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AN OPTIMAL ITÔ FORMULA FOR LÉVY PROCESSES

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Abstract

Several Itô formulas have been already established for Lévy processes. We explain according to which criteria they are not *optimal* and establish an extended Itô formula that satisfies that criteria. The interest, in particular, of this formula, is to obtain the explicit decomposition of $F(X_t, t)$, for X Lévy process and F deterministic function with locally bounded first order Radon-Nikodym derivatives, as the sum of a Dirichlet process and a bounded variation process.

1 Introduction and main results

Let X be a general real-valued Lévy process with characteristic triplet (a, σ, ν), i.e. its characteristic exponent is equal to

$$\psi(u) = iua - \sigma^2 \frac{u^2}{2} + \int_{\mathbb{R}} (e^{iuy} - 1 - iuy 1_{\{|y| \le 1\}}) \nu(dy)$$

where a and σ are real numbers and ν is a Lévy measure. We will denote by $(\sigma B_t, t \ge 0)$ the Brownian component of X. Let F be a $C^{2,1}$ function from $\mathbb{R} \times \mathbb{R}^+$ to $\mathbb{R}$. The classical Itô formula gives

$$\begin{aligned} F(X_t, t) &= F(X_0, 0) + \int_0^t \frac{\partial F}{\partial t}(X_{s-}, s) ds \\ &+ \int_0^t \frac{\partial F}{\partial x}(X_{s-}, s) dX_s + \frac{\sigma^2}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(X_s, s) ds \qquad (1.1) \\ &+ \sum_{0<s\le t} \{F(X_s, s) - F(X_{s-}, s) - \frac{\partial F}{\partial x}(X_{s-}, s) \Delta X_s\} \end{aligned}$$

This formula can be rewritten under the following form (see [8]): $(F(X_t,t), t \geq 0)$ is a semi-martingale admitting the decomposition

$$F(X_t,t) = F(X_0,0) + M_t + V_t \tag{1.2}$$

where the local martingale M and the adapted with bounded variation process V are given by

$$M_t = \sigma \int_0^t \frac{\partial F}{\partial x}(X_{s-},s)dB_s + \int_0^t \int_{\{|y|<1\}} \{F(X_{s-}+y,s) - F(X_{s-},s)\}\tilde{\mu}_X(dy,ds) \tag{1.3}$$

$$V_t = \sum_{0<s\leq t} \{F(X_s,s) - F(X_{s-},s)\}1_{\{|\Delta X_s|\geq 1\}} + \int_0^t \mathscr{A}F(X_s,s)ds \tag{1.4}$$

where $\tilde{\mu}_X(dy,ds)$ denotes the compensated Poisson measure associated to the jumps of X, and $\mathscr{A}$ is the operator associated to X defined by

$$\begin{aligned}\mathscr{A}G(x,t) \quad = \quad & \frac{\partial G}{\partial t}(x,t) + a\frac{\partial G}{\partial x}(x,s) + \frac{1}{2}\sigma^2\frac{\partial^2 G}{\partial x^2}(x,t) \\ & + \int_{\mathbb{R}} \{G(x+y,t) - G(x,t) - y\frac{\partial G}{\partial x}(x,t)\}1_{(|y|<1)}\nu(dy)\end{aligned}$$

for any function G defined on $\mathbb{R}\times\mathbb{R}^+$, such that $\frac{\partial G}{\partial x}$, $\frac{\partial G}{\partial t}$ and $\frac{\partial^2 G}{\partial x^2}$ exist as Radon-Nikodym derivatives with respect to the Lebesgue measure and the integral is well defined. The later condition is satisfied when $\frac{\partial^2 G}{\partial x^2}$ is locally bounded.

Note that the existence of locally bounded first order Radon-Nikodym derivatives alone guarantees the existence of

$$F(X_t,t) - F(X_0,0) - \int_0^t \frac{\partial F}{\partial t}(X_{s-},s)ds - \int_0^t \frac{\partial F}{\partial x}(X_{s-},s)dX_s \tag{1.5}$$

but then to say that this expression coincides with

$$\frac{\sigma^2}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(X_s,s)ds + \sum_{0<s\leq t}\{F(X_s,s) - F(X_{s-},s) - \frac{\partial F}{\partial x}(X_{s-},s)\Delta X_s\}$$

we need to assume much more on F.

