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ALMOST SURE FINITENESS FOR THE TOTAL OCCUPATION TIME OF AN (d,α,β)-superprocess

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Abstract

For $0 < \alpha \le 2$ and $0 < \beta \le 1$ let X be the (d,α,β)-superprocess, i.e. the superprocess with α-stable spatial movement in $\mathbb{R}^d$ and $(1+\beta)$-stable branching. Given that the initial measure X_0 is Lebesgue on $\mathbb{R}^d$, Iscoe conjectured in [7] that the total occupational time $\int_0^\infty X_t(B)dt$ is a.s. finite if and only if $d\beta < \alpha$, where B denotes any bounded Borel set in $\mathbb{R}^d$ with non-empty interior.

In this note we give a partial answer to Iscoe's conjecture by showing that $\int_0^\infty X_t(B)dt < \infty$ a.s. if $2d\beta < \alpha$ and, on the other hand, $\int_0^\infty X_t(B)dt = \infty$ a.s. if $d\beta > \alpha$.

For $2d\beta < \alpha$, our result can also imply the a.s. finiteness of the total occupation time (over any bounded Borel set) and the a.s. local extinction for the empirical measure process of the (d,α,β)-branching particle system with Lebesgue initial intensity measure.

1 Introduction

For $0 < \alpha \le 2$ and $0 < \beta \le 1$ the (d,α,β)-superprocess is the superprocess with symmetric α-stable spatial movement in $\mathbb{R}^d$ and spectrally positive $(1+\beta)$-stable branching. It is a measure-valued process arising as the high density limit of empirical measure for the following critical branching symmetric α-stable particle system. Independent of the others, each particle is assigned a mass n^{-1} and it branches at rate γn^β for some constant $\gamma > 0$. The offspring distribution of each particle is determined by the generating function

$$G(s) = s + (1-s)^{1+\beta}/(1+\beta),$$

which is in the domain of attraction of one-sided $(1+\beta)$-stable law; see Section 4.5 of [2] for more details.

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Before specifying the (d,α,β)-superprocess using Laplace functional we first introduce some notation. Let $\Delta_\alpha = -(-\Delta)^{\alpha/2}$ be the fractional Laplacian operator and $(S_t) := (S_t^\alpha)$ be the associated transition semigroup for α-stable process.
For $d/2 < p < (d+\alpha)/2$ write

$$\phi_p(x) := (1+|x|^2)^{-p},\ x \in \mathbb{R}^d.$$

Let $C_p(\mathbb{R}^d)$ be the space of continuous functions on $\mathbb{R}^d$ satisfying

$$\|\phi\|_p := \sup_{x\in\mathbb{R}^d} |\phi(x)/\phi_p(x)| < \infty,$$

Write

$$M_p(\mathbb{R}^d) := \{\mu \in M(\mathbb{R}^d) : \mu(\phi_p) < \infty\},$$

where $M(\mathbb{R}^d)$ denotes the collection of σ-finite measures on $\mathbb{R}^d$ and $\mu(\phi_p)$, as usual, denotes the integral of ϕ_p with respect to measure μ. We equip $M_p(\mathbb{R}^d)$ with the p-vague topology, i.e. the smallest topology such that $\mu \mapsto \mu(\phi)$ is continuous for all $\phi \in C_p(\mathbb{R}^d)$.
The (d,α,β)-superprocess X is then a càdlàg $M_p(\mathbb{R}^d)$-valued Markov process and $\int_0^\cdot X_s ds$ is the corresponding measure-valued (weighted) occupation time process. The joint Laplace functional of X and its weighted occupation time process for any $\phi, \psi \in C_p(\mathbb{R}^d)$ is determined by

$$\mathbb{E}\exp\left(-X_t(\psi) - \int_0^t X_s(\phi)ds\right) = \exp(-X_0(u_t)), \tag{1}$$

where function $u : [0,\infty)\times\mathbb{R}^d \mapsto \mathbb{R}$ with $u_t := u(t,\cdot) \in C_p(\mathbb{R}^d)$ is the unique mild solution to the following pde

$$\frac{\partial u}{\partial t} = \Delta_\alpha u - \frac{\gamma}{1+\beta}u^{1+\beta} + \phi,\ u_0 = \psi.$$

