

ON A SDE DRIVEN BY A FRACTIONAL BROWNIAN MOTION AND WITH MONOTONE DRIFT

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Abstract

Let $\{B_t^H, t \in [0, T]\}$ be a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. We prove the existence of a weak solution for a stochastic differential equation of the form $X_t = x + B_t^H + \int_0^t (b_1(s, X_s) + b_2(s, X_s)) ds$, where $b_1(s, x)$ is a Hölder continuous function of order strictly larger than $1 - \frac{1}{2H}$ in x and than $H - \frac{1}{2}$ in time and b_2 is a real bounded nondecreasing and left (or right) continuous function.

1 Introduction

Let $B^H = \{B_t^H, t \in [0, T]\}$ be a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. That is, B^H is a centered Gaussian process with covariance

$$R_H(t, s) = E(B_t^H B_s^H) = \frac{1}{2} \left\{ |t|^{2H} + |s|^{2H} - |t - s|^{2H} \right\}.$$

If $H = \frac{1}{2}$ the process B^H is a standard Brownian motion. Consider the following stochastic differential equation

$$X_t = x + B_t^H + \int_0^t (b_1(s, X_s) + b_2(s, X_s)) ds, \quad (1.1)$$

where $b_1, b_2 : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are Borel functions. The purpose of this paper is to prove, by approximation arguments, the existence of a weak solution to this equation if $H > \frac{1}{2}$, under the following weak regularity assumptions on the coefficients:

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(H₁) b_1 is Hölder continuous of order $1 > \alpha > 1 - \frac{1}{2H}$ in x and of order $\gamma > H - \frac{1}{2}$ in time:

$$|b_1(t, x) - b_1(s, y)| \leq C(|x - y|^\alpha + |t - s|^\gamma). \quad (1.2)$$

(H₂) $\sup_{s \in [0, T]} \sup_{x \in \mathbb{R}} |b_2(s, x)| \leq M < \infty$.

(H₃) $\forall s \in [0, T]$, $b_2(s, \cdot)$ is a nondecreasing and left (or right) continuous function.

The same approximation arguments can be used to consider the case where b_2 satisfies the following assumptions:

(H'₂) $\sup_{s \in [0, T]} \sup_{x \in \mathbb{R}} |b_2(s, x)| \leq M(1 + |x|)$

(H'₃) for all $s \in [0, T]$, $b_2(s, \cdot)$ is a nonincreasing and continuous function

If $b_2 \equiv 0$ and $H = \frac{1}{2}$ (the process B^H is a standard Brownian motion), the existence of a strong solution is well-known by the results of Zvonkin [18], Veretennikov [16] and Bahlali [2]. See also the work by Nakao [11] and its generalization by Ouknine [14]. In the case of Equation (1.1) driven by the fractional Brownian motion with $b_2 \equiv 0$, the weak existence and uniqueness are established in [13] using a suitable version of Girsanov theorem; the existence of a strong solution could be deduced from an extension of Yamada-Watanabe's theorem or by a direct arguments.

In the general case $H > 1/2$, to establish existence and uniqueness result, a Hölder type space-time condition is imposed on the drift. Recently, Mishura and Nualart [9] gave an existence and uniqueness result for one discontinuous function namely the *sgn* function. Their approach relies on the Novikov criterion and it is valid for $\frac{1+\sqrt{5}}{4} > H > 1/2$.

Our aim is to establish existence and uniqueness result for general monotone function including *sgn* function and $H > 1/2$.

The paper is organized as follows. In Section 2 we give some preliminaries on fractional calculus and fractional Brownian motion. In Section 3 we formulate a Girsanov theorem and show the existence of a weak solution to Equation (1.1). As a consequence we deduce the uniqueness in law and the pathwise uniqueness. Finally Section 4 discusses the existence of a strong solution.

2 Preliminaries

2.1 Fractional calculus

An exhaustive survey on classical fractional calculus can be found in [15]. We recall some basic definitions and results.

For $f \in L^1([a, b])$ and $\alpha > 0$ the *left fractional Riemann-Liouville integral* of f of order α on (a, b) is given at almost all x by

$$I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - y)^{\alpha-1} f(y) dy,$$

where Γ denotes the Euler function.

