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A FUNCTIONAL LIMIT THEOREM FOR A 2D-RANDOM WALK WITH DEPENDENT MARGINALS

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Abstract

We prove a non-standard functional limit theorem for a two dimensional simple random walk on some randomly oriented lattices. This random walk, already known to be transient, has different horizontal and vertical fluctuations leading to different normalizations in the functional limit theorem, with a non-Gaussian horizontal behavior. We also prove that the horizontal and vertical components are not asymptotically independent.

1 Introduction

The study of random walks on oriented lattices has been recently intensified with some physical motivations, e.g. in quantum information theory where the action of a noisy channel on a quantum state is related to random walks evolving on directed graphs (see [2, 3]), but they also have their own mathematical interest. A particular model where the simple random walk becomes transient on an oriented version of $\mathbb{Z}^2$ has been introduced in [3] and extended in [5] where we have proved a functional limit theorem. In this model, the simple random walk is considered on an orientation of $\mathbb{Z}^2$ where the horizontal edges are unidirectional in some i.i.d. centered random way. This extra randomness yields larger horizontal fluctuations transforming the usual normalization in $n^{1/2}$ into a normalization in $n^{3/4}$, leading to a non-Gaussian horizontal asymptotic component. The undirected vertical moves still have standard fluctuations in $n^{1/2}$ that are thus killed by the larger normalization in the result proved in [5] (Theorem 4), yielding a null vertical component in the limit. If these horizontal and vertical asymptotic components were independent, one could state this functional limit theorem with an horizontal normalization in $n^{3/4}$ and a vertical one in $n^{1/2}$, but it might not be the case.

Here, we prove this result without using independence and as a complementary result we indeed prove that these two asymptotic components are not independent.

2 Model and results

The considered lattices are oriented versions of $\mathbb{Z}^2$: the vertical lines are not oriented but the horizontal ones are unidirectional, the *orientation* at a level $y \in \mathbb{Z}$ being given by a Rademacher random variable $\epsilon_y = \pm 1$ (say left if the value is $+1$ and right if it is -1). We consider here the i.i.d. case where the random field $\epsilon = (\epsilon_y)_{y\in\mathbb{Z}}$ has a product law $\mathbb{P}_\epsilon = \otimes_{y\in\mathbb{Z}}\mathbb{P}_\epsilon^y$ defined on some probability space $(A, \mathcal{A}, \mathbb{Q})$ with marginals given by $\mathbb{P}_\epsilon^y[\pm 1] = \mathbb{Q}[\epsilon_y = \pm 1] = \frac{1}{2}$.

Definition 1 (Oriented lattices)**.** *Let $\epsilon = (\epsilon_y)_{y\in\mathbb{Z}}$ be a sequence of random variables defined as previously. The* oriented lattice $\mathbb{L}^\epsilon = (\mathbb{V}, \mathbb{A}^\epsilon)$ *is the (random) directed graph with (deterministic) vertex set $\mathbb{V} = \mathbb{Z}^2$ and (random) edge set $\mathbb{A}^\epsilon$ defined by the condition that for $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{Z}^2$, $(u, v) \in \mathbb{A}^\epsilon$ if and only if either $v_1 = u_1$ and $v_2 = u_2 \pm 1$, or $v_2 = u_2$ and $v_1 = u_1 + \epsilon_{u_2}$.*

These orientations will act as generalized random sceneries and and we denote by $W = (W_t)_{t\geq 0}$ the Brownian motion associated to it, i.e. such that

$$\Big(\frac{1}{n^{1/2}} \sum_{k=0}^{[nt]} \epsilon_k\Big)_{t\geq 0} \stackrel{\mathcal{D}}{\Longrightarrow} \big(W_t\big)_{t\geq 0}. \tag{2.1}$$

In this paper, the notation $\stackrel{\mathcal{D}}{\Longrightarrow}$ stands for weak convergence in the space $\mathcal{D} = D([0,\infty[, \mathbb{R}^n)$, for either $n = 1, 2$, of processes with càdlàg trajectories equipped with the Skorohod topology.[1]

