

## ON A THEOREM IN MULTI-PARAMETER POTENTIAL THEORY

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### Abstract

Let  $X$  be an  $N$ -parameter additive Lévy process in  $\mathbb{R}^d$  with Lévy exponent  $(\Psi_1, \dots, \Psi_N)$  and let  $\lambda_d$  denote Lebesgue measure in  $\mathbb{R}^d$ . We show that

$$E\{\lambda_d(X(\mathbb{R}_+^N))\} > 0 \iff \int_{\mathbb{R}^d} \prod_{j=1}^N \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi)} \right) d\xi < \infty.$$

This was previously proved by Khoshnevisan, Xiao and Zhong [1] under a sector condition.

## 1 Introduction and Proof

Let  $X_{t_1}^1, X_{t_2}^2, \dots, X_{t_N}^N$  be  $N$  independent Lévy processes in  $\mathbb{R}^d$  with their respective Lévy exponents  $\Psi_j, j = 1, 2, \dots, N$ . The random field

$$X_t = X_{t_1}^1 + X_{t_2}^2 + \dots + X_{t_N}^N, \quad t = (t_1, t_2, \dots, t_N) \in \mathbb{R}_+^N$$

is called the additive Lévy process. Let  $\lambda_d$  denote Lebesgue measure in  $\mathbb{R}^d$ .

**Theorem 1.1** *Let  $X$  be an additive Lévy process in  $\mathbb{R}^d$  with Lévy exponent  $(\Psi_1, \dots, \Psi_N)$ . Then*

$$E\{\lambda_d(X(\mathbb{R}_+^N))\} > 0 \iff \int_{\mathbb{R}^d} \prod_{j=1}^N \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi)} \right) d\xi < \infty. \quad (1.1)$$

Recently, Khoshnevisan, Xiao and Zhong [1] proved that if

$$\operatorname{Re} \left( \prod_{j=1}^N \frac{1}{1 + \Psi_j(\xi)} \right) \geq \theta \prod_{j=1}^N \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi)} \right) \quad (1.2)$$

for some constant  $\theta > 0$  then Theorem 1.1 holds. In fact the proof of Theorem 1.1 does not need any condition.

**Proof of Theorem 1.1:** Define

$$\mathcal{E}_\Psi(\mu) = (2\pi)^{-d} \int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 \prod_{j=1}^N \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi)} \right) d\xi$$

where  $\mu$  is a probability measure on a compact set  $F \subset \mathbb{R}^d$  and  $\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} \mu(dx)$ . Let  $F = \{0\} \subset \mathbb{R}^d$  and  $\delta_0$  be the point mass at  $0 \in \mathbb{R}^d$ . We first quote a key lemma of [1]:

**Lemma 5.5** *Suppose  $X$  is an additive Lévy process in  $\mathbb{R}^d$  that satisfies Condition (1.3), and that  $\int_{\mathbb{R}^d} \prod_{j=1}^N |1 + \Psi_j(\xi)|^{-1} d\xi < +\infty$ , where  $\Psi = (\Psi_1, \dots, \Psi_N)$  denotes the Lévy exponent of  $X$ . Then, for all compact sets  $F \subset \mathbb{R}^d$ , and for all  $r > 0$ ,*

$$E\{\lambda_d(X([0, r]^N \oplus F))\} \leq \theta^{-2} (4e^{2r})^N \cdot \mathcal{C}_\Psi(F),$$

where  $\theta > 0$  is the constant in Condition (1.3).

By reviewing the whole process of the proof of Theorem 1.1 of [1] given by Khoshnevisan, Xiao and Zhong, our Theorem 1.1 certainly follows if we instead prove the following statement:

Let  $X$  be any additive Lévy process in  $\mathbb{R}^d$ . If  $\int_{\mathbb{R}^d} \prod_{j=1}^N |1 + \Psi_j(\xi)|^{-1} d\xi < +\infty$ , then

$$E\{\lambda_d(X([0, r]^N))\} \leq \frac{c_{N,d,r}}{\mathcal{E}_\Psi(\delta_0)} \tag{1.3}$$

for some constant  $c_{N,d,r} \in (0, \infty)$  depending on  $N, d, r$  only.