This integral extends the usual n -order iterated integrals of f for $\alpha = n \in \mathbb{N}$. We have the first composition formula

$$I_{a+}^{\alpha}(I_{a+}^{\beta}f) = I_{a+}^{\alpha+\beta}f.$$

The fractional derivative can be introduced as inverse operation. We assume $0 < \alpha < 1$ and $p > 1$. We denote by $I_{a+}^{\alpha}(L^p)$ the image of $L^p([a, b])$ by the operator I_{a+}^{α} . If $f \in I_{a+}^{\alpha}(L^p)$, the function ϕ such that $f = I_{a+}^{\alpha}\phi$ is unique in L^p and it agrees with the *left-sided Riemann-Liouville derivative of f of order α* defined by

$$D_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(y)}{(x-y)^{\alpha}} dy.$$

The derivative of f has the following Weil representation:

$$D_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy \right) \mathbf{1}_{(a,b)}(x), \quad (2.1)$$

where the convergence of the integrals at the singularity $x = y$ holds in L^p -sense.

When $\alpha p > 1$ any function in $I_{a+}^{\alpha}(L^p)$ is $(\alpha - \frac{1}{p})$ -Hölder continuous. On the other hand, any Hölder continuous function of order $\beta > \alpha$ has fractional derivative of order α . That is, $C^{\beta}([a, b]) \subset I_{a+}^{\alpha}(L^p)$ for all $p > 1$.

Recall that by construction for $f \in I_{a+}^{\alpha}(L^p)$,

$$I_{a+}^{\alpha}(D_{a+}^{\alpha}f) = f$$

and for general $f \in L^1([a, b])$ we have

$$D_{a+}^{\alpha}(I_{a+}^{\alpha}f) = f.$$

If $f \in I_{a+}^{\alpha+\beta}(L^1)$, $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta \leq 1$ we have the second composition formula

$$D_{a+}^{\alpha}(D_{a+}^{\beta}f) = D_{a+}^{\alpha+\beta}f.$$

2.2 Fractional Brownian motion

Let $B^H = \{B_t^H, t \in [0, T]\}$ be a fractional Brownian motion with Hurst parameter $0 < H < 1$ defined on the probability space (Ω, \mathcal{F}, P) . For each $t \in [0, T]$ we denote by $\mathcal{F}_t^{B^H}$ the σ -field generated by the random variables B_s^H , $s \in [0, t]$ and the sets of probability zero.

We denote by \mathcal{E} the set of step functions on $[0, T]$. Let \mathcal{H} be the Hilbert space defined as the closure of \mathcal{E} with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).$$

The mapping $\mathbf{1}_{[0,t]} \rightarrow B_t^H$ can be extended to an isometry between \mathcal{H} and the Gaussian space $H_1(B^H)$ associated with B^H . We will denote this isometry by $\varphi \rightarrow B^H(\varphi)$.

The covariance kernel $R_H(t, s)$ can be written as

$$R_H(t, s) = \int_0^{t \wedge s} K_H(t, r) K_H(s, r) dr,$$

where K_H is a square integrable kernel given by (see [3]):

$$K_H(t, s) = \Gamma(H + \frac{1}{2})^{-1} (t-s)^{H-\frac{1}{2}} F(H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}, 1 - \frac{t}{s}),$$

$F(a, b, c, z)$ being the Gauss hypergeometric function. Consider the linear operator K_H^* from \mathcal{E} to $L^2([0, T])$ defined by

$$(K_H^* \varphi)(s) = K_H(T, s)\varphi(s) + \int_s^T (\varphi(r) - \varphi(s)) \frac{\partial K_H}{\partial r}(r, s) dr.$$

For any pair of step functions φ and ψ in \mathcal{E} we have (see [1])

$$\langle K_H^* \varphi, K_H^* \psi \rangle_{L^2([0, T])} = \langle \varphi, \psi \rangle_{\mathcal{H}}.$$

As a consequence, the operator K_H^* provides an isometry between the Hilbert spaces \mathcal{H} and $L^2([0, T])$. Hence, the process $W = \{W_t, t \in [0, T]\}$ defined by

$$W_t = B^H((K_H^*)^{-1}(\mathbf{1}_{[0, t]})) \tag{2.2}$$

is a Wiener process, and the process B^H has an integral representation of the form

$$B_t^H = \int_0^t K_H(t, s) dW_s, \tag{2.3}$$

because $(K_H^* \mathbf{1}_{[0, t]})(s) = K_H(t, s) \mathbf{1}_{[0, t]}(s)$.