For every realization of ϵ, one usually means by simple random walk on $\mathbb{L}^\epsilon$ the $\mathbb{Z}^2$-valued Markov chain $M = \big(M_n^{(1)}, M_n^{(2)}\big)$ defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, whose ($\epsilon$-dependent) transition probabilities are defined for all $(u, v) \in \mathbb{V}\times\mathbb{V}$ by

$$\mathbb{P}[M_{n+1} = v | M_n = u] = \begin{cases} \frac{1}{3} & \text{if (u,v)} \in \mathbb{A}^\epsilon \\ & \\ 0 & \text{otherwise.} \end{cases}$$

In this paper however, our results are also valid when the probability of an horizontal move in the direction of the orientation is $1 - p \in [0, 1]$ instead of $\frac{1}{3}$, with probabilities of moving up or down equal thus to $\frac{p}{2}$. We write then $m = \frac{1-p}{p}$ for the mean of any geometric random variable of parameter p, whose value is $m = \frac{1}{2}$ in the standard case $p = \frac{2}{3}$. We also use a self-similar process $\Delta = (\Delta_t)_{t\geq 0}$ introduced in [6] as the asymptotic limit of a random walk in a random scenery, formally defined for $t \geq 0$ by

$$\Delta_t = \int_{-\infty}^{+\infty} L_t(x) dW(x)$$

where $L = (L_t)_{t\geq 0}$ is the local time of a standard Brownian motion $B = (B_t)_{t\geq 0}$, related to the vertical component of the walk and independent of W. We also denote for all $t \geq 0$

[1] Or sometimes in its restriction $D([0, T], \mathbb{R}^n)$ for $T > 0$. We shall not precise which underlying probability space it concerns, because our final result will focus on the larger one, in a kind of annealed procedure.

$$B_t^{(m)} = \frac{1}{\sqrt{1+m}} \cdot B_t \text{ and } \Delta_{\mathrm{t}}^{(\mathrm{m})} = \frac{\mathrm{m}}{(1+\mathrm{m})^{3/4}} \cdot \Delta_{\mathrm{t}}.$$

The following functional limit theorem has been proved in [5]:

Theorem 1. *(Theorem 4 of [5]):*

$$\left(\frac{1}{n^{3/4}} M_{[nt]}\right)_{t\geq 0} \overset{\mathcal{D}}{\Longrightarrow} \left(\Delta_t^{(m)}, 0\right)_{t\geq 0}. \tag{2.2}$$

We complete here this result with the following theorem:

Theorem 2. *:*

$$\left(\frac{1}{n^{3/4}} M_{[nt]}^{(1)}, \frac{1}{n^{1/2}} M_{[nt]}^{(2)}\right)_{t\geq 0} \overset{\mathcal{D}}{\Longrightarrow} \left(\Delta_t^{(m)}, B_t^{(m)}\right)_{t\geq 0} \tag{2.3}$$

and the asymptotic components $\Delta_t^{(m)}$ and $B_t^{(m)}$ are not independent.

3 Random walk in generalized random sceneries

We suppose that there exists some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which are defined all the random variables, like e.g. the orientations ϵ and the Markov chain M.

3.1 Embedding of the simple random walk

We use the orientations to embed the 2d-random walk on $\mathbb{L}^\epsilon$ into two different components: a vertical simple random walk and an horizontal more sophisticated process.

3.1.1 Vertical embedding: simple random walk

The vertical embedding is a one dimensional simple random walk Y, that weakly converges in $\mathcal{D}$ to a standard Brownian motion B:

$$\left(\frac{1}{n^{1/2}} Y_{[nt]}\right)_{t\geq 0} \overset{\mathcal{D}}{\Longrightarrow} (B_t)_{t\geq 0}. \tag{3.4}$$

The *local time* of the walk Y is the discrete-time process $N(y) = (N_n(y))_{n\in\mathbb{N}}$ canonically defined for all $y \in \mathbb{Z}$ and $n \in \mathbb{N}$ by

$$N_n(y) = \sum_{k=0}^{n} \mathbf{1}_{Y_k=y} \tag{3.5}$$

The following result is established in [6]:

Lemma 1. *(Lemma 4 of [6])* $\lim_{n\to\infty} n^{-\frac{3}{4}} \sup_{y\in\mathbb{Z}} N_n(y) = 0$ in $\mathbb{P}-$probability.