In that sense one might say that the classical Itô formula is not *optimal*. The interest of an optimal formula is two-fold. It allows to expand $F(X_t,t)$ under minimal conditions on F but also to know explicitly the structure of the process $F(X_t,t)$. Such an optimal formula has been established in the particular case when X is a Brownian motion [4]. Indeed in that case, under the minimal assumption on F for the existence of (1.5), namely that F admits locally bounded first order Radon-Nikodym derivatives, we know that this expression coincides with

$$-\frac{1}{2}\int_0^t \int_{\mathbb{R}} \frac{\partial F}{\partial x}(x,s)dL_s^x$$

where $(L_s^x, x\in\mathbb{R}, s\geq 0)$ is the local time process of X. Moreover the process
$(\int_0^t \int_{\mathbb{R}} \frac{\partial F}{\partial x}(x,s)dL_s^x, t\geq 0)$ has a 0-quadratic energy.

In the general case, various extensions of (1.1) have been established. We will quote here only the extensions exploiting the notion of local times, we send to [4] for a more exhaustive bibliography. Meyer [9] has been the first to relax the assumption on F by introducing an integral with respect to local time, followed then by Bouleau and Yor [3], Azéma et al [1], Eisenbaum [4], [5], Ghomrasni and Peskir [7], Eisenbaum and Kyprianou [6]. In the discontinuous case, none of the obtained Itô formulas is optimal because of the presence of the expression $\sum_{0<s\leq t}\{F(X_s,s)-F(X_{s-},s)-\frac{\partial F}{\partial x}(X_{s-},s)\Delta X_s\}$.

The Itô formula for Lévy processes presented below in Theorem 1.1, is available for X admitting a Brownian component. It lightens the condition on the jumps of X required by [5], and it also lightens the condition on the first order derivatives of F required by [6]. Besides it is optimal. To introduce it we need the operator I defined on the set of locally bounded measurable functions G on $\mathbb{R}\times\mathbb{R}^+$ by

$$IG(x,t)=\int_0^x G(y,t)dy.$$

We will denote the Markov local time process of X by $(L^x_t, x\in\mathbb{R}, t\geq 0)$.

Theorem 1.1. *: Assume that $\sigma\neq 0$. Let F be a function from $\mathbb{R}\times\mathbb{R}^+$ to $\mathbb{R}$ such that $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial t}$ exist as Radon-Nikodym derivatives with respect to the Lebesgue measure and are locally bounded. Then the process $(F(X_t,t), t\geq 0)$ admits the following decomposition*

$$F(X_t,t)=F(X_0,0)+M_t+V_t+Q_t$$

with M the local martingale given by (1.3), V is the bounded variation process

$$V_t=\sum_{0\leq s\leq t}\{F(X_s,s)-F(X_{s-},s)\}1_{\{|\Delta X_s|\geq 1\}}$$

and Q the following adapted process with 0-quadratic variation

$$Q_t=-\int_0^t\int_{\mathbb{R}}\mathscr{A}IF(x,s)dL^x_s.$$

As a simple application of Theorem 1.1 consider the example of the function $F(x,s)=|x|$ in the case $\int_0^1 y\nu(dy)=+\infty$. This function does not satisfy the assumption of Theorem 3 of [6] nor X does satisfy the assumption of Theorem 2.2 in [5]. But, thanks to Theorem 1.1, we immediately obtain Tanaka's formula.

The proofs are presented in Section 2.

2 Proofs

We first remind the meaning of integration with respect to the semimartingale local time process of X denoted $(\ell^x_s, x\in\mathbb{R}, s\geq 0)$. Theorem 1.1 is expressed in terms of the Markov local time process $(L^x_s, x\in\mathbb{R}, s\geq 0)$. The two processes are connected by:

$(L^x_s, x\in\mathbb{R}, s\geq 0)=(\frac{1}{\sigma^2}\ell^x_s, x\in\mathbb{R}, s\geq 0)$.

