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CUT TIMES FOR SIMPLE RANDOM WALK

Gregory F. Lawler

Department of Mathematics
Box 90320
Duke University
Durham, NC 27708-0320

jose@math.duke.edu

Abstract: Let $S(n)$ be a simple random walk taking values in Z^d. A time n is called a cut time if

$$S[0, n] \cap S[n + 1, \infty) = \emptyset.$$

We show that in three dimensions the number of cut times less than n grows like $n^{1-\zeta}$ where $\zeta = \zeta_d$ is the intersection exponent. As part of the proof we show that in two or three dimensions

$$P\{S[0, n] \cap S[n + 1, 2n] = \emptyset\} \asymp n^{-\zeta},$$

where $\asymp$ denotes that each side is bounded by a constant times the other side.

Keywords: Random walk, cut points, intersection exponent

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Cut Times
for Simple Random Walk

Gregory F. Lawler [1]
Department of Mathematics
Box 90320
Duke University
Durham, NC 27708-0320

1 Introduction

Let $S(j)$ be a simple random walk taking values in $\mathbf{Z}^d$. An integer n is called a *cut time* for S if

$$S[0,n] \cap S[n+1,\infty) = \emptyset,$$

where $S[0,n] = \{S(j) : 0 \leq j \leq n\}$. If $d \leq 2$, then with probability one the path has no cut times. However, if $d \geq 3$, the path has cut times with positive probability. In fact, with probability one the paths have infinitely many cut times. This can be proved by considering the random time

$$\xi_n = \inf\{j : |S(j)| \geq n\},$$

and showing that with probability one ξ_n is a cut time for infinitely many values of n (see [8] for details). In this paper we show that the number of cut times along a path is uniform at least up to logarithms. The emphasis will be on $d = 3$ because this is the most difficult, but we start with a quick discussion of higher dimensions. Let J_n be the indicator function of the event $\{S[0,n] \cap S[n+1,\infty) = \emptyset\}$ and let $R_n = R(n) = \sum_{j=0}^{n} J_j$.

If $d \geq 5$ (see [9]), then

$$\lim_{n\to\infty} \mathbf{P}\{S[0,n] \cap S[n+1,\infty) = \emptyset\} = p = p(d) > 0.$$

One can show that with probability one

$$\lim_{n\to\infty} \frac{1}{n} R_n = p. \tag{1}$$

Perhaps the easiest way to see this is to introduce a second simple walk $\tilde{S}$, independent of S, and let $\bar{S}$ be the "two-sided" walk

$$\bar{S}(j) = \begin{cases} S(j), & j \geq 0, \\ \tilde{S}(-j), & j \leq 0. \end{cases}$$

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If $\bar{J}_n$ is the indicator function of the event $\{\bar{S}(-\infty, n] \cap \bar{S}[n+1, \infty) = \emptyset\}$, then $\bar{J}_n$ is a stationary process. The ergodic theorem [2, Theorem 6.21] states that

$$\frac{1}{n} \sum_{j=0}^{n} \bar{J}_j \to p$$

with probability one, and from this it is not difficult to conclude (1).

For $d = 4$ [10] there is a constant $c_3 > 0$ such that

$$\mathbf{P}\{S[0, n] \cap S[n+1, \infty) = \emptyset\} \sim c_3 (\ln n)^{-1/2}.$$

(In this paper, we use $c, c_1, c_2, \ldots$ to denote arbitrary constants that depend only on the dimension d. The values of c, c_1, c_2 may change from place to place, but the values of $c_3, c_4, \ldots$ will not change.) Therefore,

$$\mathbf{E}(R_n) \sim c_3 n (\ln n)^{-1/2}.$$

By the methods in [9, Chapter 7], it can be proved that $(c_3 n)^{-1} (\ln n)^{1/2} R_n$ converges in probability to the constant 1. However, the convergence is not with probability one. There exists a $c > 0$ [9, Theorem 4.3.5] such that for all sufficiently large n,

$$\mathbf{P}\{S[\frac{n}{4}, \frac{n}{2}] \cap S[n+1, 2n) \neq \emptyset\} \geq c (\ln n)^{-1}.$$

By considering $n = 2^k$, we can easily see that this implies

$$\mathbf{P}\{S[\frac{n}{4}, \frac{n}{2}] \cap S[n+1, 2n) \neq \emptyset \text{ i.o.}\} = 1.$$

But clearly $R_n = R_{n/2}$ on the event $\{S[n/4, n/2] \cap S[n+1, 2n) \neq \emptyset\}$, and hence convergence with probability one is impossible. However it can be shown that with probability one

$$\lim_{n \to \infty} \frac{\ln R_n}{\ln n} = 1.$$

(There are a number of ways to do this. One way is to use an argument similar to the one in the final section of this paper for $d = 3$.)

For the remainder of this paper we will consider $d \leq 3$. As $n \to \infty$ [3],

$$\mathbf{P}\{S[0, n] \cap S[n+1, 2n] = \emptyset\} \approx n^{-\zeta}, \tag{2}$$

where $\approx$ denotes that the logarithms of both sides are asymptotic and $\zeta = \zeta_d$ is the intersection exponent. The intersection exponent is defined by taking independent Brownian motions $B^1(t), B^2(t)$ starting distance one apart and defining ζ by

$$\mathbf{P}\{B^1[0, n] \cap B^2[0, n] = \emptyset\} \approx n^{-\zeta}, \quad n \to \infty.$$

It is not too difficult to show that such a ζ exists for Brownian motion although it takes more work to show that (2) holds. Cranston and Mountford [6] have shown that (2) holds for all mean zero, finite variance, truly d-dimensional random walks. It is a standard estimate that $\zeta_1 = 1$. The values of ζ_2 and ζ_3 are unknown. The best rigorous estimates [4] are

$$\zeta_2 \in [\frac{1}{2} + \frac{1}{8\pi}, \frac{3}{4}), \quad \zeta_3 \in [\frac{1}{4}, \frac{1}{2}).$$

Duplantier and Kwon [7] have conjectured from a nonrigorous conformal field theory argument that $\zeta_2 = 5/8$. This value agrees with simulations [5, 13], and simulations suggest that ζ_3 is between .28 and .29.

One of the goals of this paper is to improve the convergence in (2). We show that for $d = 2, 3$, there are constants c_4, c_5 such that for all n

$$c_4 n^{-\zeta} \leq \mathbf{P}\{S[0,n] \cap S[n+1, 2n] = \emptyset\} \leq c_5 n^{-\zeta}, \tag{3}$$

and for $d = 3$,

$$c_4 n^{-\zeta} \leq \mathbf{P}\{S[0,n] \cap S[n+1, \infty) = \emptyset\}.$$

The relation (3) also holds for $d = 1$, but this is a well known result for random walks related to the "gambler's ruin" estimate. Let

$$\xi_n = \inf\{j : |S(j)| \geq n\},$$

and let $K_{j,n}$ be the indicator function of the event

$$\{S[0,j] \cap S[j+1, \xi_n] = \emptyset\}.$$

Let

$$Q_n = \sum_{j=0}^{\xi_n - 1} K_{j,n}.$$

We will prove the following two theorems.

Theorem 1.1 *If $d = 2, 3$, there exists a $c_6 > 0$ such that for all n*

$$\mathbf{P}\{Q_n \geq c_6 n^{2(1-\zeta)}\} \geq c_6.$$

Theorem 1.2 *If $d = 3$, with probability one,*

$$\lim_{n \to \infty} \frac{\ln R_n}{\ln n} = 1 - \zeta.$$

We expect that $n^{\zeta - 1} R_n$ converges in distribution to a nondegenerate random variable, but we have no proof of this. The main technical tool in the proofs of Theorems 1.1 and 1.2 is the estimate (3). Let B^1, B^2 be independent Brownian motions in $\mathbf{R}^d$ ($d = 2, 3$) and let

$$T_n^i = \inf\{t : |B^i(t)| = n\}.$$

In [11] it was shown that there exist constants c_1, c_2 such that

$$c_1 n^{-2\zeta} \leq \sup_{|x|,|y|=1} \mathbf{P}^{x,y}\{B^1[0, T_n^1] \cap B^2[0, T_n^2] = \emptyset\} \leq c_2 n^{-2\zeta}, \tag{4}$$

where $\mathbf{P}^{x,y}$ indicates probabilities assuming $B^1(0) = x, B^2(0) = y$. The fact that the probability of no intersection is logarithmicly asymptotic to $n^{-2\zeta}$ follows easily from subadditivity and scaling. The importance of the above result is that the probability equals $n^{-2\zeta}$ up to a multiplicative constant. In this paper we prove the analogue of this for random walk.

