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GEOMETRIC INTERPRETATION OF HALF-PLANE CAPACITY

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Abstract

Schramm-Loewner Evolution describes the scaling limits of interfaces in certain statistical mechanical systems. These interfaces are geometric objects that are not equipped with a canonical parametrization. The standard parametrization of SLE is via half-plane capacity, which is a conformal measure of the size of a set in the reference upper half-plane. This has useful harmonic and complex analytic properties and makes SLE a time-homogeneous Markov process on conformal maps. In this note, we show that the half-plane capacity of a hull A is comparable up to multiplicative constants to more geometric quantities, namely the area of the union of all balls centered in A tangent to $\mathbb{R}$, and the (Euclidean) area of a 1-neighborhood of A with respect to the hyperbolic metric.

1 Introduction

Suppose A is a bounded, relatively closed subset of the upper half plane $\mathbb{H}$. We call A a compact $\mathbb{H}$-hull if A is bounded and $\mathbb{H}\setminus A$ is simply connected. The *half-plane capacity* of A, $\mathrm{hcap}(A)$, is defined in a number of equivalent ways (see [1], especially Chapter 3). If g_A denotes the unique conformal

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transformation of $\mathbb{H} \setminus A$ onto $\mathbb{H}$ with $g_A(z) = z + o(1)$ as $z \to \infty$, then g_A has the expansion

$$g_A(z) = z + \frac{\mathrm{hcap}(A)}{z} + O(|z|^{-2}), \quad z \to \infty.$$

Equivalently, if B_t is a standard complex Brownian motion and $\tau_A = \inf\{t \geq 0 : B_t \notin \mathbb{H} \setminus A\}$,

$$\mathrm{hcap}(A) = \lim_{y \to \infty} y \, \mathbb{E}^{iy} \left[\mathrm{Im}(B_{\tau_A}) \right].$$

Let $\mathrm{Im}[A] = \sup\{\mathrm{Im}(z) : z \in A\}$. Then if $y \geq \mathrm{Im}[A]$, we can also write

$$\mathrm{hcap}(A) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathbb{E}^{x+iy} \left[\mathrm{Im}(B_{\tau_A}) \right] dx.$$

These last two definitions do not require $\mathbb{H} \setminus A$ to be simply connected, and the latter definition does not require A to be bounded but only that $\mathrm{Im}[A] < \infty$.

For $\mathbb{H}$-hulls (that is, for relatively closed A for which $\mathbb{H} \setminus A$ is simply connected), the half-plane capacity is comparable to a more geometric quantity that we define. This is not new (the second author learned it from Oded Schramm in oral communication), but we do not know of a proof in the literature[3]. In this note, we prove the fact giving (nonoptimal) bounds on the constant. We start with the definition of the geometric quantity.

Definition 1. *For an $\mathbb{H}$-hull A, let* hsiz(A) *be the* 2*-dimensional Lebesgue measure of the union of all balls centered at points in A that are tangent to the real line. In other words*

$$\mathrm{hsiz}(A) = \mathrm{area}\left[\bigcup_{x+iy \in A} \mathcal{B}(x+iy, y) \right],$$

where $\mathcal{B}(z, \epsilon)$ denotes the disk of radius ϵ about z.

In this paper, we prove the following.

Theorem 1. *For every $\mathbb{H}$-hull A,*

$$\frac{1}{66} \mathrm{hsiz}(A) < \mathrm{hcap}(A) < \frac{7}{2\pi} \mathrm{hsiz}(A).$$

2 Proof of Theorem 1

It suffices to prove this for weakly bounded $\mathbb{H}$-hulls, by which we mean $\mathbb{H}$-hulls A with $\mathrm{Im}(A) < \infty$ and such that for each $\epsilon > 0$, the set $\{x + iy : y > \epsilon\}$ is bounded. Indeed, for $\mathbb{H}$-hulls that are not weakly bounded, it is easy to verify that $\mathrm{hsiz}(A) = \mathrm{hcap}(A) = \infty$.

We start with a simple inequality that is implied but not explicitly stated in [1]. Equality is achieved when A is a vertical line segment.

Lemma 1. *If A is an $\mathbb{H}$-hull, then*

$$\mathrm{hcap}(A) \geq \frac{\mathrm{Im}[A]^2}{2}. \tag{1}$$

[3]After submitting this article, we learned that a similar result was recently proved by Carto Wong as part of his Ph.D. research.

