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Expansions For Gaussian Processes And Parseval Frames

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Abstract

We derive a precise link between series expansions of Gaussian random vectors in a Banach space and Parseval frames in their reproducing kernel Hilbert space. The results are applied to pathwise continuous Gaussian processes and a new optimal expansion for fractional Ornstein-Uhlenbeck processes is derived.

Key words: Gaussian process, series expansion, Parseval frame, optimal expansion, fractional Ornstein-Uhlenbeck process.

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1 Introduction

Series expansions is a classical issue in the theory of Gaussian measures (see [2], [9], [17]). Our motivation for a new look on this issue finds its origin in recent new expansions for fractional Brownian motions (see [14], [1], [5], [6], [7]).

Let $(E, ||\cdot||)$ be a real separable Banach space and let $X : (\Omega, A, P) \to E$ be a centered Gaussian random vector with distribution P_X. In this article we are interested in series expansions of X of the following type. Let $\xi_1, \xi_2, \ldots$ be i.i.d. $N(0,1)$-distributed real random variables. A sequence $(f_j)_{j\geq 1} \in E^{\mathbb{N}}$ is called *admissible* for X if

$$\sum_{j=1}^{\infty} \xi_j f_j \text{ converges a.s. in } E \tag{1.1}$$

and

$$X \stackrel{d}{=} \sum_{j=1}^{\infty} \xi_j f_j. \tag{1.2}$$

By adding zeros finite sequences in E may be turned into infinite sequences and thus also serve as admissible sequences.

We observe a precise link to frames in Hilbert spaces. A sequence $(f_j)_{j\geq 1}$ in a real separable Hilbert space $(H, (\cdot,\cdot))$ is called *Parseval frame* for H if $\sum_{j=1}^{\infty}(f_j, h) f_j$ converges in H and

$$\sum_{j=1}^{\infty}(f_j, h) f_j = h \tag{1.3}$$

for every $h \in H$. Again by adding zeros, finite sequences in H may also serve as frames. For the background on frames the reader is referred to [4]. (Parseval frames correspond to tight frames with frame bounds equal to 1 in [4].)

Theorem 1. *Let $(f_j)_{j\geq 1} \in E^{\mathbb{N}}$. Then (f_j) is admissible for X if and only if (f_j) is a Parseval frame for the reproducing kernel Hilbert space of X.*

We thus demonstrate that the right notion of a "basis" in connection with expansions of X is a Parseval frame and not an orthonormal basis for the reproducing kernel Hilbert space of X. The first notion provides the possibility of redundancy and is more flexible as can be seen $e.g.$ from wavelet frames. It also reflects the fact that "sums" of two (or more) suitable scaled expansions of X yield an expansion of X.

The paper is organized as follows. In Section 2 we investigate the general Banach space setting in the light of frame theory and provide the proof of Theorem 1. Section 3 contains applications to pathwise continuous processes $X = (X_t)_{t\in I}$ viewed as $C(I)$-valued random vectors where I is a compact metric space. >Furthermore, we comment on optimal expansions. Fractional Brownian motions serve as illustration. Section 4 contains a new optimal expansion for fractional Ornstein-Uhlenbeck processes.

It is convenient to use the symbols $\sim$ and $\approx$ where $a_n \sim b_n$ means $a_n/b_n \to 1$ and $a_n \approx b_n$ means $0 < \liminf a_n/b_n \leq \limsup a_n/b_n < \infty$.

2 The Banach space setting

Let $(E, \|\cdot\|)$ be a real separable Banach space. For $u \in E^*$ and $x \in E$, it is convenient to write

$$\langle u, x \rangle$$

in place of $u(x)$. Let $X : (\Omega, \mathscr{A}, P) \to E$ be a centered Gaussian random vector with distribution P_X. The covariance operator $C = C_X$ of X is defined by

$$C : E^* \to E,\ Cu := E\langle u, X\rangle X. \tag{2.1}$$

This operator is linear and (norm-)continuous. Let $H = H_X$ denote the reproducing kernel Hilbert space (Cameron Martin space) of the symmetric nonnegative definite kernel

$$E^* \times E^* \to R, (u, v) \mapsto \langle u, Cv\rangle$$

(see [17], Propositions III.1.6. and III.1.7). Then H is a Hilbert subspace of E, that is $H \subset E$ and the inclusion map is continuous. The reproducing property reads

$$(h, Cu)_H = \langle u, h\rangle, u \in E^*, h \in H \tag{2.2}$$

where $(\cdot,\cdot)_H$ denotes the scalar product on H and the corresponding norm is given by

$$||h||_H = \sup\{|\ \langle u, h\rangle\ | : u \in E^*, \langle u, Cu\rangle \leq 1\}. \tag{2.3}$$

In particular, for $h \in H$,

$$||h|| \leq \sup_{||u|| \leq 1} \langle u, Cu\rangle^{1/2} ||h||_H = \|C\|^{1/2} ||h||_H. \tag{2.4}$$

The $\|\cdot\|_H$-closure of $A \subset H$ is denoted by $\overline{A}^{(H)}$. Furthermore, H is separable, $C(E^*)$ is dense in $(H, \|\cdot\|_H)$, the unit ball

$$U_H := \{h \in H : ||h||_H \leq 1\}$$

of H is a compact subset of E,

$$\text{supp}(P_X) = (\ker C)^{\perp} := \{x \in E :\ \langle u, x\rangle = 0 \text{ for every } u \in \ker C\} = \overline{H} \text{ in } E$$

and

$$H = \{x \in E :\ ||x||_H < \infty\} \tag{2.5}$$

where $||x||_H$ is formally defined by (2.3) for every $x \in E$. As for the latter fact, it is clear that $||h||_H < \infty$ for $h \in H$. Conversely, let $x \in E$ with $||x||_H < \infty$. Observe first that $x \in \overline{H}$. Otherwise, by the Hahn-Banach theorem, there exists $u \in E^*$ such that $u|\overline{H} = 0$ and $\langle u, x\rangle > 0$. Since, $\langle u, Cu\rangle = 0$ this yields

$$||x||_H \geq \sup_{a>0} a\langle u, x\rangle = \infty,$$

a contradiction. Now consider $C(E^*)$ as a subspace of $(H, ||\cdot||_H)$ and define $\varphi : C(E^*) \to R$ by $\varphi(Cu) := \langle u, x\rangle$. If $Cu_1 = Cu_2$, then using $(\ker C)^{\perp} = \overline{H}, \langle u_1 - u_2, x\rangle = 0$. Therefore, φ is well defined. The map φ is obviously linear and it is bounded since

$$||\varphi|| = \sup\{|\varphi(Cu)| : u \in E^*, ||Cu||_H \leq 1\} = ||x||_H < \infty$$

by (2.2). By the Hahn-Banach theorem there exists a linear bounded extension $\tilde{\varphi} : \overline{C(E^*)}^{(H)} \to R$ of φ. Then, since $\overline{C(E^*)}^{(H)} = H$, by the Riesz theorem there exists $g \in H$ such that $\tilde{\varphi}(h) = (h, g)_H$ for every $h \in H$. Consequently, using (2.2),

$$\langle u, x \rangle = \varphi(Cu) = (Cu, g)_H = \langle u, g \rangle$$

for every $u \in E^*$ which gives $x = g \in H$.

The key is the following characterization of admissibility. It relies on the Ito-Nisio theorem. Condition (v) is an abstract version of Mercer's theorem (cf. [15]. p. 43). Recall that a subset $G \subset E^*$ is said to be separating if for every $x, y \in E, x \neq y$ there exists $u \in G$ such that $\langle u, x \rangle \neq \langle u, y \rangle$.

Lemma 1. *Let $(f_j)_{j \geq 1} \in E^{\mathbb{N}}$. The following assertions are equivalent.*

(i) *The sequence $(f_j)_{j \geq 1}$ is admissible for X.*

(ii) *There is a separating linear subspace G of E^* such that for every $u \in G$,*

$$(\langle u, f_j \rangle)_{j \geq 1} \text{ is admissible for } \langle u, X \rangle.$$

(iii) *There is a separating linear subspace G of E^* such that for every $u \in G$,*

$$\sum_{j=1}^{\infty} \langle u, f_j \rangle^2 = \langle u, Cu \rangle.$$

(iv) *For every $u \in E^*$,*

$$\sum_{j=1}^{\infty} \langle u, f_j \rangle f_j = Cu.$$

(v) *For every $a > 0$,*

$$\sum_{j=1}^{\infty} \langle u, f_j \rangle \langle v, f_j \rangle = \langle u, Cv \rangle$$

uniformly in $u, v \in \{y \in E^* : \|y\| \leq a\}$.