Theorem 1.3 *There exist constants c_7, c_8 such that if S^1, S^2 are simple random walks starting at the origin in $\mathbf{Z}^d$ $(d = 2, 3)$,*

$$c_7 n^{-2\zeta} \leq \mathbf{P}\{S^1[0, \xi_n] \cap S^2(0, \xi_n] = \emptyset\} \leq c_8 n^{-2\zeta}, \tag{5}$$

$$c_7 n^{-\zeta} \leq \mathbf{P}\{S^1[0, n] \cap S^2(0, n] = \emptyset\} \leq c_8 n^{-\zeta}. \tag{6}$$

Since (5) is the key estimate in this paper, let us describe briefly the idea used in the proof. We use the standard Skorohod construction to define simple random walks S^1, S^2 and Brownian motions B^1, B^2 on the same probability space so that with high probability, the paths of S^i are very close to those of B^i. We have a good estimate, (4), for the probability that the Brownian motions do not intersect. The lower bound in (5) is the easier esimate. We first show that Brownian motions conditioned not to intersect have a good chance of being reasonably far apart, and conclude that the corresponding simple walks are also far apart (and hence do not intersect).

The upper bound is somewhat trickier. We first need to prove some estimates that say intuitively "random walks that get close are very likely to intersect." If we were only interested in $d = 2$, we could skip these estimates and rely on the discrete Beurling estimate (see [9, Theorem 2.5.2]); however, we need to do the work for $d = 3$. Let

$$b_n = \mathbf{P}\{S^1[0, \xi^1_{2^n}] \cap S^2(0, \xi^2_{2^n}] = \emptyset\}.$$

(We actually use a slightly different definition of b_n in the proof, but this definition will do for the heuristic description.) We give an inequality for b_n in terms of $b_j, j < n$. We do this by considering the Brownian motions B^1, B^2 associated with the random walks. Either the Brownian motions do not intersect (we can estimate the probability of this using (4)), or there is a smallest j such that the Brownian motions do not have any intersection after reaching the sphere of radius 2^j. The probability that the random walks do not intersect and that a given j is the smallest index as above is bounded essentially by the product of: the probabilty that the random walks do not intersect up to the ball of radius 2^{j-1}; the probabilty that between $\xi^i_{2^{j-1}}$ and $\xi^i_{2^j}$, the Brownian motions intersect but the random walks do not; and the probability that the Brownian motions do not intersect after hitting the sphere of radius 2^j. The last probability can be estimated easily using (4) and Brownian scaling.

With little more than Theorem 1.3, we are able to give moment estimates

$$\mathbf{E}(Q_n) \geq c_1 n^{2(1-\zeta)},$$

$$\mathbf{E}(Q_n^2) \leq c_2 n^{4(1-\zeta)}.$$

If X is any nonnegative random variable with $\mu = \mathbf{E}(X)$, then

$$\begin{aligned}\mathbf{E}(X^2) &\geq \mathbf{P}\{X > \mu/2\}\mathbf{E}(X^2 \mid X > \mu/2) \\ &\geq \mathbf{P}\{X > \mu/2\}[\mathbf{E}(X \mid X > \mu/2)]^2 \\ &\geq \frac{[\mathbf{E}(X; X > \mu/2)]^2}{\mathbf{P}\{X > \mu/2\}} \\ &\geq \frac{(\mu/2)^2}{\mathbf{P}\{X > \mu/2\}},\end{aligned}$$

and hence

$$\mathbf{P}\{X > \mu/2\} \geq \frac{\mu^2}{4\mathbf{E}(X^2)}.$$

Hence the moment estimates immediately give Theorem 1.1.

The paper is organized as follows. Section 2 gives some preliminary lemmas about Brownian motions and simple random walks. In particular, it is shown that Brownian motions that are conditioned not to intersect are likely to stay a good distance apart. In Section 3 we review the strong approximation of Brownian motion by simple random walk derived from the Skorokhod embedding. This is a well known construction; however, it is useful to describe the construction here. The proof of Theorem 1.3 is given in Section 4. The idea of the proof is similar to that in [3, 6, 12]; however, things must be done somewhat more carefully to make sure that the estimates can be done up to multiplicative constants. The last section contains the proofs of Theorems 1.1 and 1.2. I would like to thank the referee and Chad Fargason for corrections to an earlier version this paper. This paper was written while the author was visiting the University of British Columbia.

2 Preliminary Results

In this section we prove some lemmas about Brownian motion and simple random walk. Let $d = 2$ or 3, and let B^1, B^2 be independent Brownian motions in $\mathbf{R}^d$ starting at x, y respectively with $|x| = |y| = 1$. We start by stating the main estimate from [11]. Let

$$T_n^i = \inf\{t : |B^i(t)| = n\},$$

and write $\mathbf{P}^{x,y}$ to denote probabilities assuming $B^1(0) = x, B^2(0) = y$. Let $\mathcal{B}(x, r)$ denote the open ball of radius r about x. Let A_n denote the event

$$A_n = \{B^1[0, T_n^1] \cap B^2[0, T_n^2] = \emptyset\}.$$

Lemma 2.1 *[11] There exists a $c_9 < \infty$ and an increasing function $f : (0, 2] \to (0, \infty)$ such that if $|x| = |y| = 1$, then for all $n \geq 1$*

$$f(|x - y|)n^{-2\zeta} \leq \mathbf{P}^{x,y}(A_n) \leq c_9 n^{-2\zeta}.$$

It was shown in [11] (see Corollary 3.11, Corollary 3.12) that Brownian paths conditioned not to intersect have a reasonable probability of being not too close together at the endpoints, i.e., there is an $\epsilon > 0$ such that the conditional probability that

$$\text{dist}(B^i(T_n^i), B^{3-i}[0, T_n^{3-i}]) \geq \epsilon n, \quad i = 1, 2,$$

given A_n is at least ϵ. If we take Brownian paths until they reach distance $n/4$ and condition them to have no intersection up to that time, then with probability at least ϵ the distance between $B^i(T_{n/4}^i)$ and $B^{3-i}[0, T_{n/4}^{3-i}]$ will be at least $\epsilon n/4$. We can now continue the paths up through distance n and we can be sure that there is a positive probability (independent of n) that the paths will separate. In fact we can condition the paths from $T_{n/4}^i$ to T_n^i to do almost anything that has a positive probability (independent of n) of occuring. This idea can be used to prove the next lemma.

Lemma 2.2 *[11, Corollary 3.11, Corollary 3.12] Let*

$$\mathcal{H}_n = \{(x^1, \ldots, x^d) \in \mathbf{R}^d : x^1 \geq \frac{n}{4}, |x| \leq n\},$$

$$-\mathcal{H}_n = \{x \in \mathbf{R}^d : -x \in \mathcal{H}_n\}.$$

Let

$$G_n = \{B^1[0, T_n^1] \subset \mathcal{B}(0, \frac{n}{2}) \cup \mathcal{H}_n; B^2[0, T_n^2] \subset \mathcal{B}(0, \frac{n}{2}) \cup -\mathcal{H}_n\},$$

$$F_n = \{ \operatorname{dist}(B^i[T_{n/2}^i, T_n^i], B^{3-i}[0, T_n^{3-i}]) \geq \frac{n}{10}, \ \ i = 1, 2\}.$$

For any $\rho > 0$, let

$$E_n = E_n(\rho) = \{B^i[0, T_n^i] \cap \mathcal{B}(0, 1) \subset \mathcal{B}(B^i(0), \rho), \ i = 1, 2\}.$$

Then for every $\rho > 0$ there is a $u > 0$ such that for all $n \geq 8$, and all $|x| = |y| = 1$ with $|x - y| \geq 2\rho$,

$$\mathbf{P}^{x,y}(A_n \cap E_n \cap F_n \cap G_n) \geq u n^{-2\zeta}.$$

The next lemmas are needed to formalize the statement "if two Brownian motions or two random walks get close to each other then they are likely to intersect." If we were only interested in $d = 2$, we would not need these lemmas, but rather could use the Beurling projection theorem, either continuous or discrete (see [1] for the continuous version and [9] for the discrete version). However, there is no useful analogue of this theorem for $d = 3$. Since the proofs below work equally well for two or three dimensions, we will just use these lemmas and not bother with the Beurling estimates. Let B be a third Brownian motion independent of B^1, B^2 and let

$$T_n = \inf\{t : |B_t| = n\}.$$

Let

$$Y_n^i = \sup_{|z| \leq 1} \mathbf{P}^z\{B[0, T_n] \cap B^i[0, T_n^i] = \emptyset \mid B^i[0, T_n^i]\},$$

where $\mathbf{P}^z$ denotes probabilities assuming $B(0) = z$. This notation is a little ambiguous; since we will use similar notation below, let us clarify. We should just write

$$Y_n^i = \sup_{|z| \leq 1} \mathbf{P}^z\{B[0, T_n] \cap B^i[0, T_n^i] = \emptyset\}.$$

However we choose the conditional expectation notation to emphasize that the $\mathbf{P}^z$ refers to B and that Y_n^i is a function of the path $B^i[0, T_n^i]$. The first lemma was proved in [11].

