quad u_i = \exp(v_i), \quad v_i \in \mathcal{U}, \quad i = 1, 2, \dots$$

We have therefore

$$(17) \quad u_j^{z_1 z_2 \dots z_{j-1}} = \exp \left(\text{Ad}(z_1 \dots z_{j-1}) v_j \right), \quad j \geq 2.$$

Let us fix e_1, e_2, \dots, e_m a basis of $\mathcal{U} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}$ adapted to the decomposition (4). For $x = x_1 e_1 + x_2 e_2 + \dots + x_m e_m \in \mathcal{U} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}$, we set

$$\|x\| = \max_{1 \leq i \leq m} |x_i|.$$

Since μ is compactly supported we can suppose that the v_i 's in (16) satisfy

$$(18) \quad \|v_j\| \leq C, \quad j = 1, 2, \dots$$

where $C > 0$ is an appropriate positive constant.

Let us equip $\text{End}_{\overline{\mathbb{Q}_p}} \left(\mathcal{U} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p} \right)$ with the norm

$$\| \|T\| \| = \sup_{\|v\| \leq 1} \|Tv\|, \quad T \in \text{End}_{\overline{\mathbb{Q}_p}} \left(\mathcal{U} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p} \right).$$

It is clear that $\| \| \cdot \| \|$ satisfies the ultrametric property

$$(19) \quad \| \|T + T'\| \| \leq \max(\| \|T\| \|, \| \|T'\| \|).$$

Let $1 \leq l \leq r$. For

$$z = \pi_1^{n_1} \dots \pi_d^{n_d} \tilde{\tau} = \tilde{z} \tilde{\tau} \in A \times S \cong \mathbb{Z}^d \times \tilde{S}$$

we have

$$(20) \quad \text{Ad}(z)|_{W_l} = \text{Ad}(\tilde{\tau})|_{W_l} \circ \left(\tilde{\chi}_l(z)I + \mathcal{T}(z) \right),$$

where $\mathcal{T}(z)$ denotes an upper triangular matrix and where

$$\tilde{\chi}_l(z) = \chi_l(\pi_1^{n_1} \dots \pi_d^{n_d}) = \chi_l(\pi_1)^{n_1} \dots \chi_l(\pi_d)^{n_d}.$$

On the other hand it is clear that

$$(21) \quad |\tilde{\chi}_l(z)|'_p, |\tilde{\chi}_l(z)^{-1}|'_p \leq p^{C|\tilde{z}|_{\mathbb{Z}^d}}, \quad z \in A \times S,$$

where $|\cdot|_{\mathbb{Z}^d}$ denotes the Euclidean norm on \mathbb{Z}^d . We have also

$$(22) \quad \| \| \text{Ad}(z) \| \| \leq C \| \| \text{Ad}(\pi_1) \| \|^{n_1} \dots \| \| \text{Ad}(\pi_d) \| \|^{n_d} \leq Cp^{C|\tilde{z}|_{\mathbb{Z}^d}}, \quad z \in A \times S.$$

Combining (20), (21), (22) we deduce that the triangular matrix $\mathcal{T}(z)$ that appears in (20) satisfies

$$(23) \quad \| \| \mathcal{T}(z) \| \| C \leq p^{C|\tilde{z}|_{\mathbb{Z}^d}}, \quad z \in A \times S.$$

Let now $z_1, z_2, \dots, z_k \in A \times S$ and $1 \leq l \leq r$. We have

$$\begin{aligned} Ad(z_1 z_2 \dots z_k)|_{W_l} &= Ad(\tilde{\tau}_1 \tilde{\tau}_2 \dots \tilde{\tau}_k)|_{W_l} \circ \prod_{j=1}^k (\tilde{\chi}_l(z_j)I + \mathcal{F}(z_j)) \\ &= \tilde{\chi}_l(z_1 z_2 \dots z_k) Ad(\tilde{\tau}_1 \tilde{\tau}_2 \dots \tilde{\tau}_k)|_{W_l} \\ &\quad \circ \sum_{\alpha, l_j} (\tilde{\chi}_l(z_{i_1}) \dots \tilde{\chi}_l(z_{i_\alpha}))^{-1} \mathcal{F}(z_{i_1}) \dots \mathcal{F}(z_{i_\alpha}). \end{aligned}$$

Using the fact that in the last sum all the terms corresponding to indexes $\alpha > n$ vanish and combining this with (21), (23) and the ultrametric property (19) we deduce that

$$\|Ad(z_1 z_2 \dots z_k)|_{W_l}\| \leq C \left| \tilde{\chi}_l(z_1 z_2 \dots z_k) \right|_p P^{C \max_{1 \leq j \leq k} |\tilde{z}_j|_{\mathbb{Z}^d}}.$$