Let σB be the Brownian component of X. Defined the norm $||.||$ of a measurable function f from $\mathbb{R}\times\mathbb{R}_+$ to $\mathbb{R}$ by

$$||f||=2\mathbb{E}(\int_0^1 f^2(X_s,s)ds)^{1/2}+\mathbb{E}(\int_0^1|f(X_s,s)|\frac{|B_s|}{s}ds).$$

In [6], integration with respect to ℓ of locally bounded mesurable function f has been defined by

$$\int_0^t \int_{\mathbb{R}} f(x,s)d\ell_s^x = \sigma \int_0^t f(X_{s-},s)dB_s + \sigma \int_{1-t}^1 f(\hat{X}_{s-},1-s)d\hat{B}_s, \ \ 0 \le t \le 1 \tag{2.1}$$

where $\hat{B}$ and $\hat{X}$ are the time reversal at 1 of B and X.
We have the following properties:
(i) $\mathbb{E}(|\int_0^t \int_{\mathbb{R}} f(x,s)d\ell_s^x|) \le |\sigma|||f||$.
(ii) If f admits a locally bounded Radon-Nikodym derivative with respect to x, then: $\int_0^t \int_{\mathbb{R}} f(x,s)d\ell_s^x = -\sigma^2 \int_0^t \frac{\partial f}{\partial x}(X_s,s)ds$.
(iii) The process $(\int_0^t \int_{\mathbb{R}} f(x,s)d\ell_s^x, 0 \le t \le 1)$ has 0-quadratic variation.

Proof of Theorem 1.1 : We start by assuming that F and $\frac{\partial F}{\partial x}$ are bounded. We set

$$F_n(x,t) = \int\int_{\mathbb{R}^2} F(x-y/n, t-s/n)f(y)h(s)dyds$$

where f and h are nonnegative C^∞ functions with compact supports such that : $\int_{\mathbb{R}} f(x)dx = \int_{\mathbb{R}} h(x)dx = 1$. Thanks to the usual Itô formula we have:

$$\begin{aligned} F_n(X_t,t) = \quad & F_n(0,0) + \sigma \int_0^t \frac{\partial F_n}{\partial x}(X_{s-},s)dB_s + \int_0^t \frac{\partial F_n}{\partial t}(X_s,s)ds \\ & + a \int_0^t \frac{\partial F_n}{\partial x}(X_s,s)ds + \sum_{0\le s\le t} \{F_n(X_s,s) - F_n(X_{s-},s)\} 1_{\{|\Delta X_s|\ge 1\}} \\ & + \int_0^t \int_{\mathbb{R}} \{F_n(X_{s-}+y,s) - F_n(X_{s-},s)\} 1_{\{|y|<1\}} \tilde{\mu}(ds,dy) \\ & + \frac{\sigma^2}{2} \int_0^t \frac{\partial^2 F_n}{\partial x^2}(X_s,s)ds \\ & + \int_0^t \int_{-1}^1 \{F_n(X_s+y,s) - F_n(X_s,s) - \frac{\partial F_n}{\partial x}(X_s,s)y\} \nu(dy)ds \end{aligned} \tag{2.2}$$

With the same arguments as in the proof of Theorem 2.2 of [5], we see that as n tends to ∞, $F_n(X_t,t)$ and each of the first five terms of the RHS of (2.2) converges at least in probability to the corresponding expression with F replacing F_n. Besides we note that

$$\begin{aligned} \int_0^t \frac{\partial F}{\partial t}(X_s,s)ds &= -\frac{1}{\sigma^2} \int_0^t \int_{\mathbb{R}} (\int_0^x \frac{\partial F}{\partial t}(y,s)dy)d\ell_s^x \\ &= -\int_0^t \int_{\mathbb{R}} (\frac{\partial}{\partial t} \int_0^x F(y,s)dy)dL_s^x \end{aligned}$$

since $\frac{\partial F}{\partial t}$ is locally bounded. Hence we have:

$$\int_0^t \frac{\partial F}{\partial t}(X_s,s)ds = -\int_0^t \int_{\mathbb{R}} \frac{\partial (IF)}{\partial t}(x,s)dL_s^x. \tag{2.3}$$

