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A NOTE ON DIRECTED POLYMERS IN GAUSSIAN ENVIRONMENTS

YUEYUN HU
Département de Mathématiques, Université Paris XIII, 99 avenue J-B Clément, 93430 Villetaneuse, France
email: yueyun@math.univ-paris13.fr

QI-MAN SHAO
Department of Mathematics, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China
email: maqmshao@ust.hk

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Abstract

We study the problem of directed polymers in gaussian environments in $\mathbb{Z}^d$ from the viewpoint of a gaussian family indexed by the set of random walk paths. In the zero-temperature case, we give a numerical bound on the maximum of the Hamiltonian, whereas in the finite temperature case, we establish an equivalence between the *very strong disorder* and the growth rate of the entropy associated to the model.

1 Introduction and main results

1.1 Finite temperature case

Let $(g(i,x))_{i\ge 0, x\in\mathbb{Z}^d}$ be i.i.d. standard real-valued gaussian variables. We denote by **P** and **E** the corresponding probability and expectation with respect to $g(\cdot,\cdot)$. Let $\{S_k, k\ge 0\}$ be a simple symmetric random walk on $\mathbb{Z}^d$, independent of $g(\cdot,\cdot)$. We denote by $\mathbb{P}_x$ the probability measure of $(S_n)_{n\in\mathbb{N}}$ starting at $x\in\mathbb{Z}^d$ and by $\mathbb{E}_x$ the corresponding expectation. We also write $\mathbb{P}=\mathbb{P}_0$ and $\mathbb{E}=\mathbb{E}_0$.

The directed polymer measure in a gaussian random environment, denoted by $\langle\cdot\rangle^{(n)}$, is a random probability measure defined as follows: Let Ω_n be the set of nearest neighbor paths of length n: $\Omega_n \stackrel{\text{def}}{=} \Big\{\gamma : \{1,...,n\}\to\mathbb{Z}^d, |\gamma_k-\gamma_{k-1}|=1, k=2,\dots,n, \gamma_0=0\Big\}$. For any function $F:\Omega_n\to\mathbb{R}_+$,

$$\langle F(S)\rangle^{(n)} \stackrel{\text{def}}{=} \frac{1}{Z_n}\,\mathbb{E}\Big(F(S)e^{\beta H_n(g,S)-\frac{\beta^2 n}{2}}\Big), \quad H_n(g,\gamma)\stackrel{\text{def}}{=}\sum_{i=1}^n g(i,\gamma_i),$$

where $\beta > 0$ denotes the inverse of temperature and Z_n is the partition function:

$$Z_n = Z_n(\beta, g) = \mathbb{E}\Big(e^{\beta H_n(g,S) - \frac{\beta^2 n}{2}}\Big).$$

We refer to Comets, Shiga and Yoshida [3] for a review on directed polymers. It is known (see e.g. [2], [3]) that the so-called free energy, the limit of $\frac{1}{n}\log Z_n$ exists almost surely and in L^1:

$$p(\beta) := \lim_{n\to\infty} \frac{1}{n} \log Z_n,$$

$p(\beta)$ is some constant and $p(\beta) \le 0$ by Jensen's inequality since $\mathbf{E}Z_n = 1$.

A problem in the study of directed polymer is to determine the region of $\{\beta > 0 : p(\beta) < 0\}$, also called the region of *very strong disorder*. It is an important problem, for instance, $p(\beta) < 0$ yields interesting information on the localization of the polymer itself.

By using the F-K-G inequality, Comets and Yoshida [4] showed the monotonicity of $p(\beta)$, therefore the problem is to determine

$$\beta_c := \inf\{\beta > 0 : p(\beta) < 0\}.$$

It has been shown by Imbrie and Spencer [8] that for $d \ge 3$, $\beta_c > 0$ (whose exact value remains unknown). Comets and Vargas [5] proved that (for a wide class of random environments)

$$\beta_c = 0, \qquad \text{if} \qquad d = 1. \tag{1.1}$$

Recently, Lacoin [10], skilfully used the ideas developed in pinned models and solved the problem in the two-dimensional case:

$$\beta_c = 0, \qquad \text{if} \qquad d = 2. \tag{1.2}$$

Moreover, Lacoin [10] gave precise bounds on $p(\beta)$ when $\beta \to 0$ both in one-dimensional and two-dimensional cases.