If we use the linear forms L_l defined by (5) and (6) we then deduce

$$\|Ad(z_1 z_2 \dots z_k)|_{W_l}\| \leq C p^{\max_{1 \leq l \leq s} L_l(\tilde{z}_1 + \dots + \tilde{z}_k) + C \max_{1 \leq j \leq k} |\tilde{z}_j|_{\mathbb{Z}^d}}.$$

By the above considerations we have finally proved that

$$\|Ad(z_1 z_2 \dots z_k)\| \leq C p^{\max_{1 \leq l \leq s} L_l(\zeta_1 + \dots + \zeta_k) + C \max_{1 \leq j \leq n} |\zeta_j|_{\mathbb{Z}^d}},$$

$$\zeta_j = p(z_j), \quad z_j \in A \times S, \quad j = 1, \dots, k, \quad k = 1, 2, \dots$$

If we apply this estimate to $u_j^{z_1 z_2 \dots z_{j-1}} = \exp(Ad(z_1 \dots z_{j-1})v_j)$ where the u_j 's, v_j 's and z_j 's are as in (16), (17), (18) we then deduce

$$\|Ad(z_1 \dots z_{j-1})v_j\| \leq C p^{\max_{1 \leq l \leq s} L_l(\zeta_1 + \dots + \zeta_{j-1}) + C \max_{1 \leq k \leq j-1} |\zeta_k|_{\mathbb{Z}^d}},$$

$$\zeta_j = p(z_j), \quad j \geq 2.$$

Observe that the measure that controls the random walk $(S_j)_{j \in \mathbb{N}}$, defined by $S_j = \zeta_0 + \zeta_1 + \dots + \zeta_j$ ($\zeta_0 = 0$) is the symmetric measure $\check{\mu}$ defined by (8). The random variables ζ_j are in particular compactly supported and we have

$$(24) \quad \|Ad(z_1 \dots z_{j-1})v_j\| \leq C p^{\max_{1 \leq l \leq r} L_l(\zeta_1 + \dots + \zeta_j)}, \quad \zeta_j = p(z_j), \quad j \geq 2.$$

Let us consider, for $n = 1, 2, \dots$, the event E_n defined by

$$(25) \quad E_n = \left(L_l(\zeta_1 + \dots + \zeta_j) \leq C, \quad j = 1, \dots, n, \quad l = 1, \dots, s; \right. \\ \left. |\zeta_1 + \dots + \zeta_n|_{\mathbb{Z}^d} \leq C \sqrt{n} \right),$$

where $C > 0$ denotes an appropriate large constant. Using (15), Campbell-Hausdorff, (24) and the ultrametric property (19) we see that the event E_n verifies

$$E_n \subset [X_n \in B_n]$$

where B_n is defined by

$$B_n = \exp(\{u \in \mathcal{U}, \|u\| \leq C\}) \cdot \{z \in A \times S, |p(z)|_{\mathbb{Z}^d} \leq An^{1/2}\}.$$

It is easy to see that $d^r g$ shows that

$$(26) \quad |B_n| \leq Cn^{\frac{d}{2}}, \quad n = 1, 2, \dots$$

It remains to estimate the probability of the event (25). Let Π denote the polyhedral region in \mathbb{Z}^d defined by

$$\Pi = \{z \in \mathbb{Z}^d, L_j(z) \leq C, \quad j = 1, \dots, s\},$$

where C denotes the same constant as in (25). Let $h_n(x, y)$, $n = 1, 2, \dots$, $x, y \in \Pi$, denote the transition kernel corresponding to the random walk $(S_j)_{j \in \mathbb{N}}$ with killing outside of Π . Precise lower estimates for the kernel $h_n(x, y)$ can be obtained thanks to the results of [26] and [27]. To write down these estimates we need the following notations. Let us assume that $d \geq 2$ and let $0 < u_0(\sigma)$, $\sigma \in \Pi_\Sigma$, denote the eigenfunction corresponding to the first Dirichlet eigenvalue of the region Π_Σ defined by (10). Let $u(x)$ be the function defined on $\overline{\Pi}_\varphi$ by

$$u(x) = u(r, \sigma) = r^a u_0(\sigma), \quad x = (r, \sigma) \in \mathbb{R}_+^* \times \Pi_\Sigma$$

where

$$(27) \quad a = \frac{\sqrt{(d-2)^2 + 4\lambda} - (d-2)}{2},$$

and where (r, σ) denote the polar coordinates on $\mathbb{R}_+^* \times \Sigma$. The function u defined in this way is a positive function harmonic in Π_φ which vanishes on $\partial\Pi_\varphi$ (cf. [24]). In the case where $d = 1$, the function u is defined by $u(x) \equiv x$, $x \in \mathbb{R}_+^*$.