For any reals $a < b$, the fraction of time spent by the process $\left(\frac{Y_{[nt]}}{\sqrt{n}}\right)_{t\geq 0}$ in the interval $[a, b)$, during the time interval $\big[0, t\big]$, is defined by

$$T_t^{(n)}(a,b) := \frac{1}{n} \sum_{a\leq n^{-\frac{1}{2}} y<b} N_{[nt]}(y).$$

One is then particularly interested in analogous quantities for the Brownian motion $(B_t)_{t\geq 0}$, i.e. in a local time $L_t(x)$ and in a fraction of time spent in $[a,b)$ before t. If one defines naturally the former fraction of time to be

$$\Lambda_t(a,b) = \int_0^t \mathbf{1}_{[a\leq B_s<b]}\, ds$$

then ([7]) one can define for all $x\in\mathbb{R}$ such a process $\big(L_t(x)\big)_{t>0}$, jointly continuous in t and x, and such that,

$$\mathbb{P}-\text{a.s., } \Lambda_t(a,b) = \int_a^b L_t(x)dx.$$

To prove convergence of the finite-dimensional distributions in Theorem 2, we need a more precise relationship between these quantities and consider the joint distribution of the fraction of time and the random walk itself, whose marginals are not necessarily independent.

Lemma 2. *For any distinct* $t_1,\dots,t_k\geq 0$ *and any* $-\infty<a_j<b_j<\infty$ *(*$j=1,\dots,k$*),*

$$\Big(T_{t_j}^{(n)}(a_j,b_j), \frac{Y_{[nt_j]}}{\sqrt{n}}\Big)_{1\leq j\leq k} \stackrel{\mathcal{L}}{\Longrightarrow} \Big(\Lambda_{t_j}(a_j,b_j), B_{t_j}\Big)_{1\leq j\leq k}$$

where $\stackrel{\mathcal{L}}{\Longrightarrow}$ *means convergence in distribution when* $n\longrightarrow+\infty$*.*

Proof: To prove this lemma, remark first that in our context one can replace $T_t^n(a,b)$ by[2]

$$\int_0^t \mathbf{1}_{[a\leq n^{-1/2}Y_{[ns]}<b]}ds.$$

For $t\geq 0$, define the projection π_t from $\mathcal{D}$ to $\mathbb{R}$ as $\pi_t(x)=x_t$. From [6], the map

$$x\in\mathcal{D}\longrightarrow \int_0^t \mathbf{1}_{[a\leq x_s<b]}\, ds$$

is continuous on $D([0,T])$ in the Skorohod topology for any $T\geq t$ for almost any sample point of the process $(B_t)_{t\geq 0}$. Moreover, since almost all paths of the Brownian motion $(B_t)_{t\geq 0}$ are continuous at t, the map $x\rightarrow\pi_t(x)$ is continuous at a.e. sample points of the process $(B_t)_{t\geq 0}$. So, for any $t\geq 0$, for any $a,b\in\mathbb{R}$ and any $\theta_1\in\mathbb{R},\theta_2\in\mathbb{R}$, the map

$$x\in\mathcal{D}\longrightarrow \theta_1\int_0^t \mathbf{1}_{[a\leq x_s<b]}\, ds + \theta_2\pi_t(x)$$

is continuous on $D([0,T])$ for any $T\geq t$ at almost all sample points of $(B_t)_{t\geq 0}$. The weak convergence of $\left(\frac{Y_{[nt]}}{\sqrt{n}}\right)_{t\geq 0}$ to the process $(B_t)_{t\geq 0}$ implies then the convergence of the law of

$$\sum_{i=1}^k \theta_i^{(1)} T_{t_i}^{(n)}(a_i,b_i) + n^{-1/2}\sum_{i=1}^k \theta_i^{(2)} Y_{[nt_i]}$$

to that of $\sum_{i=1}^k \theta_i^{(1)}\Lambda_{t_i}(a_i,b_i)+\sum_{i=1}^k\theta_i^{(2)}B_{t_i}$. This proves the lemma using the characteristic function criterion for convergence in distribution. $\diamond$

[2]The two expressions are not equal but their difference is bounded by $C/\sqrt{n}$ for some $C>0$.