Proof. Set $X_n := \sum_{j=1}^{n} \xi_j f_j$. (i) ⇒ (v). X_n converges a.s. in E to some E-valued random vector Y, say, with $X \stackrel{d}{=} Y$. It is well known that this implies $X_n \to Y$ in L^2_E. Therefore,

$$| \sum_{j=1}^{n} \langle u, f_j \rangle \langle v, f_j \rangle - \langle u, Cv \rangle | = | E\langle u, X_n \rangle \langle v, X_n \rangle - E\langle u, Y \rangle \langle v, Y \rangle |$$

$$= | E\langle u, Y - X_n \rangle \langle v, Y - X_n \rangle | \leq a^2 E\|Y - X_n\|^2 \to 0 \quad \text{as} \quad n \to \infty$$

uniformly in $u, v \in \{y \in E^* : \|y\| \leq a\}$. (v) ⇒ (iv) ⇒ (iii) is obvious. (iii) ⇒ (i). For every $u \in G$,

$$E \exp(i\langle u, X_n \rangle) = \exp(-\sum_{j=1}^{n} \langle u, f_j \rangle^2 / 2) \to \exp(-\langle u, Cu \rangle / 2) = E \exp(i\langle u, X \rangle).$$

The assertion (i) follows from the Ito-Nisio theorem (cf. [17], p. 271). (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is obvious. $\square$

Note that the preceding lemma shows in particular that $(f_j)_{j\geq 1}$ is admissible for X if and only if $(f_{\sigma(j)})_{j\geq 1}$ is admissible for X for (some) every permutation σ of N so that $\sum \xi_j f_j$ converges unconditionally a.s. in E for such sequences and all the $a.s.$ limits under permuations of N have distribution P_X.

It is also an immediate consequence of Lemma 1(v) that admissible sequences (f_j) satisfy $\|f_j\| \to 0$ since by the Cauchy criterion, $\lim_{j\to\infty} \sup_{||u||\leq 1} \langle u, f_j \rangle^2 = 0$.

The corresponding lemma for Parseval frames reads as follows.

Lemma 2. *Let $(f_j)_{j\geq 1}$ be a sequence in a real separable Hilbert space $(K, (\cdot,\cdot)_K)$. The following assertions are equivalent.*

(i) The sequence (f_j) is a Parseval frame for K.

(ii) For every $k \in K$,

$$\lim_{n\to\infty} || \sum_{j=1}^{n} (k, f_j)_K f_j ||_K = ||k||_K.$$

(iii) There is a dense subset G of K such that for every $k \in G$,

$$\sum_{j=1}^{\infty} (k, f_j)_K^2 = ||k||_K^2.$$

(iv) For every $k \in K$,

$$\sum_{j=1}^{\infty} (k, f_j)_K^2 = ||k||_K^2.$$

Proof. (i) $\Rightarrow$ (ii) is obvious. (ii) $\Rightarrow$ (iv). For every $k \in K, n \in N$,

$$\begin{aligned} 0 &\leq || \sum_{j=1}^{n} (k, f_j)_K f_j - k ||_K^2 \\ &= || \sum_{j=1}^{n} (k, f_j)_K f_j ||_K^2 - 2 \sum_{j=1}^{n} (k, f_j)_K^2 + ||k||_K^2 \end{aligned}$$

so that

$$2 \sum_{j=1}^{n} (k, f_j)_K^2 \leq || \sum_{j=1}^{n} (k, f_j)_K f_j ||_K^2 + ||k||_K^2.$$

Hence

$$\sum_{j=1}^{\infty} (k, f_j)_K^2 \leq ||k||_K^2.$$

Using this inequality we obtain conversely for $k \in K, n \in N$

$$\begin{aligned} \|\sum_{j=1}^{n}(k,f_j)_K f_j\|_K^2 &= \sup_{\|g\|_K\le 1}(g,\sum_{j=1}^{n}(k,f_j)_K f_j)_K^2 \\ &= \sup_{\|g\|_K\le 1}(\sum_{j=1}^{n}(k,f_j)_K(g,f_j)_K)^2 \\ &\le \sum_{j=1}^{n}(k,f_j)_K^2 \sup_{\|g\|_K\le 1}\sum_{j=1}^{n}(g,f_j)_K^2 \\ &\le \sum_{j=1}^{n}(k,f_j)_K^2. \end{aligned}$$

Hence

$$\|k\|_K^2 \le \sum_{j=1}^{\infty}(k,f_j)_K^2.$$

(iv) $\Rightarrow$ (iii) is obvious. (iii) $\Rightarrow$ (i). Since G is dense in K, for $k \in K$ there exist $k_n \in G$ satisfying $k_n \to k$ so that $\lim_{n\to\infty}(k_n,f_j)_K^2 = (k,f_j)_K^2$ for every j. Fatou's lemma for the counting measure in N implies

$$\begin{aligned} \sum_{j=1}^{\infty}(k,f_j)_K^2 &\le \liminf_{n\to\infty}\sum_{j=1}^{\infty}(k_n,f_j)_K^2 \\ &= \lim_{n\to\infty}\|k_n\|_K^2 = \|k\|_K^2. \end{aligned}$$

Therefore, one easily checks that $\sum_{j=1}^{\infty} c_j f_j$ converges in K for every $c = (c_j) \in l_2(N)$ and

$$T : l_2(N) \to K, T(c) := \sum_{j=1}^{\infty} c_j f_j$$

is linear and continuous (see [4], Theorem 3.2.3). Consequently, the frame operator

$$TT^* : K \to K, TT^*k = \sum_{j=1}^{\infty}(k,f_j)_K f_j$$

is linear and continuous. By (ii),

$$(TT^*k,k)_K = \sum_{j=1}^{\infty}(k,f_j)_K^2 = \|k\|_K^2$$

for every $k \in G$ and thus $(TT^*k,k)_K = \|k\|_K^2$ for every $k \in K$. This implies $TT^*k = k$ for every $k \in K$. □

The preceeding lemma shows that the series (1.3) converges unconditionally. Note further that a Parseval frame (f_j) for K satisfies $\{f_j : j \ge 1\} \subset U_K$, since

$$\|f_m\|_K^4 + \sum_{j\ne m}(f_m,f_j)_K^2 = \sum_{j=1}^{\infty}(f_m,f_j)_K^2 = \|f_m\|_K^2,$$

$\overline{span}\{f_j : j \geq 1\} = K$ and it is an orthonormal basis for K if and only if $||f_j||_K = 1$ for every j.

Proof of Theorem 1. The "if" part is an immediate consequence of the reproducing property (2.2) and Lemmas 1 and 2 since for $u \in E^*$,

$$\sum_{j=1}^{\infty} \langle u, f_j \rangle^2 = \sum_{j=1}^{\infty} (Cu, f_j)_H^2 = ||Cu||_H^2 = \langle u, Cu \rangle.$$

The "only if" part. By Lemma 1,

$$||f_j||_H = \sup\{|\langle u, f_j \rangle| : \langle u, Cu \rangle \leq 1\} \leq 1$$

so that by (2.5), $\{f_j : j \geq 1\} \subset H$. Again the assertion follows immediately from (2.2) and Lemmas 1 and 2 since $C(E^*)$ is dense in H. □

The covariance operator admits factorizations $C = SS^*$, where $S : K \to E$ is a linear continuous operator and $(K, (\cdot, \cdot)_K)$ a real separable Hilbert space, which provide a useful tool for expansions. It is convenient to allow that S is not injective. One gets

$$\begin{aligned} S(K) &= H, \qquad (2.6)\\ \ast[.4em](Sk_1, Sk_2)_H &= (k_1, k_2)_K, k_1 \in K, k_2 \in (\mathrm{ker} S)^{\perp},\\ \ast[.4em]\|S\| &= \|S^*\| = \|C\|^{1/2},\\ \ast[.4em]\overline{S^*(E^*)} &= (\mathrm{ker} S)^{\perp} \text{ in } K,\\ \ast[.4em](\mathrm{ker} S^*)^{\perp} &:= \{x \in E : \langle u, x \rangle = 0 \ \forall u \in \mathrm{ker} S^*\} = \overline{H} \text{ in } E. \end{aligned}$$

Notice that factorizations of C correspond to linear continuous operators $T : K \to H$ satisfying $TT^* = I$ via $S = JT$, where $J : H \to E$ denotes the inclusion map.

A sequence (e_j) in K is called *Parseval frame sequence* if it is a Parseval frame for $\overline{span}\{e_j : j \geq 1\}$.