It follows easily from [27] (cf. estimate (2) p. 359) and [26] (cf. Theorem 1 and estimate (0.3.4)) that there exists $C > 0$ such that

$$h_n(0, y) \geq \frac{u(-y)}{Cn^{a+d/2}}, \quad |y| \leq C\sqrt{n}, \quad n \geq C.$$

We have therefore

$$\mathbb{P}(E_n) \geq \frac{1}{Cn^{a+d/2}} \sum_{y \in \Pi, |y| \leq C\sqrt{n}} u(-y).$$

Using the homogeneity of u we deduce then that

$$(28) \quad \mathbb{P}(E_n) \geq \frac{1}{Cn^{a+d/2}} \int_0^{C\sqrt{n}} r^{a+d-1} dr = \frac{1}{Cn^{a/2}}.$$

The lower estimate (11) is an immediate consequence of (13), (26), (27) and (28). This completes the proof of Theorem 1.

4 Metabelian p -adic groups

Our aim in this section is to show that in the case of metabelian groups the lower estimate (11) can be complemented with a similar upper bound. We keep the notation of §2. We shall denote by $\mathbb{Z}_p^* = \{x \in \mathbb{Q}_p^*, |x|_p = 1\}$ and by $\mathbb{Z}_p = \{x \in \mathbb{Q}_p, |x|_p \leq 1\}$. Let dx denote the Haar measure on \mathbb{Q}_p normalized by $dx(\mathbb{Z}_p) = 1$ and let d^*x denote the Haar measure on \mathbb{Q}_p^* normalized by $d^*x(\mathbb{Z}_p^*) = 1$. Let us fix $k, l \geq 2$ and consider

$$(29) \quad G = \mathbb{Q}_p^k \rtimes_{\sigma} \left(\mathbb{Q}_p^*\right)^l$$

the semi-direct product of $\left(\mathbb{Q}_p^*\right)^l$ with the vector space \mathbb{Q}_p^k where $\left(\mathbb{Q}_p^*\right)^l$ acts on \mathbb{Q}_p^k by

$$x = (x_1, \dots, x_k) \longrightarrow \sigma(y)x = (\chi_1(y)x_1, \dots, \chi_k(y)x_k), \quad y \in \left(\mathbb{Q}_p^*\right)^l,$$

where χ_1, \dots, χ_k denote k morphisms

$$(30) \quad \chi_1, \dots, \chi_k : \left(\mathbb{Q}_p^*\right)^l \longrightarrow \mathbb{Q}_p^*.$$

More precisely, we assume that the multiplication in G is given by

$$\begin{aligned} g \cdot g' &= (x; y) \cdot (x'; y') = (x + \sigma(y)x'; y \cdot y') \\ &= (x_1 + \chi_1(y)x'_1, x_2 + \chi_2(y)x'_2, \dots, x_k + \chi_k(y)x'_k; y_1 \cdot y'_1, y_2 \cdot y'_2, \dots, y_l \cdot y'_l) \\ g &= (x, y), \quad g' = (x', y') \in G; \quad x = (x_1, \dots, x_k), \quad x' = (x'_1, \dots, x'_k) \in \mathbb{Q}_p^k; \\ & \quad y = (y_1, \dots, y_l), \quad y' = (y'_1, \dots, y'_l) \in \left(\mathbb{Q}_p^*\right)^l. \end{aligned}$$

We shall denote by

$$d^r g = dx d^*y = dx_1 \dots dx_k d^*y_1 \dots d^*y_l; \quad d^l g = dg = \delta(g)^{-1} d^r g$$

the right and the left invariant Haar measure on G . The modular function $\delta(g)$ is given by

$$(31) \quad \delta(g) = \delta(y) = |\chi_1(y)| \dots |\chi_k(y)|, \quad g = (x, y) \in G.$$

Let $d\mu(g) = \varphi(g) d^r(g)$ denote the symmetric, compactly supported, probability measure on G defined by

$$(32) \quad \begin{aligned} \varphi(g) &= \alpha \delta^{-1/2}(g) I_{\mathbb{Z}_p}(x_1) \dots I_{\mathbb{Z}_p}(x_k) I_{\mathbb{Z}_p}(\chi_1(y)^{-1} x_1) \dots I_{\mathbb{Z}_p}(\chi_k(y)^{-1} x_k) \\ & \quad \times I_{(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_1) \dots I_{(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_l), \\ & \quad g = (x_1, \dots, x_k; y_1, \dots, y_l) \in G. \end{aligned}$$

The notation I_A is used here to denote the indicator of a subset A and $\alpha > 0$ denotes an appropriate positive constant.