On the other hand, the operator K_H on $L^2([0, T])$ associated with the kernel K_H is an isomorphism from $L^2([0, T])$ onto $I_{0+}^{H+1/2}(L^2([0, T]))$ and it can be expressed in terms of fractional integrals as follows (see [3]):

$$(K_H h)(s) = I_{0+}^{2H} s^{\frac{1}{2}-H} I_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} h, \text{ if } H \leq 1/2, \tag{2.4}$$

$$(K_H h)(s) = I_{0+}^1 s^{H-\frac{1}{2}} I_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} h, \text{ if } H \geq 1/2, \tag{2.5}$$

where $h \in L^2([0, T])$.

We will make use of the following definition of \mathcal{F}_t -fractional Brownian motion.

2.1 Definition. Let $\{\mathcal{F}_t, t \in [0, T]\}$ be a right-continuous increasing family of σ -fields on (Ω, \mathcal{F}, P) such that \mathcal{F}_0 contains the sets of probability zero. A fractional Brownian motion $B^H = \{B_t^H, t \in [0, T]\}$ is called an \mathcal{F}_t -fractional Brownian motion if the process W defined in (2.2) is an \mathcal{F}_t -Wiener process.

3 Existence of strong solution for SDE with monotone drift.

In this section we are interested by the special case $b_1 \equiv 0$. We will prove by approximation arguments that there is a strong solution of equation (1.1). We will discuss two cases:

- 1) $b_2(s, \cdot)$ satisfies **(H₂)** and **(H₃)**.
- 2) $b_2(s, \cdot)$ satisfies **(H'₂)** and **(H'₃)**.

1- The first case:

To treat the first situation, let us suppose that $b_2(s, \cdot)$ is nondecreasing and left continuous function. We will use the following approximation lemma:

3.1 Lemma. Let $b : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, a bounded measurable function such that for any $s \in [0, T]$, $b(s, \cdot)$ is a nondecreasing and left continuous function. Then there exists a family of measurable functions

$$\{b_n(s, x); n \geq 1, s \in [0, T], x \in \mathbb{R}\}$$

such that

$$\left\{ \begin{array}{l} \bullet \text{ For any sequence } x_n \text{ increasing to } x \in \mathbb{R}, \text{ we have} \\ \lim_{n \rightarrow \infty} b_n(s, x_n) = b(s, x), \quad ds \text{ a.e.} \\ \bullet x \mapsto b_n(s, x) \text{ is nondecreasing, for all } n \geq 1, s \in [0, T] \\ \bullet n \mapsto b_n(s, x) \text{ is nondecreasing, for all } x \in \mathbb{R}, s \in [0, T] \\ \bullet |b_n(s, x) - b_n(s, y)| \leq 2nM|x - y| \text{ for all } n \geq 1, s \in [0, T] \\ \bullet \sup_{n \geq 1} \sup_{s \in [0, T]} \sup_{x \in \mathbb{R}} |b_n(s, x)| \leq M. \end{array} \right.$$

Proof. First assume that $b(s, \cdot)$ is left continuous and let us choose for any $n \geq 1$

$$b_n(s, x) = n \int_{x - \frac{1}{n}}^x b(s, y) dy.$$

Since $b(s, \cdot)$ is nondecreasing then $b_n(s, \cdot)$ is also a nondecreasing function for any fixed $n \geq 1$. Let $x, y \in \mathbb{R}$, we clearly have for any $n \geq 1$,

$$|b_n(s, x) - b_n(s, y)| \leq 2nM|x - y|. \quad (3.1)$$

Obviously, we get that b_n is uniformly bounded by the constant M . Let $n < m$, $s \in [0, T]$ and $x \in \mathbb{R}$, we have

$$\begin{aligned} b_m(s, x) - b_n(s, x) &= (m - n) \int_{x - \frac{1}{m}}^x b(s, y) dy - n \int_{x - \frac{1}{n}}^{x - \frac{1}{m}} b(s, y) dy \\ &\geq (m - n) \int_{x - \frac{1}{m}}^x b(s, y) dy - \frac{m - n}{m} b(s, x - \frac{1}{m}), \\ &= (m - n) \int_{x - \frac{1}{m}}^x \left(b(s, y) - b(s, x - \frac{1}{m}) \right) dy \geq 0. \end{aligned}$$