Proof. Due to the continuity of hcap with respect to the Hausdorff metric on $\mathbb{H}$-hulls, it suffices to prove the result for $\mathbb{H}$-hulls that are path-connected. For two $\mathbb{H}$-hulls $A_1 \subseteq A_2$, it can be seen using the Optional stopping theorem that $\mathrm{hcap}(A_1) \le \mathrm{hcap}(A_2)$. Therefore without loss of generality, A can be assumed to be of the form $\eta(0,T]$ where η is a simple curve with $\eta(0+) \in \mathbb{R}$, parameterized so that $\mathrm{hcap}[\eta(0,t]) = 2t$. In particular, $T = \mathrm{hcap}(A)/2$. If $g_t = g_{\eta(0,t]}$, then g_t satisfies the Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{2}$$

where $U : [0,T] \to \mathbb{R}$ is continuous. Suppose $\mathrm{Im}(z)^2 > 2\,\mathrm{hcap}(A)$ and let $Y_t = \mathrm{Im}[g_t(z)]$. Then (2) gives

$$-\partial_t Y_t^2 \le \frac{4Y_t}{|g_t(z) - U_t|^2} \le 4,$$

which implies

$$Y_T^2 \ge Y_0^2 - 4T > 0.$$

This implies that $z \notin A$, and hence $\mathrm{Im}[A]^2 \le 2\,\mathrm{hcap}(A)$. □

The next lemma is a variant of the Vitali covering lemma. If $c > 0$ and $z = x + iy \in \mathbb{H}$, let

$$\mathscr{I}(z,c) = (x - cy, x + cy),$$

$$\mathscr{R}(z,c) = \mathscr{I}(z,c) \times (0,y] = \{x' + iy' : |x' - x| < cy, 0 < y' \le y\}.$$

Lemma 2. *Suppose A is a weakly bounded $\mathbb{H}$-hull and $c > 0$. Then there exists a finite or countably infinite sequence of points $\{z_1 = x_i + iy_1, z_2 = x_2 + iy_2, , \ldots\} \subset A$ such that:*

- $y_1 \ge y_2 \ge y_3 \ge \cdots$;
- *the intervals $\mathscr{I}(x_1,c), \mathscr{I}(x_2,c), \ldots$ are disjoint;*
- $$A \subset \bigcup_{j=1}^{\infty} \mathscr{R}(z_j, 2c). \tag{3}$$

Proof. We define the points recursively. Let $A_0 = A$ and given $\{z_1, \ldots, z_j\}$, let

$$A_j = A \setminus \left[\bigcup_{k=1}^{j} \mathscr{R}(z_j, 2c) \right].$$

If $A_j = \emptyset$ we stop, and if $A_j \ne \emptyset$,we choose $z_{j+1} = x_{j+1} + iy_{j+1} \in A$ with $y_{j+1} = \mathrm{Im}[A_j]$. Note that if $k \le j$, then $|x_{j+1} - x_k| \ge 2c\,y_k \ge c\,(y_k + y_{j+1})$ and hence $\mathscr{I}(z_{j+1},c) \cap \mathscr{I}(z_k,c) = \emptyset$. Using the weak boundedness of A, we can see that $y_j \to 0$ and hence (3) holds. □

Lemma 3. *For every $c > 0$, let*

$$\rho_c := \frac{2\sqrt{2}}{\pi} \arctan\left(e^{-\theta}\right), \quad \theta = \theta_c = \frac{\pi}{4c}.$$

Then, for any $c > 0$, if A is a weakly bounded $\mathbb{H}$-hull and $x_0 + iy_0 \in A$ with $y_0 = \mathrm{Im}(A)$, then

$$\mathrm{hcap}(A) \ge \rho_c^2\, y_0^2 + \mathrm{hcap}\left[A \setminus \mathscr{R}(z, 2c)\right].$$