Lemma 2.3 *[11, Lemma 3.4] For every $M < \infty$, there exist $\delta > 0$ and $a < \infty$ such that if $|x| \leq 1$,*

$$\mathbf{P}^x\{Y_n^i \geq n^{-\delta}\} \leq a n^{-M}.$$

Lemma 2.4 *For every $\epsilon > 0$, $b < \infty$, let*

$$Z_n^i = Z_n^i(\epsilon, b) = \sup \mathbf{P}^z\{B[0, T_{2n}] \cap B^i[0, T_{2n}^i] = \emptyset \mid B^i[0, T_{2n}^i]\},$$

where the supremum is over all z with $|z| \leq n$ and

$$\operatorname{dist}(z, B^i[0, T_{2n}^i]) \leq b n^{1-\epsilon}.$$

Then for every $M < \infty, \epsilon > 0, b < \infty$, there exist $\delta > 0$ and $a < \infty$ such that for $|x| \leq n$,

$$\mathbf{P}^x\{Z_n^i \geq n^{-\delta}\} \leq a n^{-M}.$$

Proof. We will assume $i = 1$ and write Z_n for Z_n^1. Without loss of generality we will assume that $b \geq 1, \epsilon < 1/2$. Cover the ball of radius n by $K = K_n \leq cn^3$ balls of radius 1, $V_1, \ldots, V_K$. For $j = 1, \ldots, K$, let

$$\tau_j = \inf\{t : B^1(t) \in V_j\}.$$

Let

$$G_j = G_j(n, b, \epsilon) = \sup_{|z - B^1(\tau_j)| \leq 4bn^{1-\epsilon}} P^z\{B[0, T_{2n}] \cap B^1[\tau_j, T_{2n}^1] = \emptyset \mid B^1[0, T_{2n}^1]\}.$$

By Lemma 2.3, the strong Markov property, and Brownian scaling, we can find a δ and an a such that for each j

$$\mathbf{P}^x\{G_j \geq n^{-\delta}, \tau_j < T_{2n}^1\} \leq an^{-M-3}.$$

Let

$$\tilde{G} = \tilde{G}(n, b, \epsilon) = \sup G_j,$$

where the supremum is over all j with $\tau_j < T_{2n}^1$. Since $K_n \leq cn^3$,

$$P\{\tilde{G} \geq n^{-\delta}\} \leq acn^{-M}.$$

But every z with $|z| \leq n$ and $\mathrm{dist}(z, B^1[0, T_{2n}^1]) \leq bn^{1-\epsilon}$ is within distance $bn^{1-\epsilon} + 2 \leq 4bn^{1-\epsilon}$ of $B^1(\tau_j)$ for some j with $\tau_j < T_{2n}^1$. Hence $Z_n \leq \tilde{G}$. $\square$

We will need the corresponding results for simple random walk. Let S^1, S^2 denote independent simple random walks in $\mathbf{Z}^d$ and let

$$\xi_n^i = \inf\{j : |S^i(j)| \geq n\}.$$

Let S be another simple random walk independent of S^1, S^2 and let ξ_n denote the corresponding stopping time for S. For any $m < n$, let

$$X^i(m, n) = \sup_{|z| \leq m} \mathbf{P}^z\{S[0, \xi_n] \cap S^i[0, \xi_n^i] = \emptyset \mid S^i[0, \xi_n]\}.$$

Here $\mathbf{P}^z$ denotes probabilities assuming $S(0) = z$ and $X^i(m, n)$ is considered as a function of $S^i[0, \xi_n^i]$.

Lemma 2.5 *For every $M < \infty$ there exist $\delta > 0$ and $a < \infty$ such that if $|S^i(0)| \leq m$,*

$$\mathbf{P}\{X^i(m, n) \geq (\frac{m}{n})^\delta\} \leq a(\frac{m}{n})^M.$$

Proof. We will assume $i = 1$. Assume $k \geq m$ and let

$$Z(k) = \inf_{|z| \leq k} \mathbf{P}^z\{S[0, \xi_{2k}] \cap S^1[\xi_k^1, \xi_{2k}^1] \neq \emptyset \mid S^1[\xi_k^1, \xi_{2k}^1]\},$$

where as before $\mathbf{P}^z$ denotes probabilities assuming $S(0) = z$ and $Z(k)$ is a function of $S^1[\xi_k, \xi_{2k}]$. We claim that for every $\epsilon > 0$ there is a $\delta > 0$ such that for all k,

$$\mathbf{P}\{Z(k) < \delta \mid S^1(j), 0 \leq j \leq \xi_k\} \leq \epsilon. \tag{7}$$

Once we have (7), the proof proceeds identically to the proof of Lemma 2.3, so we will only prove (7).

By [9, Theorem 3.3.2], if $S^1(0)=0$, there is a $u_1>0$ such that if $|z|\leq 5n/4$,

$$\mathbf{P}^z\{S[0,\xi_{2n}]\cap S^1[0,\xi^1_{n/2}]\neq\emptyset\}\geq u_1. \tag{8}$$

Let

$$Y^z=\mathbf{P}^z\{S[0,\xi_{2n}]\cap S^1[0,\xi^1_{n/2}]\neq\emptyset\mid S^1[0,\xi^1_{n/2}]\},$$
$$Y=\inf_{3n/4\leq|z|\leq 5n/4} Y^z.$$

By the discrete Harnack inequality [9, Theorem 1.7.2], there is a $u_2>0$ such that for all $3n/4\leq|z|\leq 5n/4$,

$$Y\geq u_2 Y^z.$$

Also (8) implies for all $|z|\leq 5n/4$,

$$\mathbf{P}\{Y^z\geq\frac{u_1}{2}\}\geq\frac{u_1}{2}.$$

Hence if $\beta=u_1u_2/2$,

$$\mathbf{P}\{Y\geq\beta\}\geq\frac{u_1}{2}.$$

For any positive integer j, let $r_i=r_{i,j}=1+(i/4j)$, $i=1,\ldots,j$. Let $x_i=x_{i,j,k}=S^1(\xi^1_{r_ik})$, $\delta=\delta_{j,k}=k/(8j)$. Define

$$\xi_s(x)=\inf\{l:|S(l)-x|\geq s\},$$

and similarly for $\xi^1_s(x)$. Let

$$Y_i=Y_{i,j,k}=\inf_{3\delta/4\leq|z-x_i|\leq 5\delta/4}\mathbf{P}^z\{S[0,\xi_{2\delta}(x_i)]\cap S^1[\xi^1_{r_ik},\xi^1_{\delta/2}(x_i)]\neq\emptyset\mid S^1[0,\xi^1_{\delta/2}(x_i)]\}.$$

Then $Y_1,\ldots,Y_j$ are independent, identically distributed, independent of $\{S^1(j);0\leq j\leq\xi_k\}$, and

$$\mathbf{P}\{Y_i\geq\beta\}\geq\frac{u_1}{2}.$$

Hence

$$\mathbf{P}\{\sup\{Y_1,\ldots,Y_j\}<\beta\}\leq(1-\frac{u_1}{2})^j.$$

However, if $Y_i\geq\beta$ for some i, the strong Markov property gives that $Z(k)\geq\lambda_j\beta$, where

$$\lambda_j=\inf_k\inf_{|x|\leq 3k/2}\mathbf{P}^0\{S[0,\xi_{2k}]\cap\mathcal{B}(x,\delta)\neq\emptyset\}.$$

By a standard estimate (using, e.g., the invariance principle), $\lambda_j>0$ and hence

$$\mathbf{P}\{Z(k)<\lambda_j\beta\}\leq(1-\frac{u_1}{2})^j\quad\square$$

The following can be concluded from Lemma 2.5 in the same way that Lemma 2.4 was concluded from Lemma 2.3.