Now let $x_0 \in \mathbb{R}$ and take an increasing sequence of real numbers x_n converging to x_0 . We want to show that for any $s \in [0, T]$, $\lim_{n \rightarrow \infty} b_n(s, x_n) = b(s, x_0)$. It is enough to prove that there exists a subsequence $b_{\varphi(n)}(s, x_{\varphi(n)})$ which converges to $b(s, x_0)$. To do this, remark first that since $b(s, \cdot)$ is left continuous we have $\lim_{n \rightarrow \infty} b_n(s, x_0) = b(s, x_0)$. Now let us consider any strictly increasing sequence x'_n converging to x_0 such that $x_0 - x'_n = o(\frac{1}{n})$. We clearly get by (3.1)

$$\forall s \in [0, T], \lim_{n \rightarrow \infty} b_n(s, x'_n) = b(s, x_0). \quad (3.2)$$

We may choose a sequence $\varphi(n) \geq n$ such that $x'_n \leq x_{\varphi(n)}$. Since $(b_n(s, x))_{n \geq 1}$ is increasing and for any fixed $n \geq 1$ the function $b_n(s, \cdot)$ is nondecreasing, we have

$$b_n(s, x'_n) \leq b_{\varphi(n)}(s, x_{\varphi(n)}) \leq b(s, x_{\varphi(n)}). \quad (3.3)$$

We deduce by (3.2) and the left continuity of $b(s, \cdot)$,

$$\lim_{n \rightarrow \infty} b_{\varphi(n)}(s, x_{\varphi(n)}) = b(s, x_0).$$

Which ends the proof. \square

Let $(B^H)_{t \geq 1}$ be a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. We consider the following SDE

$$X_t = x + B_t^H + \int_0^t b_2(s, X_s) ds, \quad 0 \leq t \leq T. \quad (3.4)$$

3.2 Theorem. *Suppose that b_2 satisfies the assumptions (\mathbf{H}_2) and (\mathbf{H}_3) . Then there exists a strong solution to the equation (3.4).*

Proof. Assume that $b_2 : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable and bounded function which is nondecreasing and left continuous with respect to the space variable x . For $n \geq 1$, let b_n be as in lemma 3.1 and consider the following SDE

$$X_t^n = x + B_t^H + \int_0^t b_n(s, X_s^n) ds, \quad 0 \leq t \leq T. \quad (3.5)$$

By standard Picard's iteration argument, one may show that for any $n \geq 1$, the equation (3.5) has a strong solution which we denote by X^n .

Let $n > m$, we denote by $\Delta_t = X_t^n - X_t^m$. Using the monotony argument on b_n , we have

$$\begin{aligned} \Delta_t &\geq \int_0^t b_m(s, X_s^n) - b_m(s, X_s^m) ds, \\ &\geq \int_0^t (b_m(s, X_s^n) - b_m(s, X_s^m)) I_{\{\Delta_s \leq 0\}} ds, \\ &\geq 2mM \int_0^t \Delta_s I_{\{\Delta_s \leq 0\}} ds \geq -2mM \int_0^t \Delta_s^- ds. \end{aligned} \quad (3.6)$$

We then get

$$\Delta_t^- \leq 2mM \int_0^t \Delta_s^- ds. \quad (3.7)$$

By Gronwall's lemma, we have for almost all w and for any $t \in [0, T]$, the sequence $(X_t^n(w))$ is a nondecreasing function of n which is bounded since b_n is. Therefore it has a limit when $n \rightarrow \infty$ and we set

$$\lim_{n \rightarrow \infty} X_t^n(\omega) = X_t(\omega),$$

which entails in particular that X is $\mathcal{F}_t^{B^H}$ -adapted. Applying the convergence result in Lemma 3.1 and the boundedness of b_n we get by Lebesgue's dominated convergence theorem,

$$X_t = x + B_t^H + \int_0^t b_2(s, X_s) ds.$$

\square

3.1 Remark. To show that Equation (3.4) has a weak solution, a continuity condition is imposed on the drift in [13]. Here, the function b_2 may have a countable set of discontinuity points. The solution constructed in Theorem 3.2 is the minimal one.

