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LOCALIZATION FOR (1+1)-DIMENSIONAL PINNING MODELS WITH $(\nabla + \Delta)$-INTERACTION

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Abstract

We study the localization/delocalization phase transition in a class of directed models for a homogeneous linear chain attracted to a defect line. The self-interaction of the chain is of mixed gradient and Laplacian kind, whereas the attraction to the defect line is of δ-pinning type, with strength $\varepsilon \geq 0$. It is known that, when the self-interaction is purely Laplacian, such models undergo a *non-trivial* phase transition: to localize the chain at the defect line, the reward ε must be greater than a strictly positive critical threshold $\varepsilon_c > 0$. On the other hand, when the self-interaction is purely gradient, it is known that the transition is *trivial*: an arbitrarily small reward $\varepsilon > 0$ is sufficient to localize the chain at the defect line ($\varepsilon_c = 0$). In this note we show that in the mixed gradient and Laplacian case, under minimal assumptions on the interaction potentials, the transition is always trivial, that is $\varepsilon_c = 0$.

1 Introduction

We consider a simple directed model for a homogeneous linear chain $\{(i, \varphi_i)\}_{0 \leq i \leq N}$, such as a polymer, which is randomly distributed in space and is attracted to the line $\{(i, 0)\}_{0 \leq i \leq N}$ through a *pinning interaction*, see Figure 1. We will often refer to $\{\varphi_i\}_i$ as the *field*. We discuss the localization properties of the model as a function of the attraction strength $\varepsilon \geq 0$ and of the characteristics of the chains, that are embodied in two potentials V_1 and V_2.

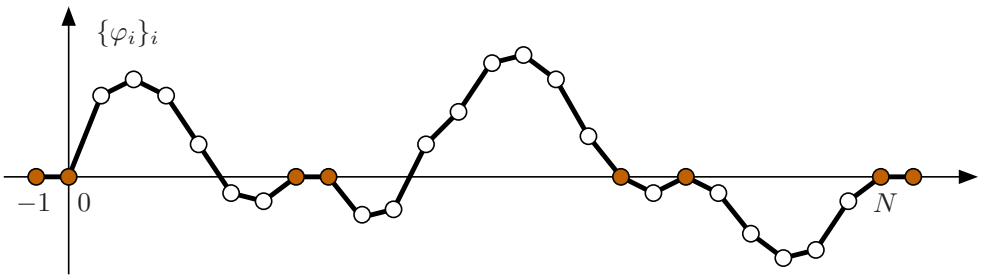

Figure 1: A sample trajectory of the model $\mathbb{P}_{\varepsilon,N}$.

1.1 The model

We first define the *Hamiltonian*, which describes the self-interaction of the field $\varphi = \{\varphi_i\}_i$:

$$\mathscr{H}_{[-1,N+1]}(\varphi) \;=\; \mathscr{H}_{[-1,N+1]}(\varphi_{-1},...,\varphi_{N+1}) \;:=\; \sum_{i=1}^{N+1} V_1(\nabla\varphi_i) + \sum_{i=0}^{N} V_2(\Delta\varphi_i), \tag{1}$$

where N represents the length of the chain. The discrete gradient and Laplacian of the field are defined respectively by $\nabla\varphi_i := \varphi_i - \varphi_{i-1}$ and $\Delta\varphi_i := \nabla\varphi_{i+1} - \nabla\varphi_i = \varphi_{i+1} + \varphi_{i-1} - 2\varphi_i$. The precise assumptions on the potentials V_1 and V_2 are stated below.
Given the strength of the pinning attraction $\varepsilon \geq 0$ between the chain and the defect line, we define our model $\mathbb{P}_{\varepsilon,N}$ as the following probability measure on $\mathbb{R}^{N-1}$:

$$\mathbb{P}_{\varepsilon,N}(\mathrm{d}\varphi_1,\dots,\mathrm{d}\varphi_{N-1}) \;:=\; \frac{\exp(-\mathscr{H}_{[-1,N+1]}(\varphi))}{\mathscr{Z}_{\varepsilon,N}} \prod_{i=1}^{N-1}(\varepsilon\delta_0(\mathrm{d}\varphi_i) + \mathrm{d}\varphi_i) \tag{2}$$

where we denote by $\delta_0(\cdot)$ the Dirac mass at zero, by $\mathrm{d}\varphi_i = \mathit{Leb}(\mathrm{d}\varphi_i)$ the Lebesgue measure on $\mathbb{R}$ and we choose for simplicity zero boundary conditions: $\varphi_{-1} = \varphi_0 = \varphi_N = \varphi_{N+1} = 0$ (see Figure 1). The normalization constant $\mathscr{Z}_{\varepsilon,N}$ appearing in (2) plays an important role, as we are going to see in a moment: it is called *partition function* and is given by

$$\mathscr{Z}_{\varepsilon,N} \;=\; \int_{\mathbb{R}^{N-1}} e^{-\mathscr{H}_{[-1,N+1]}(\varphi)} \prod_{i=1}^{N-1}(\varepsilon\delta_0(\mathrm{d}\varphi_i) + \mathrm{d}\varphi_i)\,. \tag{3}$$

We assume that the potentials $V_1, V_2 : \mathbb{R} \to \mathbb{R}$ appearing in (1) are measurable functions satisfying the following conditions:

(C1) V_1 is bounded from below ($\inf_{x\in\mathbb{R}} V_1(x) > -\infty$), symmetric ($V_1(x) = V_1(-x)$ for every $x \in \mathbb{R}$), such that $\lim_{|x|\to\infty} V_1(x) = +\infty$ and $\int_{\mathbb{R}} e^{-2V_1(x)}\,\mathrm{d}x < \infty$.

(C2) V_2 is bounded from below ($\inf_{x\in\mathbb{R}} V_2(x) > -\infty$), bounded from above in a neighborhood of zero ($\sup_{|x|\leq\gamma} V_2(x) < \infty$ for some $\gamma > 0$) and such that $\int_{\mathbb{R}} |x|\, e^{-V_2(x)}\,\mathrm{d}x < \infty$.

We stress that no continuity assumption is made. The symmetry of V_1 ensures that there is no "local drift" for the gradient of the field (remarkably, no such assumption on V_2 is necessary; see also Remark 7 below). We point out that the hypothesis that both V_1 and V_2 are finite everywhere

could be relaxed, allowing them to take the value $+\infty$ outside some interval $(-M, M)$, but we stick for simplicity to the above stated assumptions.

The model $\mathbb{P}_{\varepsilon,N}$ is an example of a *random polymer model*, more precisely a (homogeneous) *pinning model*. A lot of attention has been devoted to this class of models in the recent mathematical literature (see [8, 7] for two beautiful monographs).
The main question, for models like ours, is whether the pinning reward $\varepsilon \geq 0$ is strong enough to localize the field at the defect line for large N. The case when the self-interaction of the field is of purely gradient type, i.e., when $V_2 \equiv 0$ in (1), has been studied in depth [1, 3, 6, 2], as well as the purely Laplacian case when $V_1 \equiv 0$, cf. [4, 5]. We now consider the mixed case when both $V_1 \not\equiv 0$ and $V_2 \not\equiv 0$, which is especially interesting from a physical viewpoint, because of its direct relevance in modeling *semiflexible polymers*, cf. [9]. Intuitively, the gradient interaction penalizes large elongations of the chain while the Laplacian interaction penalizes curvature and bendings.