Lemma 2.6 *For every $\epsilon > 0$, $b < \infty$, let*

$$Z_n^i = Z_n^i(\epsilon, b) = \sup \mathbf{P}^z\{S[0,\xi_{2n}] \cap S^i[0,\xi_{2n}^i] = \emptyset \mid S^i[0,\xi_{2n}^i]\},$$

where the supremum is over all z with $|z| \leq n$ and

$$\mathrm{dist}(z, S^i[0,\xi_{2n}^i]) \leq bn^{1-\epsilon}.$$

Then for every $M < \infty, \epsilon > 0, b < \infty$, there exist $\delta > 0$ and $a < \infty$ such that if $|x| \leq n$,

$$\mathbf{P}^x\{Z_n^i \geq n^{-\delta}\} \leq an^{-M}.$$

In the next two lemmas we prove that two Brownian motions, conditioned to avoid each other, actually stay a reasonable distance apart. For positive integer n we let A^n be the event

$$A^n = A_{2^n} = \{B^1[0,T^1_{2^n}] \cap B^2[0,T^2_{2^n}] = \emptyset\}.$$

Lemma 2.7 *Let $D_j^i = D_j^i(b,\epsilon)$ be the event*

$$D_j^i = \{|B^i(s) - B^{3-i}(t)| \leq b(2^j)^{1-\epsilon} \text{ for some } s \in [T^i_{2^{j-1}}, T^i_{2^j}], t \in [0, T^{3-i}_{2^j}]\}.$$

Then for every $\epsilon > 0, b < \infty$, there exist $\delta > 0$ and $a < \infty$ such that if $|x|, |y| \leq 2^m$, and $m < j \leq n$,

$$\mathbf{P}^{x,y}(A^{n+1} \cap D_j^i) \leq a(2^j)^{-\delta}(2^{n-m})^{-2\zeta}.$$

Proof. We will assume $i = 1$ and let $D_j = D_j^1$. Assume $|x|, |y| \leq 2^m$ and let

$$\tau = \tau(j,b,\epsilon) = \inf\{s \geq T^1_{2^{j-1}} : |B^1(s) - B^2(t)| \leq b(2^j)^{1-\epsilon} \text{ for some } t \leq T^2_{2^j}\},$$

$$\sigma = \sigma(j,b,\epsilon) = \inf\{t : |B^2(t) - B^1(\tau)| \leq b(2^j)^{1-\epsilon})\},$$

$$\rho = \rho(j,b,\epsilon) = \inf\{t \geq \sigma : |B^2(t)| = 2^{j+1}\}.$$

Then,

$$A^{n+1} \cap D_j \subset \{B^1[0,T^1_{2^{j-1}}] \cap B^2[0,T^2_{2^{j-1}}] = \emptyset; \tau \leq T^1_{2^j};$$
$$B^1[0,T_{2^{j+1}}] \cap B^2[\sigma,\rho] = \emptyset; B^1[T^1_{2^{j+1}}, T^1_{2^{n+1}}] \cap B^2[T^2_{2^{j+1}}, T^2_{2^{n+1}}] = \emptyset\}.$$

By Lemma 2.1,

$$\mathbf{P}^{x,y}\{B^1[0,T^1_{2^{j-1}}] \cap B^2[0,T^2_{2^{j-1}}] = \emptyset\} \leq c_9(2^{j-m-1})^{-2\zeta}.$$

Also by Lemma 2.1 and the strong Markov property,

$$\mathbf{P}^{x,y}\{B^1[T^1_{2^{j+1}}, T^1_{2^{n+1}}] \cap B^2[T^2_{2^{j+1}}, T^2_{2^{n+1}}] = \emptyset \mid B^1[0,T^1_{2^{j-1}}] \cap B^2[0,T^2_{2^{j-1}}] = \emptyset;$$
$$\tau \leq T^1_{2^j}; B^1[0,T^1_{2^{j+1}}] \cap B^2[\sigma,\rho] = \emptyset\} \leq c_9(2^{n-j})^{-2\zeta}.$$

For the middle term, we choose δ so that

$$\mathbf{P}\{\tilde{G} \geq (2^j)^{-\delta}\} \leq a(2^j)^{-4\zeta},$$

where $\tilde{G} = \tilde{G}(j, b, \epsilon)$ is as defined in the proof of Lemma 2.4 for B^2 (rather than for B^1 as in the proof). Then by the strong Markov property applied to the stopping time τ,

$$\mathbf{P}^{x,y}\{B^1[0, T^1_{2^{j+1}}] \cap B^2[\sigma, \rho] = \emptyset \mid B^1[0, T^1_{2^{j-1}}] \cap B^2[0, T^2_{2^{j-1}}] = \emptyset, \tau \leq T^1_{2^j}, \tilde{G} \leq (2^j)^{-\delta}\} \leq (2^j)^{-\delta}.$$

Hence,

$$\begin{aligned}\mathbf{P}^{x,y}\{B^1[0, T^1_{2^{j-1}}] \cap B^2[0, T^2_{2^{j-1}}] = \emptyset; \tau \leq T^1_{2^j}; B^1[0, T^1_{2^{j+1}}] \cap B^2[\sigma, \rho] = \emptyset;\\ \tilde{G} \leq (2^j)^{-\delta}; B^1[T^1_{2^{j+1}}, T^1_{2^{n+1}}] \cap B^2[T^2_{2^{j+1}}, T^2_{2^{n+1}}] = \emptyset\} \leq c(2^j)^{-\delta}(2^{n-m})^{-2\zeta}.\end{aligned}$$

But, $\mathbf{P}\{\tilde{G} \geq (2^j)^{-\delta}\} \leq a(2^j)^{-4\zeta}$. Hence again by the strong Markov property,

$$\begin{aligned}\mathbf{P}^{x,y}\{\tilde{G} \geq (2^j)^{-\delta}; B^1[T^1_{2^{j+1}}, T^1_{2^{n+1}}] \cap B^2[T^2_{2^{j+1}}, T^2_{2^{n+1}}] = \emptyset\} &\leq c(2^j)^{-4\zeta}(2^{n-j+1})^{-2\zeta}\\ &\leq c(2^j)^{-2\zeta}(2^{n-m})^{-2\zeta}.\end{aligned}$$

This completes the proof. □

Lemma 2.8 *Let $D^i_j = D^i_j(b, \epsilon)$ be defined as in Lemma 2.7. Let*

$$D = D(m, n, b, \epsilon) = \bigcup_{j=m+1}^{n} (D^1_j \cup D^2_j).$$

For $m \leq n, \rho > 0$, let

$$E^n = E^n(m, \rho) = \{B^i[0, T^i_{2^n}] \cap \mathcal{B}(0, 2^m) \subset \mathcal{B}(B^i(0), \rho 2^m), i = 1, 2\},$$

and $G^n = G_{2^n}$ as defined in Lemma 2.2. For every b, ϵ, ρ there exist $M < \infty$ and $a > 0$ such that if $M \leq m < n < \infty$, $|x| = |y| = 2^m$, $|x - y| \geq 2^{m+1}\rho$,

$$\mathbf{P}^{x,y}(A^n \cap D^c \cap E^n \cap G^n) \geq a(2^{n-m})^{-2\zeta}.$$

Proof. Suppose b, ϵ, ρ are given. Let $F^n = F_{2^n}$ as defined in Lemma 2.2. By Lemma 2.2, there is a $u_1 = u_1(b, \rho, \epsilon) > 0$ such that for all $|x| = |y| = 2^m$, $|x - y| \geq 2^{m+1}\rho$,

$$\mathbf{P}^{x,y}(A^n \cap E^n \cap F^n \cap G^n) \geq u_1(2^{n-m})^{-2\zeta}. \tag{9}$$

Note that for m sufficiently large, $n \geq m$,

$$F^n \cap (D^1_n \cup D^2_n) = \emptyset. \tag{10}$$

By Lemma 2.7, there exist $u_2 = u_2(b, \rho, \epsilon) < \infty, \delta = \delta(b, \rho, \epsilon) > 0$, such that if $|x| = |y| = 2^m, m < j < n$,

$$\mathbf{P}^{x,y}[A^n \cap (D^1_j \cup D^2_j)] \leq u_2(2^j)^{-\delta}(2^{n-m})^{-2\zeta}.$$

By summing over j, we can find an M such that if $m \geq M$,

$$\mathbf{P}^{x,y}[A^n \cap (\bigcup_{j=m+1}^{n-1} (D^1_j \cup D^2_j))] \leq \frac{u_1}{2}(2^{n-m})^{-2\zeta}. \tag{11}$$

Therefore, by (9) - (11), for M sufficiently large,

$$\mathbf{P}^{x,y}[A^n \cap E^n \cap F^n \cap G^n \cap D^c] \geq \frac{u_1}{2}(2^{n-m})^{-2\zeta}. \quad \square$$

3 Skorokhod Embedding

Let $X(t)$ be a one-dimensional Brownian motion starting at the origin. Let $\tau_0 = 0$, and for $n > 0$,

$$\tau_n = \inf\{t > \tau_{n-1} : |X(t) - X(\tau_{n-1})| = 1\},$$

$$Y(n) = X(\tau_n).$$

This is the well known Skorokhod embedding of a simple random walk $Y(n)$ in a Brownian motion. It is easy to check that $E(\tau_1) = 1$ and $E(e^{t\tau_1}) < \infty$ for some $t > 0$. Standard exponential estimates give that for every $\epsilon > 0$ there is a $\delta > 0$ and an $a < \infty$ such that

$$\mathbf{P}\{\sup_{0 \le i \le n} |\tau_i - i| \ge n^{(1/2)+\epsilon}\} \le ae^{-n^\delta}.$$

Similar exponential estimates for the Brownian motion give

$$\mathbf{P}\{\sup_{0 \le t \le n} \sup_{0 \le s \le n^{(1/2)+\epsilon}} |X(t) - X(t+s)| \ge n^{(1/4)+\epsilon}\} \le ae^{-n^\delta},$$

for perhaps different values of δ and a (we will allow the values of δ and a to vary in this section). If we define $Y(t) = Y([t])$ for noninteger t, this implies

$$\mathbf{P}\{\sup_{0 \le t \le n} |Y(t) - X(t)| \ge n^{(1/4)+\epsilon}\} \le ae^{-n^\delta}.$$