We shall assume that the χ_j 's in (30) verify the condition

$$|\chi_j| \neq 1, \quad j = 1, \dots, k.$$

This guarantees (cf. §2) that the group G defined by (29) is compactly generated and it is easy to see that $\text{supp}(\mu)$ is a generating set of the group G . We shall denote, as in §2, by L_j , $j = 1, \dots, k$, the k linear forms on \mathbb{Z}^l induced by the χ_j 's and we shall assume that these linear forms induce a "Weyl chamber" $\Pi = \{x \in \mathbb{R}^l, \tilde{L}_j(x) > 0, j = 1, 2, \dots, k\}$ as in §2.. We shall denote by $\Sigma = \{x \in \mathbb{R}^l, |x| = 1\}$ the unit sphere in \mathbb{R}^l , by $\Pi_\Sigma = \Sigma \cap \Pi$ and by λ be the first Dirichlet eigenvalue of the region Π_Σ . We have then the following:

Theorem 2. *Let $G = \mathbb{Q}_p^k \rtimes_\sigma (\mathbb{Q}_p^*)^l$ and let $\mu \in \mathbf{P}(G)$ be as above. Let $d\mu^{*n}(g) = d(\mu * \dots * \mu)(g) = \varphi_n(g)d^r g$ denote the n^{th} convolution power of μ . Then there exists $C > 0$ such that*

$$\varphi_n(e) \leq \frac{C}{n^{1 + \frac{\sqrt{(l-2)^2 + 4\lambda}}{2}}}, \quad n = 1, 2, \dots$$

The first step to prove Theorem 2 consists in establishing an explicit formula for $\varphi_n(g)$, $g \in G$.

In what follows we shall use the notation $x_i = (x_{i,1}, \dots, x_{i,k})$ (resp $y_i = (y_{i,1}, \dots, y_{i,l})$) for $x_i \in \mathbb{Q}_p^k$, $i = 1, 2, \dots$ (resp. $y_i \in (\mathbb{Q}_p^*)^l$, $i = 1, 2, \dots$). Let us fix $g = (\xi, \zeta) = (\xi_1, \dots, \xi_k; \zeta_1, \dots, \zeta_l) \in G$ and $n = 1, 2, \dots$. By definition of convolution product we have:

$$\begin{aligned} \varphi_{n+1}(g) &= \int_G \varphi_n(h)\varphi_1(h^{-1}g)dh \\ &= \int_G \varphi_1(h^{-1}g)\delta(h^{-1})\varphi_n(h)d^r h \\ &= \int_G \varphi_1(h^{-1}g)\delta(h^{-1})d\mu^{*n}(h) \\ &= \int_G \dots \int_G \varphi_1\left((g_1 \dots g_n)^{-1}g\right)\delta(g_1 \dots g_n)^{-1}d\mu(g_1) \dots d\mu(g_n) \\ &= \int_G \dots \int_G \varphi_1\left(-(\sigma(y_1 \dots y_n)^{-1}x_1 + \sigma(y_2 \dots y_n)^{-1}x_2 + \dots + \sigma(y_n)^{-1}x_n) \right. \\ &\quad \left. + \sigma(y_1 \dots y_n)^{-1}\xi, (y_1 \dots y_n)^{-1}\zeta\right)\varphi_1(x_1, y_1) \dots \varphi(x_n, y_n) \\ &\quad \delta(y_1 \dots y_n)^{-1}dx_1 d^*y_1 \dots dx_n d^*y_n. \end{aligned}$$

An obvious change of variables combined with Fubini gives

$$(33) \quad \varphi_{n+1}(g) = \int_{(\mathbb{Q}_p^*)^l} \cdots \int_{(\mathbb{Q}_p^*)^l} \left[\int_{\mathbb{Q}_p^k} \cdots \int_{\mathbb{Q}_p^k} \left[\varphi_1 \left(- (x_1 + x_2 + \cdots + x_n) \right. \right. \right. \\ \left. \left. \left. + \sigma(y_1 \cdots y_n)^{-1} \xi, (y_1 \cdots y_n)^{-1} \zeta \right) \varphi_1(\sigma(y_1 \cdots y_n) x_1, y_1) \right. \right. \\ \left. \left. \varphi_1(\sigma(y_2 \cdots y_n) x_2, y_2) \cdots \varphi_1(\sigma(y_n) x_n, y_n) \right] dx_1 \cdots dx_n \right] \\ \prod_{i=2}^n |\chi_1(y_i \cdots y_n)| \cdots |\chi_k(y_i \cdots y_n)| d^* y_1 \cdots d^* y_n,$$