Proof. By scaling and invariance under real translation, we may assume that $\mathrm{Im}[A] = y_0 = 1$ and $x_0 = 0$. Let $S = S_c$ be defined to be the set of all points z of the form $x + iuy$ where $x + iy \in A \setminus \mathscr{R}(i, 2c)$ and $0 < u \leq 1$.
Clearly, $S \cap A = A \setminus \mathscr{R}(i, 2c)$.
Using the capacity inequality [1, (3.10)]

$$\mathrm{hcap}(A_1 \cup A_2) - \mathrm{hcap}(A_2) \leq \mathrm{hcap}(A_1) - \mathrm{hcap}(A_1 \cap A_2), \tag{4}$$

we see that

$$\mathrm{hcap}(S \cup A) - \mathrm{hcap}(S) \leq \mathrm{hcap}(A) - \mathrm{hcap}(S \cap A).$$

Hence, it suffices to show that

$$\mathrm{hcap}(S \cup A) - \mathrm{hcap}(S) \geq \rho_c^2.$$

Let f be the conformal map of $\mathbb{H} \setminus S$ onto $\mathbb{H}$ such that $z - f(z) = o(1)$ as $z \to \infty$. Let $S^* := S \cup A$. By properties of halfplane capacity [1, (3.8)] and (1),

$$\mathrm{hcap}(S^*) - \mathrm{hcap}(S) = \mathrm{hcap}[f(S^* \setminus S)] \geq \frac{\mathrm{Im}[f(i)]^2}{2}.$$

Hence, it suffices to prove that

$$\mathrm{Im}[f(i)] \geq \sqrt{2}\,\rho = \frac{4}{\pi}\arctan\left(e^{-\theta}\right). \tag{5}$$

By construction, $S \cap \mathscr{R}(z, 2c) = \emptyset$. Let $V = (-2c, 2c) \times (0, \infty) = \{x + iy : |x| < 2c, y > 0\}$ and let τ_V be the first time that a Brownian motion leaves the domain. Then [1, (3.5)],

$$\mathrm{Im}[f(i)] = 1 - \mathbb{E}^i\left[\mathrm{Im}(B_{\tau_S})\right] \geq \mathbb{P}\left\{B_{\tau_S} \in [-2c, 2c]\right\} \geq \mathbb{P}\left\{B_{\tau_V} \in [-2c, 2c]\right\}.$$

The map $\Phi(z) = \sin(\theta z)$ maps V onto $\mathbb{H}$ sending $[-2c, 2c]$ to $[-1, 1]$ and $\Phi(i) = i\sinh\theta$. Using conformal invariance of Brownian motion and the Poisson kernel in $\mathbb{H}$, we see that

$$\mathbb{P}\left\{B_{\tau_V} \in [-2c, 2c]\right\} = \frac{2}{\pi}\arctan\left(\frac{1}{\sinh\theta}\right) = \frac{4}{\pi}\arctan\left(e^{-\theta}\right).$$

The second equality uses the double angle formula for the tangent. □

Lemma 4. *Suppose $c > 0$ and $x_1 + iy_1, x_2 + iy_2, \ldots$ are as in Lemma 2. Then*

$$\mathrm{hsiz}(A) \leq [\pi + 8c]\sum_{j=1}^{\infty} y_j^2. \tag{6}$$

If $c \geq 1$, then

$$\pi\sum_{j=1}^{\infty} y_j^2 \leq \mathrm{hsiz}(A). \tag{7}$$

Proof. A simple geometry exercise shows that

$$\mathrm{area}\left[\bigcup_{x+iy \in \mathscr{R}(z_j, 2c)} \mathscr{B}(x + iy, y)\right] = [\pi + 8c]\, y_j^2.$$

Since

$$A \subset \bigcup_{j=1}^{\infty} \mathscr{R}(z_j, 2c),$$

the upper bound in (6) follows. Since $c \geq 1$, and the intervals $\mathscr{I}(z_j, c)$ are disjoint, so are the disks $\mathscr{B}(z_j, y_j)$. Hence,

$$\text{area}\left[\bigcup_{x+iy \in A} \mathscr{B}(x+iy, y)\right] \geq \text{area}\left[\bigcup_{j=1}^{\infty} \mathscr{B}(z_j, y_j)\right] = \pi \sum_{j=1}^{\infty} y_j^2.$$