Clearly, all we have to do is to complete Eq. (5.11) of [1] without bothering ourselves with Condition (1.3) of [1]. Since  $\delta_0$  is the only probability measure on  $F = \{0\}$ , letting  $\eta \rightarrow 0, k \rightarrow \infty$ , and  $\varepsilon \rightarrow 0$  and using the integrability condition  $\int_{\mathbb{R}^d} \prod_{j=1}^N |1 + \Psi_j(\xi)|^{-1} d\xi < +\infty$  yield

$$\mathcal{E}_\Psi(\delta_0) \geq c_1 \left| \int_{\mathbb{R}^d} \operatorname{Re} \left( \prod_{i=1}^N \frac{1}{1 + \Psi_i(\xi)} \right) d\xi \right|^2 E\{\lambda_d(X([0, r]^N))\} \tag{1.4}$$

where  $c_1 \in (0, \infty)$  is a constant depending on  $N, d, r$  only.

Consider the  $2^{N-1}$  similar additive Lévy processes (including  $X_t$  itself)  $X_t^\pm = X_{t_1}^1 \pm X_{t_2}^2 \pm \dots \pm X_{t_N}^N$ . Here,  $\pm$  is merely a symbol for each possible arrangement of the minus signs; e.g.,  $X^1 - X^2 + X^3, X^1 - X^2 - X^3, X^1 + X^2 + X^3$  and so on. Let  $\Psi^\pm$  be the Lévy exponent for  $X_t^\pm$ . Since  $-X^j$  has Lévy exponent  $\overline{\Psi_j}$ ,  $\mathcal{E}_{\Psi^\pm}(\mu) = \mathcal{E}_\Psi(\mu)$  for all  $X_t^\pm$  and

$$\sum \operatorname{Re} \left( \int_{\mathbb{R}_+^N} e^{-\sum_{j=1}^N s_j \cdot \Psi^\pm(\xi)} ds \right) = 2^{N-1} \prod_{j=1}^N \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi)} \right) > 0$$

where the first summation  $\sum$  is taken over the collection of all the  $X_t^\pm$ . On the other hand,

$$Q_\mu(\xi) = \int_{\mathbb{R}_+^N} \int_{\mathbb{R}_+^N} e^{-\sum_{j=1}^N |t_j - s_j| \Psi_j(\operatorname{sgn}(t_j - s_j)\xi)} \mu(ds) \mu(dt)$$

remains unchanged for all  $X_t^\pm$  as long as  $\mu$  is an  $N$ -fold product measure on  $\mathbb{R}_+^N$ . Proposition 10.3 of [1] and Theorem 2.1 of [1] together state that for any additive Lévy process  $X$ ,

$$k_1 \left( \int_{\mathbb{R}^d} Q_{\lambda^r}(\xi) d\xi \right)^{-1} \leq E\{\lambda_d(X([0, r]^N))\} \leq k_2 \left( \int_{\mathbb{R}^d} Q_{\lambda^r}(\xi) d\xi \right)^{-1},$$

where  $\lambda^r$  is the restriction of the Lebesgue measure  $\lambda_N$  in  $\mathbb{R}^N$  to  $[0, r]^N$  and  $k_1, k_2 \in (0, \infty)$  are two constants depending only on  $r, N, d, \pi$ . Note that  $\lambda^r$  is an  $N$ -fold product measure on  $\mathbb{R}_+^N$ . Thus, there exists a constant  $c_2 \in (0, \infty)$  depending only on  $N$  and  $r$  such that

$$E\{\lambda_d(X([0, r]^N))\} \leq c_2 E\{\lambda_d(X^\pm([0, r]^N))\}$$

for all  $X_t^\pm$ . Since  $|1+z| = |1+\bar{z}|$  where  $z$  is a complex number,  $\int_{\mathbb{R}^d} \prod_{j=1}^N |1+\Psi_j^\pm(\xi)|^{-1} d\xi < +\infty$  as well. Therefore, by (1.4),