More precisely, u solves the associated integral equation

$$u_t = S_t\psi + \int_0^t S_{t-s}\phi\, ds - \frac{\gamma}{1+\beta}\int_0^t S_{t-s}u_s^{1+\beta} ds.$$

It follows readily that

$$\mathbb{E}X_t(\psi) = X_0(S_t\psi).$$

In addition, the extinction probability for X is given by the following estimate

$$\mathbb{P}\{X_t(\mathbb{R}^d) = 0\} = \exp\left(-\left(\frac{1+\beta}{\gamma\beta t}\right)^{1/\beta} X_0(\mathbb{R}^d)\right), \tag{2}$$

and for $X_0(\mathbb{R}^d) < \infty$ the asymptotic survival probability as $t \to \infty$ is given by

$$\mathbb{P}\{X_t(\mathbb{R}^d) \neq 0\} \sim \left(\frac{1+\beta}{\gamma\beta t}\right)^{1/\beta} X_0(\mathbb{R}^d). \tag{3}$$

Let X be the (d,α,β)-superprocess with Lebesgue initial measure on $\mathbb{R}^d$. For $\alpha = 2$ and any bounded set B with non-empty interior it was proved in [7] that $\int_0^\infty X_s(B)ds < \infty$ a.s. if and only if $d\beta < 2$. The proof was based on analyzing the asymptotic behavior of the solution $u_t(x)$ to (1) (with $\psi = 0$) by first letting $t \to \infty$ and then letting $|x| \to \infty$. It was further conjectured that $\int_0^\infty X_s(B)ds < \infty$ a.s. if and only if $d\beta < \alpha$; see Theorem 4.3 and Remark 4.5 of [7]. To the author's best knowledge this problem is still open.

Related to the problem considered in this note, the space-time-mass scaling limit of the occupation time process is considered in [3] for the (d,α,β)-superprocess X. Let $(Y^{(k)})$ be the sequence of scaled processes defined by

$$Y_t^{(k)}(B) := \frac{1}{k^{1+\beta}} \int_0^{k^\beta t} X_s\left(k^{1/d}B\right) ds$$

for any Borel set B. It is shown that for $d < \alpha/\beta$, the process $Y^{(k)}$ converges in distribution to a measure-valued (motionless) continuous state branching process whenever the sequence of scaled initial measures $(X_0^{(k)})$ converges to some random measure as $k \to \infty$, where $X_0^{(k)}(B) := k^{-1}X_0(k^{1/d}B)$.

The fluctuation limits of the re-scaled occupation time processes have been studied extensively for the empirical measure processes of the so called (d,α,β,γ)-branching particle systems, which is an (d,α,β)-branching systems with those particles' initial locations described by a Poisson random measure of intensity $dx/(1+|x|^\gamma), \gamma \geq 0$. For such a counting-measure-valued process denoted by N, after centering and time-mass re-scaling the limiting distributions (as $T \to \infty$) of the following occupation time fluctuation process

$$X_T(t) := \frac{1}{F_T} \int_0^{Tt} (N_s - \mathbb{E}N_s)ds, \ \ t \geq 0,$$

are found in [1] and in a series of papers by the same authors. Here F_T denotes a norming function of T which guarantees that the limit is nontrivial.

Depending on the different choices of values for d,α,β and γ the above-mentioned limiting processes behave differently. In low dimensions the limiting process is typically a continuous real-valued stable process with long-range dependence. In high dimensions either the limiting process is distribution-valued with independent and non-stationary increments or it is identically a distribution-valued stable random variable for all $t > 0$. For the corresponding (d,α,β)-superprocesses with initial measure $dx/(1+|x|^\gamma)$, similar scaling limits are also obtained for their occupation time processes, which generalize the previous results in [7]. See [1] for more details in this respect. We also refer to [1] and the references therein for further asymptotic results on re-scaled occupation time processes of the (d,α,β,γ)-branching particle systems and for results on more general branching particle systems.