3.1.2 Horizontal embedding: generalized random walk in a random scenery

The horizontal embedding is a random walk with geometric jumps: consider a doubly infinite family $(\xi_i^{(y)})_{i\in\mathbb{N}^*,y\in\mathbb{Z}}$ of independent geometric random variables of mean $m = \frac{1-p}{p}$ and define the embedded horizontal random walk $X = (X_n)_{n\in\mathbb{N}}$ by $X_0 = 0$ and for $n \geq 1$,

$$X_n = \sum_{y\in\mathbb{Z}} \epsilon_y \sum_{i=1}^{N_{n-1}(y)} \xi_i^{(y)} \tag{3.6}$$

with the convention that the last sum is zero when $N_{n-1}(y) = 0$. Define now for $n \in \mathbb{N}$ the random time T_n to be the instant just after the n^{th} vertical move,

$$T_n = n + \sum_{y\in\mathbb{Z}} \sum_{i=1}^{N_{n-1}(y)} \xi_i^{(y)}. \tag{3.7}$$

Precisely at this time, the random walk $(M_n)_{n\in\mathbb{N}}$ on $\mathbb{L}^\epsilon$ coincides with its embedding. The following lemma has been proved in [3] and [5]:

Lemma 3. *1.* $M_{T_n} = (X_n, Y_n)$, $\forall n \in \mathbb{N}$.

2.

$$\frac{T_n}{n} \overset{n\to\infty}{\longrightarrow} 1 + m, \quad \mathbb{P}-\text{almost surely}.$$

3.2 Random walk in a random scenery

We call X a generalized random walk in a random scenery because it is a geometric distorsion of the following *random walk in a random scenery* $Z = (Z_n)_{n\in\mathbb{N}}$ introduced in Theorem 1.1 of [6] with

$$Z_n = \sum_{k=0}^{n} \epsilon_{Y_k} = \sum_{y\in\mathbb{Z}} \epsilon_y N_n(y).$$

From the second expression in terms of the local time of the simple random walk Y, it is straightforward to see that its variance is of order $n^{3/2}$, justifying the normalization in $n^{3/4}$ in the functional limit theorem established in [6]. There, the limiting process $\Delta = (\Delta_t)_{t\geq 0}$ of the sequence of stochastic processes $\left(n^{-\frac{3}{4}} Z_{[nt]}\right)_{t\geq 0}$ is the process obtained from the random walk in a random scenery when $\mathbb{Z}$ is changed into $\mathbb{R}$, the random walk Y into a Brownian motion $B = (B_t)_{t\geq 0}$ and the random scenery $(\epsilon_y)_{y\in\mathbb{Z}}$ into a white noise, time derivative in the distributional sense of a Brownian motion $(W(x))_{x\in\mathbb{R}}$. Formally replacing $N_n(x)$ by $L_t(x)$, the process Δ can be represented by the stochastic integral

$$\Delta_t = \int_{-\infty}^{+\infty} L_t(x)\, dW(x).$$

Since the random scenery is defined on the whole $\mathbb{Z}$ axis, the Brownian motion $(W(x))_{x\in\mathbb{R}}$ is to be defined with real time. Therefore, one introduces a pair of independent Brownian motions (W_+, W_-) so that the limiting process can be rewritten

$$\Delta_t = \int_0^{+\infty} L_t(x)\, dW_+(x) + \int_0^{+\infty} L_t(-x)\, dW_-(x). \tag{3.8}$$

In addition to its existence, Kesten and Spitzer have also proved the

Theorem 3. *(Theorem 1.1 of [6]):*

$$\Big(\frac{1}{n^{3/4}} Z_{[nt]}\Big)_{t\geq 0} \stackrel{\mathcal{D}}{\Longrightarrow} (\Delta_t)_{t\geq 0}.$$