where we used (31). Let us consider the product

$$\prod_{i=1}^n \varphi_1(\sigma(y_i \cdots y_n) x_i, y_i)$$

that appears in equation (33). By (32), this product is equal to

$$\alpha^n \prod_{i=1}^n \delta^{-1/2}(y_i) \prod_{i=1}^n \left(I_{\mathbb{Z}_p}(\chi_1(y_i \cdots y_n) x_{i,1}) \cdots I_{\mathbb{Z}_p}(\chi_k(y_i \cdots y_n) x_{i,k}) \right) \\ \times \prod_{i=1}^{n-1} \left(I_{\mathbb{Z}_p}(\chi_1(y_{i+1} \cdots y_n) x_{i,1}) \cdots I_{\mathbb{Z}_p}(\chi_k(y_{i+1} \cdots y_n) x_{i,k}) \right) \\ \times I_{\mathbb{Z}_p}(x_{n,1}) \cdots I_{\mathbb{Z}_p}(x_{n,k}) \prod_{i=1}^n I_{(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_{i,1}) \cdots I_{(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_{i,l}) \\ = \alpha^n \prod_{i=1}^n \delta^{-1/2}(y_i) \prod_{i=1}^{n-1} \left(I_{|\chi_1(y_{i+1} \cdots y_n)| \max(1, |\chi_1(y_i)|) \mathbb{Z}_p}(x_{i,1}) \right. \\ \left. \cdots I_{|\chi_k(y_{i+1} \cdots y_n)| \max(1, |\chi_k(y_i)|) \mathbb{Z}_p}(x_{i,k}) \right) \\ I_{\max(1, |\chi_1(y_n)|) \mathbb{Z}_p}(x_{n,1}) \cdots I_{\max(1, |\chi_k(y_n)|) \mathbb{Z}_p}(x_{n,k}) \\ \prod_{i=1}^n I_{(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_{i,1}) \cdots I_{(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_{i,l}).$$

We also have

$$\varphi_1 \left(- (x_1 + x_2 + \cdots + x_n) + \sigma(y_1 \cdots y_n)^{-1} \xi, (y_1 \cdots y_n)^{-1} \zeta \right) \\ = \alpha \delta^{-1/2} \left((y_1 \cdots y_n)^{-1} \zeta \right) \\ \times I_{\max(1, |\chi_1(y_1 \cdots y_n) \zeta^{-1}|) \mathbb{Z}_p} \left(x_{1,1} + x_{2,1} + \cdots + x_{n,1} - \chi_1(y_1 \cdots y_n)^{-1} \xi_1 \right) \\ \cdots I_{\max(1, |\chi_k(y_1 \cdots y_n)^{-1} \zeta^{-1}|) \mathbb{Z}_p} \left(x_{1,k} + x_{2,k} + \cdots + x_{n,k} - \chi_k(y_1 \cdots y_n)^{-1} \xi_k \right) \\ \times I_{(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)} \left((y_{1,1} \cdots y_{n,1})^{-1} \zeta_1 \right) \cdots I_{(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)} \left((y_{1,l} \cdots y_{n,l})^{-1} \zeta_l \right).$$

On the other hand we shall set

$$(34) \quad m(y) = \alpha \delta^{-1/2}(y) \prod_{j=1}^k \min(1, |\chi_j(y)|) I_{(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_1) \dots I_{(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_l),$$

$$y = (y_1, \dots, y_l) \in (\mathbb{Q}_p^*)^l.$$

What motivates this notation is the fact that

$$(35) \quad \int_{\mathbb{Q}_p^k} \varphi(x, y) dx = m(y), \quad y \in (\mathbb{Q}_p^*)^l.$$

Setting

$$(36) \quad A(y_1, y_2, \dots, y_n) = m((y_1 \dots y_n)^{-1} \zeta) \prod_{j=1}^k \max(1, |\chi_j(y_1 \dots y_n) \zeta^{-1}|)$$

$$\times \prod_{i=1}^n \max(1, |\chi_1(y_i)|^{-1}) \dots \max(1, |\chi_k(y_i)|^{-1}) m(y_i),$$

$$(37) \quad B_j(x_1, x_2, \dots, x_n; y_1, y_2, \dots, y_n) = B_j(x_1, x_2, \dots, x_n)$$

$$= I_{\max(1, |\chi_j(y_1 \dots y_n) \zeta^{-1}|) \mathbb{Z}_p} \left(x_{1,j} + x_{2,j} + \dots \right.$$

$$\left. + x_{n,j} - \chi_j(y_1 \dots y_n)^{-1} \xi_j \right) I_{\max(1, |\chi_j(y_n)|) \mathbb{Z}_p}(x_{n,j})$$

$$\times \prod_{i=1}^{n-1} I_{|\chi_j(y_{i+1} \dots y_n)| \max(1, |\chi_j(y_i)|) \mathbb{Z}_p}(x_{i,j}),$$