1.2 Free energy and localization properties

The standard way to capture the localization properties of models like ours is to look at the exponential rate of growth (Laplace asymptotic behavior) as $N \to \infty$ of the partition function $\mathscr{Z}_{\varepsilon,N}$. More precisely, we define the *free energy* $F(\varepsilon)$ of our model as

$$F(\varepsilon) := \lim_{N\to\infty} \frac{1}{N} \log \left(\frac{\mathscr{Z}_{\varepsilon,N}}{\mathscr{Z}_{0,N}} \right), \tag{4}$$

where the limit is easily shown to exist by a standard super-additivity argument [8].
The function $\varepsilon \mapsto \mathscr{Z}_{\varepsilon,N}$ is non-decreasing for fixed N (cf. (3)), hence $\varepsilon \mapsto F(\varepsilon)$ is non-decreasing too. Recalling that $F(0) = 0$, we define the *critical value* ε_c as

$$\varepsilon_c := \sup\{\varepsilon \geq 0 : F(\varepsilon) = 0\} = \inf\{\varepsilon \geq 0 : F(\varepsilon) > 0\} \in [0, \infty], \tag{5}$$

and we say that our model $\{\mathbb{P}_{\varepsilon,N}\}_{N\in\mathbb{N}}$ is

- *delocalized* if $\varepsilon < \varepsilon_c$;
- *localized* if $\varepsilon > \varepsilon_c$.

This seemingly mysterious definition of localization and delocalization does correspond to sharply different behaviors of the typical trajectories of our model. More precisely, denoting by $\ell_N := \#\{1 \leq i \leq N-1 : \varphi_i = 0\}$ the number of contacts between the linear chain and the defect line, it is easily shown by convexity arguments that

- if $\varepsilon < \varepsilon_c$, for every $\delta > 0$ there exists $c_\delta > 0$ such that

$$\mathbb{P}_{\varepsilon,N}(\ell_N/N > \delta) \leq e^{-c_\delta N}, \qquad \text{for all } N \in \mathbb{N}; \tag{6}$$

- if $\varepsilon > \varepsilon_c$, there exists $\delta_\varepsilon > 0$ and $c_\varepsilon > 0$ such that

$$\mathbb{P}_{\varepsilon,N}(\ell_N/N < \delta_\varepsilon) \leq e^{-c_\varepsilon N}, \qquad \text{for all } N \in \mathbb{N}. \tag{7}$$

In words: if the model is delocalized then typically $\ell_N = o(N)$, while if the model is localized then typically $\ell_N \geq \delta_\varepsilon N$ with $\delta_\varepsilon > 0$. We refer, e.g., to [4, Appendix A] for the proof of these facts. We

point out that the behavior of the model at the critical point is a much more delicate issue, which is linked to the regularity of the free energy.

Coming back to the critical value, it is quite easy to show that $\varepsilon_c < \infty$ (it is a by-product of our main result), that is, the localized regime is non-empty. However, it is not *a priori* clear whether $\varepsilon_c > 0$, i.e. whether the delocalized regime is non-empty. For instance, in the purely Laplacian case ($V_1 \equiv 0$, cf. [4]), one has $\varepsilon_c^{\Delta} > 0$. On the other hand, in the purely gradient case ($V_2 \equiv 0$, cf. [2]) one has $\varepsilon_c^{\nabla} = 0$ and the model is said to undergo a *trivial phase transition*: an arbitrarily small pinning reward is able to localize the linear chain.

The main result of this note is that in the general case of mixed gradient and Laplacian interaction the phase transition is always trivial.

Theorem 1. *For any choice of the potentials V_1, V_2 satisfying assumptions* (C1) *and* (C2) *one has $\varepsilon_c = 0$, i.e., $F(\varepsilon) > 0$ for every $\varepsilon > 0$.*

Generally speaking, it may be expected that the gradient interaction terms should dominate over the Laplacian ones, at least when V_1 and V_2 are comparable functions. Therefore, having just recalled that $\varepsilon_c^{\nabla} = 0$, Theorem 1 does not come as a surprise. Nevertheless, our assumptions (C1) and (C2) are very general and allow for strikingly different asymptotic behavior of the potentials: for instance, one could choose V_1 to grow only logarithmically and V_2 exponentially fast (or even more). The fact that the gradient interaction dominates even in such extreme cases is quite remarkable.

Remark 2. Our proof yields actually an explicit lower bound on the free energy, which is however quite poor. This issue is discussed in detail in Remark 9 in section 3 below.

Remark 3. Theorem 1 was first proved in the Ph.D. thesis [1] in the special case when both interaction potentials are quadratic: $V_1(x) = \frac{\alpha}{2} x^2$ and $V_2(x) = \frac{\beta}{2} x^2$, for any $\alpha, \beta > 0$. We point out that, with such a choice for the potentials, the free model $\mathbb{P}_{0,N}$ is a Gaussian law and several explicit computations are possible.

1.3 Organization of the paper

The rest of the paper is devoted to the proof of Theorem 1, which is organized in two parts:

- in section 2 we give a basic representation of the free model ($\varepsilon = 0$) as the bridge of an integrated Markov chain, and we study some asymptotic properties of this hidden Markov chain;
- in section 3 we give an explicit lower bound on the partition function $\mathcal{Z}_{\varepsilon,N}$ which, together with the estimates obtained in section 2, yields the positivity of the free energy $F(\varepsilon)$ for every $\varepsilon > 0$, hence the proof of Theorem 1.

Some more technical points are deferred to Appendix A.

1.4 Some recurrent notation and basic results

We set $\mathbb{R}^+ = [0, \infty)$, $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$. We denote by *Leb* the Lebesgue measure on $\mathbb{R}$.

We denote by $L^p(\mathbb{R})$, for $p \in [1, \infty]$, the Banach space of (equivalence classes of) measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that $\|f\|_p < \infty$, where $\|f\|_p := (\int_{\mathbb{R}} |f(x)|^p \, \mathrm{d}x)^{1/p}$ for $p \in [1, \infty)$ and $\|f\|_\infty := \operatorname{ess\,sup}_{x \in \mathbb{R}} |f(x)| = \inf\{M > 0 : \mathit{Leb}\{x \in \mathbb{R} : |f(x)| > M\} = 0\}$.

Given two measurable functions $f, g : \mathbb{R} \to \mathbb{R}^+$, their convolution is denoted as usual by $(f * g)(x) := \int_{\mathbb{R}} f(x-y)\, g(y)\, \mathrm{d}y$. We recall that if $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$ then $f * g$ is bounded and continuous, cf. Theorem D.4.3 in [11].

2 A Markov chain viewpoint

We are going to construct a Markov chain which will be the basis of our analysis. Consider the linear integral operator $f \mapsto \mathscr{K} f$ defined (for a suitable class of functions f) by

$$(\mathscr{K} f)(x) := \int_{\mathbb{R}} k(x,y) f(y)\, \mathrm{d}y\,, \qquad \text{where} \quad k(x,y) := e^{-V_1(y) - V_2(y-x)}\,. \tag{8}$$

The idea is to modify $k(x,y)$ with boundary terms to make $\mathscr{K}$ a probability kernel.