$$\begin{aligned} & 2^{N-1} \sqrt{c_2} \sqrt{\frac{\mathcal{E}_\Psi(\delta_0)}{E\{\lambda_d(X([0, r]^N))\}}} \\ & \geq \sum \sqrt{\frac{\mathcal{E}_{\Psi^\pm}(\delta_0)}{E\{\lambda_d(X^\pm([0, r]^N))\}}} \\ & \geq \sqrt{c_1} \sum \left| \int_{\mathbb{R}^d} \operatorname{Re} \left( \int_{\mathbb{R}_+^N} e^{-\sum_{j=1}^N s_j - s \cdot \Psi^\pm(\xi)} ds \right) d\xi \right| \\ & \geq \sqrt{c_1} \left| \sum \int_{\mathbb{R}^d} \operatorname{Re} \left( \int_{\mathbb{R}_+^N} e^{-\sum_{j=1}^N s_j - s \cdot \Psi^\pm(\xi)} ds \right) d\xi \right| \\ & = 2^{N-1} \sqrt{c_1} \int_{\mathbb{R}^d} \prod_{j=1}^N \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi)} \right) d\xi \\ & = 2^{N-1} \sqrt{c_1} (2\pi)^d \mathcal{E}_\Psi(\delta_0). \end{aligned}$$

(1.3) follows, so does the theorem.  $\square$

## 2 Applications

### 2.1 The Range of An Additive Lévy Process

As the first application, we use Theorem 1.1 to compute  $\dim_H X(\mathbb{R}_+^N)$ . Here,  $\dim_H$  denotes the Hausdorff dimension. To begin, we introduce the standard  $d$ -parameter additive  $\alpha$ -stable Lévy process in  $\mathbb{R}^d$  for  $\alpha \in (0, 1)$ :

$$S_t^\alpha = S_{t_1}^1 + S_{t_2}^2 + \dots + S_{t_d}^d,$$

that is, the  $S^j$  are independent standard  $\alpha$ -stable Lévy processes in  $\mathbb{R}^d$  with the common Lévy exponent  $|\xi|^\alpha$ .

**Theorem 2.1** *Let  $X$  be any  $N$ -parameter additive Lévy process in  $\mathbb{R}^d$  with Lévy exponent  $(\Psi_1, \dots, \Psi_N)$ . Then*

$$\dim_H X(\mathbb{R}_+^N) = \sup \left\{ \beta \in (0, d) : \int_{\mathbb{R}^d} |\xi|^{\beta-d} \prod_{j=1}^N \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi)} \right) d\xi < \infty \right\} \text{ a.s.} \quad (2.1)$$

**Proof** Let  $\mathcal{C}_\beta$  denote the Riesz capacity. By Theorem 7.2 of [1], for all  $\beta \in (0, d)$  and  $S^{1-\beta/d}$  independent of  $X$ ,

$$EC_\beta(X(\mathbb{R}_+^N)) > 0 \iff E\{\lambda_d(S^{1-\beta/d}(\mathbb{R}_+^d) + X(\mathbb{R}_+^N))\} > 0. \tag{2.2}$$

Note that  $S^{1-\beta/d} + X$  is a  $(d + N, d)$ -additive Lévy process. Thus, by Theorem 1.1 and the fact that  $\beta < d$  and  $\text{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) \in (0, 1]$ , we have for all  $\beta \in (0, d)$ ,

$$EC_\beta(X(\mathbb{R}_+^N)) > 0 \iff \int_{\mathbb{R}^d} |\xi|^{\beta-d} \prod_{j=1}^N \text{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) d\xi < \infty. \tag{2.3}$$

Thanks to the Frostman theorem, it remains to show that  $\mathcal{C}_\beta(X(\mathbb{R}_+^N)) > 0$  is a trivial event. Let  $\mathcal{E}_\beta$  denote the Riesz energy. By Plancherel’s theorem, given any  $\beta \in (0, d)$ , there is a constant  $c_{d,\beta} \in (0, \infty)$  such that

$$\mathcal{E}_\beta(\nu) = c_{d,\beta} \int_{\mathbb{R}^d} |\hat{\nu}(\xi)|^2 |\xi|^{\beta-d} d\xi \tag{2.4}$$

holds for all probability measures  $\nu$  in  $\mathbb{R}^d$ ; see Mattila [3; Lemma 12.12]. Consider the 1-killing occupation measure

$$O(A) = \int_{\mathbb{R}_+^N} 1(X_t \in A) e^{-\sum_{j=1}^N t_j} dt, \quad A \subset \mathbb{R}^d.$$

Clearly,  $O$  is a probability measure supported on  $X(\mathbb{R}_+^N)$ . It is easy to verify that

$$E|\widehat{O}(\xi)|^2 = \prod_{j=1}^N \text{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right).$$