$$j = 1, 2, \dots, k,$$

we rewrite (33) as

$$(38) \quad \varphi_{n+1}(g) = \int_{(\mathbb{Q}_p^*)^l} \dots \int_{(\mathbb{Q}_p^*)^l} \left[\int_{\mathbb{Q}_p^k} \dots \int_{\mathbb{Q}_p^k} \prod_{j=1}^k B_j(x_1, x_2, \dots, x_n) dx_1 \dots dx_n \right]$$

$$A(y_1, y_2, \dots, y_n) \prod_{i=2}^n |\chi_1(y_i \dots y_n)| \dots |\chi_k(y_i \dots y_n)|$$

$$d^* y_1 \dots d^* y_n.$$

The next step is to calculate

$$\int_{\mathbb{Q}_p} \dots \int_{\mathbb{Q}_p} B_j(x_1, x_2, \dots, x_n) dx_{1,j} dx_{2,j} \dots dx_{n,j}, \quad j = 1, \dots, k.$$

Towards this we observe that:

$$\begin{aligned}
& \int_{\mathbb{Q}_p} \dots \int_{\mathbb{Q}_p} I_{\max(1, |\chi_j(y_1 \dots y_n) \zeta^{-1}|)_{\mathbb{Z}_p}}(x_{1,j} + \dots + x_{n,j} - \chi_j(y_1 \dots y_n)^{-1} \xi_j) \\
& I_{\max(1, |\chi_j(y_n)|)_{\mathbb{Z}_p}}(x_{n,j}) \prod_{i=1}^{n-1} I_{|\chi_j(y_{i+1} \dots y_n)| \max(1, |\chi_j(y_i)|)_{\mathbb{Z}_p}}(x_{i,j}) dx_{1,j} dx_{2,j} \dots dx_{n,j} \\
& = \int_{\mathbb{Q}_p} \left(I_{|\chi_j(y_2 \dots y_n)| \max(1, |\chi_j(y_1)|)_{\mathbb{Z}_p}} * \dots * I_{|\chi_j(y_n)| \max(1, |\chi_j(y_{n-1})|)_{\mathbb{Z}_p}} * I_{\max(1, |\chi_j(y_n)|)_{\mathbb{Z}_p}} \right) (x) \\
& \quad I_{\max(1, |\chi_j(y_1 \dots y_n) \zeta^{-1}|)_{\mathbb{Z}_p}}(x - \chi_j(y_1 \dots y_n)^{-1} \xi_j) dx
\end{aligned}$$

where $*$ denotes the usual convolution in \mathbb{Q}_p (cf. [21]). Taking Fourier transform (cf. [21]) we obtain

$$\begin{aligned}
& \int_{\mathbb{Q}_p} \dots \int_{\mathbb{Q}_p} B_j(x_1, x_2, \dots, x_n) dx_{1,j} dx_{2,j} \dots dx_{n,j} \\
& = \text{Min} \left[\min_{1 \leq i \leq n-1} \left(|\chi_j(y_{i+1} \dots y_n)| \max(1, |\chi_j(y_i)|) \right); \max(1, |\chi_j(y_n)|) \right] \\
& \quad \min(1, |\chi_j(y_n)|^{-1}) \prod_{i=1}^{n-1} |\chi_j(y_{i+1} \dots y_n)|^{-1} \min(1, |\chi_j(y_i)|^{-1}) \\
& \quad \times \left(I_{\text{Min} \left[\min_{1 \leq i \leq n-1} (|\chi_j(y_{i+1} \dots y_n)| \max(1, |\chi_j(y_i)|)); \max(1, |\chi_j(y_i)|) \right]_{\mathbb{Z}_p}} * \right. \\
& \quad \left. I_{\max(1, |\chi_j(y_1 \dots y_n) \zeta^{-1}|)_{\mathbb{Z}_p}} \right) (\chi_j(y_1 \dots y_n)^{-1} \xi_j).
\end{aligned}$$

Let us denote by $S_j^\zeta(y_1, \dots, y_n) \subset \mathbb{Q}_p$, $y_1, \dots, y_n \in \mathbb{Q}_p^*$ the subset defined by

$$\begin{aligned}
S_j^\zeta(y_1, \dots, y_n) = & \text{Min} \left[\min_{1 \leq i \leq n-1} \left(|\chi_j(y_{i+1} \dots y_n)| \max(1, |\chi_j(y_i)|) \right); \right. \\
& \left. \max(1, |\chi_j(y_n)|) \right] \wedge \max(1, |\chi_j(y_1 \dots y_n) \zeta^{-1}|)_{\mathbb{Z}_p}.
\end{aligned}$$