Since : $F(x,s)=\frac{\partial(IF)}{\partial x}(x,s)$, we immediately obtain:

$$a\int_0^t \frac{\partial F}{\partial x}(X_s,s)ds = -\int_0^t\int_{\mathbb{R}} a\frac{\partial(IF)}{\partial x}(x,s)dL_s^x. \tag{2.4}$$

The convergence in L^2 of the sixth term of the RHS is obtained with the same proof as in [6]. The limit is equal to

$$\int_0^t\int_{\mathbb{R}}\{F(X_{s-}+y,s)-F(X_{s-},s)\}1_{\{|y|<1\}}\tilde{\mu}(ds,dy) \tag{2.5}$$

For the seventh term of the RHS of (2.2), we note that :
$\frac{\sigma^2}{2}\int_0^t\frac{\partial^2F_n}{\partial x^2}(X_s,s)ds=-\frac{1}{2}\int_0^t\int_{\mathbb{R}}\frac{\partial F_n}{\partial x}(x,s)d\ell_s^x$. Thanks to the properties (i) and (ii) of the integration with respect to the local times, this expression converges in L^1 to $-\frac{1}{2}\int_0^t\int_{\mathbb{R}}\frac{\partial F}{\partial x}(x,s)d\ell_s^x$. We can obviously write:

$$-\frac{1}{2}\int_0^t\int_{\mathbb{R}}\frac{\partial F}{\partial x}(x,s)d\ell_s^x = -\frac{\sigma^2}{2}\int_0^t\int_{\mathbb{R}}\frac{\partial^2(IF)}{\partial x^2}(x,s)dL_s^x. \tag{2.6}$$

We now study the convergence of the last term of the RHS of (2.2). We have:

$$\begin{aligned}\int_0^t\int_{-1}^1 & \{F_n(X_s+y,s)-F_n(X_s,s)-\frac{\partial F_n}{\partial x}(X_s,s)y\}\nu(dy)ds\\ = & -\int_0^t\int_{\mathbb{R}}H_n(x,s)dL_s^x \end{aligned}\tag{2.7}$$

where: $H_n(x,s)=\int_0^x\int_{-1}^1\{F_n(z+y,s)-F_n(z,s)-\frac{\partial F_n}{\partial x}(z,s)y\}\nu(dy)dz$. We have:

$$\begin{aligned}|F_n(z+y,s) & -F_n(z,s)-\frac{\partial F_n}{\partial x}(z,s)y|1_{\{|y|<1\}}\\ & =|\int_z^{z+y}\frac{\partial F_n}{\partial x}(v,t)-\frac{\partial F_n}{\partial x}(z,t)dv|1_{\{|y|<1\}}\\ & \le y^2\sup|\frac{\partial^2F_n}{\partial x^2}|1_{\{|y|<1\}}.\end{aligned}$$

Noting that: $\frac{\partial^2F_n}{\partial x^2}(x,t)=n^2\int\int_{\mathbb{R}^2}F(x-y/n,t-s/n)f''(y)h(s)dyds$, we obtain $|F_n(z+y,s)-F_n(z,s)-\frac{\partial F_n}{\partial x}(z,s)y|1_{\{|y|<1\}}\le cste\, n^2y^21_{\{|y|<1\}}\sup|F|$
Consequently :

$$\begin{aligned}H_n(x,s) & =\int_{-1}^1\int_0^x\{F_n(z+y,s)-F_n(z,s)-\frac{\partial F_n}{\partial x}(z,s)y\}dz\nu(dy)\\ & = \int_{-1}^1\{\int_0^{x+y}F_n(z,s)dz-\int_0^xF_n(z,s)dz-yF_n(x,s)+yF_n(0,s)-\int_0^yF_n(z,s)dz\}\nu(dy)\\ & = G_n(x,s)+\int_{-1}^1(yF_n(0,s)-\int_0^yF_n(z,s)dz)\nu(dy)\end{aligned}$$

where $G_n(x,s)=\int_{-1}^{1}(IF_n(x+y,s)-IF_n(x,s)-yF_n(x,s))\nu(dy)$. Thanks to Corollary 8 of [6], we know that

$$\int_0^t\int_{\mathbb{R}} H_n(x,s)dL_s^x=\int_0^t\int_{\mathbb{R}} G_n(x,s)dL_s^x. \tag{2.8}$$