2.1 Integrated Markov chain

By assumption (C1) we have $\|e^{-2V_1}\|_1 < \infty$. It also follows by assumption (C2) that $e^{-V_2} \in L^1(\mathbb{R})$, because we can write

$$\|e^{-V_2}\|_1 \;=\; \int_{\mathbb{R}} e^{-V_2(x)}\, \mathrm{d}x \;\le\; 2 \sup_{x \in [-1,1]} e^{-V_2(x)} + \int_{\mathbb{R}\setminus[-1,1]} |x|\, e^{-V_2(x)}\, \mathrm{d}x \;<\; \infty\,.$$

Since we also have $e^{-V_2} \in L^\infty(\mathbb{R})$, again by (C2), it follows that $e^{-V_2} \in L^p(\mathbb{R})$ for all $p \in [1,\infty]$, in particular $\|e^{-2V_2}\|_1 < \infty$. We then obtain

$$\int_{\mathbb{R}\times\mathbb{R}} k(x,y)^2\, \mathrm{d}x\, \mathrm{d}y \;=\; \int_{\mathbb{R}} e^{-2V_1(y)} \left(\int_{\mathbb{R}} e^{-2V_2(y-x)}\, \mathrm{d}x \right) \mathrm{d}y \;=\; \|e^{-2V_1}\|_1\, \|e^{-2V_2}\|_1 \;<\; \infty\,.$$

This means that $\mathscr{K}$ is Hilbert-Schmidt, hence a compact operator on $L^2(\mathbb{R})$. Since $k(x,y) \ge 0$ for all $x, y \in \mathbb{R}$, we can then apply an infinite dimensional version of the celebrated Perron-Frobenius Theorem. More precisely, Theorem 1 in [13] ensures that the spectral radius $\lambda > 0$ of $\mathscr{K}$ is an isolated eigenvalue, with corresponding right and left eigenfunctions $v, w \in L^2(\mathbb{R})$ satisfying $w(x) > 0$ and $v(x) > 0$ for almost every $x \in \mathbb{R}$:

$$v(x) \;=\; \frac{1}{\lambda} \int_{\mathbb{R}} k(x,y)\, v(y)\, \mathrm{d}y\,, \qquad w(x) \;=\; \frac{1}{\lambda} \int_{\mathbb{R}} w(y)\, k(y,x)\, \mathrm{d}y\,. \tag{9}$$

These equations give a canonical definition of $v(x)$ and $w(x)$ (up to a multiplicative constant) for every $x \in \mathbb{R}$. Since $k(x,y) > 0$ for all $x, y \in \mathbb{R}$, it is then clear that $w(x) > 0$ and $v(x) > 0$ *for every* $x \in \mathbb{R}$. We also point out that the symmetry assumption on V_1 (cf. (C1)) entails that $w(x)$ is a constant multiple of $e^{-V_1(x)} v(-x)$, cf. Remark 7 below.
We can now define a probability kernel $\mathscr{P}(x, \mathrm{d}y)$ by setting

$$\mathscr{P}(x, \mathrm{d}y) \;:=\; p(x,y)\, \mathrm{d}y \;:=\; \frac{1}{\lambda}\, \frac{1}{v(x)}\, k(x,y)\, v(y)\, \mathrm{d}y\,. \tag{10}$$

Since $\mathscr{P}(x, \mathbb{R}) = \int_{\mathbb{R}} p(x,y)\, \mathrm{d}y = 1$ for every $x \in \mathbb{R}$, we can define a Markov chain on $\mathbb{R}$ with transition kernel $\mathscr{P}(x, \mathrm{d}y)$. More precisely, for $a, b \in \mathbb{R}$ let $(\Omega, \mathscr{A}, \mathrm{P}^{(a,b)})$ be a probability space on which is defined a Markov chain $Y = \{Y_i\}_{i \in \mathbb{N}_0}$ on $\mathbb{R}$ such that

$$Y_0 \;=\; a\,, \qquad \mathrm{P}^{(a,b)}(Y_{n+1} \in \mathrm{d}y \,|\, Y_n = x) \;=\; \mathscr{P}(x, \mathrm{d}y)\,, \tag{11}$$

and we define the corresponding *integrated Markov chain* $W=\{W_i\}_{i\in\mathbb{N}_0}$ setting

$$W_0 \;=\; b\,, \qquad W_n \;=\; b+Y_1+\ldots+Y_n\,. \tag{12}$$

The reason for introducing such processes is that they are closely related to our model, as we show in Proposition 5 below. We first need to compute explicitly the finite dimensional distributions of the process W.

Proposition 4. *For every $n\in\mathbb{N}$, setting $w_{-1}:=b-a$ and $w_0:=b$, we have*

$$\mathrm{P}^{(a,b)}\big((W_1,...,W_n)\in(\mathrm{d}w_1,...,\mathrm{d}w_n)\big) \;=\; \frac{\nu(w_n-w_{n-1})}{\lambda^n\,\nu(a)}\,e^{-\mathscr{H}_{[-1,n]}(w_{-1},...,w_n)}\prod_{i=1}^{n}\mathrm{d}w_i\,. \tag{13}$$

Proof. Since $Y_i=W_i-W_{i-1}$ for all $i\geq 1$, the law of $(W_1,...,W_n)$ is determined by the law of $(Y_1,...,Y_n)$. If we set $y_i:=w_i-w_{i-1}$ for $i\geq 2$ and $y_1:=w_1-b$, it then suffices to show that the right hand side of equation (13) is a probability measure under which the variables $(y_i)_{i=1,...,n}$ are distributed like the first n steps of a Markov chain starting at a with transition kernel $p(x,y)$. To this purpose, the Hamiltonian can be rewritten as

$$\mathscr{H}_{[-1,n]}(w_{-1},...,w_n) \;=\; V_1(y_1)+V_2(y_1-a)+\sum_{i=2}^{n}\big(V_1(y_i)+V_2(y_i-y_{i-1})\big)\,.$$

Therefore, recalling the definitions (8) of $k(x,y)$ and (10) of $p(x,y)$, we can write

$$\begin{aligned}\frac{\nu(w_n-w_{n-1})}{\lambda^n\,\nu(a)}e^{-\mathscr{H}_{[-1,n]}(w_{-1},...,w_n)} \;&=\; \frac{\nu(y_n)}{\lambda^n\,\nu(a)}\,k(a,y_1)\prod_{i=2}^{n}k(y_{i-1},y_i)\\ &=\; p(a,y_1)\prod_{i=2}^{n}p(y_{i-1},y_i)\,,\end{aligned}$$

which is precisely the density of $(Y_1,...,Y_n)$ under $\mathrm{P}^{(a,b)}$ with respect to the Lebesgue measure $\mathrm{d}y_1\cdots\mathrm{d}y_n$. Since the map from $(w_i)_{i=1,...,n}$ to $(y_i)_{i=1,...,n}$ is linear with determinant one, the proof is completed. □

For $n\geq 2$ we denote by $\varphi_n^{(a,b)}(\cdot,\cdot)$ the density of the random vector (W_{n-1},W_n):

$$\varphi_n^{(a,b)}(w_1,w_2) \;:=\; \frac{\mathrm{P}^{(a,b)}\big((W_{n-1},W_n)\in(\mathrm{d}w_1,\mathrm{d}w_2)\big)}{\mathrm{d}w_1\mathrm{d}w_2}\,, \qquad \text{for } w_1,w_2\in\mathbb{R}\,. \tag{14}$$

We can now show that our model $\mathbb{P}_{\varepsilon,N}$ in the free case, that is for $\varepsilon=0$, is nothing but a bridge of the integrated Markov chain W.