□

Proof of Theorem 1. Let $V_j = A \cap \mathscr{R}(z_j, c)$. Lemma 3 tells us that

$$\text{hcap}\left[\bigcup_{k=j}^{\infty} V_j\right] \geq \rho_c^2 \, y_j^2 + \text{hcap}\left[\bigcup_{k=j+1}^{\infty} V_j\right],$$

and hence

$$\text{hcap}(A) \geq \rho_c^2 \sum_{j=1}^{\infty} y_j^2. \tag{8}$$

Combining this with the upper bound in (6) with any $c > 0$ gives

$$\frac{\text{hcap}(A)}{\text{hsiz}(A)} \geq \frac{\rho_c^2}{\pi + 8c}.$$

Choosing $c = \frac{8}{5}$ gives us

$$\frac{\text{hcap}(A)}{\text{hsiz}(A)} > \frac{1}{66}.$$

For the upper bound, choose a covering as in Lemma 2. Subadditivity and scaling give

$$\text{hcap}(A) \leq \sum_{j=1}^{\infty} \text{hcap}\left[\mathscr{R}(z_j, 2c\,y_j)\right] = \text{hcap}[\mathscr{R}(i, 2c)] \sum_{j=1}^{\infty} y_j^2. \tag{9}$$

Combining this with the lower bound in (6) with $c = 1$ gives

$$\frac{\text{hcap}(A)}{\text{hsiz}(A)} \leq \frac{\text{hcap}[\mathscr{R}(i,2)]}{\pi}.$$

Note that $\mathscr{R}(i,2)$ is the union of two real translates of $\mathscr{R}(i,1)$, $\text{hcap}[\mathscr{R}(i,2)] \leq 2\,\text{hcap}[\mathscr{R}(i,1)]$ whose intersection is the interval $(0, i]$. Using (4), we see that

$$\text{hcap}(\mathscr{R}(i,2)) \leq 2\,\text{hcap}(\mathscr{R}(i,1)) - \text{hcap}((0,i]) = 2\,\text{hcap}(\mathscr{R}(i,1)) - \frac{1}{2}.$$

But $\mathscr{R}(i,1)$ is strictly contained in $A' := \{z \in \mathbb{H} : |z| \leq \sqrt{2}\}$, and hence

$$\text{hcap}[\mathscr{R}(i,1)] < \text{hcap}(A') = 2.$$

The last equality can be seen by considering $h(z) = z + 2z^{-1}$ which maps $\mathbb{H} \setminus A'$ onto $\mathbb{H}$. Therefore,

$$\text{hcap}[\mathscr{R}(i,2)] < \frac{7}{2},$$

and hence

$$\frac{\text{hcap}(A)}{\text{hsiz}(A)} < \frac{7}{2\pi}.$$

□

An equivalent form of this result can be stated[4] in terms of the area of the 1-neighborhood of A (denoted hyp(A)) in the hyperbolic metric. The unit hyperbolic ball centered at a point $x + \iota y$ is the Euclidean ball with respect to which $x + \iota y/e)$ and $x = \iota y e$ are diametrically opposite boundary points. For any c, choosing a covering as in Lemma 2,

$$\text{hyp}(A) < \left(\left(\frac{e}{2}\right)^2 \pi + 4ec \right) \sum_{j=1}^{\infty} y_j^2.$$

So by (8),

$$\frac{\text{hcap}(A)}{\text{hyp}(A)} > \rho_c^2 \left(\left(\frac{e}{2}\right)^2 \pi + 4ec \right)^{-1}.$$

Setting c to $\frac{8}{5}$,

$$\frac{\text{hcap}(A)}{\text{hyp}(A)} > \frac{1}{100}.$$

For any $c > \frac{e-e^{-1}}{2}$,

$$\text{hyp}(A) \geq \pi \left(\frac{e - e^{-1}}{2} \right)^2 \sum_{j=1}^{\infty} y_j^2.$$

So by (9),

$$\frac{\text{hcap}(A)}{\text{hyp}(A)} < \frac{\text{hcap}[\mathscr{R}(i,3)]}{\pi \left(\frac{e-e^{-1}}{2}\right)^2}.$$

$$\text{hcap}(\mathscr{R}(i,3)) \leq \text{hcap}(\mathscr{R}(i,1)) + \text{hcap}(\mathscr{R}(i,2)) - \text{hcap}((0,i]) \leq 5.$$

Therefore,

$$\frac{1}{100} < \frac{\text{hcap}(A)}{\text{hyp}(A)} < \frac{20}{\pi(e - e^{-1})^2}.$$

References

[1] G. Lawler, *Conformally Invariant Processes in the Plane*, American Mathematical Society, 2005. MR2129588

[4] This formulation was suggested to us by Scott Sheffield and the anonymous referee.