Proposition 1. *Let $C = SS^*, S : K \to E$ be a factorization of C and let (e_j) be a Parseval frame sequence in K satisfying $(kerS)^{\perp} \subset \overline{span}\{e_j : j = 1, 2, \ldots\}$. Then $(S(e_j))$ is admissible for X. Conversely, if (f_j) is admissible for X then there exists a Parseval frame sequence (e_j) in K satisfying $(kerS)^{\perp} = \overline{span}\{e_j : j = 1, 2, \ldots\}$ such that $S(e_j) = f_j$ for every j.*

Proof. Let $K_0 := \overline{\mathrm{span}}\{e_j : j = 1, 2, \ldots\}$. Since by (2.6)

$$S^*(E^*) \subset (\mathrm{ker} S)^{\perp} \subset K_0,$$

one obtains for every $u \in E^*$, by Lemma 2,

$$\sum_j \langle u, Se_j \rangle^2 = \sum_j (S^*u, e_j)_K^2 = \|S^*u\|_K^2 = \langle u, Cu \rangle.$$

The assertion follows from Lemma 1. Conversely, if (f_j) is admissible for X then (f_j) is a Parseval frame for H by Theorem 1. Set $e_j := (S|(\mathrm{ker} S)^{\perp})^{-1}(f_j) \in (\mathrm{ker} S)^{\perp}$. Then by (2.6) and Lemma 2, for every $k \in (\mathrm{ker} S)^{\perp}$,

$$\sum_j (k, e_j)_K^2 = \sum_j (Sk, f_j)_H^2 = ||Sk||_H^2 = ||k||_K^2$$

so that again by Lemma 2, (e_j) is a Parseval frame for $(\ker S)^{\perp}$. □

EXAMPLES • Let $S : H \to E$ be the inclusion map. Then $C = SS^*$.
• Let K be the closure of E^* in $L^2(P_X)$ and $S : K \to E, Sk = Ek(X)X$. Then $S^* : E^* \to K$ is the natural embedding. Thus $C = SS^*$ and S is injective (see (2.6)). (K is sometimes called the energy space of X.) One obtains

$$H = S(K) = \{Ek(X)X : k \in K\}$$

and

$$(Ek_1(X)X, Ek_2(X)X)_H = \int k_1 k_2 dP_X.$$

• Let E be a Hilbert space, $K = E$ and $S = C^{1/2}$. Then $C = SS^* = S2$ and $(\ker S)^{\perp} = \overline{H}$. Consequently, if (e_j) is an orthonormal basis of the Hilbert subspace $\overline{H}$ of E consisting of eigenvectors of C and (λ_j) the corresponding nonzero eigenvalues, then $(\sqrt{\lambda_j} e_j)$ is admissible for X and an orthonormal basis of $(H, (\cdot, \cdot)_H)$ (Karhunen-Loève basis).

Admissible sequences for X can be charaterized as the sequences $(Se_j)_{j \geq 1}$ where (e_j) is a fixed orthonormal basis of K and S provides a factorization of C. That every sequence (Se_j) of this type is admissible follows from Proposition 1.

Theorem 2. *Assume that $(f_j)_{j \geq 1}$ is admissible for X. Let K be an infinite dimensional real separable Hilbert space and $(e_j)_{j \geq 1}$ an orthonormal basis of K. Then there is a factorization $C = SS^*, S : K \to E$ such that $S(e_j) = f_j$ for every j.*

Proof. First, observe that $\sum_{j=1}^{\infty} c_j f_j$ converges in E for every $(c_j)_j \in l_2(N)$. In fact, using Lemma 1,

$$\begin{aligned} \|\sum_{j=n}^{n+m} c_j f_j\|^2 &= \sup_{\|u\| \leq 1} \langle u, \sum_{j=n}^{n+m} c_j f_j \rangle^2 \\ &\leq \sum_{j=n}^{n+m} c_j^2 \sup_{\|u\| \leq 1} \sum_{j=1}^{\infty} \langle u, f_j \rangle^2 \\ &= \sum_{j=n}^{n+m} c_j^2 \sup_{\|u\| \leq 1} \langle u, Cu \rangle \\ &= \sum_{j=n}^{n+m} c_j^2 \|C\| \to 0, \ n, m \to \infty \end{aligned}$$

and thus the sequence is Cauchy in E. Now define $S : K \to E$ by

$$S(k) := \sum_{j=1}^{\infty} (k, e_j)_K f_j$$

where $\sum (k, e_j)_K f_j$ converges in E since $((k, e_j)_K)_j \in l_2(N)$. S is obviously linear. Moreover, for $k \in K$, using again Lemma 1,

$$\begin{aligned} \|Sk\|^2 &= \sup_{\|u\| \leq 1} \langle u, Sk \rangle^2 \\ &= \sup_{\|u\| \leq 1} (\sum_{j=1}^{\infty} (k, e_j)_K \langle u, f_j \rangle)^2 \\ &\leq \|k\|_K^2 \|C\|. \end{aligned}$$

Consequently, S is continuous and $S(e_j) = f_j$ for every j. (At this place one needs orthonormality of (e_j).) Finally, $S^*(u) = \sum_{j=1}^{\infty} \langle u, f_j \rangle e_j$ and hence

$$SS^*u = \sum_{j=1}^{\infty} \langle u, f_j \rangle f_j = Cu$$

for every $u \in E^*$ by Lemma 1. □

It is an immediate consequence of the preceding theorem that an admissible sequence (f_j) for X is an orthonormal basis for H if and only if (f_j) is l_2-independent, that is $\sum_{j=1}^{\infty} c_j f_j = 0$ for some $(c_j) \in l_2(N)$ implies $c_j = 0$ for every j. In fact, l_2-independence of (f_j) implies that the operator S in Theorem 2 is injective.

Let F be a further separable Banach space and $V : E \to F$ a P_X-*measurable linear transfromation*, that is, V is Borel measurable and linear on a Borel measurable subspace D_V of E with $P_X(D_V) = 1$. Then $H_X \subset D_V$, the operator $VJ_X : H_X \to F$ is linear and continuous, where $J_X : H_X \to E$ denotes the inclusion map and $V(X)$ is centered Gaussian with covariance operator

$$C_{V(X)} = VJ_X(VJ_X)^* \tag{2.7}$$

(see [11], [2], Chapter 3.7). Consequently, by (2.6)

$$\begin{aligned} H_{V(X)} &= V(H_X), \qquad (2.8) \\ *[.4em](Vh_1, Vh_2)_{H_{V(X)}} &= (h_1, h_2)_{H_X}, h_1 \in H_X, h_2 \in (\ker(V \mid H_X))^{\perp}. \end{aligned}$$

Note that the space of P_X-measurable linear transfromation $E \to F$ is equal to the $L_F^p(P_X)$-closure of the space of linear continuous operators $E \to F, p \in [1, \infty)$ (see [11]).

>From Theorem 1 and Proposition 1 one may deduce the following proposition.

Proposition 2. *Assume that $V : E \to F$ is a P_X-measurable linear transformation. If $(f_j)_{j \geq 1}$ is admissible for X in E, then $(V(f_j))_{j \geq 1}$ is admissible for $V(X)$ in F. Conversely, if $V|H_X$ is injective and $(g_j)_{j \geq 1}$ an admissible sequence for $V(X)$ in F, then there exists a sequence $(f_j)_{j \geq 1}$ in E which is admissible for X such that $V(f_j) = g_j$ for every j.*

Example Let X and Y be jointly centered Gaussian random vectors in E and F, respectively. Then $E(Y|X) = V(X)$ for some P_X-measurable linear transformation $V : E \to F$. The cross covariance operator $C_{YX} : E^* \to F$, $C_{YX}u = E\langle u, x \rangle Y$ can be factorized as $C_{YX} = U_{YX} S_X^*$, where $C_X = S_X S_X^*$ is the energy factorization of C_X with K_X the closure of E^* in $L^2(P_X)$ and $U_{YX} : K_X \to F, U_{YX}k = Ek(X)Y$. Then

$$V = U_{YX} S_X^{-1} \text{ on } H_X$$

(see [11]). Consequently, if $(f_j)_{j \geq 1}$ is admissible for X in E then $(U_{YX} S_X^{-1} f_j)_{j \geq 1}$ is admissible for $E(Y|X)$ in F.