It follows from (2.4) that

$$E\mathcal{E}_\beta(O) = c_{d,\beta} \int_{\mathbb{R}^d} |\xi|^{\beta-d} \prod_{j=1}^N \text{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) d\xi < \infty$$

when  $EC_\beta(X(\mathbb{R}_+^N)) > 0$ . Therefore,  $\mathcal{E}_\beta(O) < \infty$  a.s. Hence,  $\mathcal{C}_\beta(X(\mathbb{R}_+^N)) > 0$  a.s.  $\square$

*2.2 The Set of  $k$ -Multiple Points*

First, we mention a  $q$ -potential density criterion: Let  $X$  be an additive Lévy process and assume that  $X$  has an a.e. positive  $q$ -potential density on  $\mathbb{R}^d$  for some  $q \geq 0$ . Then for all Borel sets  $F \subset \mathbb{R}^d$ ,

$$P\left\{F \cap X((0, \infty)^N) \neq \emptyset\right\} > 0 \iff E\{\lambda_d(F - X((0, \infty)^N))\} > 0. \tag{2.5}$$

The argument is elementary but crucially hinges on the property:  $X_{b+t} - X_b, t \in \mathbb{R}_+^N$  (independent of  $X_b$ ) can be replaced by  $X$  for all  $b \in \mathbb{R}_+^N$ ; moreover, the second condition “a.e. positive on  $\mathbb{R}^d$ ” is absolutely necessary for the direction  $\Leftarrow$  in (2.5); see for example Proposition 6.2 of [1].

Let  $X^1, \dots, X^k$  be  $k$  independent Lévy processes in  $\mathbb{R}^d$ . Define

$$Z_t = (X_{t_2}^2 - X_{t_1}^1, \dots, X_{t_k}^k - X_{t_{k-1}}^{k-1}), \quad t = (t_1, t_2, \dots, t_k) \in \mathbb{R}_+^k.$$

$Z$  is a  $k$ -parameter additive Lévy process taking values in  $\mathbb{R}^{d(k-1)}$ .

**Theorem 2.2** *Let  $(X^1; \Psi_1), \dots, (X^k; \Psi_k)$  be  $k$  independent Lévy processes in  $\mathbb{R}^d$  for  $k \geq 2$ . Assume that  $Z$  has an a.e. positive  $q$ -potential density for some  $q \geq 0$ . [A special case is that if for each  $j = 1, \dots, k$ ,  $X^j$  has a one-potential density  $u_j^1 > 0$ ,  $\lambda_d$ -a.e., then  $Z$  has an a.e. positive 1-potential density on  $\mathbb{R}^{d(k-1)}$ .] Then*

$$P\left(\bigcap_{j=1}^k X^j((0, \infty)) \neq \emptyset\right) > 0 \iff \int_{\mathbb{R}^{d(k-1)}} \prod_{j=1}^k \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi_j - \xi_{j-1})} \right) d\xi_1 \cdots d\xi_{k-1} < \infty \tag{2.6}$$

with  $\xi_0 = \xi_k = 0$ .

**Proof** For any  $\mathbb{R}^d$ -valued random variable  $X$  and  $\xi_1, \xi_2 \in \mathbb{R}^d$ ,  $e^{i[(\xi_1, \xi_2) \cdot (X, -X)]} = e^{i(\xi_1 - \xi_2) \cdot X}$ . In particular, the Lévy process  $(X^j, -X^j)$  has Lévy exponent  $\Psi_j(\xi_1 - \xi_2)$ . It follows that the corresponding integral in (1.1) for  $Z$  equals

$$\int_{\mathbb{R}^{d(k-1)}} \prod_{j=1}^k \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi_j - \xi_{j-1})} \right) d\xi_1 \cdots d\xi_{k-1}$$

with  $\xi_0 = \xi_k = 0$ . Clearly,

$$P\left(\bigcap_{j=1}^k X^j((0, \infty)) \neq \emptyset\right) > 0 \iff P(0 \in Z((0, \infty)^k)) > 0.$$

Since  $Z$  has an a.e. positive  $q$ -potential density, by (2.5)

$$P(0 \in Z((0, \infty)^k)) > 0 \iff E\{\lambda_{d(k-1)}(Z((0, \infty)^k))\} > 0.$$

(2.6) now follows from Theorem 1.1.  $\square$

For each  $\beta \in (0, d)$  and  $S^{1-\beta/d}$  independent of  $X^1, \dots, X^k$ , define

$$Z_t^{S, \beta} = (X_{t_1}^1 - S_{t_0}^{1-\beta/d}, X_{t_2}^2 - X_{t_1}^1, \dots, X_{t_k}^k - X_{t_{k-1}}^{k-1}),$$

$$t = (t_0, t_1, t_2, \dots, t_k) \in \mathbb{R}_+^{d+k}, \quad t_0 \in \mathbb{R}_+^d.$$

$Z^{S, \beta}$  is a  $k + d$  parameter additive Lévy process taking values in  $\mathbb{R}^{dk}$ .