A remarkable feature for the superprocess X with $\alpha < 2$ is the propagation of support, i.e. almost surely the support for X_t is either $\emptyset$ or $\mathbb{R}^d$ for all $t > 0$; see e.g. [5] and [8]. Therefore, the integral $\int_0^t X_s(B_1)ds$ is strictly increasing in t when X_0 is an infinite measure, and the almost sure local extinction can never happen for X. But the empirical measure process N for the corresponding branching particle system behaves differently. For the $(d,\alpha,\beta,0)$-branching particle system it is shown in [6] that the process N_t becomes locally extinct in probability as t tends to infinity if and only if $d \leq \alpha/\beta$. It is further conjectured in [1] that given $\alpha < 2$ and $d < \alpha/\beta + \gamma$, almost sure local extinction occurs for N, i.e. for any bounded set $A \in \mathbb{R}^d$, almost surely $N_t(A) = 0$ for all t large enough.

In this note we are going to provide a partial answer to Iscoe's conjecture on total occupation time for the (d,α,β)-superprocess with Lebesgue initial measure. Our approaches are different from the original proof in [7] for the $(d,2,\beta)$-superprocesses. For $0 < 2\beta d < \alpha$ we exploit the local extinction property, the scaling property for (S_t^α) and the branching property for superprocess. On the other hand, for $\beta d > \alpha$ we take use of the Feynman-Kac formula and the time-space scaling for α-stable process to obtain useful estimates on lower bounds of the occupation time measure over finite time periods, and then prove the desired result via a Borel-Cantelli argument. These two results are presented in Proposition 2.1 and Proposition 2.2, respectively.

As an application of Proposition 2.1, in Corollary 2.1 we also obtain two results on the total occupation time and the almost sure local extinction of the empirical measure process for the corresponding branching particle system. In particular, the result on almost sure local extinction confirms that the previously mentioned conjecture of [1] on almost sure local extinction for the branching particle system is true for $\alpha < 2, 2d\beta < \alpha$ and $\gamma = 0$.

2 Main results

Throughout this section C and C_i denote positive constants whose values might vary from line to line.

For the d-dimensional symmetric α-stable process ξ starting off at 0, let $p_t(x)$ be the density function for ξ_t which is strictly positive, smooth, symmetric and nuimodal. It is known that for all $x \in \mathbb{R}^d$ and $t > 0$

$$p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha} x) \tag{4}$$

and

$$\frac{C_1}{1+|x|^{\alpha+d}} \le p_1(x) \le \frac{C_2}{1+|x|^{\alpha+d}},$$

where the lower bound holds for $\alpha < 2$; see e.g. Lemma 2.2 of [7] and Section 3.2 of [1]. Then we have

$$p_t(x) \le \frac{t^{-d/\alpha}}{1+|t^{-1/\alpha}x|^{\alpha+d}} \le C\left(t|x|^{-\alpha-d} \wedge t^{-d/\alpha}\right)$$

for all $x \in \mathbb{R}^d$ and $t > 0$. In addition, for $\alpha < 2$ we have

$$p_t(x) \ge \frac{C_1 t^{-d/\alpha}}{1+|t^{-1/\alpha}x|^{\alpha+d}} \ge C t|x|^{-\alpha-d} \tag{5}$$

for $t^{1/\alpha} \le |x|$.

Proposition 2.1. *Let X be the (d,α,β)-superprocess with Lebesgue initial measure. Then for $2\beta d < \alpha$ and for any bounded Borel set B we have*

$$\int_0^\infty X_t(B)dt < \infty \quad a.s. \tag{6}$$

Proof. Write $B_r = B(r)$ for the open ball in $\mathbb{R}^d$ with center 0 and radius r. We only need to prove (6) for $B = B_1$.