Now let $X^1, \ldots, X^d$ be d independent one-dimensional Brownian motions. Let Y^j be the simple random walks derived from X^j by the Skorokhod embedding and let $\tau^j(n) = \tau^j_n$ be the corresponding stopping times so that

$$Y^j(n) = X^j(\tau^j(n)).$$

Let

$$Z_n = (Z^1_n, \ldots, Z^d_n)$$

be a multinomial process independent of $X^1, \ldots, X^d$ with $Z_0 = (0, \ldots, 0)$; $\{Z_n - Z_{n-1} : n = 1, 2, \ldots\}$ independent; and

$$\mathbf{P}\{Z_n - Z_{n-1} = e_j\} = \frac{1}{d}, \quad j = 1, \ldots, d,$$

where e_j denotes the unit vector whose jth component equals 1. Let

$$B(t) = (X^1(t), \ldots, X^d(t)),$$

$$S(n) = (Y^1(Z^1_n), \ldots, Y^d(Z^d_n)) = (X^1(\tau^1(Z^1_n)), \ldots, X^d(\tau^d(Z^d_n))).$$

Then $B(t)$ is a d-dimensional Brownian motion and S is a d-dimensional simple random walk. More exponential estimates give for each $j = 1, \ldots, d$,

$$\mathbf{P}\{\sup_{0 \le i \le n} |Z^j_i - \frac{i}{d}| \ge n^{(1/2)+\epsilon}\} \le ae^{-n^\delta},$$

$$\mathbf{P}\{\sup_{0 \le i \le n} \sup_{0 \le k \le n^{(1/2)+\epsilon}} |Y^j(i) - Y^j(i+k)| \ge n^{(1/4)+\epsilon}\} \le ae^{-n^\delta}.$$

Hence we get the following.

Lemma 3.1 *Let B and S be defined as above. Then for every $\epsilon > 0$ there exist $\delta > 0$ and $a < \infty$ such that*

$$\mathbf{P}\{\sup_{0\leq t\leq n} |B(t) - S(td)| \geq n^{(1/4)+\epsilon}\} \leq ae^{-n^{\delta}}.$$

Let

$$T_n = \inf\{t : |B_t| = n\},$$
$$\xi_n = \inf\{j : |S(j)| \geq n\}.$$

More exponential estimates give

$$\mathbf{P}\{T_n \geq n^{2+\epsilon}\} \leq ae^{-n^{\delta}},$$
$$\mathbf{P}\{\xi_n \geq n^{2+\epsilon}\} \leq ae^{-n^{\delta}}.$$

Hence

Lemma 3.2 *Let B and S be defined as above. Then for every $\epsilon > 0$ there exist $\delta > 0$ and $a < \infty$ such that*

$$\mathbf{P}\{\sup_{0\leq t\leq T_{8n}} |B(t) - S(td)| \geq n^{(1/2)+\epsilon}\} \leq ae^{-n^{\delta}}.$$

In the next sections we will consider Brownian motions B and simple random walks S defined as above. We will be using the strong Markov property at times T_{2n}. One slight complication that arises is the fact that

$$\{B(t), S(td) : t \leq T_n\}$$

might contain a little information about $B(t)$ beyond time T_n. Let

$$\beta(n) = \max\{\tau^1(Z_n^1), \ldots, \tau^d(Z_n^d)\}.$$

Then $S(td)$ is measurable with respect to the σ-algebra generated by

$$\{Z_1, Z_2, \ldots\} \cup \{X(t) : t \leq \beta(n)\}.$$

Another exponential estimate gives that

$$\mathbf{P}\{\beta(T_n) \geq T_{3n/2} \text{ or } \beta(\xi_n) \geq T_{3n/2}\} \leq ae^{-n^{\delta}}.$$

We can therefore derive the following lemma.

Lemma 3.3 *There exist $\delta > 0$ and $a < \infty$ such that the following holds. For each n, there is an event Γ_n which is measurable with respect to the σ-algebra generated by*

$$\{B(t) : t \leq T_{2n}\} \cup \{Z_n : n = 1, 2, \ldots\},$$

with

$$P(\Gamma_n) \geq 1 - ae^{-n\delta},$$

with the property that on the event Γ_n,

$$\{B(t) : t \leq \max\{T_n, \xi_n\}\} \cup \{S(td) : t \leq \max\{T_n, \xi_n\}\} \cup \{Z_n : n = 1, 2, \ldots\}$$

and

$$\{B(t) : t \geq T_{2n}\}$$

are conditionally independent given $B(T_{2n})$.

4 Bounds for Random Walk

In this section we will prove Theorem 1.3. We will start with the lower bound in (5). Throughout this section we will let (B^1, S^1) and (B^2, S^2) be two independent Brownian motion—random walk pairs coupled as in the previous section. Let $\rho = .1, \epsilon = .25, b = 1$ in Lemma 2.8. Let M, a be as in the conclusion of the lemma. Assume $B^1(0) = S^1(0) = 2^m e_1$, $B^2(0) = S^2(0) = -2^m e_1$, where $m \geq M$ and e_1 is the unit vector whose first component is 1. As before we let

$$A^n = \{B^1[0, T^1_{2^n}] \cap B^2[0, T^2_{2^n}] = \emptyset\}.$$

Define the event Q^i_j by

$$Q^i_j = \{|B^i(s) - S^i(s)| \geq (2^j)^{.6} \text{ for some } s \leq T^i_{2^{j+1}}\}.$$

It follows from Lemma 3.1 that (assuming $j \geq m$),

$$\mathbf{P}(Q^i_j) \leq a \exp\{-(2^j)^\delta\}.$$

If we let $\Gamma = \Gamma_{2^j}$ be the event defined in Lemma 3.3, if $n \geq j$,

$$\begin{aligned}
\mathbf{P}(Q^i_j \cap A^n) &\leq \mathbf{P}(Q^i_j \cap \Gamma)\mathbf{P}(A^n \mid Q^i_j \cap \Gamma) + \mathbf{P}(\Gamma^c)\mathbf{P}(A^n \mid \Gamma^c) \\
&\leq a \exp\{-(2^j)^\delta\} c_9 (2^{n-j-1})^{-2\zeta} + a \exp\{-(2^j)^\delta\} c_9 (2^{n-j-1})^{-2\zeta} \\
&\leq a \exp\{-(2^j)^\delta\} (2^n)^{-2\zeta},
\end{aligned}$$

for perhaps different values of a and δ. By summing over all values of j and $i = 1, 2$ we can therefore conclude the following lemma.

Lemma 4.1 *There exist $c_{10} < \infty$ and $\delta_1 > 0$ such that the following holds. Let Q^i_j be defined as above and let*

$$Q = Q(m, n) = \bigcup_{i=1,2} \bigcup_{j=m}^{n-1} Q^i_j.$$

Then if $B^1(0) = S^1(0) = x$, $B^2(0) = S^2(0) = y$ with $|x| = |y| = 2^m$, then

$$\mathbf{P}(Q \cap A^n) \leq c_{10} \exp\{-(2^m)^{\delta_1}\} (2^n)^{-2\zeta}.$$

From Lemmas 2.8 and 4.1 we immediately get the following. Define events

$$\Delta^n = \{S^1[0, \xi_{2^n}] \cap S^2[0, \xi_{2^n}] = \emptyset\},$$

$$\Theta^n = \Theta^n(m, \rho) = \{S^i[0, \xi_{2^n}] \cap \mathcal{B}(0, 2^m) \subset \mathcal{B}(S^i(0), \rho 2^m), i = 1, 2\}$$

$$\begin{aligned}
\Phi^n = \{&S^1[0, \xi_{2^n}] \subset \mathcal{B}(0, 2^{n-1}) \cup \{(x^1, \ldots, x^d) : x^1 \geq \frac{2^n}{8}\}, \\
&S^2[0, \xi_{2^n}] \subset \mathcal{B}(0, 2^{n-1}) \cup \{(x^1, \ldots, x^d) : x^1 \leq -\frac{2^n}{8}\}\}.
\end{aligned}$$

We also let C_n be the discrete ball of radius n,

$$C_n = \{z \in \mathbf{Z}^d : |z| < n\},$$

with boundary

$$\partial C_n = \{z \in \mathbf{Z}^d \setminus C_n : |z - y| = 1 \text{ for some } y \in C_n\}.$$

Corollary 4.2 *For every $\rho > 0$, there exist $M < \infty$ and $u > 0$ such that if $M \leq m \leq n$, $x, y \in \partial C_{2^m}$, $|x-y| \geq \rho 2^{m+2}$, then*

$$\mathbf{P}^{x,y}(\Delta^n \cap \Theta^n \cap \Phi^n) \geq u(2^{n-m})^{-2\zeta}.$$

Once we have this corollary we can start two simple random walks at the origin. If we force S^1 to go directly to $2^m e_1$ along a straight line and similarly force S^2 to go directly to $-2^m e_1$, we can conclude the following.