In this note, we study this problem from the point of view of entropy (see also Birkner [1]). Let

$$e_n(\beta) := \mathbf{E}\Big(Z_n \log Z_n\Big) - (\mathbf{E}Z_n)\log(\mathbf{E}Z_n) = \mathbf{E}\Big(Z_n \log Z_n\Big)$$

be the entropy associated to Z_n (recalling $\mathbf{E}Z_n = 1$).

Theorem 1.1. Let $\beta > 0$. The following limit exits

$$\tilde{e}_\infty(\beta) := \lim_{n\to\infty} \frac{e_n(\beta)}{n} = \inf_{n\ge 1} \frac{e_n(\beta)}{n} \ge 0. \tag{1.3}$$

There is some numerical constant $c_d > 0$, only depending on d, such that the following assertions are equivalent:

(a) $\tilde{e}_\infty(\beta) > 0$.

(b) $\limsup_{n\to\infty} \frac{e_n(\beta)}{\log n} > c_d$.

(c) $p(\beta) < 0$.

The proof of the implication **(b)** $\Longrightarrow$ **(c)** relies on a criterion of $p(\beta) < 0$ (cf. Fact 3.1 in Section 3) developed by Comets and Vargas [5] in a more general settings.

We can easily check **(b)** in the one-dimensional case: In fact, we shall show in the sequel (cf. (3.7)) that in any dimension and for any $\beta > 0$,

$$e_n(\beta) \geq \frac{\beta^2}{2}\,\mathbb{E}(L_n(S^1, S^2)),$$

where S^1 and S^2 are two independent copies of S and $L_n(\gamma,\gamma') = \sum_{k=1}^n 1_{(\gamma_k = \gamma'_k)}$ is the number of common points of two paths γ and γ'. It is well known that $\mathbb{E}(L_n(S^1, S^2))$ is of order $n^{1/2}$ when $d = 1$ and of order $\log n$ when $d = 2$. Therefore **(b)** holds in $d = 1$ and by the implication **(b)** $\Longrightarrow$ **(c)**, we recover Comets and Vargas' result (1.1) in the one-dimensional gaussian environment case.

1.2 Zero temperature case

When $\beta \to \infty$, the problem of directed polymers boils down to the problem of first-passage percolation. Let

$$H_n^* = H_n^*(g) := \max_{\gamma \in \Omega_n} H_n(g,\gamma), \qquad H_n(g,\gamma) = \sum_1^n g(i,\gamma_i),$$

where as before $\Omega_n = \{\gamma : [0,n] \to \mathbb{Z}^d, |\gamma_i - \gamma_{i-1}| = 1, i = 2,\dots,n,\, \gamma_0 = 0\}$.

The problem is to characterize these paths γ which maximize $H_n(g,\gamma)$. See Johansson [9] for the solution of the Poisson points case. We limit here our attention to some explicit bounds on H_n^*. An easy subadditivity argument (see Lemma 2.2) shows that

$$\frac{H_n^*}{n} \to \sup_{n\geq 1} \frac{\mathbf{E}H_n^*}{n} \stackrel{\text{def}}{=} c_d^*, \qquad \text{both } a.s. \text{ and in } L^1.$$

By Slepian's inequality ([12]),

$$\mathbf{E}H_n^* \leq \sqrt{n}\,\mathbf{E}\max_{\gamma\in\Omega_n} Y_\gamma,$$

where $(Y_\gamma)_{\gamma\in\Omega_n}$ is a family of i.i.d. centered gaussian variables of variance 1. Since $\#\Omega_n = (2d)^n$, it is a standard exercise from extreme value theory that

$$\frac{1}{\sqrt{n}}\mathbf{E}\max_{\gamma\in\Omega_n} Y_\gamma \to \sqrt{2\log(2d)}.$$

Hence

$$c_d^* \leq \sqrt{2\log(2d)}.$$

It is a natural problem to ask whether this inequality is strict; In fact, a strict inequality means that the gaussian family $\{H_n(g,\gamma), \gamma \in \Omega_n\}$ is sufficiently correlated to be significantly different from the independent one, exactly as the problem to determine whether $p(\beta) < 0$.

We prove that the inequality is strict by establishing a numerical bound:

Theorem 1.2. For any $d \geq 1$, we have

$$c_d^* \leq \sqrt{2\log(2d) - \frac{(2d-1)}{2d}\Big(1 - \frac{2d-1}{5\pi d}\Big)(1 - \Phi(\sqrt{\log(2d)}))},$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-u^2/2}du$ is the partition function of a standard gaussian variable.