3.2 Remark. Let $b_2 : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a bounded measurable function, which is nondecreasing and right continuous. In this case we consider a decreasing sequence of Lipschitz continuous functions which approximate the drift. One may take

$$b_n(s, x) = n \int_x^{x + \frac{1}{n}} b_2(s, y) dy.$$

For any fixed $(s, x) \in [0, T] \times \mathbb{R}$, the sequence $(b_n(s, x))_{n \geq 1}$ is nonincreasing and for any fixed $n \geq 1$ and $s \in [0, T]$ the function $b_n(s, \cdot)$ is nondecreasing. The same arguments as in Lemma 3.1 can be used to prove that for any sequence $(x_n)_{n \geq 1}$ decreasing to x , we have

$$\lim_{n \rightarrow \infty} b_n(s, x_n) = b_2(s, x).$$

This allows us to construct the maximal solution to the equation (3.4).

2– The second case:

In this case we use the following lemma:

3.3 Lemma. *Let $b(\cdot, \cdot) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function with linear growth, that is there exists a constant $M < \infty$ such that $\forall (s, x) \in [0, T] \times \mathbb{R}$, $|b(s, x)| \leq M(1 + |x|)$. Then the sequence of functions*

$$b_n(s, x) = \sup_{y \in Q} (b(s, y) - n|x - y|),$$

is well defined for $n \geq M$ and it satisfies

$$\left\{ \begin{array}{l} \bullet \text{ For any sequence } x_n \text{ converging to } x \in \mathbb{R}, \text{ we have} \\ \quad \lim_{n \rightarrow \infty} b_n(s, x_n) = b(s, x), \\ \bullet n \mapsto b_n(s, x) \text{ is nonincreasing, for all } x \in \mathbb{R}, s \in [0, T] \\ \bullet |b_n(s, x) - b_n(s, y)| \leq n|x - y| \text{ for all } n \geq M, s \in [0, T], x, y \in \mathbb{R} \\ \bullet |b_n(s, x)| \leq M(1 + |x|), \text{ for all } (s, x) \in [0, T] \times \mathbb{R}, n \geq M. \end{array} \right.$$

For the proof of this lemma we refer for example to [8].

3.4 Theorem. *Assume that b_2 satisfies conditions \mathbf{H}_2' and \mathbf{H}_3' . Then there exists a unique strong solution to the equation (3.4).*

Proof. For any $n \geq 1$, let b_n be as in Lemma 3.3. Since b_n is Lipschitz and linear growth, the result in [13] assures the existence of a strong solution X^n to the equation

$$X_t^n = x + B_t^H + \int_0^t b_n(s, X_s^n) ds.$$

Since $(b_n)_{n \geq 1}$ is nonincreasing, comparison theorem entails that $(X^n)_{n \geq 1}$ is a.s nonincreasing. By the linear growth condition on b_n and Gronwall's lemma we may deduce that X^n converges

a.s to X , which is clearly a strong solution to the SDE (3.4). Moreover, if X^1 and X^2 are two solutions of (3.4), using the fact that $b_2(s, \cdot)$ is nonincreasing, we get by applying Tanaka's formula to the continuous semi-martingale $X^1 - X^2$,

$$(X_t^1 - X_t^2)^+ = \int_0^t \text{sign}(X_s^1 - X_s^2) (b_2(s, X_s^1) - b_2(s, X_s^2)) ds \leq 0.$$

Then we have the pathwise uniqueness of the solution. \square

4 Existence of a weak solution

4.1 Girsanov transform

As in the previous section, let B^H be a fractional Brownian motion with Hurst parameter $0 < H < 1$ and denote by $\{\mathcal{F}_t^{B^H}, t \in [0, T]\}$ its natural filtration.

Given an adapted process with integrable trajectories $u = \{u_t, t \in [0, T]\}$ and consider the transformation

$$\tilde{B}_t^H = B_t^H + \int_0^t u_s ds. \quad (4.1)$$

We can write

$$\begin{aligned} \tilde{B}_t^H &= B_t^H + \int_0^t u_s ds = \int_0^t K_H(t, s) dW_s + \int_0^t u_s ds \\ &= \int_0^t K_H(t, s) d\tilde{W}_s, \end{aligned}$$

where

$$\tilde{W}_t = W_t + \int_0^t \left(K_H^{-1} \left(\int_0^\cdot u_s ds \right) (r) \right) dr. \quad (4.2)$$