With this notation we have

$$\begin{aligned}
(39) \quad & \int_{\mathbb{Q}_p} \dots \int_{\mathbb{Q}_p} B_j(x_1, x_2, \dots, x_n) dx_{1,j} dx_{2,j} \dots dx_{n,j} \\
& = \min \left(1; \text{Min} \left(\min_{1 \leq i \leq n-1} \left(|\chi_j(y_{i+1} \dots y_n)| \max(1, |\chi_j(y_i)|) \right); \right. \right. \\
& \quad \left. \left. \max(1, |\chi_j(y_n)|) \right) \min(1, |\chi_j(y_1 \dots y_n)^{-1} \zeta|) \right) \min(1, |\chi_j(y_n)|^{-1}) \\
& \quad \prod_{i=1}^{n-1} |\chi_j(y_{i+1} \dots y_n)|^{-1} \min(1, |\chi_j(y_i)|^{-1}) I_{S_j^\zeta(y_1, \dots, y_n)}(\chi_j(y_1 \dots y_n)^{-1} \xi_j),
\end{aligned}$$

$$j = 1, \dots, k.$$

Putting together (33), (36), (37), (38) and (39) and taking into account the fact that

$$\begin{aligned} & \prod_{j=1}^k \min(1, |\chi_j(y_n)|^{-1}) \prod_{i=1}^{n-1} \left(\prod_{j=1}^k \min(1, |\chi_j(y_i)|^{-1}) \right) \\ & \times \prod_{i=1}^n \max(1, |\chi_1(y_i)|^{-1}) \dots \max(1, |\chi_k(y_i)|^{-1}) \\ & = \prod_{j=1}^k |\chi_j(y_1 \dots y_n)|^{-1} = \delta^{-1}(y_1 \dots y_n), \end{aligned}$$

and that

$$\begin{aligned} & \delta^{-1}(y_1 \dots y_n) \prod_{j=1}^k \max(1, |\chi_j(y_1 \dots y_n) \zeta^{-1}|) m((y_1 \dots y_n)^{-1} \zeta) \\ & = \alpha \delta^{-1/2}(y_1 \dots y_n) \delta^{-1/2}(\zeta) I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}((y_{1,1} \dots y_{n,1})^{-1} \zeta_1) \\ & \quad \dots I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}((y_{1,l} \dots y_{n,l})^{-1} \zeta_l), \end{aligned}$$

we finally deduce the formula

$$\begin{aligned} (40) \quad \varphi_{n+1}(g) & = \alpha \delta^{-1/2}(\zeta) \int_{(\mathbb{Q}_p^*)^l} \dots \int_{(\mathbb{Q}_p^*)^l} \prod_{j=1}^k \text{Min} \left(1; \right. \\ & \quad \left. \min \left(\min_{1 \leq i \leq n-1} \left(|\chi_j(y_{i+1} \dots y_n)| \max(1, |\chi_j(y_i)|) \right) \right); \right. \\ & \quad \left. \max(1, |\chi_j(y_n)|) \right) \min(1, |\chi_j(y_1 \dots y_n)^{-1} \zeta|) \Big) \\ & \quad \delta^{-1/2}(y_1 \dots y_n) \prod_{j=1}^k I_{S_j^\zeta(y_1, \dots, y_n)}(\chi_j(y_1 \dots y_n)^{-1} \xi_j) \\ & \quad \prod_{j=1}^l I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}((y_{1,j} \dots y_{n,j})^{-1} \zeta_j) \\ & \quad m(y_1) \dots m(y_n) d^* y_1 \dots d^* y_n. \end{aligned}$$

The next step is to give a probabilistic interpretation of (40). Towards that let us consider a sequence of independent identically $(\mathbb{Q}_p^*)^l$ -valued random variables Y_1, Y_2, \dots with distribution on $(\mathbb{Q}_p^*)^l$ given by

$$(41) \quad \mathbb{P}[Y_j \in dy] = m(y) d^* y$$

where $m(y)$ is as in (34). Observe that by (35) we have $\int m(y)d^*y = 1$. The formula (40) can be rewritten

$$\begin{aligned} \varphi_{n+1}(g) = & \alpha\delta^{-1/2}(\zeta)\mathbb{E}\left[\prod_{j=1}^k \text{Min}\left(1; \min\left(\min_{1\leq i\leq n-1}\left(|\chi_j(Y_{i+1}\dots Y_n)| \max(1, |\chi_j(Y_i)|)\right)\right.\right.\right. \\ & \left.\left.\left.\max(1, |\chi_j(Y_n)|)\right)\right) \min(1, |\chi_j(Y_1\dots Y_n)^{-1}\zeta|)\right) \\ & \delta^{-1/2}(Y_1\dots Y_n)\prod_{j=1}^k I_{S_j^\zeta(Y_1,\dots,Y_n)}(\chi_j(Y_1\dots Y_n)^{-1}\xi_j) \\ & \prod_{j=1}^l I_{(p^{-1}\mathbb{Z}_p^*\cup\mathbb{Z}_p^*\cup p\mathbb{Z}_p^*)}((Y_{1,j}\dots Y_{n,j})^{-1}\zeta_j)\right]. \end{aligned}$$