The proofs of Theorems 1.1 and 1.2 are presented in two separate sections.

2 Proof of Theorem 1.2

We begin with several preliminary results. Recall at first the following concentration of measure property of Gaussian processes (see Ibragimov and al. [7]).

Fact 2.1. Consider a function $F : \mathbb{R}^M \to \mathbb{R}$ and assume that its Lipschitz constant is at most A, i.e.

$$|F(x) - F(y)| \le A||x - y|| \qquad (x, y \in \mathbb{R}^M),$$

where $||x||$ denotes the euclidean norm of x. Then if $g = (g_1, \dots, g_M)$ are i.i.d. $\mathcal{N}(0,1)$ we have

$$\mathbf{P}\Big(|F(g) - \mathbf{E}(F(g))| \ge u\Big) \le \exp(-\frac{u^2}{2A^2}) \qquad (u > 0).$$

Lemma 2.2. There exists some positive constant c_d^* such that

$$\frac{H_n^*}{n} \to c_d^* = \sup_{n\ge1} \mathbf{E}(\frac{H_n^*}{n}), \qquad \text{a.s. and in } L^1. \tag{2.1}$$

Moreover,

$$\mathbf{P}\Big(|H_n^* - \mathbf{E}(H_n^*)| > \lambda\Big) \le e^{-\frac{\lambda^2}{2n}}, \qquad \lambda > 0. \tag{2.2}$$

Proof: We prove at first the concentration inequality. Define a function $F : \mathbb{R}^m \to \mathbb{R}$ by

$$F(\mathbf{z}) = \max_{\gamma\in\Omega_n} \sum_{i=1}^n \sum_{|x|\le n} \mathbf{z}_{i,x}\, 1_{(\gamma_i = x)}, \qquad \mathbf{z} = (\mathbf{z}_{i,x})_{1\le i\le n, x\in\mathbb{Z}^d, |x|\le n} \in \mathbb{R}^m.$$

By the Cauchy-Schwarz inequality,

$$\Big|\sum_{i=1}^n \sum_{|x|\le n} z_{i,x}\, \mathbf{1}_{(S_i=x)} - \sum_{i=1}^n \sum_{|x|\le n} z'_{i,x}\, \mathbf{1}_{(S_i=x)}\Big| \le n^{1/2} \Big(\sum_{i=1}^n \sum_{|x|\le n} (z_{i,x} - z'_{i,x})^2\Big)^{1/2}.$$

Hence F is a Lipschitz function: $|F(\mathbf{z^1}) - F(\mathbf{z^2})| \le \sqrt{n}||\mathbf{z^1} - \mathbf{z^2}||$. Note that $H_n^* = F((g_{i,x})_{i,x})$. By the Gaussian concentration inequality Fact 2.1, we get (2.2).

Now we prove that $n \to \mathbf{E}H_n^*$ is superadditive: for $n, k \ge 1$, let $\gamma^* \in \Omega_n$ be a path such that $H_n(g, \gamma^*) = H_n^*$, then

$$H_{n+k}^* \ge H_n(g, \gamma^*) + \max_{\gamma\in\Omega_k} \sum_{i=1}^k g(i+n, \gamma_i + \gamma_n^*),$$

hence by conditioning on $\sigma\{g(i,\cdot), i \le n\}$, we get that

$$\mathbf{E}(H_{n+k}^*) \ge \mathbf{E}(H_n(g,\gamma^*)) + \mathbf{E}(H_k^*) = \mathbf{E}(H_n^*) + \mathbf{E}(H_k^*),$$

which in view of concentration (2.2) implies (2.1). □

For $x, y \in \mathbb{Z}^d$ and $n \ge 0$, we shall denote by $y - x \hookleftarrow n$ when $\mathbb{P}_x(S_n = y) = \mathbb{P}(S_n = y - x) > 0$. Observe that

$$x \hookleftarrow n \text{ if and only if } n - \sum_1^d x_j \equiv 0 \text{ (mod 2) and } \sum_1^d |x_j| \le n,$$

with $x = (x_1, ..., x_d) \in \mathbb{Z}^d$. We shall also write $\sum_{x\hookleftarrow n}$ or $\max_{x\hookleftarrow n}$ to mean that the sum or maximum is taken over those x such that $x \hookleftarrow n$.

