Elect. Comm. in Probab. **5** (2000) 91–94

ELECTRONIC COMMUNICATIONS in PROBABILITY

STRICT FINE MAXIMA

P. J. FITZSIMMONS
Department of Mathematics, 0112
University of California, San Diego
9500 Gilman Drive
La Jolla, CA 92093–0112, USA
email: `pfitz@math.ucsd.edu`

submitted May 3, 2000 *Final version accepted* June 15, 2000

AMS 1991 Subject classification: 60J45, 60J65, 31C15
Brownian motion, fine topology, local maxima, optional projection.

Abstract

We provide a simple probabilistic proof of a result of J. Král and I. Netuka: If f is a measurable real-valued function on $\mathbf{R}^d$ ($d \geq 2$) then the set of points at which f has a strict fine local maximum value is polar.

Let $X = (X_t)_{t \geq 0}$ be standard Brownian motion in $\mathbf{R}^d$ ($d \geq 2$), and recall that the fine topology on $\mathbf{R}^d$ is the coarsest topology with respect to which all superharmonic functions are continuous. Here is an alternative description of the fine topology: A set $V \subset \mathbf{R}^d$ is a fine neighborhood of $x \in V$ if and only if there is a Borel set U such that $x \in U \subset V$ and $x \notin (\mathbf{R}^d \setminus U)^r$. Here $B^r := \{x \in \mathbf{R}^d : \mathbf{P}^x[T_B = 0] = 1\}$ is the set of regular points for B, $\mathbf{P}^x$ is the law of X started at $X_0 = x$, and $T_B := \inf\{t > 0 : X_t \in B\}$ is the hitting time of B.

It is well known that the set of points in $\mathbf{R}^d$ at which a function $f : \mathbf{R}^d \to \mathbf{R}$ has a strict local maximum value is at most countable. Of course, "local" here refers to the usual topology on $\mathbf{R}^d$. The situation changes if the fine topology is substituted for the usual topology. For example, suppose $d \geq 3$ and let $M \subset \mathbf{R}^d$ be a subspace of dimension $d-2$. Then M is uncountable, but the indicator function 1_M admits a strict local maximum value (with respect to the fine topology!) at each point of M. To see this simply notice that if $x \in M$ then $\{x\} \cup M^c$ is a fine neighborhood of x, because M is polar.

Our purpose in this note is to give a new proof of a result of J. Král and I. Netuka [7], to the effect that the preceding example is the worst case scenario. We say that a Borel measurable function $f : \mathbf{R}^d \to \mathbf{R}$ has a *strict fine local maximum value* at $x \in \mathbf{R}^d$ provided there is a fine neighborhood V of x such that $f(y) < f(x)$ for all $y \in V \setminus \{x\}$. If we define

$$A_f(x) := \{y \in \mathbf{R}^d : f(y) \geq f(x)\} \tag{1}$$

then, because singletons are polar for X, f has a strict fine local maximum value at x if and only if $x \notin A_f(x)^r$.

Theorem. *The set* $M(f) := \{x \in \mathbf{R}^d : x \notin A_f(x)^r\}$, *of locations of strict fine local maxima for* f, *is polar. That is,* $\mathbf{P}^x[T_{M(f)} < \infty] = 0$ *for all* $x \in \mathbf{R}^d$.

Remarks.

(a) Our proof is quite different from the one given by Král and Netuka, and perhaps more accessible to a probabilistic audience. The proof in [7] is based on a deep result of A. Ancona [1], which states that every non-polar set contains a compact set K with $K^r = K$. We use the strong Markov property and the invariance of Brownian motion under time reversal.

(b) Our method is quite robust and applies to any (transient) symmetric strong Markov process for which singletons are polar. The method can even be applied (with somewhat modified conclusions) to non-symmetric processes for which semipolar sets (and singletons) are polar. We leave these extensions to the interested reader.

(c) Král and Netuka show that even if f is not Borel measurable the set $M(f)$ is inner polar, in that each of its Borel subsets is polar. □

Proof of the Theorem. We assume $d \geq 3$; the recurrent case $d = 2$ can be handled similarly, by killing X at the first exit time from a large enough ball. We refer the reader to [7] for a proof that the function $\varphi(x) := \mathbf{P}^x[T_{A_f(x)} > 0]$ and the set $M(f) = \{x : \varphi(x) = 1\}$ are Borel measurable.

Suppose, contrary to the assertion of the Theorem, that $M(f)$ is non-polar. Then, by Choquet's capacitability theorem there exists a compact non-polar set K contained in $M(f)$. Because X is transient, the hitting probability $x \mapsto \mathbf{P}^x[T_K < \infty]$ is the Green potential of a measure π—the equilibrium measure of K. This measure is carried by K, charges no polar set, and has total mass equal to the Newtonian capacity of K.

Let $Y = (Y_t)_{t \in \mathbf{R}}$ be the stationary Brownian motion in $\mathbf{R}^d$. This is a strong Markov process with the same transition probabilities as X, defined for all times $t \in \mathbf{R}$, and with one-dimensional distributions given by Lebesgue measure on $\mathbf{R}^d$. Associated with the equilibrium measure π is a homogeneous random measure κ of Y characterized by

$$\mathbf{E} \int_{\mathbf{R}} g(Y_t, t)\, \kappa(dt) = \int_{\mathbf{R}^d} \int_{\mathbf{R}} g(x, t)\, dt\, \pi(dx) \tag{2}$$

for all Borel functions $g : \mathbf{R}^d \times \mathbf{R} \to [0, \infty)$. (Here $\mathbf{E}$ denotes expectation with respect to Y.) In fact, κ is the dual predictable projection of the random measure placing a unit mass at the last exit time from K. The random measure κ is diffuse: $\kappa\{t\} = 0$ for all $t \in \mathbf{R}$, almost surely. See [8], [2], [6], [5; §5.1], and [4; §5].