Proposition 5. *For every $N\in\mathbb{N}$ the following relations hold:*

$$\mathbb{P}_{0,N}(.) \;=\; \mathrm{P}^{(0,0)}\big((W_1,...,W_{N-1})\in\cdot\,\big|\,W_N=W_{N+1}=0\big)\,, \tag{15}$$

$$\mathscr{Z}_{0,N} \;=\; \lambda^{N+1}\,\varphi_{N+1}^{(0,0)}(0,0)\,. \tag{16}$$

Proof. By Proposition 4, for every measurable subset $A\subseteq\mathbb{R}^{N-1}$ we can write

$$\begin{aligned}&\mathrm{P}^{(0,0)}\big((W_1,...,W_{N-1})\in A\,\big|\,W_N=W_{N+1}=0\big)\\ &\qquad=\; \frac{1}{\lambda^{N+1}\,\varphi_{N+1}^{(0,0)}(0,0)}\int_A e^{-\mathscr{H}_{[-1,N+1]}(w_{-1},...,w_{N+1})}\prod_{i=1}^{N-1}\mathrm{d}w_i\,,\end{aligned} \tag{17}$$

where we set $w_{-1} = w_0 = w_N = w_{N+1} = 0$. Choosing $A = \mathbb{R}^{N-1}$ and recalling the definition (3) of the partition function $\mathscr{Z}_{\varepsilon,N}$, we obtain relation (16). Recalling the definition (2) of our model $\mathbb{P}_{\varepsilon,N}$ for $\varepsilon = 0$, we then see that (17) is nothing but (15). □

2.2 Some asymptotic properties

We now discuss some basic properties of the Markov chain $Y = \{Y_i\}_{i\in\mathbb{N}_0}$, defined in (11). We recall that the underlying probability measure is denoted by $\mathrm{P}^{(a,b)}$ and we have $a = Y_0$. The parameter b denotes the starting point W_0 of the integrated Markov chain $W = \{W_i\}_{i\in\mathbb{N}_0}$ and is irrelevant for the study of Y, hence we mainly work under $\mathrm{P}^{(a,0)}$.

Since $p(x,y) > 0$ for all $x, y \in \mathbb{R}$, cf. (10) and (8), the Markov chain Y is *φ-irreducible* with $\varphi = Leb$: this means (cf. [11, §4.2]) that for every measurable subset $A \subseteq \mathbb{R}$ with $Leb(A) > 0$ and for every $a \in \mathbb{R}$ there exists $n \in \mathbb{N}$, possibly depending on a and A, such that $\mathrm{P}^{(a,0)}(Y_n \in A) > 0$. In our case we can take $n = 1$, hence the chain Y is also *aperiodic*.
Next we observe that $\int_{\mathbb{R}} v(x)w(x)\,\mathrm{d}x \le \|v\|_2 \|w\|_2 < \infty$, because $v, w \in L^2(\mathbb{R})$ by construction. Therefore we can define the probability measure π on $\mathbb{R}$ by

$$\pi(\mathrm{d}x) := \frac{1}{c}\, v(x)w(x)\,\mathrm{d}x\,, \qquad \text{where} \qquad c := \int_{\mathbb{R}} v(x)w(x)\,\mathrm{d}x\,. \tag{18}$$

The crucial observation is that π is an invariant probability for the transition kernel $\mathscr{P}(x,\mathrm{d}y)$: from (10) and (9) we have

$$\begin{aligned} \int_{x\in\mathbb{R}} \pi(\mathrm{d}x)\mathscr{P}(x,\mathrm{d}y) &= \int_{x\in\mathbb{R}} \frac{v(x)w(x)}{c}\,\mathrm{d}x\,\frac{k(x,y)v(y)}{\lambda v(x)}\,\mathrm{d}y \\ &= \frac{w(y)v(y)}{c}\,\mathrm{d}y \;=\; \pi(\mathrm{d}y)\,. \end{aligned} \tag{19}$$

Being φ-irreducible and admitting an invariant probability measure, the Markov chain $Y = \{Y_i\}_{i\in\mathbb{N}_0}$ is *positive recurrent*. For completeness, we point out that Y is also Harris recurrent, hence it is a *positive Harris chain*, cf. [11, §10.1], as we prove in Appendix A (where we also show that *Leb* is a maximal irreducibility measure for Y).

Next we observe that the right eigenfunction *v is bounded and continuous*: in fact, spelling out the first relation in (9), we have

$$v(x) \;=\; \frac{1}{\lambda}\int_{\mathbb{R}} e^{-V_2(y-x)}\, e^{-V_1(y)}\, v(y)\,\mathrm{d}y \;=\; \frac{1}{\lambda}\big(e^{-V_2} * (e^{-V_1}\, v)\big)(x)\,. \tag{20}$$

By construction $v \in L^2(\mathbb{R})$ and by assumption (C1) $e^{-V_1} \in L^2(\mathbb{R})$, hence $(e^{-V_1}\, v) \in L^1(\mathbb{R})$. Since $e^{-V_2} \in L^\infty(\mathbb{R})$ by assumption (C2), it follows by (20) that v, being the convolution of a function in $L^\infty(\mathbb{R})$ with a function in $L^1(\mathbb{R})$, is bounded and continuous. In particular, $\inf_{|x|\le M} v(x) > 0$ for every $M > 0$, because $v(x) > 0$ for every $x \in \mathbb{R}$, as we have already remarked (and as it is clear from (20)).

Next we prove a suitable *drift condition* on the kernel $\mathscr{P}$. Consider the function

$$U(x) \;:=\; \frac{|x|\, e^{V_1(x)}}{v(x)}\,, \tag{21}$$

and note that

$$\begin{aligned}(\mathscr{P}U)(x) &= \int_{\mathbb{R}} p(x,y)\,U(y)\,\mathrm{d}y \;=\; \frac{1}{\lambda\,\nu(x)} \int_{\mathbb{R}} e^{-V_2(y-x)}\,|y|\,\mathrm{d}y \\ &= \frac{1}{\lambda\,\nu(x)} \int_{\mathbb{R}} e^{-V_2(z)}\,|z+x|\,\mathrm{d}z \;\le\; \frac{c_0 + c_1\,|x|}{\lambda\,\nu(x)}\,,\end{aligned} \tag{22}$$

where $c_0 := \int_{\mathbb{R}} |z|\,e^{-V_2(z)}\,\mathrm{d}z < \infty$ and $c_1 := \int_{\mathbb{R}} e^{-V_2(z)}\,\mathrm{d}z < \infty$ by our assumption (C2). Then we fix $M \in (0,\infty)$ such that

$$U(x) - (\mathscr{P}U)(x) \;=\; \frac{|x|\,e^{V_1(x)}}{\nu(x)} - \frac{c_1\,|x| + c_0}{\lambda\,\nu(x)} \;\ge\; \frac{1+|x|}{\nu(x)}\,, \qquad \text{for } |x| > M\,.$$

This is possible because $V_1(x) \to \infty$ as $|x| \to \infty$, by assumption (C1). Next we observe that

$$b \;:=\; \sup_{|x|\le M} \big((\mathscr{P}U)(x) - U(x)\big) \;<\; \infty\,,$$

as it follows from (21) and (22) recalling that ν is bounded and $\inf_{|x|\le M} \nu(x) > 0$ for all $M > 0$. Putting together these estimates, we have shown in particular that

$$(\mathscr{P}U)(x) - U(x) \;\le\; -\frac{1+|x|}{\nu(x)} + b\,1_{[-M,M]}(x)\,. \tag{23}$$

This relation is interesting because it allows to prove the following result.