Define

$$\phi_n(\lambda) = \log\left(\sum_{x \hookleftarrow n} \mathbf{E} e^{\lambda H^*_{n,x}}\right), \quad \phi^*_n(a) = \sup_{\lambda>0}\big(a\lambda - \phi_n(\lambda)\big).$$

Lemma 2.3. For any $n \geq 1$, Let $\zeta_n \stackrel{\text{def}}{=} \inf\{c > 0 : \phi^*_n(cn) > 0\}$. We have

$$c^*_d \leq \zeta_n \leq c^*_d + 2\sqrt{\frac{d\log(2n+1)}{n}}.$$

Proof: Let $\tau_{n,x}$ be the time and space shift on the environment:

$$g \circ \tau_{n,x}(\cdot,\cdot) = g(n+\cdot, x+\cdot). \tag{2.3}$$

We have for any n, k,

$$H^*_{n+k} = \max_{x \hookleftarrow n}\Big\{\max_{\gamma\in\Omega_n:\gamma_n=x} H_n(g,\gamma) + H^*_k(g\circ\tau_{n,x})\Big\}.$$

Write for simplification $H^*_{n,x} := \max_{\gamma\in\Omega_n:\gamma_n=x} H_n(g,\gamma)$. Then for any $\lambda \in \mathbb{R}$,

$$\begin{aligned}
\mathbf{E} e^{\lambda H^*_{n+k}} &= \mathbf{E} e^{\lambda \max_{x\hookleftarrow n}(H^*_{n,x} + H^*_k(g\circ\tau_{n,x}))} \\
&\leq \mathbf{E}\left(\sum_{x\hookleftarrow n} e^{\lambda H^*_{n,x}} e^{\lambda H^*_k(g\circ\tau_{n,x})}\right) \\
&= \mathbf{E}\left(\sum_{x\hookleftarrow n} e^{\lambda H^*_{n,x}}\right)\mathbf{E} e^{\lambda H^*_k}.
\end{aligned}$$

We get

$$\mathbf{E} e^{\lambda H^*_{jn}} \leq e^{j\phi_n(\lambda)}, \qquad j, n \geq 1, \lambda \in \mathbb{R}.$$

Chebychev's inequality implies that

$$\mathbf{P}\Big(H^*_{jn} > cjn\Big) \leq e^{-j\phi^*_n(cn)}, \qquad c > 0,$$

where $\phi^*_n(a) = \sup_{\lambda>0}\big(a\lambda - \phi_n(\lambda)\big)$. Then for any n and c such that $\phi^*_n(cn) > 0$, $\limsup_{j\to\infty} \dfrac{H^*_{jn}}{jn} \leq c$, a.s. It follows that

$$c^*_d \leq \zeta_n, \qquad \forall n \geq 1.$$

On the other hand, by using the concentration inequality (2.2) and the fact that $\mathbf{E}(H^*_n) \leq nc^*_d$, we get

$$\mathbf{E} e^{\lambda H^*_n} \leq e^{\lambda \mathbf{E} H^*_n} e^{\lambda^2 n/2} \leq e^{\lambda n c^*_d + \lambda^2 n/2}, \quad \lambda > 0.$$

It follows that for any $\lambda > 0$,

$$\phi_n(\lambda) \leq \log \sum_{x\hookleftarrow n} \mathbf{E}^{\lambda H^*_n} \leq \lambda n c^*_d + \frac{\lambda^2}{2} n + 2d\log(2n+1).$$

Choosing $\lambda = 2\sqrt{\frac{d\log(2n+1)}{n}}$, we get $\phi_n(\lambda) < \lambda n(c^*_d + a)$ for any $a > 2\sqrt{\frac{d\log(2n+1)}{n}}$, which means that $\phi^*_n((c^*_d + a)n) > 0$. Hence $\zeta_n \leq c^*_d + a$ and the lemma follows. $\square$