Let $Z_t(\omega)$ be the indicator of the event that the function $s \mapsto f(Y_s(\omega))$ has a strict local maximum at time $t \in \mathbf{R}$. Classically, the set $\{t \in \mathbf{R} : Z_t(\omega) = 1\}$ is countable for each sample point ω. Thus if ϕ is a strictly positive Borel function on $\mathbf{R}$ with finite Lebesgue integral, then

$$\mathbf{E} \int_R \phi(t) Z_t\, \kappa(dt) = 0. \tag{3}$$

Let ΠZ denote the optional projection of Z with respect to the filtration $\mathcal{F}_t := \sigma\{Y_s : s \leq t\}$, $t \in \mathbf{R}$; that is, ΠZ is the unique $(\mathcal{F}_t)$ optional process such that $\mathbf{E}(Z_T | \mathcal{F}_T) = (\Pi Z)_T$ a.s. for every $(\mathcal{F}_t)$ stopping time T. Observe that

$$Z_t(\omega) = Z_t^-(\omega) \cdot H(Y_t(\omega), \omega_t^+), \tag{4}$$

where $\omega_t^+ = (Y_{t+s}(\omega); s > 0)$, $H(x, \omega_t^+)$ is the indicator of the event that $T_{A_f(x)}(\omega_t^+) > 0$, and $Z_t^-(\omega)$ is the indicator of the event that there is an $\epsilon(\omega) > 0$ such that $f(Y_s(\omega)) < f(Y_t(\omega))$ for all $-\epsilon(\omega) < s < 0$. If T is an $(\mathcal{F}_t)$ stopping time then Z_T^- is $\mathcal{F}_T$ measurable, so by the strong Markov property of Y we have

$$\mathbf{E}[Z_T|\mathcal{F}_T] = Z_T^- \cdot \mathbf{E}[H(Y_T, \omega_T^+)|\mathcal{F}_T] = Z_T^- \cdot \mathbf{P}^x[T_{A_f(x)} > 0]\Big|_{x=Y_T} = Z_T^- \cdot \varphi(Y_T). \tag{5}$$

It follows that

$$(\Pi Z)_t = Z_t^- \varphi(Y_t), \qquad t \in \mathbf{R}. \tag{6}$$

Because the random measure κ is $(\mathcal{F}_t)$ optional,

$$\mathbf{E}\int_{\mathbf{R}} \phi(t) Z_t \,\kappa(dt) = \mathbf{E}\int_{\mathbf{R}} \phi(t)(\Pi Z)_t \,\kappa(dt). \tag{7}$$

See, for example, Théorème VI.2.66 in [3].
The time reversed process $\hat{Y} : t \mapsto Y_{-t}$ has the same law as Y. Therefore the optional projection $\hat{\Pi}(\Pi Z)$ of ΠZ with respect to the reverse filtration $\hat{\mathcal{F}}_t := \sigma\{Y_s : s \geq t\}$, $t \in \mathbf{R}$, is given by

$$(\hat{\Pi}(\Pi Z))_t = (\hat{\Pi} Z^-)_t \cdot \varphi(Y_t) = \varphi(Y_t)\varphi(Y_t) = \varphi(Y_t), \tag{8}$$

because φ is $\{0,1\}$-valued by Blumenthal's 0–1 law. Now κ is also adapted to $(\hat{\mathcal{F}}_t)$, so

$$\mathbf{E}\int_{\mathbf{R}} \phi(t)(\Pi Z)_t \,\kappa(dt) = \mathbf{E}\int_{\mathbf{R}} \phi(t)(\hat{\Pi}(\Pi Z))_t \,\kappa(dt). \tag{9}$$

Taken together, (3), (7), (9), (8), and (2) yield

$$0 = \mathbf{E}\int_{\mathbf{R}} \phi(t)\varphi(Y_t)\,\kappa(dt) = \int_{\mathbf{R}} \phi(t)\,dt \int_{\mathbf{R}^d} \varphi(x)\,\pi(dx), \tag{10}$$

which is absurd because $\{\varphi > 0\} = M(f)$ and $\pi(M(f)) \geq \pi(K) > 0$. □

References

[1] A. Ancona: Sur une conjecture concernant la capacité et l'effilement. *Lecture Notes in Math.* **1096**, Théorie du potentiel (Orsay, 1983), pp. 34–68. Springer-Verlag, Berlin-New York, 1984.

[2] K. L. Chung: Probabilistic approach in potential theory to the equilibrium problem. *Ann. Inst. Fourier (Grenoble)* **23** (1973) 313–322.

[3] C. Dellacherie and P.-A. Meyer: *Probabilités et potentiel, Chapitres V à VIII, Théorie des martingales.* Hermann, Paris, 1980.

[4] P. J. Fitzsimmons: Homogeneous random measures and a weak order for the excessive measures of a Markov process. *Trans. Amer. Math. Soc.* **303** (1987) 431–478.

[6] R. K. Getoor and M. J. Sharpe: Last exit times and additive functionals. *Ann. Probability* **1** (1973) 550–569.

[5] M. Fukushima, Y. Ōshima and M. Takeda: *Dirichlet Forms and Symmetric Markov Processes.* Walter de Gruyter & Co., Berlin, 1994.

[7] J. Král and I. Netuka: Fine topology in potential theory and strict maxima of functions. *Exposition. Math.* **5** (1987) 185–191.

[8] H. P. McKean, Jr.: A probabilistic interpretation of equilibrium charge distributions. *J. Math. Kyoto Univ.* **4** (1964/1965) 617–625.