Proposition 6. *There exists a constant $C \in (0,\infty)$ such that for all $n \in \mathbb{N}$ we have*

$$\mathrm{E}^{(0,0)}\big(|Y_n|\big) \;\le\; C\,, \qquad \mathrm{E}^{(0,0)}\left(\frac{1}{\nu(Y_n)}\right) \le\; C\,. \tag{24}$$

Proof. In Appendix A we prove that $Y = \{Y_i\}_{i\in\mathbb{N}_0}$ is a T-chain (see Chapter 6 in [11] for the definition of T-chains). It follows by Theorem 6.0.1 in [11] that for irreducible T-chains every compact set is petite (see §5.5.2 in [11] for the definition of petiteness). We can therefore apply Theorem 14.0.1 in [11]: relation (23) shows that condition (iii) in that theorem is satisfied by the function U. Since $U(x) < \infty$ for every $x \in \mathbb{R}$, this implies that *for every starting point* $x_0 \in \mathbb{R}$ and for every measurable function $g : \mathbb{R} \to \mathbb{R}$ with $|g(x)| \le (const.)(1+|x|)/\nu(x)$ we have

$$\lim_{n\to\infty} \mathrm{E}^{(x_0,0)}\big(g(Y_n)\big) \;=\; \int_{\mathbb{R}} g(z)\,\pi(\mathrm{d}z) \;<\; \infty\,. \tag{25}$$

The relations in (24) are obtained by taking $x_0 = 0$ and $g(x) = |x|$ or $g(x) = 1/\nu(x)$. □

As a particular case of (25), we observe that for every measurable subset $A \subseteq \mathbb{R}$ and for every $x_0 \in \mathbb{R}$ we have

$$\lim_{n\to\infty} \mathrm{P}^{(x_0,0)}(Y_n \in A) \;=\; \pi(A) \;=\; \frac{1}{c} \int_A \nu(x)\,w(x)\,\mathrm{d}x\,. \tag{26}$$

This is actually a consequence of the classical ergodic theorem for aperiodic Harris recurrent Markov chains, cf. Theorem 113.0.1 in [11].

Remark 7. Although we do not use this fact explicitly, it is interesting to observe that the invariant probability π is symmetric. To show this, we set $\widetilde{\nu}(x) := e^{-V_1(x)}\nu(-x)$ and we note that by the first relation in (9), with the change of variables $y \mapsto -y$, we can write

$$\widetilde{\nu}(x) \;=\; \frac{1}{\lambda}\int_{\mathbb{R}} e^{-V_1(x)}\,k(-x,y)\,\nu(y)\,\mathrm{d}y \;=\; \frac{1}{\lambda}\int_{\mathbb{R}} e^{-V_1(x)}\,k(-x,-y)\,e^{V_1(y)}\,\widetilde{\nu}(y)\,\mathrm{d}y\,.$$

However $e^{-V_1(x)}\,k(-x,-y)\,e^{V_1(y)} = k(y,x)$, as it follows by (8) and the symmetry of V_1 (recall our assumption (C1)). Therefore $\widetilde{\nu}$ satisfies the same functional equation $\widetilde{\nu}(x) = \frac{1}{\lambda}\int_{\mathbb{R}} \widetilde{\nu}(y)\,k(y,x)\,\mathrm{d}y$ as the right eigenfunction w, cf. the second relation in (9). Since the right eigenfunction is uniquely determined up to constant multiples, there must exist $C > 0$ such that $w(x) = C\,\widetilde{\nu}(x)$ for all $x \in \mathbb{R}$. Recalling (18), we can then write

$$\pi(\mathrm{d}x) \;=\; \frac{1}{\widetilde{c}}\,e^{-V_1(x)}\,\nu(x)\,\nu(-x)\,\mathrm{d}x\,, \qquad \widetilde{c} := \frac{c}{C}\,, \tag{27}$$

from which the symmetry of π is evident. From the symmetry of π and (25) it follows in particular that $\mathrm{E}^{(0,0)}(Y_n) \to 0$ as $n \to \infty$, whence the integrated Markov chain $W = \{W_i\}_{i\in\mathbb{N}_0}$ is somewhat close to a random walk with zero-mean increments.
We stress that the symmetry of π follows just by the symmetry of V_1, with no need of an analogous requirement on V_2. Let us give a more intuitive explanation of this fact. When V_1 is symmetric, one can easily check from (10) and (8) that the transition density $p(x,y)$ (or equivalently $k(x,y)$) is invariant under the joint application of *time reversal and space reflection*: by this we mean that for all $n \in \mathbb{N}$ and $x_1,\dots,x_n \in \mathbb{R}$

$$p(x_1,x_2)\cdots p(x_{n-1},x_n)\cdot p(x_n,x_1) \;=\; p(-x_n,-x_{n-1})\cdots p(-x_2,-x_1)\cdot p(-x_1,-x_n)\,. \tag{28}$$

Note that V_2 plays no role for the validity of (28). The point is that, whenever relation (28) holds, the invariant measure of the kernel $p(x,y)$ is symmetric. In fact, (28) implies that the function $h(x) := p(\overline{x},x)/p(-x,-\overline{x})$, where $\overline{x} \in \mathbb{R}$ is an arbitrary fixed point, satisfies

$$h(x)\,p(x,y) \;=\; h(y)\,p(-y,-x)\,, \qquad \forall x,y \in \mathbb{R}\,. \tag{29}$$

It is then an immediate consequence of (29) that $h(-x) = h(x)$ for all $x \in \mathbb{R}$ and that the measure $h(x)\mathrm{d}x$ is invariant. (For our model one computes easily $h(x) = (const.)\,e^{-V_1(x)}\,\nu(x)\,\nu(-x)$, in accordance with (27).)

2.3 Some bounds on the density

We close this section with some bounds on the behavior of the density $\varphi_n^{(0,0)}(x,y)$ at $(x,y) = (0,0)$.

Proposition 8. *There exist positive constants C_1, C_2 such that for all odd $N \in \mathbb{N}$*

$$\frac{C_1}{N} \;\le\; \varphi_N^{(0,0)}(0,0) \;\le\; C_2\,. \tag{30}$$

The restriction to odd values of N is just for technical convenience. We point out that neither of the bounds in (30) is sharp, as the conjectured behavior (in analogy with the pure gradient case, cf. [3]) is $\varphi_N^{(0,0)}(0,0) \sim (const.)N^{-1/2}$.