Proof of Theorem 1.2: Let $n=2$ in Lemma 2.3, we want to estimate $\phi_2(\lambda)$.
Let $g, g_0, g_1, g_2, \cdots$ be iid standard gaussian variables. Observe that the possible choices of γ_1 are $(\pm 1, 0, \cdots, 0)$, $(0, \pm 1, 0, \cdots, 0)$, $\cdots$, $(0, \cdots, 0, \pm 1)$, and the possible choices of γ_2 are $(0, 0, \cdots, 0)$, $(\pm 2, 0, \cdots, 0)$, $(\pm 1, \pm 1, 0 \cdots, 0)$, $(\pm 1, 0, \pm 1, 0, \cdots, 0)$, $\cdots$, $(\pm 1, 0, \cdots, 0, \pm 1)$, $\cdots$, $(0, \pm 1, \pm 1, 0, \cdots, 0)$, $\cdots$, $(0, \cdots, \pm 1, \pm 1)$. Therefore

$$
\begin{aligned}
&\sum_{x \leftarrow 2} \mathbf{E} e^{\lambda H^*_{2,x}} \\
&= \mathbf{E} e^{\lambda(g_0 + \max_{1 \le i \le 2d} g_i)} + 2d\mathbf{E} e^{\lambda(g_0 + g_1)} + 4(d-1+d-2+\cdots+1)\mathbf{E} e^{\lambda(g_0 + \max(g_1, g_2))} \\
&= \mathbf{E} e^{\lambda(g_0 + \max_{1 \le i \le 2d} g_i)} + 2d\mathbf{E} e^{\lambda(g_0 + g_1)} + 2d(d-1)\mathbf{E} e^{\lambda(g_0 + \max(g_1, g_2))} \\
&\le \sum_{i=1}^{d} \mathbf{E} e^{\lambda(g_0 + \max(g_{2i-1}, g_{2i}))} + 2dEe^{\lambda(g_0 + g_1)} + 2d(d-1)\mathbf{E} e^{\lambda(g_0 + \max(g_1, g_2))} \\
&= 2d\mathbf{E} e^{\lambda(g_0 + g_1)} + d(2d-1)\mathbf{E} e^{\lambda(g_0 + \max(g_1, g_2))} \\
&= 2de^{\lambda^2} + d(2d-1)e^{\lambda^2/2}\mathbf{E} e^{\lambda \max(g_1, g_2)} \\
&= 2de^{\lambda^2} + d(2d-1)e^{\lambda^2/2} 2e^{\lambda^2/2}\Phi(\lambda/\sqrt{2}) \\
&= e^{\lambda^2}(2d + 2d(2d-1)\Phi(\lambda/\sqrt{2})),
\end{aligned} \tag{2.4}
$$

where we use the fact that

$$
Ee^{\lambda \max(g_1, g_2)} = 2e^{\lambda^2/2}\Phi(\lambda/\sqrt{2}).
$$

In fact, since $\max(g_1, g_2) = (1/2)(g_1 + g_2 + |g_1 - g_2|)$ and $g_1 + g_2$ and $g_1 - g_2$ are independent, we have

$$
\mathbf{E} e^{\lambda \max(g_1, g_2)} = \mathbf{E} e^{\lambda(g_1 + g_2 + |g_1 - g_2|)/2} = e^{\lambda^2/4}\mathbf{E} e^{\lambda |g_1|/\sqrt{2}} = 2e^{\lambda^2/2}\Phi(\lambda/\sqrt{2}),
$$

where we use

$$
\mathbf{E} e^{\lambda |g|} = \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{\lambda x - x^2/2} dx = \frac{2e^{\lambda^2/2}}{\sqrt{2\pi}} \int_0^\infty e^{-(x-\lambda)^2/2} dx = 2e^{\lambda^2/2}\Phi(\lambda).
$$

We conclude from (2.4) that

$$
\begin{aligned}
\phi_2(\lambda) &\le \lambda^2 + \log\Big(2d + 2d(2d-1)\Phi(\lambda/\sqrt{2})\Big) \\
&= \lambda^2 + 2\log(2d) + \log\Big(1 - \frac{(2d-1)}{2d}(1 - \Phi(\lambda/\sqrt{2}))\Big) \\
&\le \lambda^2 + 2\log(2d) - \frac{(2d-1)}{2d}(1 - \Phi(\lambda/\sqrt{2})) \stackrel{\text{def}}{=} h(\lambda)
\end{aligned}
$$