By dominated convergence, we have as n tends to ∞ for every (x,s)

$$IF_n(x+y,s)-IF_n(x,s)-yF_n(x,s)\to IF(x+y,s)-IF(x,s)-yF(x,s).$$

Besides, for every n : $|IF_n(x+y,s)-IF_n(x,s)-yF_n(x,s)|\le y^2 1_{\{|y|<1\}}\sup|\frac{\partial F}{\partial x}|$, hence for every (x,s) : $G_n(x,s)$ tends to $G(x,s)$, where

$$G(x,s)=\int_{\mathbb{R}}(IF(x+y,s)-IF(x,s)-yF(x,s))1_{\{|y|<1\}}\nu(dy).$$

By dominated convergence, $(G_n)_{n>0}$ converges for the norm $||.||$ to G. Consequently the limit of the last term of the RHS of (2.2) is equal by (2.7) and (2.8) to

$$-\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}(IF(x+y,s)-IF(x,s)-yF(x,s))1_{\{|y|<1\}}\nu(dy)dL_s^x. \tag{2.9}$$

Summing all the limits (2.3), (2.4), (2.5), (2.6) and (2.9), we finally obtain

$$\begin{aligned} F(X_t,t) &= F(X_0,0)+\sigma\int_0^t\frac{\partial F}{\partial x}(X_{s-},s)dB_s \\ &+ \int_0^t\int_{\mathbb{R}}\{F(X_{s-}+y,s)-F(X_{s-},s)\}1_{\{|y|<1|\}}\tilde{\mu}(ds,dy) \\ &+ \sum_{0<s\le t}\{F(X_s,s)-F(X_{s-},s)\}1_{\{|\Delta X_s|\ge 1\}} \\ &- \int_0^t\int_{\mathbb{R}}\{\frac{\partial(IF)}{\partial t}(x,s)+a\frac{\partial(IF)}{\partial x}(x,s)+\frac{\sigma^2}{2}\frac{\partial^2(IF)}{\partial x^2}(x,s)\}dL_s^x \\ &- \int_0^t\int_{\mathbb{R}}\{\int\{IF(x+y,s)-IF(x,s)-yF(x,s)\}1_{\{|y|<1\}}\nu(dy)\}dL_s^x. \end{aligned} \tag{2.10}$$

which summarizes in

$$\begin{aligned} F(X_t,t) &= F(X_0,0)+\sigma\int_0^t\frac{\partial F}{\partial x}(X_{s-},s)dB_s \\ &\quad+\int_0^t\int_{\mathbb{R}}\{F(X_{s-}+y,s)-F(X_{s-},s)\}1_{\{|y|<1|\}}\tilde{\mu}(ds,dy) \\ &\quad+\sum_{0<s\le t}\{F(X_s,s)-F(X_{s-},s)\}1_{\{|\Delta X_s|\ge 1\}}-\int_0^t\int_{\mathbb{R}}\mathscr{A}IF(x,s)dL_s^x. \end{aligned}$$

In the general case, we set:

$$\tilde{F}_n(x,s)=F(x,s)1_{[a_n,b_n]}(x)+F(a_n,s)1_{(-\infty,a_n)}(x)+F(b_n,s)1_{(b_n,\infty)}(x)$$