We complete this result and consider the (non-independent) coupling between the simple vertical random walk and the random walk in a random scenery and prove:

Theorem 4. *:*

$$\Big(\frac{1}{n^{3/4}} Z_{[nt]}, \frac{1}{n^{1/2}} Y_{[nt]}\Big)_{t\geq 0} \stackrel{\mathcal{D}}{\Longrightarrow} \big(\Delta_t, B_t\big)_{t\geq 0}.$$

4 Proofs

4.1 Strategy

The main strategy is to relate the simple random walk on the oriented lattice $\mathbb{L}^\epsilon$ to the random walk in random scenery Z using the embedded process (X, Y). We first prove the functional limit Theorem 4 by carefully carrying the strategy of [6], used to prove Theorem 3, for a possibly non independent couple (Z, Y). This result extends to the embedded process (X, Y) due to an asymptotic equivalence in probability of X with a multiple of Z. Theorem 2 is then deduced from it using nice convergence properties of the random times (3.7) and self-similarity. Eventually, we prove that the asymptotic horizontal components of these two-dimensional processes are not independent, using stochastic calculus techniques.

4.2 Proof of Theorem 4

We focus on the convergence of finite dimensional distributions, because we do not really need the tightness to prove our main result Theorem 2. It could nevertheless be proved in the similar way as the tightness in Lemma 7, see next section.

Proposition 1. *The finite dimensional distributions of* $\Big(\frac{1}{n^{3/4}} Z_{[nt]}, \frac{1}{n^{1/2}} Y_{[nt]}\Big)_{t\geq 0}$ *converge to those of* $(\Delta_t, B_t)_{t\geq 0}$*, as* $n \to \infty$.

Proof: We first identify the finite dimensional distributions of $\big(\Delta_t, B_t\big)_{t\geq 0}$.

Lemma 4. *For any distinct* $t_1, \ldots, t_k \geq 0$ *and* $\theta_1, \ldots, \theta_k \in \mathbb{R}^2$*, the characteristic function of the corresponding linear combination of* $\big(\Delta_t, B_t\big)$ *is given by*

$$\mathbb{E}\Big[\exp\Big(i\sum_{j=1}^k (\theta_j^{(1)}\Delta_{t_j} + \theta_j^{(2)} B_{t_j})\Big)\Big] = \mathbb{E}\Big[\exp\Big(-\frac{1}{2}\int_{\mathbb{R}} (\sum_{j=1}^k \theta_j^{(1)} L_{t_j}(x))^2\, dx\Big) \exp\Big(i\sum_{j=1}^k \theta_j^{(2)} B_{t_j}\Big)\Big].$$

Proof: The function $x \to \sum_{j=1}^k \theta_j^{(1)} L_{t_j}(x)$ being continuous, almost surely with compact support, for almost all fixed sample of the random process $(B_t)_t$, the stochastic integrals

$$\int_0^{+\infty} \sum_{j=1}^k \theta_j^{(1)} L_{t_j}(x)\, dW_+(x) \quad \text{and} \quad \int_0^{+\infty} \sum_{j=1}^k \theta_j^{(1)} L_{t_j}(-x)\, dW_-(x)$$

are independent Gaussian random variables, centered, with variance

$$\int_0^{+\infty}\Big(\sum_{j=1}^k \theta_j^{(1)} L_{t_j}(x)\Big)^2\,dx \quad\text{and}\quad \int_0^{+\infty}\Big(\sum_{j=1}^k \theta_j^{(1)} L_{t_j}(-x)\Big)^2\,dx.$$