Now consider the function

$$
h^*(2c) := \sup_{\lambda > 0}(2c\lambda - h(\lambda)), \qquad c > 0.
$$

Clearly $h^*(2c) \le \phi^*(2c)$ for any $c > 0$. Let us study the $h^*(2c)$: The maximal achieves when

$$
2c = h'(\lambda)
$$

That is

$$
2c = 2\lambda + \frac{(2d-1)}{2d\sqrt{2\pi}} e^{-\lambda^2/4}
$$

Now choose c so that

$$2c\lambda = h(\lambda) = \lambda^2 + 2\log(2d) - \frac{(2d-1)}{2d}(1-\Phi(\lambda/\sqrt{2})) \tag{2.5}$$

Let

$$a = \frac{(2d-1)}{4d\sqrt{2\pi}}e^{-\lambda^2/4},\ b = \frac{(2d-1)}{2d}(1-\Phi(\lambda/\sqrt{2})).$$

Then

$$2c = 2\lambda + 2a, 2c\lambda = \lambda^2 + 2\log(2d) - b,$$

which gives

$$\lambda^2 + 2a\lambda = 2\log(2d) - b$$

or

$$c^2 = (\lambda + a)^2 = 2\log(2d) - b + a^2.$$

It is easy to see that

$$(1-\Phi(t)) \geq \frac{5}{16}e^{-t^2},\ \text{for } t > 0$$

Hence

$$\frac{16}{5}\Big(\frac{2d-1}{4d\sqrt{2\pi}}\Big)^2(1-\Phi(\lambda/\sqrt{2})) \geq a^2$$

and

$$\begin{aligned}
c^2 &= 2\log(2d) - \frac{(2d-1)}{2d}(1-\Phi(\lambda/\sqrt{2})) + a^2 \\
&\leq 2\log(2d) - \Big(\frac{(2d-1)}{2d} - \Big(\frac{2d-1}{4d\sqrt{2\pi}}\Big)\frac{16}{5}\Big)(1-\Phi(\lambda/\sqrt{2})) \\
&= 2\log(2d) - \frac{(2d-1)}{2d}\Big(1 - \frac{2d-1}{5\pi d}\Big)(1-\Phi(\lambda/\sqrt{2})) \\
&\leq 2\log(2d) - \frac{(2d-1)}{2d}\Big(1 - \frac{2d-1}{5\pi d}\Big)(1-\Phi(\sqrt{\log(2d)})) \overset{\text{def}}{=} \widetilde{c}^2,
\end{aligned}$$

Here in the last inequality, we used the fact that $\lambda \leq \sqrt{2\log(2d)}$. Recall (c,λ) satisfying (2.5). For any $\varepsilon > 0$, $\phi_2^*(2(\widetilde{c}+\varepsilon)) \geq h^*(2(\widetilde{c}+\varepsilon)) \geq 2(c+\varepsilon)\lambda - h(\lambda) = 2\varepsilon\lambda > 0$. It follows $\zeta_2 \leq \widetilde{c} + \varepsilon$ and hence $c_d^* \leq \widetilde{c}$. □

3 Proof of Theorem 1.1

Let

$$Z_m(x) := \mathbb{E}\Big(1_{(S_m=x)}e^{\beta H_m(g,S)-\beta^2 m/2}\Big), \qquad m \geq 1, x \in \mathbb{Z}^d.$$

Fact 3.1 (Comets and Vargas [5]). If there exists some $m \geq 1$ such that

$$\mathbf{E}\Big(\sum_x Z_m(x)\log Z_m(x)\Big) \geq 0, \tag{3.1}$$

then $p(\beta) < 0$.

In fact, the case $\mathbf{E}\Big(\sum_x Z_m(x)\log Z_m(x)\Big) = 0$ follows from their Remark 3.5 in Comets and Vargas (2006), whereas if $\mathbf{E}\Big(\sum_x Z_m(x)\log Z_m(x)\Big) > 0$, which means the derivative of $\theta \to \mathbf{E}\sum_x Z_m^\theta(x)$ at 1 is positif, hence for some $\theta < 1$, $\mathbf{E}\sum_x Z_m^\theta(x) < 1$ and again by Comets and Vargas (2006) (lines before Remark 3.4), we have $p(\beta) < 0$.