Choose $\lambda > 0$ such that

$$d\beta < \lambda < \frac{\alpha}{2}. \tag{7}$$

Write L_A for the Lebesgue measure restricted to set $A \subset \mathbb{R}$. For any $r > 1$ and $n = 1, 2, \dots$ by the branching property we can find independent (α, β)-superprocesses $(Y_{n+m})_{m=0}^{\infty}$ and Z_n such that

$$X = \sum_{m=0}^{\infty} Y_{n+m} + Z_n$$

with respective initial measures

$$Y_{n+m}(0) = L_{n+m} := L_{B(r^{n+m+1})-B(r^{n+m})}$$

and

$$Z_n(0) = L_{B(r^n)}.$$

We first note from the extinction probability (3) that

$$\begin{aligned} \mathbb{P}\{A_n\} &:= \mathbb{P}\{Z_n(r^{n\lambda})(\mathbb{R}) \neq 0\} \\ &\leq \left(\frac{1+\beta}{\gamma\beta r^{n\lambda}}\right)^{1/\beta} C r^{dn} \\ &= C\left(\frac{1+\beta}{\gamma\beta}\right)^{1/\beta} r^{-n(\frac{\lambda}{\beta}-d)}. \end{aligned} \tag{8}$$

Then by (7)

$$\lim_{r\to\infty} \mathbb{P}\{\cup_{n=1}^{\infty} A_n\} = 0. \tag{9}$$

For the α-stable process ξ with “initial law” L_n and for any $r^{n\lambda} \leq t < r^{(n+1)\lambda}$,

$$\begin{aligned} \mathbb{E}Y_n(t)(B_1) &= \int_{L_n} \mathbb{P}\{\xi_t^x \in B_1\} dx \\ &\leq C t\, (r^n)^{-\alpha-d}\, r^{d(n+1)} \\ &= C t r^{d-n\alpha}. \end{aligned} \tag{10}$$

It follows that

$$\begin{aligned} &\mathbb{E}\int_{r^{n\lambda}}^{r^{(n+1)\lambda}} Y_{n+m}(t)(B_1) dt \\ &\leq C r^{(n+1)\lambda} r^{(n+1)\lambda} r^{d-(n+m)\alpha} \\ &= C r^{2\lambda+d-n(\alpha-2\lambda)-m\alpha}. \end{aligned} \tag{11}$$

Therefore,

$$\begin{aligned} &\mathbb{E}\left[1\left((\cup_{n=1}^{\infty} A_n)^c\right) \int_{r^\lambda}^{\infty} X_t(B_1) dt\right] \\ &= \mathbb{E}\left[1((\cup_{n=1}^{\infty} A_n)^c) \sum_{n=1}^{\infty} \int_{r^{n\lambda}}^{r^{(n+1)\lambda}} \sum_{m=0}^{\infty} Y_{n+m}(t)(B_1) dt\right] \\ &\leq \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \mathbb{E}\int_{r^{n\lambda}}^{r^{(n+1)\lambda}} Y_{n+m}(t)(B_1) dt \\ &< \infty, \end{aligned} \tag{12}$$

where we needs (7) for the last inequality. Hence,

$$1\left((\cup_{n=1}^{\infty}A_n)^c\right)\int_{r^\lambda}^{\infty}X_t(B_1)dt<\infty,\ \ a.s..$$

The Lebesgue measure is invariant for the α-stable process. Clearly,

$$\mathbb{E}\left[\int_0^{r^\lambda}X_t(B_1)dt\right]<\infty,$$

which implies

$$\int_0^{r^\lambda}X_t(B_1)dt<\infty\ \ a.s..$$

The desired result follows from (9). □

Proposition 2.1 together with Lemmas A and B of [1] immediately implies the following result on the empirical measure process N for the branching particle system introduced in Section 1. Note that the almost sure local extinction for N was proved in [1] for $\alpha=2$ and $d<2/\beta+\gamma$.

Corollary 2.1. *Given the empirical measure process N of the $(d,\alpha,\beta,0)$-branching particle system with $2d\beta<\alpha$, for any bounded Borel set B we have*

$$\int_0^{\infty}N_t(B)dt<\infty\ a.s.$$

In addition, almost surely $N_t(B)=0$ for all t large enough.

If $\alpha=2$, the approach in Proposition 2.1 can be pushed further to give another proof for Iscoe's original result on superBrownian motion in [7] for $d\beta<\alpha=2$. Since this approach requires the property of exponential decay for normal tail distribution, it appears difficult to generalize such an argument to the case $\alpha<2$ in a straightforward fashion.