where $(-a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are two positive real sequences increasing to ∞.
We write (2.10) for $\tilde{F}_n$ and stop the process $(\tilde{F}_n(X_s,s), 0 \le s \le 1)$ at $T_m = 1 \wedge \inf\{s \ge 0 : |X_s| > m\}$. We let n tend to ∞ and then m tend to ∞. The behavior of two terms deserves specific explanations, the other terms converging respectively to the expected expressions.
The first one is : $\int_0^{t\wedge T_m}\int_{\mathbb{R}}\{\int\{I\tilde{F}_n(x+y,s)-I\tilde{F}_n(x,s)-y\tilde{F}_n(x,s)\}1_{\{|y|<1\}}\nu(dy)\}dL_s^x$. Thanks to the definition of the integral with respect to local time (2.1), it is equal to

$$\frac{1}{\sigma}\int_0^{t\wedge T_m}\tilde{H}_n(X_{s-},s)dB_s+\frac{1}{\sigma}\int_{1-(t\wedge T_m)}^1\tilde{H}_n(\hat{X}_{s-},s)d\hat{B}_s \tag{2.11}$$

where $\tilde{H}_n(x,s)=\int\{I\tilde{F}_n(x+y,s)-I\tilde{F}_n(x,s)-y\tilde{F}_n(x,s)\}1_{\{|y|<1\}}\nu(dy)$.
We set $H(x,s)=\int\{IF(x+y,s)-IF(x,s)-yF(x,s)\}1_{\{|y|<1\}}\nu(dy)$.
We can choose n big enough to have $|a_n|$ and b_n bigger than $m+1$. Hence (2.11) is equal to

$$\frac{1}{\sigma}\int_0^{t\wedge T_m}H(X_{s-},s)dB_s+\frac{1}{\sigma}\int_{1-(t\wedge T_m)}^1 H(\hat{X}_{s-},s)d\hat{B}_s.$$

For every $\epsilon > 0$

$$\begin{aligned}\mathbb{P}(\sup_{0\le t\le 1}|\int_{1-(t\wedge T_m)}^1 H(\hat{X}_{s-},s)d\hat{B}_s \quad &- \quad \int_{1-t}^1 H(\hat{X}_{s-},s)d\hat{B}_s|\ge\epsilon)\\ &\le \quad \mathbb{P}(T_m<1)\\ &= \quad \mathbb{P}(\sup_{0\le t\le 1}|X_t|>m)\end{aligned}$$

which shows that as m tends to ∞, $\int_{1-(t\wedge T_m)}^1 H(\hat{X}_{s-},s)d\hat{B}_s$ converges in probability uniformly on $[0,1]$ to $\int_{1-t}^1 H(\hat{X}_{s-},s)d\hat{B}_s$. Similarly $\int_0^{t\wedge T_m}H(X_{s-},s)dB_s$ converges in probability to $\int_0^t H(X_{s-},s)dB_s$. Consequently as m tends to ∞, (2.11) converges to

$$\int_0^t\int_{\mathbb{R}}\{\int\{IF(x+y,s)-IF(x,s)-yF(x,s)\}1_{\{|y|<1\}}\nu(dy)\}dL_s^x.$$

The second term is : $\int_0^t\int_{\mathbb{R}}\{\tilde{F}_n(X_{s-}+y,s)-\tilde{F}_n(X_{s-},s)\}1_{\{s<T_m\}}1_{\{|y|<1|\}}\tilde{\mu}(ds,dy)$. For n big enough such that $|a_n|$, $b_n > m$, this term is equal to
$\int_0^t\int_{\mathbb{R}}\{F(X_{s-}+y,s)-F(X_{s-},s)\}1_{\{s<T_m\}}1_{\{|y|<1|\}}\tilde{\mu}(ds,dy)$. As Ikeda and Watanabe [8], we then denote by $(\int_0^t\int_{\mathbb{R}}\{F(X_{s-}+y,s)-F(X_{s-},s)\}1_{\{|y|<1|\}}\tilde{\mu}(ds,dy), 0\le t\le 1)$ the local martingale $(Y_t, 0\le t\le 1)$ defined by :

$$Y_{t\wedge T_m}=\int_0^t\int_{\mathbb{R}}\{\tilde{F}(X_{s-}+y,s)-\tilde{F}(X_{s-},s)\}1_{\{s<T_m\}}1_{\{|y|<1|\}}\tilde{\mu}(ds,dy).\square$$

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