Notice that $K_H^{-1} \left(\int_0^\cdot u_s ds \right)$ belongs a.s to $L^2([0, T])$ if and only if $\int_0^\cdot u_s ds \in I_{0+}^{H+1/2}(L^2([0, T]))$. As a consequence we deduce the following version of the Girsanov theorem for the fractional Brownian motion, which has been obtained in [3, Theorem 4.9]:

4.1 Theorem. *Consider the shifted process (4.1) defined by a process $u = \{u_t, t \in [0, T]\}$ with integrable trajectories. Assume that:*

- i) $\int_0^\cdot u_s ds \in I_{0+}^{H+1/2}(L^2([0, T]))$, almost surely.
- ii) $E(\xi_T) = 1$, where

$$\xi_T = \exp \left(- \int_0^T \left(K_H^{-1} \int_0^\cdot u_s ds \right) (s) dW_s - \frac{1}{2} \int_0^T \left(K_H^{-1} \int_0^\cdot u_s ds \right)^2 (s) ds \right).$$

Then the shifted process \tilde{B}^H is an $\mathcal{F}_t^{B^H}$ -fractional Brownian motion with Hurst parameter H under the new probability \tilde{P} defined by $\frac{d\tilde{P}}{dP} = \xi_T$.

Proof. By the standard Girsanov theorem applied to the adapted and square integrable process $K_H^{-1}(\int_0^\cdot u_s ds)$ we obtain that the process \widetilde{W} defined in (4.2) is an $\mathcal{F}_t^{B^H}$ -Brownian motion under the probability \widetilde{P} . Hence, the result follows. \square

From (2.5) the inverse operator K_H^{-1} is given by

$$K_H^{-1}h = s^{H-\frac{1}{2}}D_{0+}^{H-\frac{1}{2}}s^{\frac{1}{2}-H}h', \text{ if } H > 1/2 \quad (4.3)$$

for all $h \in I_{0+}^{H+\frac{1}{2}}(L^2([0, T]))$. Then if $H > \frac{1}{2}$ we need $u \in I_{0+}^{H-1/2}(L^2([0, T]))$, and a sufficient condition for i) is the fact that the trajectories of u are Hölder continuous of order $H - \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$.

4.2 Existence of a weak solution

Consider the stochastic differential equation:

$$X_t = x + B_t^H + \int_0^t (b_1(s, X_s) + b_2(s, X_s)) ds, \quad 0 \leq t \leq T, \quad (4.4)$$

where b_1 and b_2 are Borel functions on $[0, T] \times \mathbb{R}$ satisfying the conditions \mathbf{H}_1 for b_1 and \mathbf{H}_2 and \mathbf{H}_3 (resp. \mathbf{H}_2' and \mathbf{H}_3') for b_2 . By a *weak solution* to equation (4.4) we mean a couple of adapted continuous processes (B^H, X) on a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t, t \in [0, T]\})$, such that:

- i) B^H is an \mathcal{F}_t -fractional Brownian motion in the sense of Definition 2.1.
- ii) X and B^H satisfy (4.4).

4.2 Theorem. *Suppose that b_1 and b_2 are Borel functions on $[0, T] \times \mathbb{R}$ satisfying the conditions \mathbf{H}_1 for b_1 , \mathbf{H}_2 and \mathbf{H}_3 (resp. \mathbf{H}_2' and \mathbf{H}_3') for b_2 . Then Equation (4.4) has a weak solution.*

Proof. Let X^2 be the strong solution of (3.4) and set $\widetilde{B}_t^H = B_t^H - \int_0^t b_1(s, X_s^2) ds$. We claim that the process $u_s = -b_1(s, X_s^2)$ satisfies conditions i) and ii) of Theorem 4.1. If this claim is true, under the probability measure \widetilde{P} , \widetilde{B}^H is an $\mathcal{F}_t^{B^H}$ -fractional Brownian motion, and (\widetilde{B}^H, X^2) is a weak solution of (4.4) on the filtered probability space $(\Omega, \mathcal{F}, \widetilde{P}, \{\mathcal{F}_t^{B^H}, t \in [0, T]\})$.