Corollary 2.2. *For the superprocess X in Proposition 2.1, inequality (6) holds a.s. given $\beta d<\alpha=2$.*

Proof. Choose λ satisfying $d\beta<\lambda<2$ and define processes (Y_n) and (Z_n) in the the same way as those in the proof for Proposition 2.1. For large time t the superBrownian motion support can not propagate much faster that $\sqrt{t}$. By Lemma 2.2 of [11] we have for all n large enough satisfying $2n/(n+1)>\lambda$,

$$\lim_{r\to\infty}r^n\mathbb{P}\left\{\int_{r^{n\lambda}}^{r^{(n+1)\lambda}}\sum_{m=0}^{\infty}Y_{n+m}(t)(B_1)dt>0\right\}=0.$$

Then for large r by the Borel-Cantelli lemma,

$$\mathbb{P}\left\{\int_{r^{n\lambda}}^{r^{(n+1)\lambda}}\sum_{m=0}^{\infty}Y_{n+m}(t)(B_1)dt>0\ \text{i.o.}\right\}=0. \tag{13}$$

Note that $\int_0^t X_s(B_1)ds$ is almost surely finite for any finite t. The inequality (6) thus follows from a combination of (9) and (13) together with the branching property. □

To prove the next result on infiniteness of total occupation time we need the following Feynman-Kac formula for α-stable process; see Theorem 4.2 of [9] and Chapter III.19 of [10] for similar results.

Lemma 2.1. *Let $v : [0,\infty) \times \mathbb{R}^d \mapsto \mathbb{R}$ be the mild solution to*

$$\frac{\partial v}{\partial t} = \Delta_\alpha v - kv + g,\ v_0 = \psi, \tag{14}$$

where $k : [0,\infty) \times \mathbb{R}^d \mapsto [0,\infty)$ is any nonnegative function, $g : [0,\infty) \times \mathbb{R}^d \mapsto \mathbb{R}$ is any bounded function and ψ is any bounded continuous function on $\mathbb{R}^d$. Then we have

$$v_t(x) = \mathbb{E}^x \left[\psi(\xi_t) e^{-\int_0^t k(t-s,\xi_s)ds} + \int_0^t g(t-\theta,\xi_\theta) e^{-\int_0^\theta k(t-s,\xi_s)ds} d\theta \right], \tag{15}$$

where ξ is the d-dimensional α-stable process.

Proof. For $\lambda > 0$ put

$$\psi^{(\lambda)} := \lambda \int_0^\infty e^{-\lambda s} S_s \psi ds.$$

Then $\psi^{(\lambda)}$ is in the domain of generator Δ_α. Let $v^{(\lambda)}$ be the (strong) solution to pde (14) with initial condition $v_0^{(\lambda)} = \psi^{(\lambda)}$. It follows from Itô formula (applied to process $v_{t-\theta}^{(\lambda)}(\xi_\theta) e^{-\int_0^\theta k(t-s,\xi_s)ds}$, $0 \le \theta \le t$) and (14) that

$$\begin{aligned} M_\theta := {} & v_{t-\theta}^{(\lambda)}(\xi_\theta) e^{-\int_0^\theta k(t-s,\xi_s)ds} - v_t^{(\lambda)}(\xi_0) \\ & - \int_0^\theta \Big(-\dot{v}_{t-\theta'}^{(\lambda)}(\xi_{\theta'}) e^{-\int_0^{\theta'} k(t-s,\xi_s)ds} + \Delta_\alpha v_{t-\theta'}^{(\lambda)}(\xi_{\theta'}) e^{-\int_0^{\theta'} k(t-s,\xi_s)ds} \\ & \qquad - k(t-\theta',\xi_{\theta'}) v_{t-\theta'}^{(\lambda)}(\xi_{\theta'}) e^{-\int_0^{\theta'} k(t-s,\xi_s)ds} \Big) d\theta' \\ = {} & v_{t-\theta}^{(\lambda)}(\xi_\theta) e^{-\int_0^\theta k(t-s,\xi_s)ds} - v_t^{(\lambda)}(x) + \int_0^\theta g(t-\theta',\xi_{\theta'}) e^{-\int_0^{\theta'} k(t-s,\xi_s)ds} d\theta' \end{aligned}$$