Corollary 4.3 *There is a constant $c_7 > 0$ such that if S^1, S^2 are simple random walks starting at the origin, then*

$$\mathbf{P}\{S^1[0,\xi_n] \cap S^2(0,\xi_n] = \emptyset\} \geq c_7 n^{-2\zeta}.$$

We will now prove the upper bound for the nonintersection probability for random walks. Define b_n by

$$\tilde{b}_n = \sup_m \sup_{|x|,|y| \leq 2^m} (2^n)^{2\zeta} \mathbf{P}^{x,y}(\Delta^{m+n}),$$

$$b_n = \max\{\tilde{b}_0, \ldots, \tilde{b}_n\}.$$

To prove the upper bound it suffices to show that b_n is a bounded sequence. Let $(B^1, S^1), (B^2, S^2)$ be independent Brownian motion – random walk pairs starting at $|x|, |y| \leq 2^m$. For each n, let

$$\gamma = \gamma(n) = \inf\{j : B^1[T^1_{2^j}, T^1_{2^n}] \cap B^2[T^2_{2^j}, T^2_{2^n}] = \emptyset\},$$

$$J_j = J_{j,n} = \{\gamma = j\}.$$

Lemma 4.4 *There exist $c_{12} < \infty, \delta_2 > 0$ such that if $|x|, |y| \leq 2^m$, $m < j \leq n$,*

$$\mathbf{P}^{x,y}(J_j \cap \Delta^{n+1}) \leq c_{12} b_{j-m} (2^{n-m})^{-2\zeta} (2^j)^{-\delta_2}.$$

Proof. Assume $|x|, |y| \leq 2^m, m < j \leq n$. In this proof we will write $\mathbf{P}$ for $\mathbf{P}^{x,y}$. Note that

$$J_j \subset L^1 \cup L^2,$$

where

$$L^i = L^i(j,n) = \{B^i[T^i_{2^{j-1}}, T^i_{2^j}] \cap B^{3-i}[0, T^{3-i}_{2^j}] \neq \emptyset; B^1[T^1_{2^j}, T^1_{2^n}] \cap B^2[T^2_{2^j}, T^2_{2^n}] = \emptyset\}.$$

We will prove the estimate for $L^1 \cap \Delta^{n+1}$; a similar argument holds for $L^2 \cap \Delta^{n+1}$. Let

$$\tau = \tau(j) = \inf\{l \geq T^1_{2^{j-1}} : |S^1(l) - S^2(k)| \leq (2^j)^{.6} \text{ for some } k \leq T^2_{2^j}\},$$

$$\sigma = \sigma(j) = \inf\{k : |S^1(\tau) - S^2(k)| \leq (2^j)^{.6}\}.$$

Let

$$Q_j = \{|S^i(k) - B^i(k)| \geq (2^j)^{.6}/3 \text{ for some } k \leq T^i_{2^{j+1}}, i = 1,2\},$$

Let $\Gamma^1_{2^{j+1}}, \Gamma^2_{2^{j+1}}$ be the events given in Lemma 3.3 for (B^1, S^1) and (B^2, S^2), respectively, and let $\Gamma = \Gamma^1_{2^{j+1}} \cup \Gamma^2_{2^{j+1}}$. Let

$$V_j = V_{j,n} = \{B^1[T^1_{2^{j+2}}, T^1_{2^n}] \cap B^2[T^2_{2^{j+2}}, T^2_{2^n}] = \emptyset\}.$$

Then $L^1 \cap \Delta^{n+1} \subset W_j \cap V_j$ where

$$W_j = \{\Gamma \cap [Q_j \cup (\Delta^{j+1} \cap \{\tau < T^1_{2^j}\})]\} \cup \Gamma^c.$$

By Lemmas 3.3 and 2.1,

$$\mathbf{P}(V_j \mid W_j) \le c_9(2^{n-j-2})^{-2\zeta}.$$

Hence it suffices to prove that

$$\mathbf{P}(W_j) \le c b_{j-m}(2^{j-m})^{-2\zeta}(2^j)^{-\delta}, \tag{12}$$

for some appropriately chosen c, δ. By Lemma 3.3,

$$\mathbf{P}(\Gamma^c) \le a \exp\{-(2^j)^\delta\},$$

and by Lemma 3.2

$$\mathbf{P}(Q_j) \le a \exp\{-(2^j)^\delta\};$$

hence, we need only consider $\Delta^{j+1} \cap \{\tau < T^1_{2^j}\}$,

Let $Z = Z^1_j$ be defined as in Lemma 2.6 with $\epsilon = .1, b = 2$. By the lemma we can find a δ so that

$$\mathbf{P}\{Z \ge (2^j)^{-\delta}\} \le c(2^j)^{-4\zeta}.$$

But

$$\mathbf{P}(\Delta^{j+1} \cap \{\tau < T^1_{2^j}\}) \le \mathbf{P}\{Z \ge (2^j)^{-\delta}\} + \mathbf{P}(\Delta^{j-1})\mathbf{P}(\Delta^{j+1} \mid \Delta^{j-1}, \tau < T^1_{2^j}, Z \le (2^j)^{-\delta}).$$

By the strong Markov property, the second term on the right is bounded by

$$b_{j-m}(2^{j-m-1})^{-2\zeta}(2^j)^{-\delta}.$$

The first term is bounded by

$$c(2^j)^{-4\zeta}.$$

and hence

$$\mathbf{P}(\Delta^{j+1} \cap \{\tau < T^1_{2^j}\}) \le c b_{j-m}(2^{j-m})^{-2\zeta}(2^j)^{-\delta}.$$

(We have assumed without loss of generality that $\zeta > \delta$.) This completes the proof. $\Box$.

If $m \le n$ and $|x|, |y| \le 2^m$,

$$\mathbf{P}^{x,y}(\Delta^{n+1}) = \sum_{j=m}^{n} \mathbf{P}^{x,y}(\Delta^{n+1} \cap J_j),$$

where $J_j = J_{j,n}$ is as above. Note that

$$\mathbf{P}^{x,y}(\Delta^{n+1} \cap J_m) \le \mathbf{P}^{x,y}(J_m) \le c(2^{n-m})^{-2\zeta}.$$

Hence by Lemma 4.4, if $|x|, |y| \le 2^m$,

$$\mathbf{P}^{x,y}(\Delta^{n+1}) \le c(2^{n-m})^{-2\zeta} + c \sum_{j=m+1}^{n} b_{j-m}(2^{n-m})^{-2\zeta}(2^j)^{-\delta_2},$$

and hence,

$$b_{n+1} \leq c \sum_{j=0}^{n} b_j u^j,$$

where $u = 2^{-\delta_2} < 1$. To finish the proof of (5) we need only prove the following simple lemma about sequences of positive numbers.

Lemma 4.5 *Let $b_0, b_1, b_2, \ldots$ be a sequence of positive numbers. Suppose there exist $a < \infty$ and $u < 1$ such that for all $n \geq 1$,*

$$b_n \leq a \sum_{j=0}^{n-1} b_j u^j.$$

Then there exists an $M = M(a,u) < \infty$ such that for all n,

$$b_n \leq M b_0.$$

Proof. Without loss of generality we will assume $b_0 = 1$ and

$$b_n = a \sum_{j=0}^{n-1} b_j u^j,$$

for all $n \geq 1$. Let

$$r_n = \max_{0 \leq j \leq n} b_j u^j.$$

Then $b_n \leq a n r_{n-1}$ and hence

$$r_n \leq \max\{a n u^n r_{n-1}, r_{n-1}\}.$$

If we choose m sufficiently large so that $amu^m < 1$ and let $k = r_m$, then we see that $r_n \leq k$ for all n. Therefore

$$b_n = a \sum_{j=0}^{n-1} b_j u^j \leq akn,$$

for all n. Iterating again, we see this implies that

$$b_n = a \sum_{j=0}^{n-1} b_j u^j \leq a \sum_{j=0}^{\infty} akju^j = M < \infty. \quad \Box$$

Corollary 4.6 *There exists a $c_8 < \infty$ such that*

$$\mathbf{P}\{S^1[0,\xi_n] \cap S^2(0,\xi_n] = \emptyset\} \leq c_8 n^{-2\zeta}.$$

Moreover, for all $m \leq n$,

$$\sup_{|x|,|y| \leq m} \mathbf{P}^{x,y}\{S^1[0,\xi_n^1] \cap S^2[0,\xi_n^2] = \emptyset\} \leq c_8 (\frac{m}{n})^{2\zeta}.$$

The proof of (6) from (5) is essentially the same as the proofs of Proposition 3.14 and Proposition 3.15 in [11]. Since the proofs are nearly identical, we will not give them but will just state the results. Let

$$\Delta_n = \{S^1[0,\xi_n] \cap S^2(0,\xi_n] = \emptyset\}.$$

We will write $\mathbf{P}$ for $\mathbf{P}^{0,0}$.