Proof of Proposition 8. We start with the lower bound. By Proposition 5 and equation (3), we have

$$\varphi^{(0,0)}_{2N+1}(0,0) \;=\; \frac{1}{\lambda^{2N+1}}\,\mathscr{Z}_{0,2N} \;=\; \frac{1}{\lambda^{2N+1}}\int_{\mathbb{R}^{2N-1}} e^{-\sum_{i=1}^{2N+1}V_1(\nabla\varphi_i)-\sum_{i=0}^{2N}V_2(\Delta\varphi_i)}\prod_{i=1}^{2N-1}\mathrm{d}\varphi_i\,,$$

where we recall that the boundary conditions are $\varphi_{-1}=\varphi_0=\varphi_{2N}=\varphi_{2N+1}=0$. To get a lower bound, we restrict the integration on the set

$$C^1_N \;:=\; \Big\{(\varphi_1,\dots,\varphi_{2N-1})\in\mathbb{R}^{2N-1}:\ |\varphi_N-\varphi_{N-1}|<\frac{\gamma}{2}\,,\ |\varphi_N-\varphi_{N+1}|<\frac{\gamma}{2}\Big\}\,,$$

where $\gamma>0$ is the same as in assumption (C2). On C^1_N we have $|\nabla\varphi_{N+1}|<\gamma/2$ and $|\Delta\varphi_N|<\gamma$, therefore $V_2(\Delta\varphi_N)\le M_\gamma:=\sup_{|x|\le\gamma}V_2(x)<\infty$. Also note that $V_1(\nabla\varphi_{2N+1})=V_1(0)$ due to the boundary conditions. By the symmetry of V_1 (recall assumption (C1)), setting $C^2_N(\varphi_N):=\{(\varphi_1,\dots,\varphi_{N-1})\in\mathbb{R}^{N-1}:\ |\varphi_N-\varphi_{N-1}|<\gamma/2\}$, we can write

$$\begin{aligned}
&\varphi^{(0,0)}_{2N+1}(0,0)\\
&\ge\ \frac{e^{-(M_\gamma+V_1(0))}}{\lambda^{2N+1}}\int_{C^1_N} e^{-\sum_{i=1}^{N}V_1(\nabla\varphi_i)-\sum_{i=0}^{N-1}V_2(\Delta\varphi_i)}\,e^{-\sum_{i=N+1}^{2N+1}V_1(\nabla\varphi_i)-\sum_{i=N+1}^{2N}V_2(\Delta\varphi_i)}\prod_{i=1}^{2N-1}\mathrm{d}\varphi_i\\
&=\ \frac{e^{-(M_\gamma+V_1(0))}}{\lambda^{2N+1}}\int_{\mathbb{R}}\mathrm{d}\varphi_N\left[\int_{C^2_N(\varphi_N)} e^{-\sum_{i=1}^{N}V_1(\nabla\varphi_i)-\sum_{i=0}^{N-1}V_2(\Delta\varphi_i)}\prod_{i=1}^{N-1}\mathrm{d}\varphi_i\right]^2.
\end{aligned}$$

For a given $c_N>0$, we restrict the integration over $\varphi_N\in[-c_N,c_N]$ and we apply Jensen's inequality, getting

$$\begin{aligned}
\varphi^{(0,0)}_{2N+1}(0,0)\ &\ge\ \frac{e^{-(M_\gamma+V_1(0))}}{\lambda\cdot 2c_N}\left[\frac{1}{\lambda^N}\int_{-c_N}^{c_N}\mathrm{d}\varphi_N\int_{C^2_N(\varphi_N)} e^{-\sum_{i=1}^{N}V_1(\nabla\varphi_i)-\sum_{i=0}^{N-1}V_2(\Delta\varphi_i)}\prod_{i=1}^{N-1}\mathrm{d}\varphi_i\right]^2\\
&\ge\ \frac{e^{-(M_\gamma+V_1(0))}}{\lambda\cdot 2c_N}\,\frac{\nu(0)^2}{\|\nu\|^2_\infty}\left[\frac{1}{\lambda^N}\int_{-c_N}^{c_N}\mathrm{d}\varphi_N\int_{C^2_N(\varphi_N)}\frac{\nu(\varphi_N-\varphi_{N-1})}{\nu(0)}\right.\\
&\qquad\qquad\left.\cdot\,e^{-\sum_{i=1}^{N}V_1(\nabla\varphi_i)-\sum_{i=0}^{N-1}V_2(\Delta\varphi_i)}\prod_{i=1}^{N-1}\mathrm{d}\varphi_i\right]^2\\
&=\ \frac{e^{-(M_\gamma+V_1(0))}}{\lambda\cdot 2c_N}\,\frac{\nu(0)^2}{\|\nu\|^2_\infty}\Big[\mathrm{P}^{(0,0)}(|W_N|\le c_N\,,\ |W_N-W_{N-1}|\le\gamma/2)\Big]^2,
\end{aligned}\tag{31}$$

where in the last equality we have used Proposition 4. Now we observe that

$$\begin{aligned}
\mathrm{P}^{(0,0)}(|W_N|\le c_N\,,\ |Y_N|\le\gamma/2)\ &\ge\ 1-\mathrm{P}^{(0,0)}(|W_N|>c_N)-\mathrm{P}^{(0,0)}(|Y_N|>\gamma/2)\\
&\ge\ 1-\frac{1}{c_N}\,\mathrm{E}^{(0,0)}[|W_N|]-\mathrm{P}^{(0,0)}(|Y_N|>\gamma/2)\,.
\end{aligned}\tag{32}$$

By (26), as $N\to\infty$ we have $\mathrm{P}^{(0,0)}(|Y_N|>\gamma/2)\to\pi(\mathbb{R}\setminus(-\frac{\gamma}{2},\frac{\gamma}{2}))=:1-3\eta$, with $\eta>0$, therefore $\mathrm{P}^{(0,0)}(|Y_N|>\gamma/2)\le 1-2\eta$ for N large enough. On the other hand, by Proposition 6 we have

$$\mathrm{E}^{(0,0)}[|W_N|]\ \le\ \sum_{n=1}^{N}\mathrm{E}^{(0,0)}[|Y_n|]\ \le\ C\,N\,.\tag{33}$$

If we choose $c_N := CN/\eta$, from (31), (32) and (33) we obtain

$$\varphi^{(0,0)}_{2N+1}(0,0) \geq \frac{e^{-(M_\gamma + V_1(0))}}{2\lambda C} \frac{\nu(0)^2}{\|\nu\|^2_\infty} \eta^3 \frac{1}{N} = \frac{(const.)}{N},$$

which is the desired lower bound in (30).

The upper bound is easier. By assumptions (C1) and (C2) both V_1 and V_2 are bounded from below, therefore we can replace $V_1(\nabla\varphi_{2N+1})$, $V_1(\nabla\varphi_{2N})$, $V_2(\Delta\varphi_{2N})$ and $V_2(\Delta\varphi_{2N-1})$ by the constant $\widetilde{c} := \inf_{x\in\mathbb{R}} \min\{V_1(x), V_2(x)\} \in \mathbb{R}$ getting the upper bound:

$$\begin{aligned}\varphi^{(0,0)}_{2N+1}(0,0) &= \frac{1}{\lambda^{2N+1}} \int_{\mathbb{R}^{2N-1}} e^{-\mathscr{H}_{[-1,2N+1]}(\varphi)} \prod_{i=1}^{2N-1} \mathrm{d}\varphi_i \\ &\leq \frac{e^{-4\widetilde{c}}}{\lambda^{2N+1}} \int_{\mathbb{R}^{2N-1}} e^{-\mathscr{H}_{[-1,2N-1]}(\varphi)} \prod_{i=1}^{2N-1} \mathrm{d}\varphi_i\,.\end{aligned}$$

Recalling Proposition 4 and Proposition 6, we obtain

$$\begin{aligned}\varphi^{(0,0)}_{2N+1}(0,0) &\leq \frac{e^{-4\widetilde{c}}}{\lambda^2} \int_{\mathbb{R}^2} \frac{\nu(0)}{\nu(\varphi_{2N-1} - \varphi_{2N-2})}\, \mathrm{P}^{(0,0)}(W_{2N-2} \in \mathrm{d}\varphi_{2N-2},\, W_{2N-1} \in \mathrm{d}\varphi_{2N-1}) \\ &= \frac{\nu(0)}{\lambda^2} e^{-4\widetilde{c}}\, \mathrm{E}^{(0,0)}\left(\frac{1}{\nu(Y_{2N-1})}\right) \leq \frac{\nu(0)}{\lambda^2} e^{-4\widetilde{c}}\, C = (const.),\end{aligned}$$

which completes the proof of (30). □

3 A lower bound on the partition function

We are going to give an explicit lower bound on the partition function in terms of a suitable renewal process. First of all, we rewrite equation (3) as

$$\mathscr{Z}_{\varepsilon,N} = \sum_{k=0}^{N-1} \varepsilon^k \sum_{\substack{A\subseteq\{1,\dots,N-1\}\\ |A|=k}} \int e^{-\mathscr{H}_{[-1,N+1]}(\varphi)} \prod_{m\in A} \delta_0(\mathrm{d}\varphi_m) \prod_{n\in A^c} \mathrm{d}\varphi_n\,, \tag{34}$$

where we set $A^c := \{1,\dots,N-1\} \setminus A$ for convenience.