3 Continuous Gaussian processes

Now let I be a compact metric space and $X = (X_t)_{t\in I}$ be a real pathwise continuous centered Gaussian process. Let $E := \mathscr{C}(I)$ be equipped with the sup-norm $\|x\| = \sup_{t\in I} |x(t)|$ so that the norm dual $\mathscr{C}(I)^*$ coincides with the space of finite signed Borel measures on I by the Riesz theorem. Then X can be seen as a $\mathscr{C}(I)$-valued Gaussian random vector and the covariance operator $C : \mathscr{C}(I)^* \to \mathscr{C}(I)$ takes the form

$$\begin{aligned} Cu(t) &= \langle \delta_t, Cu\rangle = \langle C\delta_t, u\rangle \\ &= \langle EX_t X, u\rangle = \int_I EX_t X_s du(s). \end{aligned} \tag{3.1}$$

Corollary 1. *Let* $(f_j)_{j\geq 1} \in \mathscr{C}(I)^{\mathbb{N}}$.
(a) If

$$EX_s X_t = \sum_{j=1}^{\infty} f_j(s) f_j(t) \text{ for every } s,t \in I$$

then (f_j) *is admissible for* X.

(b) If

$$\sum_{j=1}^{\infty} f_j(t)^2 < \infty \text{ for every } t \in I$$

and if the process Y *with* $Y_t = \sum_{j=1}^{\infty} \xi_j f_j(t)$ *has a pathwise continuous modification* X*, then* (f_j) *is admissible for* X *and* $X = \sum_{j=1}^{\infty} \xi_j f_j$ *a.s.*

Proof. (a) For $u \in G := \text{span}\,\{\delta_t : t \in I\}, u = \sum_{i=1}^{m} \alpha_i \delta_{t_i}$ we have

$$\langle u, Cu\rangle = \sum_{i=1}^{m}\sum_{k=1}^{m} \alpha_i \alpha_k EX_{t_i} X_{t_k}$$

and

$$\sum_{j=1}^{n} \langle u, f_j\rangle^2 = \sum_{i=1}^{m}\sum_{k=1}^{m} \alpha_i \alpha_k \sum_{j=1}^{n} f_j(t_i) f_j(t_k)$$

so that

$$\sum_{j=1}^{\infty} \langle u, f_j\rangle^2 = \langle u, Cu\rangle.$$

Since G is a separating subspace of $\mathscr{C}(I)^*$ the assertion follows from Lemma 1.
(b) Notice that $\sum \xi_j f_j(t)$ converges a.s. in R and Y is a centered Gaussian process. Hence X is centered Gaussian. Since

$$EX_s X_t = EY_s Y_t = \sum_{j=1}^{\infty} f_j(s) f_j(t) \text{ for every } s,t \in I,$$

the assertion follows from (a). □

>Factorizations of C can be obtained as follows. For Hilbert spaces K_i, let $\oplus_{i=1}^m K_i$ denote the Hilbertian (or l_2-)direct sum.

Lemma 3. *For $i \in \{1, \dots, m\}$, let K_i be a real separable Hilbert space. Assume the representation*

$$EX_sX_t = \sum_{i=1}^{m} (g_s^i, g_t^i)_{K_i}, s, t \in I$$

for vectors $g_t^i \in K_i$. Then

$$S : \oplus_{i=1}^m K_i \to \mathscr{C}(I), Sk(t) := \sum_{i=1}^{m} (g_t^i, k_i)_{K_i}$$

is a linear continuous operator, $(kerS)^{\perp} = \overline{span}\{(g_t^1, \dots, g_t^m) : t \in I\}$ and $C = SS^*$.

Proof. Let $K := \oplus_{i=1}^m K_i$ and $g_t := (g_t^1, \dots, g_t^m)$. Then $EX_sX_t = (g_s, g_t)_K$ and $Sk(t) = (g_t, k)_K$. First, observe that

$$\sup_{t \in I} \|g_t\|_K \le \|C\|^{1/2} < \infty.$$

Indeed, for every $t \in I$, by (3.1),

$$\|g_t\|_K^2 = EX_t^2 = \langle \delta_t, C\delta_t \rangle \le \|C\|.$$

The function Sk is continuous for $k \in$ span $\{g_s : s \in I\}$. This easily implies that Sk is continuous for every $k \in \overline{\text{span}}\{g_s : s \in I\}$ and thus for every $k \in K$. S is obviously linear and

$$\|Sk\| = \sup_{t \in I} |(g_t, k)_K| \le \|C\|^{1/2} \|k\|_K.$$

Finally, $S^*(\delta_t) = g_t$ so that

$$SS^*\delta_t(s) = Sg_t(s) = EX_sX_t = C\delta_t(s)$$

for every $s, t \in I$. Consequently, for every $u \in \mathscr{C}(I)^*, t \in I$,

$$\begin{aligned} SS^*u(t) &= \langle SS^*u, \delta_t \rangle = \langle u, SS^*\delta_t \rangle \\ &= \langle u, C\delta_t \rangle = \langle Cu, \delta_t \rangle = Cu(t) \end{aligned}$$

and hence $C = SS^*$. □

Example Let K be the first Wiener chaos, that is $K = \overline{\text{span}}\{X_t : t \in I\}$ in $L^2(P)$ and $g_t = X_t$. Then $Sk = EkX$ and S is injective. If for instance $X = W$ (Brownian motion) and $I = [0, T]$, then

$$K = \left\{ \int_0^T f(s) dW_s : f \in L^2([0, T], dt) \right\}.$$

We derive from the preceeding lemma and Proposition 1 the following corollary.

Corollary 2. *Assume the situation of Lemma 3. Let $(e^i_j)_j$ be a Parseval frame sequence in K_i satisfying $\{g^i_t : t \in I\} \subset \overline{span}\ \{e^i_j : j = 1, 2, \dots\}$. Then, $(S_i(e^i_j))_{1 \le i \le m, j}$ is admissible for X, where $S_i k(t) = (g^i_t, k)_{K_i}$.*

The next corollary implies the well known fact that the Karhunen-Loève expansion of X in some Hilbert space $L^2(I, \mu)$ already converges uniformly in $t \in I$. It appears as special case of Proposition 2.

Corollary 3. *Let μ be a finite Borel measure on I with $supp(\mu) = I$ and let $V : \mathscr{C}(I) \to L^2(I, \mu)$ denote the natural (injective) embedding. Let $(g_j)_{j \ge 1}$ be admissible for $V(X)$ in $L^2(I, \mu)$. Then there exists a sequence $(f_j)_{j \ge 1}$ in $\mathscr{C}(I)$ which is admissible for X such that $V(f_j) = g_j$ for every j.*

The admissibility feature is stable under tensor products. For $i \in \{1, \dots, d\}$, let I_i be a compact metric space and $X^i = (X^i_t)_{t \in I_i}$ a continuous centered Gaussian process. Set $I := \Pi_{i=1}^d I_i$ and let $X = (X_t)_{t \in I}$ be a continuous centered Gaussian process with covariance function

$$EX_s X_t = \Pi_{i=1}^d E X^i_{s_i} X^i_{t_i}, s, t \in I. \tag{3.2}$$

For instance, $X := \otimes_{i=1}^d X^i$ satisfies (3.2) provided $X1, \dots, X^d$ are independent. For real separable Hilbert spaces K_i, let $\widehat{\otimes_{i=1}^d} K_i$ denote the d-fold Hilbertian tensor product.

Proposition 3. *For $i \in \{1, \dots, d\}$, let $(f^i_j)_{j \ge 1}$ be an admissible sequence for X^i in $\mathscr{C}(I_i)$. Then*

$$(\otimes_{i=1}^d f^i_{j_i})_{\underline{j} = (j_1, \dots, j_d) \in \mathbb{N}^d}$$

is admissible for X with covariance (3.2) in $\mathscr{C}(I)$. Furthermore, if $C_{X^i} = S_i S_i^*, S_i : K_i \to \mathscr{C}(I_i)$ is a factorization of C_{X^i}, then $\otimes_{i=1}^d S_i : \widehat{\otimes_{i=1}^d} K_i \to \mathscr{C}(I)$ provides a factorization of C_X.

Proof. For $i \in \{1, \dots, d\}$, let K_i be a real separable Hilbert space and $(e^i_j)_j$ an orthonormal basis of K_i. Then $(\otimes_{i=1}^d e^i_{j_i})_{\underline{j}}$ is an orthonormal basis of $K := \widehat{\otimes_{i=1}^d} K_i$.