Lemma 4.7 *There exist c_{14}, c_{15} such that for every positive integer n and every $a > 0$,*

$$\mathbf{P}\{\Delta_n; \min(\xi_n^1, \xi_n^2) \le an^2\} \le c_{14} e^{-c_{15}/a} n^{-2\zeta},$$

$$\mathbf{P}\{\Delta_n; \max(\xi_n^1, \xi_n^2) \ge an^2\} \le c_{14} e^{-c_{15} a} n^{-2\zeta}. \tag{13}$$

Proposition 4.8 *There exist c_{16}, c_{17} such that*

$$\begin{aligned} c_{16} n^{-2\zeta} &\le \mathbf{P}\{S^1[0,n^2] \cap S^2(0,n^2] = \emptyset\} \\ &\le \mathbf{P}\{S^1[0,\min(\xi_n^1, n^2)] \cap S^2(0,\min(\xi_n^2, n^2)] = \emptyset\} \\ &\le c_{17} n^{-2\zeta}. \end{aligned}$$

Proposition 4.9 *There exists c_{18} such that if $|x|, |y| \le m$,*

$$\mathbf{P}^{x,y}\{S^1[0,n] \cap S^2[0,n] = \emptyset\} \le c_{18}(\frac{m^2}{n})^\zeta.$$

This completes the proof of Theorem 1.3. We will need some slight generalizations of the lemmas proved above in the next section.

Lemma 4.10 *Let*

$$V_n = \{\mathrm{dist}(S^j(\xi_n^j), S^{3-j}[0,\xi_n^{3-j}]) \ge \frac{n}{2},\ j = 1,2\}.$$

Then there exist c_1, c_2 such that

$$\mathbf{P}(V_n \cap \Delta_n; \max(\xi_n^1, \xi_n^2) \le c_2 n^2\} \ge c_1 n^{-2\zeta}.$$

Proof. It suffices to prove the result for n sufficiently large. For n sufficiently large it is easy to see that Corollary 4.2 gives

$$\mathbf{P}(V_n \cap \Delta_n) \ge u n^{-2\zeta},$$

for some $u > 0$. But from (13) we see that there is a $c_2 < \infty$ such that

$$\mathbf{P}[\Delta_n; \max(\xi_n^1, \xi_n^2) \ge c_2 n^2] \le \frac{u}{2} n^{-2\zeta}. \quad \square$$

Lemma 4.11 *There exists a $c_{19} > 0$ such that the following is true. Let*

$$E = E_n = \{S^1[0,2n^2] \cap S^2(0,\xi_{8n}] = \emptyset\},$$

$$F(j,x) = F_n(j,x) = \{S^1(j) = x \text{ or } S^1(j+1) = x\},$$

$$G = G_n = \{S^1[0, 2n^2] \subset \mathcal{B}(0, \frac{15}{8}n)\},$$

$$H = H_n = \{S^2[0, \xi_{8n}] \cap \mathcal{B}(0, \frac{5}{4}n) = \emptyset\}.$$

Then if $x \in \partial C_n$ and $11n/8 \leq |y| \leq 13n/8, n^2 \leq j \leq 2n^2$,

$$\mathbf{P}^{y,y}[E \cap F(j, x) \cap G \cap H] \geq c_{19} n^{-d/2} n^{-2\zeta}.$$

Proof. We will just sketch the proof. By Lemma 4.10, we can find an $\epsilon \in (0, 1/50)$ so that if $S^1$and S^2 start at the origin,

$$\mathbf{P}\{S^1[0, \xi^1_{\epsilon n}] \cap S^2(0, \xi^2_{\epsilon n}] = \emptyset; \operatorname{dist}(S^i(\xi^i_{\epsilon n}), S^{3-i}[0, \xi^{3-i}_{\epsilon n}]) \geq \frac{n\epsilon}{2},\ i = 1, 2,$$
$$\max(\xi^1_{\epsilon n}, \xi^2_{\epsilon n}) \leq \frac{n^2}{4}\} \geq cn^{-2\zeta}.$$

Fix such an ϵ. Now by extending the paths, it is not difficult to see that if $U_1 = U_1(n, \epsilon)$ is the event

$$U_1 = \{S^1[0, \frac{n^2}{2}] \cap S^2(0, \xi^2_{8n}] = \emptyset, S^1[0, \frac{n^2}{2}] \subset \mathcal{B}(0, \frac{15n}{8});$$
$$|S^1(\frac{n^2}{2})| \leq n; S^2[0, \xi_{8n}] \cap \mathcal{B}(0, \frac{5n}{4}) = \emptyset\},$$

then there is a $c > 0$ such that for all $11n/8 \leq |y| \leq 13n/8$,

$$\mathbf{P}^{y,y}(U_1) > cn^{-2\zeta}.$$

Finally, it is easy using the local central limit theorem to show that there is a constant $c > 0$ such that if $|x|, |z| \leq n + 1$ and $n^2/2 \leq j \leq 2n^2$, then

$$\mathbf{P}^z\{S(j) = x \text{ or } S(j + 1) = x; j \leq \xi_{3n/2}\} \geq cn^{-d/2}.$$

Hence, if

$$U_2 = U_2(j, x) = \{S^1(j) = x \text{ or } S^1(j + 1) = x; S^1[0, 2n^2] \subset \mathcal{B}(0, \frac{15}{8}n)\},$$

then for all $11n/8 \leq |y| \leq 13n/8$,

$$\mathbf{P}^{y,y}(U_1 \cap U_2) \geq cn^{-d/2} n^{-2\zeta}. \quad \Box$$

By summing over all y with $11n/8 \leq |y| \leq 13n/8$ and translating the origin, we get the following.

Corollary 4.12 *Let*

$$E = E_{j,n} = \{S[0, j] \cap S[j + 1, \xi_{8n}] = \emptyset\},$$

$$F = F_{j,n} = \{\frac{11}{8}n \leq |S(j)| \leq \frac{13}{8}n\},$$

$$G = G_{j,n} = \{S[0, j] \subset \mathcal{B}(0, \frac{15}{8}n)\},$$

$$H = H_{j,n} = \{S[j + 1, \xi_{8n}] \cap \mathcal{B}(0, \frac{5}{4}n) = \emptyset\}.$$

There exists a $c_{20} > 0$ such that if $x \in \partial C_n$, $n^2 \leq j \leq 2n^2$,

$$\mathbf{P}^x(E \cap F \cap G \cap H) \geq c_{20} n^{-2\zeta}.$$

In particular, Corollary 4.12 implies that

$$\mathbf{P}^{0,0}\{S^1[0,n^2]\cap S^2(0,\xi_{6n}]=\emptyset; S^1(0,n^2]\subset\mathcal{B}(0,2n)\}\geq cn^{-2\zeta}.$$

If $d=3$ and $|z|\geq 6n$, then

$$\mathbf{P}^z\{S[0,\infty)\cap\mathcal{B}(0,2n)=\emptyset\}\geq c.$$

Hence we can conclude for $d=3$,

$$\mathbf{P}\{S^1[0,n]\cap S^2(0,\infty)=\emptyset\}\geq cn^{-\zeta}.$$

5 Proofs of Theorems

Assume $d=2,3$, and let $J_{j,n}$ be the indicator function of the event

$$\{S[0,j]\cap S[j+1,n]=\emptyset\},$$

and let

$$Y_n=\sum_{j=0}^{n}J_{j,n}.$$

It follows from Proposition 4.8 that

$$\mathbf{E}(Y_n)\geq cn^{1-\zeta}.$$

Lemma 5.1 *There exists a $c_{21}<\infty$ such that*

$$\mathbf{E}(Y_n^2)\leq c_{21}n^{2(1-\zeta)}.$$

Proof. We will show that there exists a $c<\infty$ such that if $0\leq i\leq j\leq n$,

$$\mathbf{P}\{J_{i,n}=J_{j,n}=1\}\leq cn^{\zeta}(i+1)^{-\zeta}(j-i+1)^{-\zeta}(n-j+1)^{-\zeta}. \tag{14}$$

The lemma then follows easily by expanding the square (recall that $0<\zeta<1$ for $d=2,3$).