is a bounded $\mathbb{P}^x$ martingale with $M_0 = 0$ for all x and for $0 \le \theta \le t$. Taking expectations and by $v_0^{(\lambda)} = \psi^{(\lambda)}$ we have

$$\begin{aligned} & -v_t^{(\lambda)}(x) + \mathbb{E}^x \left[\psi^{(\lambda)}(\xi_t) e^{-\int_0^t k(t-s,\xi_s)ds} + \int_0^t g(t-\theta,\xi_\theta) e^{-\int_0^\theta k(t-s,\xi_s)ds} d\theta \right] \\ & = \mathbb{E}^x M_t = 0. \end{aligned} \tag{16}$$

Since

$$v_t^{(\lambda)} = S_t \psi^{(\lambda)} - \int_0^t S_{t-s}(k_s v_s^{(\lambda)}) ds + \int_0^t S_{t-s} g_s ds, \tag{17}$$

where $k_s(\cdot) := k(s,\cdot)$ and $g_s(\cdot) := g(s,\cdot)$, letting $\lambda \to \infty$ first in (16) and then in (17) we see that the function v given by (15) is a mild solution to (14). We thus complete the proof since the mild solution to (14) is unique. □

Proposition 2.2. *Let X be the (d,α,β)-superprocess with Lebesgue initial measure. For $\beta d>\alpha$ and for any bounded Borel set B with non-empty interior we have*

$$\int_0^\infty X_t(B)dt=\infty \quad a.s. \tag{18}$$

Proof. Again, we only prove (18) for $B=B_1$.
Let u be the mild solution to

$$\frac{\partial u}{\partial t}=\Delta_\alpha u-\frac{\gamma}{1+\beta}u^{1+\beta}+1_{B_1},\, u_0=0.$$

We first notice that for all x, $u_t(x)>0$ for $t>0$. Moreover, for all x we have that $u_t(x)$ increases in t, which can be seen from the Laplace functional (1) with $\psi:=0$, $\phi:=1_{B_1}$ and $X_0:=\delta_x$. Then we can choose constants $a>0$ and $b>0$ small enough so that for t large enough $u_t(x)\geq a$ for all $x\in[-2b,2b]^d\subset\mathbb{R}^d$. We further choose a continuous function ψ such that ψ has a compact support contained in $[-2b,2b]^d$, $\psi(x)=a$ for all $x\in[-b,b]^d$ and $0\leq\psi(x)\leq a$ for all x. It follows that $\psi\leq u_t$ for t large. Let ψ be such a function throughout the rest of the proof.
Let v be the mild solution to the following pde

$$\frac{\partial v}{\partial t}=\Delta_\alpha v-\frac{\gamma}{1+\beta}v^{1+\beta},\, v_0=\psi.$$

Let w be the mild solution to

$$\frac{\partial w}{\partial t}=\Delta_\alpha w_t-\frac{\gamma}{1+\beta}(S_t\psi)^\beta w_t,\, w_0=\psi.$$

By Lemma 2.1 we have

$$w_t(x)=\mathbb{E}^x\left[\psi(\xi_t)e^{-\int_0^t\frac{\gamma}{1+\beta}(S_{t-s}\psi(\xi_s))^\beta ds}\right], \tag{19}$$

where ξ is the α-stable process. Moreover, put $v^*:=v-w$. Since $0<v_t\leq S_t\psi$ for $t>0$ and v^* is the mild solution to

$$\frac{\partial v^*}{\partial t}=\Delta_\alpha v^*_t-\frac{\gamma}{1+\beta}(S_t\psi)^\beta v^*_t+\frac{\gamma}{1+\beta}\left((S_t\psi)^\beta-v_t^\beta\right)v_t,\, v^*_0=0,$$

by Lemma 2.1 again we have

$$\begin{aligned}&v_t(x)-w_t(x)\\&=v^*_t(x)\\&=\mathbb{E}^x\left[\int_0^t\frac{\gamma}{1+\beta}\left((S_{t-s}\psi(\xi_s))^\beta-v_{t-s}(\xi_s)^\beta\right)v_{t-s}(\xi_s)e^{-\int_0^s\frac{\gamma}{1+\beta}(S_{t-s'}\psi(\xi_{s'}))^\beta ds'}ds\right]\\&\geq 0.\end{aligned}$$