**Theorem 2.3** *Let  $(X^1; \Psi_1), \dots, (X^k; \Psi_k)$  be  $k$  independent Lévy processes in  $\mathbb{R}^d$  for  $k \geq 2$ . Assume that for each  $\beta \in (0, d)$ ,  $Z^{S, \beta}$  has an a.e. positive  $q$ -potential density on  $\mathbb{R}^{dk}$  for some  $q \geq 0$ . ( $q$  might depend on  $\beta$ .) [A special case is that if for each  $j = 1, \dots, k$ ,  $X^j$  has a*

one-potential density  $u_j^1 > 0$ ,  $\lambda_d$ -a.e., then  $Z^{S,\beta}$  has an a.e. positive 1-potential density on  $\mathbb{R}^{dk}$  for all  $\beta \in (0, d)$ .] If  $P(\bigcap_{j=1}^k X^j((0, \infty)) \neq \emptyset) > 0$ , then almost surely  $\dim_H \bigcap_{j=1}^k X^j((0, \infty))$  is a constant on  $\{\bigcap_{j=1}^k X^j((0, \infty)) \neq \emptyset\}$  and

$$\dim_H \bigcap_{j=1}^k X^j((0, \infty)) = \sup\{\beta \in (0, d) : \int_{\mathbb{R}^{dk}} (1 + |\sum_{j=1}^k \xi_j|)^{\beta-d} \prod_{j=1}^k \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi_j)} \right) d\xi_1 d\xi_2 \cdots d\xi_k < \infty\}. \tag{2.7}$$

**Proof** According to the argument, Eq. (4.96)-(4.102), in *Proof of Theorem 3.2.* of Khoshnevisan, Shieh, and Xiao [2], it suffices to show that for all  $\beta \in (0, d)$  and  $S^{1-\beta/d}$  independent of  $X^1, \dots, X^k$ ,

$$P \left[ \bigcap_{j=1}^k X^j((0, \infty)) \cap S^{1-\beta/d}((0, \infty)^d) \neq \emptyset \right] > 0 \iff \int_{\mathbb{R}^{dk}} (1 + |\sum_{j=1}^k \xi_j|)^{\beta-d} \prod_{j=1}^k \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi_j)} \right) d\xi_1 d\xi_2 \cdots d\xi_k < \infty. \tag{2.8}$$

Similarly, the corresponding integral in (1.1) for  $Z^{S,\beta}$  equals

$$\int_{\mathbb{R}^{dk}} \frac{1}{(1 + |\xi_0|^{1-\beta/d})^d} \prod_{j=1}^k \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi_j - \xi_{j-1})} \right) d\xi_0 d\xi_1 \cdots d\xi_{k-1}$$

with  $\xi_k = 0$ . Since  $Z^{S,\beta}$  has an a.e. positive  $q$ -potential density, by (2.5) and Theorem 1.1

$$P \left[ \bigcap_{j=1}^k X^j((0, \infty)) \cap S^{1-\beta/d}((0, \infty)^d) \neq \emptyset \right] > 0 \iff P(0 \in Z^{S,\beta}((0, \infty)^{k+d})) > 0 \iff E\{\lambda_{dk}(Z^{S,\beta}((0, \infty)^{k+d}))\} > 0 \iff \int_{\mathbb{R}^{dk}} \frac{1}{(1 + |\xi_0|^{1-\beta/d})^d} \prod_{j=1}^k \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi_j - \xi_{j-1})} \right) d\xi_0 d\xi_1 \cdots d\xi_{k-1} < \infty$$

with  $\xi_k = 0$ . Note that

$$\int_{\mathbb{R}^{dk}} \frac{1}{(1 + |\xi_0|^{1-\beta/d})^d} \prod_{j=1}^k \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi_j - \xi_{j-1})} \right) d\xi_0 d\xi_1 \cdots d\xi_{k-1} < \infty \iff \int_{\mathbb{R}^{dk}} (1 + |\xi_0|)^{\beta-d} \prod_{j=1}^k \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi_j - \xi_{j-1})} \right) d\xi_0 d\xi_1 \cdots d\xi_{k-1} < \infty.$$