Set

$$v_s = -K_H^{-1} \left(\int_0^\cdot b_1(r, X_r^2) dr \right) (s).$$

We will show that the process v satisfies conditions i) and ii) of Theorem 4.1. Along the proof c_H will denote a generic constant depending only on H . Let $H > \frac{1}{2}$, by (4.3), the process v is clearly adapted and we have

$$\begin{aligned} v_s &= -s^{H-\frac{1}{2}}D_{0+}^{H-\frac{1}{2}}s^{\frac{1}{2}-H}b_1(s, X_s^2) \\ &:= -c_H(\alpha(s) + \beta(s)), \end{aligned}$$

where

$$\begin{aligned}\alpha(s) &= b_1(s, X_s^2) s^{\frac{1}{2}-H} \\ &\quad + (H - \frac{1}{2}) s^{H-\frac{1}{2}} b_1(s, X_s^2) \int_0^s \frac{s^{\frac{1}{2}-H} - r^{\frac{1}{2}-H}}{(s-r)^{\frac{1}{2}+H}} dr \\ &\quad + (H - \frac{1}{2}) s^{H-\frac{1}{2}} \int_0^s \frac{b_1(s, X_s^2) - b_1(r, X_s^2)}{(s-r)^{\frac{1}{2}+H}} r^{\frac{1}{2}-H} dr.\end{aligned}$$

and

$$\beta(s) = (H - \frac{1}{2}) s^{H-\frac{1}{2}} \int_0^s \frac{b_1(r, X_s^2) - b_1(r, X_r^2)}{(s-r)^{\frac{1}{2}+H}} r^{\frac{1}{2}-H} dr.$$

Using the estimate

$$|b_1(s, X_s^2)| \leq |b(0, x)| + C \left(|s|^\gamma + |X_s^2|^\alpha \right)$$

and the equality

$$\int_0^s \frac{r^{\frac{1}{2}-H} - s^{\frac{1}{2}-H}}{(s-r)^{\frac{1}{2}+H}} dr = c_H s^{1-2H},$$

we obtain

$$\begin{aligned}|\alpha(s)| &\leq c_H \left(s^{\frac{1}{2}-H} \left[|b_1(0, x)| + C \left(|s|^\gamma + |X_s^2|^\alpha \right) \right] + C s^{\gamma+\frac{1}{2}-H} \right) \\ &\leq c_H s^{\frac{1}{2}-H} \left(C \|X^2\|_\infty^\alpha + C s^\gamma + |b_1(0, x)| \right).\end{aligned}$$

As consequence, taking into account that $\alpha < 1$, we have for any $\lambda > 1$

$$E \left(\exp \left(\lambda \int_0^T \alpha(s)^2 ds \right) \right) < \infty. \quad (4.5)$$

In order to estimate the term $\beta(s)$, we apply the Hölder continuity condition (1.2) and we get

$$\begin{aligned}|\beta(s)| &\leq c_H s^{H-\frac{1}{2}} \int_0^s \left(\frac{|X_s^2 - X_r^2|^\alpha}{(s-r)^{H+\frac{1}{2}}} + \frac{|r-s|^\gamma}{(s-r)^{\frac{1}{2}+H}} \right) r^{\frac{1}{2}-H} dr \\ &\leq c_H s^{H-\frac{1}{2}} \int_0^s \left(\frac{|B_s^H - B_r^H|^\alpha}{(s-r)^{H+\frac{1}{2}}} + (s-r)^{\alpha-H-\frac{1}{2}} + \frac{|r-s|^\gamma}{(s-r)^{\frac{1}{2}+H}} \right) r^{\frac{1}{2}-H} dr \\ &\leq c_H s^{\frac{1}{2}-H+\alpha(H-\varepsilon)} G^\alpha,\end{aligned}$$

where we have fixed $\varepsilon < H - \frac{1}{\alpha}(H - \frac{1}{2})$ and we denote

$$G = \sup_{0 \leq s < r \leq 1} \frac{|B_s^H - B_r^H|}{|s-r|^{H-\varepsilon}}.$$

By Fernique's Theorem, taking into account that $\alpha < 1$, for any $\lambda > 1$ we have

$$E \left(\exp \left(\lambda \int_0^T \beta(s)^2 ds \right) \right) < \infty,$$

and we deduce condition ii) of Theorem 4.1 by means of Novikov criterion. \square

4.3 Uniqueness in law and pathwise uniqueness

In this subsection we will prove uniqueness in law of weak solution under the condition \mathbf{H}_1 for b_1 , \mathbf{H}_2' and \mathbf{H}_3' for b_2 . The main result is