Therefore, for almost all fixed sample of the random process B, $\sum_{j=1}^k \theta_j^{(1)}\Delta_{t_j}$ is a centered Gaussian random variable with variance given by

$$\int_{\mathbb{R}}\Big(\sum_{j=1}^k \theta_j^{(1)} L_{t_j}(x)\Big)^2\,dx.$$

Then we get

$$\mathbb{E}\Big[\mathbb{E}\Big[e^{i\sum_{j=1}^k \theta_j^{(1)}\Delta_{t_j}}|B_t, t\geq 0\Big]e^{i\sum_{j=1}^k \theta_j^{(2)}B_{t_j}}\Big] = \mathbb{E}\Big[e^{-\frac{1}{2}\int_{\mathbb{R}}(\sum_{j=1}^k \theta_j^{(1)}L_{t_j}(x))^2\,dx}e^{i\sum_{j=1}^k \theta_j^{(2)}B_{t_j}}\Big].\diamond$$

Hence we have expressed the characteristic function of the linear combination of $(\Delta_t, B_t)_{t\geq 0}$ in terms of B and its local time only. We focus now on the limit of the couple $\left(\frac{1}{n^{3/4}}Z_{[nt]}, \frac{1}{n^{1/2}}Y_{[nt]}\right)_{t\geq 0}$ when n goes to infinity and introduce for distinct $t_j \geq 0$ and $\theta_j \in \mathbb{R}^2$ the characteristic function

$$\phi_n(\theta_1,\ldots,\theta_k) := \mathbb{E}\left[\exp\Big(in^{-3/4}\sum_{j=1}^k \theta_j^{(1)} Z_{[nt_j]}\Big)\ \exp\Big(in^{-1/2}\sum_{j=1}^k \theta_j^{(2)} Y_{[nt_j]}\Big)\right].$$

By independence of the random walk Y with the random scenery ϵ, one gets

$$\phi_n(\theta_1,\ldots,\theta_k) = \mathbb{E}\left[\prod_{x\in\mathbb{Z}}\lambda\Big(n^{-\frac{3}{4}}\sum_{j=1}^k \theta_j^{(1)} N_{[nt_j]}(x)\Big)\exp\Big(in^{-1/2}\sum_{j=1}^k \theta_j^{(2)} Y_{[nt_j]}\Big)\right].$$

where $\lambda(\theta) = \mathbb{E}\big[e^{i\theta\epsilon_y}\big]$ is the characteristic function of the orientation ϵ_y, defined for all $y\in\mathbb{Z}$ and for all $\theta\in\mathbb{R}$. Define now for any $\theta_j\in\mathbb{R}^2$ and $n\geq 1$,

$$\psi_n(\theta_1,\ldots,\theta_k) := \mathbb{E}\left[\exp\Big(-\frac{1}{2}\sum_{x\in\mathbb{Z}} n^{-\frac{3}{2}}(\sum_{j=1}^k \theta_j^{(1)} N_{[nt_j]}(x))^2\Big)\exp\Big(in^{-1/2}\sum_{j=1}^k \theta_j^{(2)} Y_{[nt_j]}\Big)\right].$$

Lemma 5. $\lim_{n\to\infty}\Big|\phi_n(\theta_1,\ldots,\theta_k) - \psi_n(\theta_1,\ldots,\theta_k)\Big| = 0.$

Proof : Let $\epsilon > 0$ and $A_n = \{\omega; n^{-\frac{3}{4}}\sup_{x\in\mathbb{Z}}|\sum_{j=1}^k \theta_j^{(1)} N_{[nt_j]}(x)| > \epsilon\}$. Then