Finally, use the cyclic transformation:  $\xi_j - \xi_{j-1} = \xi'_j, j = 1, \dots, k-1, \xi_{k-1} = \xi'_k$  to obtain

$$\int_{\mathbb{R}^{dk}} (1 + |\xi_0|)^{\beta-d} \prod_{j=1}^k \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi_j - \xi_{j-1})} \right) d\xi_0 d\xi_1 \cdots d\xi_{k-1} < \infty$$

$$\iff \int_{\mathbb{R}^{dk}} (1 + |\sum_{j=1}^k \xi'_j|)^{\beta-d} \prod_{j=1}^k \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi'_j)} \right) d\xi'_1 d\xi'_2 \cdots d\xi'_k < \infty. \quad \square$$

Let  $X$  be a Lévy process in  $\mathbb{R}^d$ . Fix any path  $X_t(\omega)$ . A point  $x^\omega \in \mathbb{R}^d$  is said to be a  $k$ -multiple point of  $X(\omega)$  if there exist  $k$  distinct times  $t_1, t_2, \dots, t_k$  such that  $X_{t_1}(\omega) = X_{t_2}(\omega) = \dots = X_{t_k}(\omega) = x^\omega$ . Denote by  $E_k^\omega$  the set of  $k$ -multiple points of  $X(\omega)$ . It is well known that  $E_k$  can be identified with  $\bigcap_{j=1}^k X^j((0, \infty))$  where the  $X^j$  are i.i.d. copies of  $X$ . Thus, Theorem 2.2 and Theorem 2.3 imply the next theorem.

**Theorem 2.4** *Let  $(X, \Psi)$  be any Lévy process in  $\mathbb{R}^d$ . Assume that  $X$  has a one-potential density  $u^1 > 0, \lambda_d$ -a.e. Let  $E_k$  be the  $k$ -multiple-point set of  $X$ . Then*

$$P(E_k \neq \emptyset) > 0 \iff \int_{\mathbb{R}^{d(k-1)}} \prod_{j=1}^k \operatorname{Re} \left( \frac{1}{1 + \Psi(\xi_j - \xi_{j-1})} \right) d\xi_1 \cdots d\xi_{k-1} < \infty \quad (2.9)$$

with  $\xi_0 = \xi_k = 0$ . If  $P(E_k \neq \emptyset) > 0$ , then almost surely  $\dim_H E_k$  is a constant on  $\{E_k \neq \emptyset\}$  and

$$\dim_H E_k = \sup\{\beta \in (0, d) : \int_{\mathbb{R}^{dk}} (1 + |\sum_{j=1}^k \xi_j|)^{\beta-d} \prod_{j=1}^k \operatorname{Re} \left( \frac{1}{1 + \Psi(\xi_j)} \right) d\xi_1 d\xi_2 \cdots d\xi_k < \infty\}. \quad (2.10)$$

**2.3 Intersection of Two Independent Subordinators**

Let  $X_t, t \geq 0$  be a process with  $X_0 = 0$ , taking values in  $\mathbb{R}_+$ . First, we ask this question: What is a condition on  $X$  such that for all sets  $F \subset (0, \infty)$ ,

$$P(F \cap X((0, \infty)) \neq \emptyset) > 0 \iff E\{\lambda_1(F - X((0, \infty)))\} > 0 ?$$

For subordinators, still the existence and positivity of a  $q$ -potential density ( $q \geq 0$ ) is the only known useful condition to this question.

Let  $\sigma$  be a subordinator. Take an independent copy  $\sigma^-$  of  $-\sigma$ . We then define a process  $\tilde{\sigma}$  on  $\mathbb{R}$  by  $\tilde{\sigma}_s = \sigma_s$  for  $s \geq 0$  and  $\tilde{\sigma}_s = \sigma^-_{-s}$  for  $s < 0$ . Note that  $\tilde{\sigma}$  is a process of the property:  $\tilde{\sigma}_{t+b} - \tilde{\sigma}_b, t \geq 0$  (independent of  $\tilde{\sigma}_b$ ) can be replaced by  $\sigma$  for all  $b \in \mathbb{R}$ .