If $C_{X^i} = S_i S_i^*, S_i : K_i \to \mathscr{C}(I_i)$ is a factorization of C_{X^i}, set $g^i_t := S_i^* \delta_t, t \in I_i$. Then $E X^i_s X^i_t = (g^i_s, g^i_t)_{K_i}$ and hence, by (3.2)

$$EX_s X_t = \Pi_{i=1}^d (g^i_{s_i}, g^i_{t_i})_{K_i} = (\otimes_{i=1}^d g^i_{s_i}, \otimes_{i=1}^d g^i_{t_i})_K, s, t \in I.$$

Consequently, by Lemma 3

$$U : K \to \mathscr{C}(I), Uk(t) = (\otimes_{i=1}^d g^i_{t_i}, k)_K$$

provides a factorization of C_X. Since

$$\begin{aligned} U(\otimes_{i=1}^d e^i_{j_i})(t) &= \Pi_{i=1}^d (g^i_{t_i}, e^i_{j_i})_{K_i} \\ &= \Pi_{i=1}^d S_i e^i_{j_i}(t_i) = \otimes_{i=1}^d (S_i e^i_{j_i})(t) \\ &= (\otimes_{i=1}^d S_i)(\otimes_{i=1}^d e^i_{j_i})(t), t \in I, \end{aligned}$$

we obtain $U = \otimes_{i=1}^d S_i$ and thus $\otimes_{i=1}^d S_i$ provides a factorization of C_X.

If $(f^i_j)_{j \ge 1}$ is admissible for X^i, then by Theorem 2 assuming now that K_i is infinite dimensional, there is a factorization $C_{X^i} = T_i T_i^*, T_i : K_i \to \mathscr{C}(I_i)$ such that $T_i(e^i_j) = f^i_j$ for every j. Since

$\otimes_{i=1}^{d} T_i : K \to \mathscr{C}(I)$ provides a factorization of C_X as shown above and $(\otimes_{i=1}^{d} T_i)(\otimes_{i=1}^{d} e_{j_i}^{i}) = \otimes_{i=1}^{d} f_{j_i}^{i}$, it follows from Proposition 1 that $(\otimes_{i=1}^{d} f_{j_i}^{i})_{\underline{j} \in N^d}$ is admissible for X. □

Comments on optimal expansions. For $n \in N$, let

$$l_n(X) := \inf\{E\|\sum_{j=n}^{\infty} \xi_j f_j\| : (f_j)_{j\geq 1} \in \mathscr{C}(I)^{N} \text{ admissible for } X\}. \tag{3.3}$$

Rate optimal solutions of the $l_n(X)$-problem are admissible sequences (f_j) for X in $\mathscr{C}(I)$ such that

$$E\|\sum_{j=n}^{\infty} \xi_j f_j\| \approx l_n(X) \quad \text{as} \quad n \to \infty.$$

For $I = [0, T]^d \subset R^d$, consider the covariance operator $R = R_X$ of X on $L^2(I, dt)$ given by

$$R : L^2(I, dt) \to L^2(I, dt), Rk(t) = \int_I EX_s X_t k(s) ds. \tag{3.4}$$

Using (3.1) we have $R_X = V C_X V^*$, where $V : C(I) \to L^2(I, dt)$ denotes the natural (injective) embedding. The choice of Lebesgue measure on I is the best choice for our purposes (see (A1)). Let $\lambda_1 \geq \lambda_2 \geq \ldots > 0$ be the ordered nonzero eigenvalues of R (each written as many times as its multiplicity).

Proposition 4. *Let $I = [0, T]^d$. Assume that the eigenvalues of R satisfy*

(A1) $\lambda_j \geq c_1 j^{-2\vartheta} \log(1 + j)^{2\gamma}$ for every $j \geq 1$ with $\vartheta > 1/2, \gamma \geq 0$ and $c_1 > 0$

and that X admits an admissible sequence (f_j) in $\mathscr{C}(I)$ satisfying

(A2) $\|f_j\| \leq c_2 j^{-\vartheta} \log(1 + j)^{\gamma}$ for every $j \geq 1$with $c_2 < \infty$,

(A3) f_j is a-Hölder-continuous and $[f_j]_a \leq c_3 j^b$ for every $j \geq 1$ with $a \in (0, 1], b \in R$ and $c_3 < \infty$, where

$$[f]_a = \sup_{s \neq t} \frac{|f(s) - f(t)|}{|s - t|^a}$$

(and $|t|$ denotes the l_2-norm of $t \in R^d$).
Then

$$l_n(X) \approx n^{-(\vartheta - \frac{1}{2})} (\log n)^{\gamma + \frac{1}{2}} \quad \text{as} \quad n \to \infty \tag{3.5}$$

and (f_j) is rate optimal.

Proof. The lower estimate in (3.5) follows from (A1) (see [8], Proposition 4.1) and from (A2) and (A3) follows

$$E\|\sum_{j=n}^{\infty} \xi_j f_j\| \leq c_4 n^{-(\vartheta - \frac{1}{2})} (\log(1 + n))^{\gamma + \frac{1}{2}}$$

for every $n \geq 1$, (see [13], Theorem 1). □

Concerning assumption (A3) observe that we have by (2.2) and (3.1) for $h \in H = H_X, s, t \in I$,

$$h(t) = <\delta_t, h> = (h, C\delta_t)_H$$

and

$$||C(\delta_s - \delta_t)||_H^2 = <\delta_s - \delta_t, C(\delta_s - \delta_t)> = E|X_s - X_t|^2$$

so that

$$\begin{aligned} |h(s) - h(t)| &= |(h, C(\delta_s - \delta_t))_H| \\ &\leq ||h||_H ||C(\delta_s - \delta_t)||_H \\ &= ||h||_H (E|X_s - X_t|^2)^{1/2}. \end{aligned} \tag{3.6}$$

Consequently, since admissible sequences are contained in the unit ball of H, (A3) is satisfied with $b = 0$ provided $I \to L^2(P), t \mapsto X_t$ is a-Hölder-continuous.

The situation is particularly simple for Gaussian sheets.

Corollary 4. *Assume that for $i \in \{1, \ldots, d\}$, the continuous centered Gaussian process $X^i = (X_t^i)_{t\in[0,T]}$ satisfies (A1) - (A3) for some admissible sequence $(f_j^i)_{j\geq 1}$ in $\mathscr{C}([0,T])$ with parameters $\vartheta_i, \gamma_i, a_i, b_i$ such that $\gamma_i = 0$ and let $X = (X_t)_{t\in I}, I = [0,T]^d$ be the continuous centered Gaussian sheet with covariance (3.2). Then*

$$l_n(X) \approx n^{-(\vartheta - \frac{1}{2})} (\log n)^{\vartheta(m-1) + \frac{1}{2}} \tag{3.7}$$

with $\vartheta = \min_{1\leq i\leq d} \vartheta_i$ and $m = card\{i \in \{1, \ldots, d\} : \vartheta_i = \vartheta\}$ and a decreasing arrangement of $(\otimes_{i=1}^d f_{j_i}^i)_{j\in \underline{N}^d}$ is rate optimal for X.

Proof. In view of Lemma 1 in [13] and Proposition 3, the assertions follow from Proposition 4. □

EXAMPLES The subsequent examples may serve as illustrations.

• Let $W = (W_t)_{t\in[0,T]}$ be a *standard Brownian motion*. Since $EW_sW_t = s \wedge t = \int_0^T 1_{[0,s]}(u) 1_{[0,t]}(u) du$, the (injective) operator

$$S : L^2([0,T], dt) \to \mathscr{C}([0,T]), \quad Sk(t) = \int_0^t k(s) ds$$

provides a factorization of C_W so that we can apply Corollary 2. The orthonormal basis $e_j(t) = \sqrt{2/T} \cos(\pi(j - 1/2)t/T), j \geq 1$ of $L^2([0,T], dt)$ yields the admissible sequence

$$f_j(t) = Se_j(t) = \frac{\sqrt{2T}}{\pi(j - 1/2)} \sin(\frac{\pi(j - 1/2)t}{T}),\ j \geq 1 \tag{3.8}$$

for W (Karhunen-Loève basis of H_W) and $e_j(t) = \sqrt{2/T} \sin(\pi jt/T)$ yields the admissible sequence

$$g_j(t) = \frac{\sqrt{2T}}{\pi j}(1 - \cos(\frac{\pi jt}{T})),\ j \geq 1.$$

Then

$$f_j^1(t) = \frac{1}{\sqrt{2}} f_j(t) = \frac{\sqrt{T}}{\pi(j-1/2)} \sin(\frac{\pi(j-1/2)t}{T}),\ j \geq 1 \tag{3.9}$$
$$f_j^2(t) = \frac{1}{\sqrt{2}} g_j(t) = \frac{\sqrt{T}}{\pi j}(1-\cos(\frac{\pi j t}{T})),\ j \geq 1$$

is a Parseval frame for H_W and hence admissible for W. The trigonometric basis $e_0(t) = 1/\sqrt{T}$, $e_{2j}(t) = \sqrt{2/T}\cos(2\pi jt/T), e_{2j-1}(t) = \sqrt{2/T}\sin(2\pi jt/T)$ of $L^2([0,T],dt)$ yields the admissible sequence

$$f_0(t) = \frac{t}{\sqrt{T}}, f_{2j}(t) = \frac{\sqrt{T}}{\sqrt{2}\pi j}\sin(\frac{2\pi jt}{T}), \tag{3.10}$$
$$f_{2j-1}(t) = \frac{\sqrt{T}}{\sqrt{2}\pi j}(1-\cos(\frac{2\pi jt}{T})), j \geq 1$$

(Paley-Wiener basis of H_W). By Proposition 4, all these admissible sequences for W (with $f_{2j} := f_j^1, f_{2j-1} := f_j^2$, say in (3.9)) are rate optimal.