For $\beta d>\alpha$, we proceed to find a lower bound for an integral on w_t. Since $0\leq\psi\leq a$ and ψ has a compact support, the time-space scaling (4) gives that uniformly for all $t>1$ and for any function

$h : [0,\infty) \mapsto \mathbb{R}$,

$$\begin{aligned}\int_0^t (S_{t-s}\psi(h_s))^\beta ds &= \int_0^t (S_s\psi(h_{t-s}))^\beta ds \\ &= \int_0^1 (S_s\psi(h_{t-s}))^\beta ds + \int_1^t \left(\int p_s(y-h_{t-s})\psi(y)dy\right)^\beta ds \\ &\le a^\beta + \int_1^t s^{-\frac{d\beta}{\alpha}} \left(\int p_1(s^{-1/\alpha}(y-h_{t-s}))\psi(y)dy\right)^\beta ds \\ &\le C.\end{aligned} \tag{20}$$

Then for $\alpha < 2$ and for t large enough we have from (19)

$$\begin{aligned}\int_{B_{2t}-B_t} w_{t^\alpha}(x)dx &\ge \mathbb{E}^x\left[\psi(\xi_{t^\alpha})e^{-\frac{\gamma}{1+\beta}C}\right] \\ &\ge C\int_{B_{2t}-B_t} S_{t^\alpha}\psi(x)dx \\ &\ge C\int_{B_{2t}-B_t} dx \int_{[-b,b]^d} p_{t^\alpha}(y-x)\psi(y)dy \\ &\ge a(2b)^d C\int_{B_{2t}-B_t} t^\alpha |x|^{-(\alpha+d)}dx \\ &= C > 0,\end{aligned} \tag{21}$$

where we have applied (20) for the first inequality, and we need $\psi(y) = a$ for $y \in [-b,b]^d$ and estimate (5) for the forth inequality. Note that the last constant C does not depend on t as long as t is large.

For $\alpha = 2$, we can verify directly that (21) still holds.

Now for (α,β)-superprocess X^* with initial measure $L_{B_{2t}-B_t}$, applying the Markov property and the previous estimates,

$$\begin{aligned}\mathbb{E}e^{-\int_{t^\alpha}^{t^\alpha+t} X_s^*(B_1)ds} &= \mathbb{E}e^{-X^*_{t^\alpha}(u_t)} \\ &\le \mathbb{E}e^{-X^*_{t^\alpha}(\psi)} \\ &= e^{-\int_{B_{2t}-B_t} v_{t^\alpha}(x)dx} \\ &\le e^{-\int_{B_{2t}-B_t} w_{t^\alpha}(x)dx} \\ &\le e^{-C} < 1\end{aligned} \tag{22}$$

for all t large enough, where C is the constant from the last line of (21). It follows that for some $\varepsilon > 0$

$$\liminf_{t\to\infty} \mathbb{P}\left\{\int_{t^\alpha}^{t^\alpha+t} X_s^*(B_1)ds > \varepsilon\right\} > \varepsilon. \tag{23}$$

Choosing sequence $(t_n) := (2^n t)$ and considering processes (X_n^*) with initial measures $(L_{B_{t_{n+1}}-B_{t_n}})$, the desired result follows from (23), the branching property and the Borel-Cantelli lemma. □

Remark 2.1. *Edwin Perkins suggested an alternative proof for Proposition 2.2. The idea goes as follows. Since X is persistent for $\beta d > \alpha$ and it converges to a non-degenerate weak limit X_∞ in $M_p(\mathbb{R}^d)$ as $t \to \infty$ (see Proposition 6.1 of [4]), using the Markov property and the above-mentioned equilibrium result on (d, α, β)-superprocess, one can also obtain an inequality similar to (23). Then one can derive the desired result by a Borel-Cantelli argument using the survival probability estimate (3) and the branching property. We leave the details to the interested readers.*

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