Let  $X_t, t \geq 0$  be any process in  $\mathbb{R}^d$ . Then the  $q$ -potential density is nothing but the density of the expected  $q$ -occupation measure with respected to the Lebesgue measure. (When  $q = 0$ , assume that the expected 0-occupation measure is finite on the balls.) Since the reference measure is Lebesgue, one can easily deduce that if  $u$  is a  $q$ -potential density of  $X$ , then  $u(-x)$  is a  $q$ -potential density of  $-X$ . Consequently, if we define  $\tilde{X}_s = X_s$  for  $s \geq 0$  and  $\tilde{X}_s = X^-_{-s}$  for  $s < 0$  where  $X^-$  is an independent copy of  $-X$ , then  $u(x) + u(-x)$  is a  $q$ -potential density of  $\tilde{X}$ . Conversely, if  $\tilde{X}$  has a  $q$ -potential density, then it has to be the form  $u(x) + u(-x)$ , where  $u$

is a  $q$ -potential density of  $X$ . If  $\sigma$  is a subordinator, after a little thought we can conclude that  $\tilde{\sigma}$  has an a.e. positive  $q$ -potential density on  $\mathbb{R}$  if and only if  $\sigma$  has an a.e. positive  $q$ -potential density on  $\mathbb{R}_+$ .

**Lemma 2.5** *If a subordinator  $\sigma$  has an a.e. positive  $q$ -potential density for some  $q \geq 0$  on  $\mathbb{R}_+$ , then for all Borel sets  $F \subset (0, \infty)$ ,*

$$P(F \cap \sigma((0, \infty)) \neq \emptyset) > 0 \iff E\{\lambda_1(F - \sigma((0, \infty)))\} > 0. \tag{2.11}$$

**Proof** Assume that  $E\{\lambda_1(F - \sigma((0, \infty)))\} > 0$ . From the above discussion,  $\tilde{\sigma}$  has an a.e. positive  $q$ -potential density. Moreover,  $\tilde{\sigma}$  is a process of the property:  $\tilde{\sigma}_{t+b} - \tilde{\sigma}_b$ ,  $t \geq 0$  (independent of  $\tilde{\sigma}_b$ ) can be replaced by  $\sigma$  for all  $b \in \mathbb{R}$ . It follows from the standard  $q$ -potential density argument that  $P(F \cap \tilde{\sigma}(\mathbb{R} \setminus \{0\}) \neq \emptyset) > 0$ . But  $F \subset (0, \infty)$  and  $\tilde{\sigma}((-\infty, 0]) \subset (-\infty, 0]$ . Thus,  $P(F \cap \sigma((0, \infty)) \neq \emptyset) > 0$ . The direction  $\implies$  in (2.11) is elementary since  $\sigma$  has a  $q$ -potential density.  $\square$

**Theorem 2.6** *Let  $\sigma^1$  and  $\sigma^2$  be two independent subordinators having the Lévy exponents  $\Psi_1$  and  $\Psi_2$ , respectively. Assume that  $\sigma^1$  has an a.e. positive  $q$ -potential density for some  $q \geq 0$  on  $\mathbb{R}_+$ . Then*

$$P[\sigma^1((0, \infty)) \cap \sigma^2((0, \infty)) \neq \emptyset] > 0 \iff \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{1}{\Psi_1(x)} \right) \operatorname{Re} \left( \frac{1}{1 + \Psi_2(x)} \right) dx < \infty. \tag{2.12}$$

Note that our result does not require any continuity condition on the  $q$ -potential density.

**Proof** By Lemma 2.5 and Theorem 1.1,

$$P[\sigma^1((0, \infty)) \cap \sigma^2((0, \infty)) \neq \emptyset] > 0 \iff \int_{-\infty}^{\infty} \operatorname{Re} \left( \frac{1}{1 + \Psi_1(x)} \right) \operatorname{Re} \left( \frac{1}{1 + \Psi_2(x)} \right) dx < \infty.$$

Since  $\sigma^1$  is transient,  $\int_{|x| \leq 1} \operatorname{Re} \left( \frac{1}{\Psi_1(x)} \right) dx < \infty$ . The proof is therefore completed.  $\square$