Assume that the wavelet system $2^{j/2}\psi(2^j \cdot -k), j,k \in \mathbb{Z}$ is an orthonormal basis (or only a Parseval frame) for $L^2(R,dt)$. Then the restrictions of these functions to [0,T] clearly provide a Parseval frame for $L^2([0,T],dt)$ so that the sequence

$$f_{j,k}(t) = S(2^{j/2}\psi(2^j \cdot -k))(t) = 2^{-j/2}\int_{-k}^{2^j t-k} \psi(u)du, j,k \in \mathbb{Z}$$

is admissible for W. If $\psi \in L^1(R,dt)$ and $\Psi(x) := \int_{-\infty}^x \psi(u)du$, then this admissible sequence takes the form

$$f_{j,k}(t) = 2^{-j/2}(\Psi(2^j t-k) - \Psi(-k)), j,k \in \mathbb{Z}. \tag{3.11}$$

• We consider the Dzaparidze-van Zanten expansion of the *fractional Brownian motion* $X = (X_t)_{t\in[0,T]}$ with Hurst index $\rho \in (0,1)$ and covariance function

$$EX_sX_t = \frac{1}{2}(s^{2\rho} + t^{2\rho} - |s-t|^{2\rho}).$$

These authors discovered in [5] for $T = 1$ a time domain representation

$$EX_sX_t = (g_s^1, g_t^1)_K + (g_s^2, g_t^2)_K$$

with $K = L^2([0,1],dt)$ and kernels $g_t^i \in L^2([0,1],dt)$. Hence by Lemma 3, the operator

$$S : L^2([0,1],dt) \oplus L^2([0,1],dt) \to \mathscr{C}([0,1]),\ S(k_1,k_2)(t) = \int_0^1 g_t^1(s)k_1(s)ds + \int_0^1 g_t^2(s)k_2(s)ds$$

provides a factorization of C_X so that for every pair of orthonormal bases $(e_j^1)_{j\geq 1}$ and $(e_j^2)_{j\geq 1}$ of $L^2([0,1],dt)$,

$$f_j^i(t) = \int_0^1 g_t^i(s)e_j^i(s)ds,\ j \geq 1,\ i = 1,2$$

is admissible in $\mathscr{C}([0,1])$ for X. By Corollary 2, this is a consequence of the above representation of the covariance function (and needs no extra work). Then Dzaparidze and van Zanten [5] could calculate f_j^i explicitely for the Fourier-Bessel basis of order $-\rho$ and $1-\rho$, respectively and arrived at the admissible family in $\mathscr{C}([0,1])$

$$\begin{aligned} f_j^1(t) &= \frac{c_\rho\sqrt{2}}{|J_{1-\rho}(x_j)|\, x_j^{\rho+1}}\sin(x_j t), j\geq 1 \\ f_j^2(t) &= \frac{c_\rho\sqrt{2}}{|J_{-\rho}(y_j)|\, y_j^{\rho+1}}(1-\cos(y_j t)), j\geq 1 \end{aligned}$$

where J_ν denotes the Bessel function of the first kind of order ν, $0<x_1<x_2<\dots$ are the positive zeros of $J_{-\rho}, 0<y_1<y_2<\dots$ the positive zeros of $J_{1-\rho}$ and $c_\rho^2=\Gamma(1+2\rho)\sin(\pi\rho)/\pi$. Consequently, by self-similarity of X, the sequence

$$\begin{aligned} f_j^1(t) &= \frac{T^\rho c_\rho\sqrt{2}}{|J_{1-\rho}(x_j)|\, x_j^{\rho+1}}\sin(\frac{x_j t}{T}), j\geq 1 \\ f_j^2(t) &= \frac{T^\rho c_\rho\sqrt{2}}{|J_{-\rho}(y_j)|\, y_j^{\rho+1}}(1-\cos(\frac{y_j t}{T})), j\geq 1 \end{aligned} \tag{3.12}$$

in $\mathscr{C}([0,T])$ is admissible for X. Using Lemma 1, one can deduce (also without extra work)

$$EX_sX_t=\sum_{j=1}^{\infty}f_j^1(s)f_j^1(t)+\sum_{j=1}^{\infty}f_j^2(s)f_j^2(t)$$

uniformly in $(s,t)\in[0,T]^2$. Rate optimality of (3.12) (using an arrangement like $f_{2j}:=f_j^1, f_{2j-1}:=f_j^2$) is shown in [6] based on the work [8] and is also an immediate consequence of Proposition 4 since

$$x_j\sim y_j\sim\pi j, J_{1-\rho}(x_j)\sim J_{-\rho}(y_j)\sim\frac{\sqrt{2}}{\pi}j^{-1/2}$$

(see [5]), and the eigenvalues satisfy $\lambda_j\sim c j^{-(1+2\rho)}$ as $j\to\infty$ (see [3], [12]).

In the ordinary Brownian motion case $\rho=1/2$, (3.12) coincides with (3.9). The interesting extension of (3.10) to fractional Brownian motions is discussed in [7] and extensions of the wavelet expansion (3.11) can be found in [1], [14].

• Let $X=(X_t)_{t\in[0,T]}$ be *Brownian bridge* with covariance

$$EX_sX_t=s\wedge t-\frac{st}{T}=\int_0^T(1_{[0,s]}(u)-\frac{s}{T})(1_{[0,t]}(u)-\frac{t}{T})du.$$

By Lemma 3, the operator

$$S:L^2([0,T],dt)\to\mathscr{C}([0,T]), Sk(t)=\int_0^t k(s)ds-\frac{t}{T}\int_0^T k(s)ds$$

provides a factorization of C_X and $\ker S = \text{span}\{1_{[0,T]}\}$. The choice $e_j(t) = \sqrt{2/T}\cos(\pi j t/T), j \geq 1$ of an orthonormal basis of $(\ker S)^{\perp}$ yields admissibility of

$$f_j(t) = Se_j(t) = \frac{\sqrt{2T}}{\pi j}\sin(\frac{\pi j t}{T}), j \geq 1 \tag{3.13}$$

for X (Karhunen-Loève basis of H_X). By Proposition 4, this sequence is rate optimal.

• One considers the *stationary Ornstein-Uhlenbeck process* as the solution of the Langevin equation

$$dX_t = -\alpha X_t dt + \sigma dW_t, t \in [0, T]$$

with X_0 independent of $W$and $N(0, \frac{\sigma 2}{2\alpha})$-distributed, $\sigma > 0, \alpha > 0$. It admits the explicit representation

$$X_t = e^{-\alpha t}X_0 + \sigma e^{-\alpha t}\int_0^t e^{\alpha s} dW_s$$

and

$$EX_sX_t = \frac{\sigma^2}{2\alpha}e^{-\alpha|s-t|} = \frac{\sigma^2}{2\alpha}e^{-\alpha(s+t)} + \sigma 2 e^{-\alpha(s+t)}\int_0^{s\wedge t} e^{2\alpha u}du.$$

Thus the (injective) operator

$$S : R \oplus L^2([0, T], dt) \to \mathscr{C}([0, T]), S(c, k)(t) = \frac{c\sigma}{\sqrt{2\alpha}}e^{-\alpha t} + \sigma\int_0^t e^{-\alpha(t-s)}k(s)ds$$

provides a factorization of C_X so that for every Parseval frame $(e_j)_{j\geq 1}$ for $L^2([0, T], dt)$, the functions

$$f_0(t) = \frac{\sigma}{\sqrt{2\alpha}}e^{-\alpha t},\ f_j(t) = \sigma\int_0^t e^{-\alpha(t-s)}e_j(s)ds,\ j \geq 1 \tag{3.14}$$

provide an admissible sequence for X. For instance the choice of the orthonormal basis $\sqrt{2/T}\cos(\pi(j - 1/2)t/T), j \geq 1$ implies that (3.14) is rate optimal. This follows from Lemma 1 in [13] and Proposition 4.