We try to check (3.1) in the sequel:

Lemma 3.2. There exists some constant $c_d > 0$ such that for any $m \geq 1$,

$$\mathbf{E}\Big(\sum_x Z_m(x)\log Z_m(x)\Big) \geq \mathbf{E}\Big(Z_m \log Z_m\Big) - c_d \log m.$$

Proof: Write

$$u_m(x) := \frac{Z_m(x)}{Z_m} = \langle 1_{(S_m=x)}\rangle^{(m)}. \tag{3.2}$$

We have

$$\begin{aligned}\mathbf{E}\Big(\sum_x Z_m(x)\log Z_m(x)\Big) &= \mathbf{E}\left(Z_m \sum_x u_m(x)\Big[\log Z_m + \log u_m(x)\Big]\right)\\ &= \mathbf{Q}\Big(\sum_x u_m(x)\log u_m(x)\Big) + \mathbf{Q}\Big(\log Z_m\Big),\end{aligned}$$

where the probability measure $\mathbf{Q} = \mathbf{Q}^{(\beta)}$ is defined by

$$d\mathbf{Q}|_{\mathscr{F}_n^g} = Z_n\, d\mathbf{P}|_{\mathscr{F}_n^g}, \qquad \forall\ n \geq 1, \tag{3.3}$$

with $\mathscr{F}_n = \sigma\{g(i,\cdot), i \leq n\}$. By convexity,

$$\mathbf{Q}\Big(\sum_x u_m(x)\log u_m(x)\Big) \geq \sum_x \mathbf{Q}(u_m(x))\log \mathbf{Q}(u_m(x)).$$

Note that $\mathbf{Q}(u_m(x)) = \mathbf{E}(Z_m(x)) = \mathbb{P}(S_m = x)$ and $\sum_x \mathbb{P}(S_m = x)\log \mathbb{P}(S_m = x) \sim -c_d' \log m$ for some positive constant c_d'. Assembling all the above, we get a constant $c_d > 0$ such that

$$\mathbf{E}\Big(\sum_x Z_m(x)\log Z_m(x)\Big) \geq \mathbf{Q}\Big(\log Z_m\Big) - c_d \log m$$

for all $m \geq 1$, as desired. $\square$

Let μ be a Gaussian measure on $\mathbb{R}^m$. The logarithmic Sobolev inequality says (cf. Gross [6], Ledoux [11]): for any $f : \mathbb{R}^m \to \mathbb{R}$,

$$\int f^2 \log f^2 d\mu - \left(\int f^2 d\mu\right)\log\left(\int f^2 d\mu\right) \leq 2\int \left|\nabla f\right|^2 d\mu. \tag{3.4}$$

Using the above inequality, we have

Lemma 3.3. Let S^1 and S^2 be two independent copies of S. We have

$$\begin{aligned} e_n(\beta) &\leq \frac{\beta^2}{2}\mathbf{E}\Big(Z_n \langle L_n(S^1,S^2)\rangle_2^{(n)}\Big), &(3.5)\\ \frac{d}{d\beta}e_n(\beta) &= \beta\mathbf{E}\Big(Z_n \langle L_n(S^1,S^2)\rangle_2^{(n)}\Big), &(3.6)\end{aligned}$$

where $\langle L_n(S^1,S^2)\rangle_2^{(n)} = \frac{1}{Z_n^2}\mathbb{E}\big(e^{\beta H_n(g,S^1)+\beta H_n(g,S^2)-\beta^2 n}L_n(S^1,S^2)\big)$. Consequently,

$$e_n(\beta) \geq \frac{\beta^2}{2}\,\mathbb{E}(L_n(S^1,S^2)). \tag{3.7}$$

Proof: Taking

$$f(\mathbf{z}) = \sqrt{\mathbb{E}\exp\left(\beta\sum_{i=1}^n\sum_x z(i,x)\mathbf{1}_{(S_i=x)} - \frac{\beta^2}{2}n\right)}, \qquad \mathbf{z} = (z(i,x), 1\leq i\leq n, x\hookleftarrow i).$$

Note that $Z_n = f^2(\mathbf{g})$ with $\mathbf{g} = (g(i,x), 1\leq i\leq n, x\hookleftarrow i)$. Applying the log-Sobolev inequality yields the first estimate (3.5).
The another assertion follows from the integration by parts: for a standard gaussian variable g and any derivable function ψ such that both $g\psi(g)$ and $\psi'(g)$ are integrable, we have

$$\mathbf{E}(g\psi(g)) = \mathbf{E}(\psi'(g)).$$

Elementary computations based on the above formula yield (3.6). The details are omitted. From (3.5) and (3.6), we deduce that the function $\beta\to\frac{e_n(\beta)}{\beta^2}$ is nondecreasing. On the other hand, it is elementary to check that $\lim_{\beta\to 0}\frac{e_n(\beta)}{\beta^2} = \frac{1}{2}\mathbb{E}(L_n(S^1,S^2))$, which gives (3.7) and completes the proof of the lemma. $\square$