### 2.4 A Fourier Integral Problem

This part of content can be found in Section 6 of [1]. It is an independent Fourier integral problem. Neither computing the Hausdorff dimension nor proving the existence of 1-potential density needs the discussion below. [But this Fourier integral problem might be of novelty to those who want to replace the Lévy exponent by the 1-potential density.] Let  $X$  be an additive Lévy process. Here is the question. Suppose that  $K : \mathbb{R}^d \rightarrow [0, \infty]$  is a symmetric function with  $K(x) < \infty$  for  $x \neq 0$  that satisfies  $K \in L^1$  and  $\widehat{K}(\xi) = k_1 \prod_{j=1}^N \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi)} \right)$ . Under what conditions, can

$$\int \int K(x - y) \mu(dx) \mu(dy) = k_2 \int |\hat{\mu}(\xi)|^2 \prod_{j=1}^N \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi)} \right) d\xi \tag{2.13}$$



hold for all probability measures  $\mu$  in  $\mathbb{R}^d$ ? Here,  $k_1, k_2 \in (0, \infty)$  are two constants. Consider the function  $K$  in the following example. Define  $\tilde{X}_{t_j}^j = -Y_{-t_j}^j$  for  $t_j < 0$  and  $\tilde{X}_{t_j}^j = X_{t_j}^j$  for  $t_j \geq 0$ , where  $Y^j$  is an independent copy of  $X^j$  and the  $Y^j$  are independent of each other and of  $X$  as well. Then  $\tilde{X}_t = \tilde{X}_{t_1}^1 + \tilde{X}_{t_2}^2 + \dots + \tilde{X}_{t_N}^N$ ,  $t \in \mathbb{R}^N$  is a random field on  $\mathbb{R}^N$ . Assume that  $\tilde{X}$  has a 1-potential density  $K$ . So,  $K \in L^1$  and a direct check verifies that  $K$  is symmetric. By the definition of  $K$ ,  $\hat{K}(\xi) = \int_{\mathbb{R}^N} e^{-\sum_{j=1}^N |t_j|} E e^{i\xi \cdot \tilde{X}_t} dt$ . Evaluating this integral quadrant by quadrant and using the identity  $\sum \prod_{j=1}^N \frac{1}{1+z_j^{\pm}} = 2^N \prod_{j=1}^N \operatorname{Re} \left( \frac{1}{1+z_j} \right)$  for  $\operatorname{Re}(z_j) \geq 0$  (where  $\sum$  is taken over the  $2^N$  permutations of conjugate) yield  $\hat{K}(\xi) = k_1 \prod_{j=1}^N \operatorname{Re} \left( \frac{1}{1+\Psi_j(\xi)} \right) > 0$ . If  $\hat{K} \in L^1$  (even though this case is less interesting), on one hand by Fubini,

$$\int |\hat{\mu}(\xi)|^2 \hat{K}(\xi) d\xi = \int \int \int e^{-i\xi \cdot (x-y)} \hat{K}(\xi) d\xi \mu(dx) \mu(dy)$$

and on the other hand by inversion (assuming the inversion holds everywhere by modification on a null set),

$$\int \int K(x-y) \mu(dx) \mu(dy) = (2\pi)^{-d} \int \int \int e^{-i\xi \cdot (x-y)} \hat{K}(\xi) d\xi \mu(dx) \mu(dy).$$

Thus, (2.13) holds automatically in this case. If  $K$  is continuous at 0 and  $K(0) < \infty$ , then  $\hat{K} \in L^1$ . This is a standard fact. Since  $K \in L^1$  and  $\hat{K} > 0$ , a bottom line condition needed to prove (2.13) is that  $K$  is continuous at 0 on  $[0, \infty]$ . This paper makes no attempt to solve the general case  $K(0) = \infty$ .

**Remark** Lemma 6.1 of [1] is not valid. The assumption that  $\operatorname{Re} \left( \prod_{j=1}^N \frac{1}{1+\Psi_j(\xi)} \right) > 0$  cannot justify either equation in (6.4) of [1]. Fortunately, Lemma 6.1 played no role in [1], because Theorem 7.2 of [1] is an immediate consequence of the well-known identity (2.4) of the present paper and Theorem 1.5 of [1]. Nevertheless [1] indeed showed that the 1-potential density of an isotropic stable additive process is comparable to the Riesz kernel at 0, and therefore the 1-potential density is continuous at 0 on  $[0, \infty]$ .

## References

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