Another representation is given by the Lamperti transformation $X = V(W)$ for the linear continuous operator

$$V : \mathscr{C}([0, e^{2\alpha T}]) \to \mathscr{C}([0, T],\ Vx(t) = \frac{\sigma}{\sqrt{2\alpha}}e^{-\alpha t}x(e^{2\alpha t}).$$

The admissible sequence (f_j) in $\mathscr{C}([0, e^{2\alpha T}])$ for $(W_t)_{t\in[0,e^{2\alpha T}]}$ from (4.1) yields the admissible sequence

$$\tilde{f}_j(t) = Vf_j(t) = \frac{\sigma}{\sqrt{\alpha}\pi(j - 1/2)}e^{\alpha(T-t)}\sin(\pi(j - 1/2)e^{-2\alpha(T-t)}), j \geq 1 \tag{3.15}$$

for X. By Proposition 4, the sequence (3.15) is rate optimal.

• Sheet versions can be deduced from Proposition 3 and Corollary 4 (and need no extra work).

4 Optimal expansion of a class of stationary Gaussian processes

4.1 Expansion of fractional Ornstein-Uhlenbeck processes

The fractional Ornstein-Uhlenbeck process $X^\rho = (X_t^\rho)_{t\in\mathbb{R}}$ of index $\rho \in (0,2)$ is a continuous stationary centered Gaussian process having the covariance function

$$EX_s^\rho X_t^\rho = e^{-\alpha|s-t|^\rho}, \alpha > 0. \tag{4.1}$$

We derive explicit optimal expansions of X^ρ for $\rho \le 1$. Let

$$\gamma^\rho : R \to R, \gamma^\rho(t) = e^{-\alpha|t|^\rho}$$

and for a given $T > 0$, set

$$\beta_0(\rho) := \frac{1}{2T}\int_{-T}^{T} \gamma^\rho(t)dt, \beta_j(\rho) := \frac{1}{T}\int_{-T}^{T} \gamma^\rho(t)\cos(\pi jt/T)dt, j \ge 1. \tag{4.2}$$

Theorem 3. *Let $\rho \in (0,1]$. Then $\beta_j(\rho) > 0$ for every $j \ge 0$ and the sequence*

$$f_0 = \sqrt{\beta_0(\rho)}, \quad f_{2j} = \sqrt{\beta_j(\rho)}\cos(\pi jt/T), \quad f_{2j-1}(t) = \sqrt{\beta_j(\rho)}\sin(\pi jt/T), j \ge 1 \tag{4.3}$$

is admissible for X^ρ in $\mathscr{C}([0,T])$. Furthermore,

$$l_n(X^\rho) \approx n^{-\rho/2}(\log n)^{1/2} \quad as \quad n \to \infty$$

and the sequence (4.3) is rate optimal.

Proof. Since γ^ρ is of bounded variation (and continuous) on [-T,T], it follows from the Dirichlet criterion that its (classical) Fourier series converges pointwise to γ^ρ on [-T,T], that is using symmetry of γ^ρ,

$$\gamma^\rho(t) = \beta_0(\rho) + \sum_{j=1}^{\infty} \beta_j(\rho)\cos(\pi jt/T), t \in [-T,T].$$

Thus one obtains the representation

$$EX_sX_t = \gamma^\rho(s-t) = \beta_0(\rho) + \sum_{j=1}^{\infty} \beta_j(\rho)[\cos(\pi js/T)\cos(\pi jt/T) + \sin(\pi js/T)\sin(\pi jt/T)], s,t \in [0,T]. \tag{4.4}$$

This is true for every $\rho \in (0,2)$. If $\rho = 1$, then integration by parts yields

$$\beta_0(1) = \frac{1-e^{-\alpha T}}{\alpha T}, \quad \beta_j(1) = \frac{2\alpha T(1-e^{-\alpha T}(-1)^j)}{\alpha^2T^2 + \pi^2 j2}, \ j \ge 1. \tag{4.5}$$

In particular, we obtain $\beta_j(1) > 0$ for every $j \ge 0$. If $\rho \in (0,1)$, then $\gamma^\rho_{|[0,\infty)}$ is the Laplace transform of a suitable one-sided strictly ρ-stable distribution with Lebesgue-density q_ρ. Consequently, for

$j \geq 1$,

$$\beta_j(\rho) = \frac{2}{T}\int_0^T e^{-\alpha t^\rho} \cos(\pi j t/T)dt \tag{4.6}$$
$$= \int_0^\infty \frac{2}{T}\int_0^T e^{-tx} \cos(\pi j t/T)dt q_\rho(x)dx$$
$$= \int_0^\infty \frac{2xT(1-e^{-xT}(-1)^j)}{x^2T^2+\pi^2 j^2} q_\rho(x)dx.$$

Again, $\beta_j(\rho) > 0$ for every $j \geq 0$. It follows from (4.4) and Corollary 1(a) that the sequence $(f_j)_{j\geq 0}$ defined in (4.3) is admissible for X^ρ in $\mathscr{C}([0,T])$.

Next we investigate the asymptotic behaviour of $\beta_j(\rho)$ as $j \to \infty$ for $\rho \in (0,1)$. The spectral measure of X^ρ still for $\rho \in (0,2)$ is a symmetric ρ-stable distribution with continuous density p_ρ so that

$$\gamma^\rho(t) = \int_{\mathbb{R}} e^{itx} p_\rho(x)dx$$
$$= 2\int_0^\infty \cos(tx)p_\rho(x)dx, \ t \in R$$

and the spectral density satisfies the high-frequency condition

$$p_\rho(x) \sim c(\rho)x^{-(1+\rho)} \quad \text{as} \quad x \to \infty \tag{4.7}$$

where

$$c(\rho) = \frac{\alpha\Gamma(1+\rho)\sin(\pi\rho/2)}{\pi}.$$

Since by the Fourier inversion formula

$$p_\rho(x) = \frac{1}{\pi}\int_0^\infty \gamma^\rho(t)\cos(tx)dt, \ x \in R,$$

we obtain for $j \geq 1$,

$$\beta_j(\rho) = \frac{2}{T}\int_0^T \gamma^\rho(t)\cos(\pi j t/T)dt$$
$$= \frac{2}{T}\left(\int_0^\infty \gamma^\rho(t))\cos(\pi j t/T)dt - \int_T^\infty \gamma^\rho(t)\cos(\pi j t/T)dt\right)$$
$$= \frac{2\pi}{T}p_\rho(\pi j/T) - \frac{2}{T}\int_T^\infty \gamma^\rho(t))\cos(\pi j t/T)dt$$

Integrating twice by parts yields

$$\int_T^\infty \gamma^\rho(t)\cos(\pi j t/T)dt = O(j^{-2})$$

for any $\rho \in (0,2)$ so that for $\rho \in (0,1)$

$$\begin{aligned} \beta_j(\rho) &\sim \frac{2\pi}{T} p_\rho(\pi j/T) \\ &\sim \frac{2\pi T^\rho c(\rho)}{(\pi j)^{1+\rho}} \\ &= \frac{2\alpha T^\rho \Gamma(1+\rho)\sin(\pi\rho/2)}{(\pi j)^{1+\rho}} \quad \text{as} \quad j \to \infty. \end{aligned} \tag{4.8}$$

We deduce from (4.5) and (4.8) that the admissible sequence (4.3) satisfies the conditions (A2) and (A3) from Proposition 4 with parameters $\vartheta = (1+\rho)/2$, $\gamma = 0$, $a = 1$ and $b = (1-\rho)/2$. Furthermore, by Theorem 3 in Rosenblatt [16], the asymptotic behaviour of the eigenvalues of the covariance operator of X^ρ on $L^2([0,T], dt)$ (see (3.4)) for $\rho \in (0,2)$ is as follows:

$$\lambda_j \sim \frac{2T^{1+\rho}\pi c(\rho)}{(\pi j)^{1+\rho}} \quad \text{as} \quad j \to \infty. \tag{4.9}$$

Therefore, the remaining assertions follow from Proposition 4. □

Here are some comments on the above theorem.

First, note that the admissible sequence (4.3) is not an orthonormal basis for $H = H_{X^\rho}$ but only a Parseval frame at least in case $\rho = 1$. In fact, it is well known that for $\rho = 1$,

$$||h||_H^2 = \frac{1}{2}(h(0)^2 + h(T)^2) + \frac{1}{2\alpha}\int_0^T (h^{'}(t)^2 + \alpha^2 h(t)^2)dt$$

so that *e.g.*

$$||f_{2j-1}||_H^2 = \frac{1 - e^{-\alpha T}(-1)^j}{2} < 1.$$

A result corresponding to Theorem 3 for fractional Ornstein-Uhlenbeck sheets on $[0,T]^d$ with covariance structure

$$EX_sX_t = \prod_{i=1}^d e^{-\alpha_i|s_i-t_i|^{\rho_i}},\ \alpha_i > 0,\ \rho_i \in (0,1] \tag{4.10}$$

follows from Corollary 4.