If P and Q are two probability measures on $(\Omega,\mathscr{F})$, the relative entropy is defined by

$$H(Q\,|\,P) \stackrel{\text{def}}{=} \int \log\frac{dQ}{dP}\,dQ,$$

where the expression has to be understood to be infinite if Q is not absolutely continuous with respect to P or if the logarithm of the derivative is not integrable with respect to Q. The following entropy inequality is well-known:

Lemma 3.4. For any $A\in\mathscr{F}$, we have

$$\log\frac{P(A)}{Q(A)} \geq -\frac{H(Q\,|\,P)+e^{-1}}{Q(A)}.$$

This inequality is useful only if $Q(A)\sim 1$. Recall (3.3) for the definition of $\mathbf{Q}$. Note that for any $\delta>0$, $\mathbf{Q}\Big(Z_n\geq\frac{\delta}{1+\delta}\Big)\geq\frac{1}{1+\delta}$, it follows that

$$\mathbf{P}\Big(Z_n\geq\frac{\delta}{1+\delta}\Big)\geq\frac{1}{1+\delta}e^{-(1+\delta)e^{-1}}\exp\Big(-(1+\delta)e_n(\beta)\Big). \tag{3.8}$$

Now we give the proof of Theorem 1.1:

Proof of Theorem 1.1: We prove at first (1.3) by subadditivity argument: Recalling (3.2) and (2.3). By markov property of S, we have that for all $n,m\geq 1$,

$$Z_{n+m} = Z_n\sum_x u_n(x)Z_m(x, g\circ\tau_{n,x}).$$

Let $\psi(x) = x\log x$. We have

$$\begin{aligned} Z_{n+m}\log Z_{n+m} &= \psi(Z_n)\sum_x u_n(x)Z_m(x, g\circ\tau_{n,x}) + Z_n\,\psi\left(\sum_x u_n(x)Z_m(x, g\circ\tau_{n,x})\right) \\ &\le \psi(Z_n)\sum_x u_n(x)Z_m(x, g\circ\tau_{n,x}) + Z_n\sum_x u_n(x)\psi(Z_m(x, g\circ\tau_{n,x})), \end{aligned}$$

by the convexity of ψ. Taking expectation gives (1.3).
To show the equivalence between **(a), (b), (c)**, we remark at first that the implication **(b)** $\Rightarrow$ **(c)** follows from Lemma 3.2 and (3.1). It remains to show the implication **(c)** $\Rightarrow$ **(a)**. Assume that $p(\beta) < 0$. The superadditivity says that

$$p(\beta) = \sup_{n\ge 1} p_n(\beta), \qquad \text{with } p_n(\beta) := \frac{1}{n}\mathbf{E}(\log Z_n).$$

On the other hand, the concentration of measure (cf. [2]) says that

$$\mathbf{P}\Big(\Big|\frac{1}{n}\log Z_n - p_n(\beta)\Big| > u\Big) \le \exp\Big(-\frac{nu^2}{2\beta^2}\Big), \quad \forall u > 0.$$

It turns out for $\alpha > 0$,

$$\begin{aligned} \mathbf{E}\Big(Z_n^\alpha\Big) &= e^{\alpha\mathbf{E}(\log Z_n)}\, e^{\alpha(\log Z_n - \mathbf{E}(\log Z_n))} \\ &\le e^{\alpha\mathbf{E}(\log Z_n)}\, e^{\alpha^2\beta^2 n/2} \\ &\le e^{\alpha p(\beta)n + \alpha^2\beta^2 n/2}. \end{aligned}$$

By choosing $\alpha = -p(\beta)/\beta^2$ (note that $\alpha > 0$), we deduce from the Chebychev's inequality that

$$\mathbf{P}\Big(Z_n > \frac{1}{2}\Big) \le 2^\alpha\, e^{-p^2(\beta)n/(2\beta^2)},$$

which in view of (3.8) with $\delta = 1$ imply that $\liminf_{n\to\infty}\frac{1}{n}e_n(\beta) \ge \frac{p(\beta)^2}{4\beta^2}$. □

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