3.1 A renewal process lower bound

We restrict the summation over A in (34) to the class of subsets $\mathfrak{B}_{2k}$ consisting of $2k$ points organized in k consecutive couples:

$$\mathfrak{B}_{2k} := \big\{\{t_1 - 1, t_1, \dots, t_k - 1, t_k\} \mid 0 = t_0 < t_1 < \dots < t_k \leq N-1 \text{ and } t_i - t_{i-1} \geq 2\ \forall i\big\}.$$

Plainly, $\mathfrak{B}_{2k}=\emptyset$ for $k>(N-1)/2$. We then obtain from (34)

$$\begin{aligned}\mathscr{Z}_{\varepsilon,N} &\geq \sum_{k=0}^{\lfloor (N-1)/2\rfloor}\varepsilon^{2k}\sum_{A\in\mathfrak{B}_{2k}}\int e^{-\mathscr{H}_{[-1,N+1]}(\varphi)}\prod_{m\in A}\delta_0(\mathrm{d}\varphi_m)\prod_{n\in A^c}\mathrm{d}\varphi_n\\ &= \sum_{k=0}^{\lfloor (N-1)/2\rfloor}\varepsilon^{2k}\sum_{\substack{0=t_0<t_1<\ldots<t_k<t_{k+1}=N+1\\ t_i-t_{i-1}\geq 2\ \forall i\leq k+1}}\prod_{j=1}^{k+1}\widetilde{K}(t_j-t_{j-1}),\end{aligned}\tag{35}$$

where we have set for $n\in\mathbb{N}$

$$\widetilde{K}(n):=\begin{cases}0 & \text{if } n=1\\ e^{-\mathscr{H}_{[-1,2]}(0,0,0,0)}=e^{-2V_1(0)-2V_2(0)} & \text{if } n=2\\ \left.\begin{array}{l}\displaystyle\int_{\mathbb{R}^{n-2}} e^{-\mathscr{H}_{[-1,n]}(w_{-1},\ldots,w_n)}\mathrm{d}w_1\cdots\mathrm{d}w_{n-2}\\ \text{with } w_{-1}=0, w_0=0, w_{n-1}=0, w_n=0\end{array}\right\} & \text{if } n\geq 3\end{cases}.\tag{36}$$

We stress that a factorization of the form (35) is possible because the Hamiltonian $\mathscr{H}_{[-1,N+1]}(\varphi)$ consists of two- and three-body terms and we have restricted the sum over subsets in $\mathfrak{B}_{2k}$, that consist of consecutive couples of zeros. We also note that the condition $t_i-t_{i-1}\geq 2$ is immaterial, because by definition $\widetilde{K}(1)=0$.

We now give a probabilistic interpretation to the right hand side of (35) in terms of a renewal process. To this purpose, for every $\varepsilon>0$ and for $n\in\mathbb{N}$ we define

$$K_\varepsilon(1):=0,\qquad K_\varepsilon(n):=\frac{\varepsilon^2}{\lambda^n}\widetilde{K}(n)\,e^{-\mu_\varepsilon n}=\varepsilon^2\,\varphi_n^{(0,0)}(0,0)\,e^{-\mu_\varepsilon n},\ \forall n\geq 2.$$

where the second equality follows recalling (36), Proposition 4 and the definition (14) of the density φ_n. The constant μ_ε is chosen to make K_ε a probability on $\mathbb{N}$:

$$\sum_{n\in\mathbb{N}}K_\varepsilon(n)=1,\qquad \text{that is}\qquad \sum_{n=2}^{\infty}\varphi_n^{(0,0)}(0,0)\,e^{-\mu_\varepsilon n}=\frac{1}{\varepsilon^2}.\tag{37}$$

It follows from Proposition 8 that $0<\mu_\varepsilon<\infty$ for every $\varepsilon>0$. We can therefore define a renewal process $(\{\eta_n\}_{n\geq 0},\mathscr{P}_\varepsilon)$ on $\mathbb{N}_0$ with inter-arrival law $K_\varepsilon(\cdot)$. More explicitly, $\eta_0:=0$ and the increments $\{\eta_{k+1}-\eta_k\}_{k\geq 0}$ are independent, identically distributed random variables with marginal law $\mathscr{P}_\varepsilon(\eta_{k+1}-\eta_k=n)=K_\varepsilon(n)$. Coming back to (35), we can write

$$\begin{aligned}\mathscr{Z}_{\varepsilon,N} &\geq \frac{\lambda^{N+1}e^{(N+1)\mu_\varepsilon}}{\varepsilon^2}\sum_{k=0}^{\lfloor (N-1)/2\rfloor}\sum_{0=t_0<t_1<\ldots<t_k<t_{k+1}=N+1}\prod_{j=1}^{k+1}K_\varepsilon(t_j-t_{j-1})\\ &= \frac{\lambda^{N+1}e^{(N+1)\mu_\varepsilon}}{\varepsilon^2}\sum_{k=0}^{\lfloor (N-1)/2\rfloor}\sum_{0=t_0<t_1<\ldots<t_k<t_{k+1}=N+1}\mathscr{P}_\varepsilon\big(\eta_1=t_1,\ldots,\eta_{k+1}=t_{k+1}\big)\\ &= \frac{\lambda^{N+1}e^{(N+1)\mu_\varepsilon}}{\varepsilon^2}\sum_{k=0}^{\lfloor (N-1)/2\rfloor}\mathscr{P}_\varepsilon\big(\eta_{k+1}=N+1\big)=\frac{\lambda^{N+1}e^{(N+1)\mu_\varepsilon}}{\varepsilon^2}\mathscr{P}_\varepsilon\big(N+1\in\eta\big),\end{aligned}\tag{38}$$

where in the last equality we look at $\eta=\{\eta_k\}_{k\geq 0}$ as a random subset of $\mathbb{N}_0$, so that $\{N+1\in\eta\}=\bigcup_{m=1}^{\infty}\{\eta_m=N+1\}$ (note that $\mathscr{P}_\varepsilon(\eta_{k+1}=N+1)=0$ for $k>\lfloor(N-1)/2\rfloor$).

We have thus obtained a lower bound on the partition function $\mathscr{Z}_{\varepsilon,N}$ of our model in terms of the renewal mass function (or Green function) of the renewal process $(\{\eta_n\}_{n\geq 0}\},\mathscr{P}_\varepsilon)$.