As a second comment, we wish to emphasize that, unfortunately, in the nonconvex case $\rho \in (1,2)$ it is not true that $\beta_j(\rho) \geq 0$ for every $j \geq 0$ so that the approach of Theorem 3 no longer works. In fact, starting again from

$$\beta_j(\rho) = \frac{2\pi}{T} p_\rho(\pi_j/T) - \frac{2}{T}\int_T^\infty \gamma^\rho(t)\cos(\pi j t/T)dt,$$

three integrations by parts show that

$$\begin{aligned} \beta_j(\rho) &= \tfrac{2\pi}{T} p_\rho(\pi j/T) + \tfrac{(-1)^{j+1} 2T\alpha\rho e^{-\alpha T^\rho} T^{\rho-1}}{(\pi j)^2} + O(j^{-3}) \\ &= \tfrac{(-1)^{j+1} 2T\alpha\rho e^{-\alpha T^\rho} T^{\rho-1}}{(\pi j)^2} + O(j^{-(1+\rho)}),\ j \to \infty. \end{aligned}$$

This means that for any $T > 0$, the $2T$-periodic extension of $\gamma^{\rho}_{|[-T,T]}$ is not nonnegative definite for $\rho \in (1,2)$ in contrast to the case $\rho \in (0,1]$.

4.2 Expansion of stationary Gaussian processes with convex covariance function

As concerns admissibility, it is interesting to observe that the convex function $\gamma(t) = e^{-\alpha t^{\rho}}$, $\rho \in (0,1]$ in Theorem 3 can be replaced by any convex on $(0,\infty)$ function γ going to zero at infinity. Some additional natural assumptions related on its regularity at 0 or the rate of decay of its spectral density at infinity then provide the optimality.

Namely, let $X = (X)_{t\in\mathbb{R}}$ be a continuous stationary centered Gaussian process with $EX_sX_t = \gamma(s-t)$, $\gamma : R \to R$. Then γ is continuous, symmetric and nonnegative definite. Set

$$\beta_0 := \frac{1}{2T}\int_{-T}^{T}\gamma(t)dt, \quad \beta_j := \frac{1}{T}\int_{-T}^{T}\gamma(t)\cos(\pi jt/T)dt,\ j \geq 1.$$

Then the extension of Theorem 3 reads as follows.

Theorem 4. (a) Admissiblity. *Assume that the function γ is convex, positive on $(0,\infty)$ and $\gamma(\infty) := \lim_{t\to\infty}\gamma(t) = 0$. Then X admits a spectral density p, $\beta_j \geq 0$ for every $j \geq 0$ and the sequence defined by*

$$f_0 = \sqrt{\beta_0}, \quad f_{2j}(t) = \sqrt{\beta_j}\cos(\pi jt/T), \quad f_{2j-1}(t) = \sqrt{\beta_j}\sin(\pi jt/T), \quad j \geq 1 \tag{4.11}$$

is admissible for the continuous process X in $\mathscr{C}([0,T])$.

Optimality. *Assume furthermore that one of the following two conditions is satisfied by γ for some $\delta \in (1,2]$:*

(A_δ) *$\gamma \in L^1([T,\infty))$, the spectral density $p \in L^2(R,dx)$ and the high-frequency condition*

$$p(x) \sim cx^{-\delta} \ \text{as} \ x \to \infty,$$

or

(B_δ) *$\gamma(t) = \gamma(0) - a|t|^{\delta-1} + b(t)$, $|t| \leq T$ with $a > 0$ and $b : [-T,T] \to R$ is a $(\delta+\eta)$-Hölder function for some $\eta > 0$, null at zero.*

Then,

$$l_n(X) \approx n^{-(\delta-1)/2}(\log n)^{1/2} \quad \text{as} \quad n \to \infty$$

and the sequence (4.11) is rate optimal.

Remarks. • Assumption (A_δ) is related to the spectral density whereas (B_δ) is related to the (right) regularity of γ at 0. In fact, these assumptions are somewhat similar since the rate of decay of p is closely related to the right regularity of γ at 0. So, in practice it mainly depends on which quantity is straightforwardly available for a given process.

• The notion of β-Hölder regularity of a function g means that the $[\beta]$ first derivatives of g do exist on $[-T,T]$ and that $g^{([\beta])}$ is $\beta-[\beta]$-Hölder. In fact, in [16], th enotion is still a bit more general ($g^{([\beta])}$ is assumed to $\beta-[\beta]$-Hölder in a $L^1(dt)$-sense).

Proof. (a) The function γ is positive, convex over $(0,\infty)$ and $\gamma(\infty)=0$ so that γ is in fact a Polya-type function. Hence its right derivative γ' is non-decreasing with $\gamma'(\infty)=0$, the spectral measure of X admits a Lebesgue-density p and

$$\gamma(t)=\int_{(0,\infty)}\left(1-\frac{|t|}{s}\right)^{+}d\nu(s)$$

for all $t\in R$, where ν is a finite Borel measure on $(0,\infty)$ with mass $\gamma(0)$ (see [10], Theorems 4.3.1 and 4.3.3 for details). Therefore, using Fubini's theorem, it is enough to show the positivity of the numbers β_j for functions of the type $\gamma(t)=(1-\frac{|t|}{s})^{+}$, $s\in(0,\infty)$. But in this case an integration by parts yields

$$\beta_0=\frac{T\wedge s}{T}(1-\frac{T\wedge s}{2s})\geq 0 \text{ and } \beta_j=\frac{2T}{s(\pi j)^2}(1-\cos(\pi j(T\wedge s)/T))\geq 0,\quad j\geq 1.$$

Now one proceeds along the lines of the proof of Theorem 3. Since γ is of bounded variation on $[-T,T]$, the representation (4.4) of EX_sX_t is true with $\beta_j(\rho)$ replaced by β_j so that the sequence $(f_j)_{j\geq 0}$ is admissible for X in $\mathscr{C}([0,T])$.

(b) Assume (A_δ). Using $\gamma\in L^1(R,dt)$ and the Fourier inversion formula, one gets for $j\geq 1$

$$\beta_j=\frac{2\pi}{T}p(\pi j/T)-\frac{2}{T}\int_T^\infty\gamma(t)\cos(\pi jt/T)dt.$$

Since $\gamma(\infty)=\gamma'(\infty)=0$, integrating twice by parts yields

$$\int_T^\infty\gamma(t)\cos(\pi jt/T)dt=O(j^{-2}),$$

hence

$$\beta_j=O(j^{-\delta})\quad\text{as}\quad j\to\infty$$

in view of $\delta\leq 2$. Furthermore, the assumption $p\in L^2(R,dx)$ and the high-frequency condition yield

$$\lambda_j\sim c_1j^{-\delta}\quad\text{as}\quad j\to\infty$$

for an appropriate constant $c_1\in(0,\infty)$ (see [16]). Now, one derives from Proposition 4 the remaining assertions.

Assume (B_δ). An integration by parts (using that γ is even) yields

$$\beta_j=-\frac{2}{\pi j}\int_0^T\sin\left(\frac{\pi jt}{T}\right)\gamma'(t)\,dt$$

where γ' denotes the right derivative of γ. Consequently, for $\varepsilon\in(0,T]$, using that $\gamma'\leq 0$ and is non-decreasing,

$$\beta_j=|\beta_j|\leq\frac{2}{\pi j}(\gamma(0)-\gamma(\varepsilon))+\frac{\gamma(0)-\gamma(\varepsilon)}{\varepsilon}\frac{4T}{(\pi j)^2}$$

At this stage, Assumption (B_δ) yields

$$\beta_j \le C\left(\frac{\varepsilon^{\delta-1}}{j} + \frac{\varepsilon^{\delta-2}}{j^2}\right)$$

where C is positive real constant. Setting $\varepsilon = T/j$, implies as expected $\beta_j = O(j^{-\delta})$.
On the other hand, calling upon Theorem 2 in [16], shows that the eigenvalues λ_j of the covariance operator on $L^2([0,T], dt)$ (indexed in decreasing order) satisfy

$$\lambda_j \sim 2\,\kappa\,\Gamma(\delta)\left(\frac{T}{\pi j}\right)^\delta \qquad \text{as } j \to \infty$$

where $\kappa = -a\cos\left(\frac{\pi}{2}\delta\right) > 0$ and Γ denotes the Gamma function. One concludes as above. $\square$

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