4.3 Theorem. *Suppose that b_1 and b_2 are Borel functions on $[0, T] \times \mathbb{R}$. satisfying the conditions \mathbf{H}_1 for b_1 , \mathbf{H}_2' and \mathbf{H}_3' for b_2 . Then we have the uniqueness in distribution for the solution of Equation (4.4).*

Proof. It is clear that X^2 is pathwise unique, hence the uniqueness in law holds when $b_1 \equiv 0$. Let (X, B^H) be a solution of the stochastic differential equation (4.4) defined in the filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t, t \in [0, T]\})$. Define

$$u_s = \left(K_H^{-1} \int_0^s b_1(r, X_r) dr \right) (s).$$

Let \tilde{P} defined by

$$\frac{d\tilde{P}}{dP} = \exp \left(- \int_0^T u_s dW_s - \frac{1}{2} \int_0^T u_s^2 ds \right). \quad (4.6)$$

We claim that the process u_s satisfies conditions i) and ii) of Theorem 4.1. In fact, u_s is an adapted process and taking into account that X_t has the same regularity properties as the fBm we deduce that $\int_0^T u_s^2 ds < \infty$ almost surely. Finally, we can apply again Novikov theorem in order to show that $E \left(\frac{d\tilde{P}}{dP} \right) = 1$, because by Gronwall's lemma

$$\|X\|_\infty \leq (|x| + \|B^H\|_\infty + C_1 T) e^{C_2 T},$$

and

$$|X_t - X_s| \leq |B_t^H - B_s^H| + C_3 |t - s| (1 + \|X\|_\infty)$$

for some constants $C_i, i = 1, 2, 3$.

By the classical Girsanov theorem the process

$$\tilde{W}_t = W_t + \int_0^t u_r dr$$

is an \mathcal{F}_t -Brownian motion under the probability \tilde{P} . In terms of the process \tilde{W}_t we can write

$$X_t = x + \int_0^t K_H(t, s) d\tilde{W}_s + \int_0^t b_2(s, X_s) ds,$$

Set

$$\tilde{B}_s^H = \int_0^s K_H(t, s) d\tilde{W}_s.$$

Then X satisfies the following SDE,

$$X_t = x + \tilde{B}_t^H + \int_0^t b_2(s, X_s) ds$$

As a consequence, the processes X and X^2 have the same distribution under the probability P . In fact, if Ψ is a bounded measurable functional on $C([0, T])$, we have

$$\begin{aligned}
E_P(\Psi(X)) &= \int_{\Omega} \Psi(\xi) \frac{dP}{d\tilde{P}}(\xi) d\tilde{P} \\
&= E_{\tilde{P}} \left(\Psi(X) \exp \left(\int_0^T \left(K_H^{-1} \int_0^{\cdot} b_1(r, X_r) dr \right) (s) dW_s \right. \right. \\
&\quad \left. \left. + \frac{1}{2} \int_0^T \left(K_H^{-1} \int_0^{\cdot} b_1(r, X_r) dr \right)^2 (s) ds \right) \right) \\
&= E_{\tilde{P}} \left(\Psi(X) \left(\exp \int_0^T \left(K_H^{-1} \int_0^{\cdot} b_1(r, X_r) dr \right) (s) d\tilde{W}_s \right. \right. \\
&\quad \left. \left. - \frac{1}{2} \int_0^T \left(K_H^{-1} \int_0^{\cdot} b_1(r, X_r) dr \right)^2 (s) ds \right) \right) \\
&= E_P \left(\Psi(X^2) \left(\exp \int_0^T \left(K_H^{-1} \int_0^{\cdot} b_1(r, X_r^2) dr \right) (s) dW_s \right) \right) \\
&\quad \left. - \frac{1}{2} \int_0^T \left(K_H^{-1} \int_0^{\cdot} b_1(r, X_r^2) dr \right)^2 (s) ds \right) \\
&= E_P(\Psi(X^2)).
\end{aligned}$$

In conclusion we have proved the uniqueness in law, which is equivalent to pathwise uniqueness (see [13] Theorem 5) \square

4.1 Remark. In the case $H < 1/2$, a deep study is made between stochastic differential equation with continuous coefficient and unit drift and anticipating ones (cf [4]).

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