To prove (14), we may assume without loss of generality that $i\leq n-j$. Let

$$k_1=\min\{[\frac{i}{2}],[\frac{j-i}{2}]\},$$

$$k_2=\min\{[\frac{j-i}{2}],[\frac{n-j}{2}]\}.$$

$$\tau_1=\inf\{m:\max(|S(i+m)-S(i)|,|S(i-m)-S(i)|)\geq\sqrt{k_1}\},$$

$$\tau_2=\inf\{m:\max(|S(j+m)-S(j)|,|S(j-m)-S(j)|)\geq\sqrt{k_2}\},$$

$$\rho_i=\min\{k_i,\tau_i\}.$$

Let

$$U=U(i,j,n)=\{S[i-\rho_1,i]\cap S[i+1,i+\rho_1]=\emptyset; S[j-\rho_2,j]\cap S[j+1,j+\rho_2]=\emptyset\},$$

$$V=V(i,j,n)=\{S[0,i-\rho_1]\cap S[j+\rho_2,n]=\emptyset\}.$$

Then

$$\mathbf{P}\{J_{i,n} = J_{j,n} = 1\} \le \mathbf{P}(U \cap V) = \mathbf{P}(U)\mathbf{P}(V \mid U).$$

By independence and Proposition 4.8,

$$\mathbf{P}(U) \le c(k_1+1)^{-\zeta}(k_2+1)^{-\zeta}.$$

By Proposition 4.9,

$$\mathbf{P}(V \mid U) \le c(\frac{i+1}{k_1+1})^{-\zeta}.$$

Combining these estimates gives (14) and hence the lemma. □

Now let $E_{j,n}, F_{j,n}, G_{j,n}, H_{j,n}$ be as defined in Corollary 4.12. Let $X_{j,n}$ be the indicator function of $E_{j,n} \cap F_{j,n} \cap G_{j,n} \cap H_{j,n}$ and let

$$Y_n = \sum_{n^2 \le j \le 2n^2} X_{j,n}.$$

It follows from Corollary 4.12 that

$$\mathbf{E}(Y_n) \ge c_1 n^{2(1-\zeta)}.$$

From Lemma 5.1 we know that

$$\mathbf{E}(Y_n^2) \le c_2 n^{4(1-\zeta)}.$$

Therefore, by the argument sketched at the end of Section 1, we can conclude the following. Note that Theorem 1.1 follows immediately from this corollary.

Corollary 5.2 *There exists a* $c_{21} > 0$ *such that*

$$\mathbf{P}\{Y_n \ge c_{21} n^{2(1-\zeta)}\} \ge c_{21}.$$

It remains to prove Theorem 1.2. For the remainder of this section we assume that $d = 3$. Let R_n be as defined in the first section. One direction is easy. Note that

$$\mathbf{E}(R_n) \le c n^{1-\zeta}.$$

Let $\epsilon > 0$ By Markov's inequality,

$$\mathbf{P}\{R(2^n) \ge (2^n)^{1-\zeta+\epsilon}\} \le c(2^n)^{-\epsilon}.$$

Hence by the Borel-Cantelli Lemma, with probability one, for all n sufficiently large

$$R(2^n) \le (2^n)^{1-\zeta+\epsilon},$$

and hence (since R_n is increasing in n)

$$\limsup_{n\to\infty} \frac{\ln R_n}{\ln n} \le 1 - \zeta + \epsilon.$$

Since ϵ is arbitrary, with probability one

$$\limsup_{n\to\infty} \frac{\ln R_n}{\ln n} \le 1 - \zeta.$$

Let $K(j,n) = K_{j,n}$ be as defined in Section 1. For any n define the event L^n,

$$L^n = \{ \sum_{j=0}^{\xi_{(5/6)2^n}} K(j, 2^n) \geq c_{21}(2^n)^{2(1-\zeta)}; S[0, \xi_{(5/6)2^n}] \subset \mathcal{B}(0, (11/12)2^n)\}.$$

It follows from Corollaries 4.12 and 5.2 that

$$\mathbf{P}(L^n \mid S(j) : j \leq \xi_{2^{n-1}}) \geq c_{21}.$$

Let

$$V^n = \{S[\xi_{2^n}, \infty) \cap \mathcal{B}(0, (11/12)2^n) = \emptyset\}.$$

There exists a $c_{22} > 0$ (see, e.g., [9, Proposition 5.10]) such that

$$\mathbf{P}(V^n \mid S(j), j \leq \xi_{2^n}) \geq c_{22}. \tag{15}$$

Note that on the event $L^n \cap V^n$,

$$R(\xi_{2^n}) \geq c_{21}(2^n)^{2(1-\zeta)},$$

where R_j is as defined in Section 1. We will show that there exists an $\alpha < \infty$ and a $c < \infty$ such that if

$$\Lambda^n = \Lambda^n(\alpha) = \bigcup_{n \leq j \leq n+\alpha \ln n} (L^j \cap V^j),$$

then

$$\mathbf{P}(\Lambda^n) \geq 1 - \frac{c}{n^2}. \tag{16}$$

Note that on the event Λ^n,

$$R(\xi_{2^n 2^{\alpha \ln n}}) \geq c_{21}(2^n)^{2(1-\zeta)}.$$

It follows from (16) and the Borel-Cantelli Lemma, that with probability one for all n sufficiently large Λ^n holds. It is easy to check that if Λ^n holds for all sufficiently large n, with probability one, then with probability one

$$\liminf_{n\to\infty} \frac{\ln R_n}{\ln n} \geq 1 - \zeta.$$

Hence it suffices to prove (16).

Fix n and define a (random) sequence $s_0, s_1, s_2, \ldots$ inductively as follows. Let $s_0 = n$. Suppose s_i has been defined. If $s_i = \infty$, then $s_{i+1} = \infty$. Suppose $s_i = s < \infty$. On the event $(L^s)^c$, we set $s_{i+1} = s + 1$. On the event L^s, let

$$\rho = \inf\{m \geq \xi_{2^s} : |S(m)| \leq (11/12)2^s\},$$

where $\rho = \infty$ if no such m exists. Let

$$s_{i+1} = \inf\{k : S[0, \rho] \subset \mathcal{B}(0, 2^{k-1})\},$$

and $s_{i+1} = \infty$ if $\rho = \infty$. Let

$$\hat{s} = \sup\{s_i : s_i < \infty\}.$$

Note that $L^{\hat{s}} \cap V^{\hat{s}}$ holds. Hence it suffices to prove that there is an $\alpha < \infty$ such that for all n sufficiently large

$$\mathbf{P}\{\hat{s} \geq n + \alpha \ln n\} \leq \frac{2}{n^2}. \tag{17}$$

Note that there is a $c_{23} > 0$ such that

$$\mathbf{P}\{s_{i+1} = \infty \mid s_0, \ldots, s_i\} \geq c_{23}$$

(this follows from Corollaries 4.12 and 5.2 and (15)). It is standard (see [9, Proposition 5.10]), that there is a $u < 1$ such that if m, k are positive integers, and S is a simple random walk in $\mathbf{Z}^3$ starting at $|x| \geq 2^{n+k}$, then

$$\mathbf{P}^x\{|S(j)| \leq 2^m \text{ for some } j \geq 0\} \leq u^k.$$

Hence, there is a $u < 1$ such that for all k,

$$\mathbf{P}\{s_i + k \leq s_{i+1} < \infty \mid s_0, \ldots, s_i\} \leq u^k.$$

Choose M so that

$$\sum_{k=M}^{\infty} u^k < 1 - c_{23},$$

and let p be a probability distribution on $\{0, 1, 2, \ldots\} \cup \{\infty\}$ with

$$\begin{aligned} p(\infty) &= c_{23}, \\ p(k) &= u^k, \ k \geq M, \\ p(M-1) &= 1 - c_{23} - \sum_{k=M}^{\infty} u^k. \end{aligned} \tag{18}$$

Let $N_1, N_2, \ldots$ be independent random variables from this distribution, and $\hat{N} = N_1 + \cdots + N_{l-1}$ where l is the first index with $N_l = \infty$. Then we can see that $\hat{N}$ stochastically dominates $\hat{s} - n$, i.e., for all $r > 0$,

$$\mathbf{P}\{\hat{s} - n \geq r\} \leq \mathbf{P}\{\hat{N} \geq r\}.$$

By (18), it is easy to see that there is a $\beta < \infty$ such that for all n sufficiently large

$$\mathbf{P}\{N_j < \infty : j = 1, \ldots, [\beta \ln n]\} \leq \frac{1}{n^2}.$$

Let

$$\tilde{N}_j = \begin{cases} N_j, & N_j < \infty, \\ 0, & N_j = \infty. \end{cases}$$

Then by standard large deviation estimates (using the exponential tails of $\tilde{N}_j$), there is an $\alpha < \infty$ such that for all n sufficiently large

$$\mathbf{P}\{\tilde{N}_1 + \cdots + \tilde{N}_{[\beta \ln n]} \geq \alpha \ln n\} \leq \frac{1}{n^2}.$$

Hence for all n sufficiently large

$$\begin{aligned} \mathbf{P}\{\hat{N} \geq \alpha \ln n\} &\leq \mathbf{P}\{N_j < \infty : j = 1, \ldots, [\beta \ln n]\} + \mathbf{P}\{\tilde{N}_1 + \cdots + \tilde{N}_{[\beta \ln n]} \geq \alpha \ln n\} \\ &\leq \frac{2}{n^2}. \end{aligned}$$

This gives (17) and hence proves the theorem.

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