In the case $g = e$ the above formula simplifies considerably and we obtain

$$\begin{aligned} \varphi_{n+1}(e) \leq & C\mathbb{E}\left(\prod_{j=1}^k \min\left[1, \min_{1\leq i\leq n} |\chi_j(Y_i\dots Y_n)|\right]\right) \\ & \times \prod_{j=1}^l I_{(p^{-1}\mathbb{Z}_p^*\cup\mathbb{Z}_p^*\cup p\mathbb{Z}_p^*)}((Y_{1,j}\dots Y_{n,j})^{-1}). \end{aligned}$$

Now we project the random walk $Y_1\dots Y_n$ controlled by (41) on $\mathbb{Z}^l \subset \mathbb{R}^l$ and we assume that \mathbb{R}^l is equipped with the Euclidean structure ensuring (9). We shall denote by $S_n = X_1 + X_2 + \dots + X_n$ ($n \geq 1$) the random walk obtained via this projection. We have

$$\varphi_{n+1}(e) \leq C\mathbb{E}\left(\prod_{j=1}^k \min_{1\leq i\leq n} p^{L_j(S_n - S_i)}, S_n \in K\right), \quad n = 1, 2, \dots$$

where K denotes a neighbourhood of the origin in \mathbb{Z}^l . It follows then that

$$\varphi_{n+1}(e) \leq C\mathbb{E}\left(\prod_{j=1}^k p^{-\max_{1\leq i\leq n} L_j(S_i)}, S_n \in K\right), \quad n = 1, 2, \dots$$

and therefore

$$(42) \quad \varphi_{n+1}(e) \leq C \sum_{\lambda_1 \geq 0, \dots, \lambda_k \geq 0} p^{-\sum_{i=1}^k \lambda_i} \mathbb{P}\left(\max_{0 \leq i \leq n} L_1(S_i) = \lambda_1, \dots, \max_{0 \leq i \leq n} L_k(S_i) = \lambda_k; S_n \in K\right), \quad n = 1, 2, \dots$$

Let us now fix $x_0 \in \Pi$ such that

$$L_j(x_0) > 0, \quad j = 1, \dots, k.$$

Let

$$\lambda_0 = \min_{1 \leq j \leq k} L_j(x_0).$$

We have

$$L_j(u_0) = L_j(x_0/\lambda_0) \geq 1.$$

It follows then from (42) that

$$\begin{aligned} \varphi_{n+1}(e) &\leq C \sum_{\lambda_1 \geq 0, \dots, \lambda_k \geq 0} p^{-\sum_{i=1}^k \lambda_i} \mathbb{P} \left(L_1(S_i) \leq \lambda_1 L_1(u_0), \dots, L_k(S_i) \leq \lambda_k L_k(u_0), \right. \\ &\quad \left. 0 \leq i \leq n; \quad S_n \in K \right) \\ &\leq C \sum_{\lambda_1 \geq 0, \dots, \lambda_k \geq 0} p^{-\sum_{i=1}^k \lambda_i} \mathbb{P} \left(L_1(S_i - (\max_{1 \leq i \leq k} \lambda_i + C)u_0) \leq 0, \dots, \right. \\ &\quad \left. L_k(S_i - (\max_{1 \leq i \leq k} \lambda_i + C)u_0) \leq 0, \quad 0 \leq i \leq n; \quad S_n \in K \right). \end{aligned}$$

Choosing $C > 0$ large enough, we can apply Theorem 4 of [26] and we deduce

$$\begin{aligned} \varphi_{n+1}(e) &\leq \frac{C}{n^{1 + \frac{\sqrt{(l-2)^2 + 4\lambda}}{2}}} \sum_{\lambda_1 \geq 0, \dots, \lambda_k \geq 0} \left(\max_{1 \leq i \leq k} \lambda_i + 1 \right)^C p^{-\sum_{i=1}^k \lambda_i} \\ &\leq \frac{C}{n^{1 + \frac{\sqrt{(l-2)^2 + 4\lambda}}{2}}} \sum_{\lambda_1 \geq 0, \dots, \lambda_k \geq 0} \left(\sum_{i=1}^k \lambda_i + 1 \right)^C p^{-\sum_{i=1}^k \lambda_i} \\ &\leq \frac{C}{n^{1 + \frac{\sqrt{(l-2)^2 + 4\lambda}}{2}}}. \end{aligned}$$

This completes the proof of Theorem 2.

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