3.2 Proof of Theorem 1

Recall the free energy from definition 4

$$F(\varepsilon) = \lim_{N\to\infty} \frac{1}{N} \log \frac{\mathscr{Z}_{\varepsilon,N}}{\mathscr{Z}_{0,N}} \,.$$

From now on, the limits $N \to \infty$ will be implicitly taken along the odd numbers. Observe that by Proposition 5 and both bounds in Proposition 8

$$\lim_{N\to\infty} \frac{1}{N} \log \mathscr{Z}_{0,N} \;=\; \lim_{N\to\infty} \frac{1}{N} \Big((N+1)\log\lambda + \log \varphi^{(0,0)}_{N+1}(0,0) \Big) \;=\; \log\lambda \,.$$

Therefore for every $\varepsilon > 0$ by (38) we obtain

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \log \frac{\mathscr{Z}_{\varepsilon,N}}{\mathscr{Z}_{0,N}} \;&\geq\; \limsup_{N\to\infty} \frac{1}{N} \log \left[\frac{\lambda^{N+1}\, e^{\mu_\varepsilon (N+1)}}{\varepsilon^2} \, \mathscr{P}_\varepsilon(N+1 \in \eta) \right] - \log\lambda \\ &\geq\; \mu_\varepsilon + \limsup_{N\to\infty} \frac{1}{N} \log \mathscr{P}_\varepsilon(N+1 \in \eta) \,. \end{aligned} \tag{39}$$

Since $\mathscr{P}_\varepsilon(\eta_1 = n) > 0$ for all $n \in \mathbb{N}$ with $n \geq 2$, the renewal process $(\{\eta_k\}_{k\geq 0}, \mathscr{P}_\varepsilon)$ is aperiodic and by the classical renewal theorem $\mathscr{P}_\varepsilon(N+1 \in \eta) \to \frac{1}{m_\varepsilon}$ as $N \to \infty$, where

$$m_\varepsilon \;=\; \sum_{n\geq 2} n\, K_\varepsilon(n) \;=\; \varepsilon^2 \sum_{n\geq 2} n\, \varphi^{(0,0)}_n(0,0)\, e^{-\mu_\varepsilon n} \;<\; \infty \,.$$

by Proposition 8. Therefore from (39) we get $F(\varepsilon) \geq \mu_\varepsilon$. As we already mentioned above, we have $\mu_\varepsilon > 0$, hence $F(\varepsilon) > 0$, for all $\varepsilon > 0$. This shows that our model exhibit a trivial phase transition. □

Remark 9. We have just shown that $F(\varepsilon) \geq \mu_\varepsilon$. Recalling the definition (37) of μ_ε, it is clear that the lower bound in (30) on $\varphi^{(0,0)}_N(0,0)$ yields a corresponding lower bound on μ_ε, hence on $F(\varepsilon)$. Unfortunately, this lower bound is very poor: in fact, by standard Tauberian theorems, from (30) we get $\mu_\varepsilon \geq \exp(-(const.)/\varepsilon^2)$, which vanishes as $\varepsilon \downarrow 0$ faster than any polynomial. On the other hand, the conjectured correct behavior of the free energy, in analogy with the purely gradient case, should be $F(\varepsilon) \sim (const.)\,\varepsilon^2$.
One could hope to sharpen the lower bound on μ_ε by improving the one on $\varphi^{(0,0)}_N(0,0)$. This is possible, but only to a certain extent: even the conjectured sharp lower bound $\varphi^{(0,0)}_N(0,0) \geq (const.)/\sqrt{N}$ (in analogy with the purely gradient case) would yield only $\mu_\varepsilon \geq (const.)\,\varepsilon^4$. This discrepancy is a limitation of our lower bound technique: in order to have a genuine renewal structure, the chain is forced to visit the defect line at *couples of neighboring points*, which are rewarded ε^2 instead of ε. If one could replace $1/\varepsilon^2$ by $1/\varepsilon$ in (37), the lower bound $\varphi^{(0,0)}_N(0,0) \geq (const.)/\sqrt{N}$ would yield $\mu_\varepsilon \geq (const.')\,\varepsilon^2$, as expected.

A Some recurrence properties

We have already remarked that $Y = \{Y_i\}_{i\in\mathbb{N}_0}$ is *Leb*-irreducible, hence it is also π-irreducible, see (18), because π is absolutely continuous with respect to *Leb*. By Proposition 4.2.2 in [11], a maximal irreducibility measure for Y is $\psi(\mathrm{d}x) := \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} (\pi \mathscr{P}^n)(\mathrm{d}x)$, where we set $(\pi\mathscr{Q})(\mathrm{d}x) :=$

$\int_{z\in\mathbb{R}} \pi(\mathrm{d}z)\mathscr{Q}(z,\mathrm{d}x)$ for any kernel $\mathscr{Q}$ and we use the standard notation $\mathscr{P}^0(z,\mathrm{d}x) := \delta_z(\mathrm{d}x)$, $\mathscr{P}^1 = \mathscr{P}$ (we recall (10)) and for $n \geq 1$

$$\mathscr{P}^{n+1}(z,\mathrm{d}x) := \int_{y\in\mathbb{R}} \mathscr{P}^n(z,\mathrm{d}y)\mathscr{P}(y,\mathrm{d}x).$$

Since the law π is invariant for the kernel $\mathscr{P}$, see (19), we have $\pi\mathscr{P}^n = \pi$ for all $n \geq 0$, therefore the maximal irreducibility measure ψ is nothing but π itself. Since a maximal irreducibility measure is only defined up to equivalent measures (in the sense of Radon-Nikodym), it follows that *Leb*, which is equivalent to π, is a maximal irreducibility measure.
(As a matter of fact, it is always true that if a φ-irreducible Markov chain admits an invariant measure π, then π is a maximal irreducibility measure, cf. Theorem 5.2 in [12].)

Next we prove that Y is a T-chain, as it is defined in Chapter 6 of [11]. To this purpose, we first show that Y is a Feller chain, that is, for every bounded and continuous function $f : \mathbb{R} \to \mathbb{R}$ the function $(\mathscr{P}f)(x) := \int_{\mathbb{R}} \mathscr{P}(x,\mathrm{d}y)f(y)$ is bounded and continuous. We recall that the function v is continuous, as we have shown in §2.2. We then write

$$\begin{aligned}(\mathscr{P}f)(x) &:= \int_{\mathbb{R}} \mathscr{P}(x,\mathrm{d}y)f(y) = \frac{1}{\lambda v(x)} \int_{\mathbb{R}} e^{-V_1(y)-V_2(y-x)} v(y)f(y)\,\mathrm{d}y \\ &= \frac{1}{\lambda v(x)} \big(e^{-V_2} * (e^{-V_1} v f)\big)(x),\end{aligned}$$

from which the continuity of $\mathscr{P}f$ follows, because $e^{-V_2} \in L^\infty(\mathbb{R})$ and $(e^{-V_1} v f) \in L^1(\mathbb{R})$ and we recall that the convolution of a function in $L^\infty(\mathbb{R})$ with a function in $L^1(\mathbb{R})$ is bounded and continuous. Since Y is a *Leb*-irreducible Feller chain, it follows from Theorem 6.0.1 (iii) in [11] that Y is a *Leb*-irreducible T-chain.
Finally, we observe that from the drift condition (23) it follows that Y is a Harris recurrent chain. For this it suffices to apply Theorem 9.1.8 in [11], observing that the function U defined in (21) is *coercive*, i.e. $\lim_{|x|\to\infty} U(x) = +\infty$, hence it is "unbounded off petite sets" (cf. [11, §8.4.2]) because every compact set is petite for irreducible T-chains, by Theorem 6.0.1 (ii) in [11].

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