$$\begin{aligned}
&|\phi_n(\theta_1,\ldots,\theta_k) - \psi_n(\theta_1,\ldots,\theta_k)| \\
\leq\ & \int_{A_n}\left|\prod_{x\in\mathbb{Z}}\lambda\Big(n^{-\frac{3}{4}}\sum_{j=1}^k \theta_j^{(1)} N_{[nt_j]}(x)\Big) - \exp\Big(-\frac{1}{2}\sum_{x\in\mathbb{Z}} n^{-\frac{3}{2}}(\sum_{j=1}^k \theta_j^{(1)} N_{[nt_j]}(x))^2\Big)\right|d\mathbb{P} \\
+\ & \int_{A_n^c}\left|\prod_{x\in\mathbb{Z}}\lambda\Big(n^{-\frac{3}{4}}\sum_{j=1}^k \theta_j^{(1)} N_{[nt_j]}(x)\Big) - \exp\Big(-\frac{1}{2}\sum_{x\in\mathbb{Z}} n^{-\frac{3}{2}}(\sum_{j=1}^k \theta_j^{(1)} N_{[nt_j]}(x))^2\Big)\right|d\mathbb{P}. \\
\leq\ & 2\mathbb{P}(A_n) + \int_{A_n^c}\left|\prod_{x\in\mathbb{Z}}\lambda\Big(n^{-\frac{3}{4}}\sum_{j=1}^k \theta_j^{(1)} N_{[nt_j]}(x)\Big) - \prod_{x\in\mathbb{Z}}\exp\Big(-\frac{1}{2}n^{-\frac{3}{2}}(\sum_{j=1}^k \theta_j^{(1)} N_{[nt_j]}(x))^2\Big)\right|d\mathbb{P}.
\end{aligned}$$

The first term tends to zero as $n \to +\infty$ in virtue of Lemma 1.
Let us prove that we can choose ϵ in order to the second term be arbitrary small. Denote by $U_n(x)$ the random variables defined by

$$U_n(x) = n^{-3/4} \sum_{j=1}^{k} \theta_j^{(1)} N_{[nt_j]}(x) \quad , \quad x \in \mathbb{Z}$$

With these notations, it is equivalent to prove that

$$\lim_{n\to\infty} \mathbb{E}\left[\mathbf{1}_{A_n^c}\left(\prod_{x\in\mathbb{Z}} \lambda\left(U_n(x)\right) - \prod_{x\in\mathbb{Z}} \exp\left(-U_n^2(x)/2\right)\right)\right] = 0.$$

Note that the products although indexed by $x \in \mathbb{Z}$ have only a finite number of factors different from 1. And furthermore, all factors are complex numbers in $\bar{\mathbb{D}} = \{z \in \mathbb{C} |\ |z| \leq 1\}$. We use the following inequality : let $(z_i)_{i\in I}$ and $(z'_i)_{i\in I}$ two families of complex numbers in $\bar{\mathbb{D}}$ such that all terms are equal to one, except a finite number of them. Then

$$\left|\prod_{i\in I} z'_i - \prod_{i\in I} z_i\right| \leq \sum_{i\in I} |z'_i - z_i|\,.$$

This yields

$$\begin{aligned} &\left| \prod_{x\in\mathbb{Z}} \lambda\left(U_n(x)\right) - \prod_{x\in\mathbb{Z}} \exp\left(-U_n^2(x)/2\right) \right| \\ \leq \quad &\sum_{x\in\mathbb{Z}} \left|\ \lambda\left(U_n(x)\right) - \exp\left(-U_n^2(x)/2\right)\ \right| = \sum_{x\in\mathbb{Z}} |U_n(x)|^2 g(U_n(x)) \end{aligned} \tag{4.9}$$

where g is the function defined by $g(0) = 0$ and

$$g(v) = |v|^{-2} \left|\lambda(v) - \exp(-\frac{v^2}{2})\right| \quad , \quad v \neq 0.$$

Remark that since $\lambda(\theta) = e^{-\frac{\theta^2}{2}} + o(|\theta|^2)$, the function g is continuous and bounded. Define the function $\tilde{g} : [0, +\infty) \to [0, +\infty)$ by

$$\tilde{g}(u) = \sup_{|v|\leq u} g(v).$$

Note that $\tilde{g}$ is not decreasing so that using (4.9) we obtain that

$$\mathbb{E}\left[\mathbf{1}_{A_n^c}\left|\prod_{x\in\mathbb{Z}} \lambda\left(U_n(x)\right) - \prod_{x\in\mathbb{Z}} \exp\left(-U_n^2(x)/2\right)\right|\right] \leq \tilde{g}(\epsilon)\mathbb{E}\left(\sum_{x\in\mathbb{Z}} |U_n(x)|^2\right).$$

We know (see for instance Lemma 3.3 in [4]) that $\sum_{x\in\mathbb{Z}} |U_n(x)|^2$ is bounded in $\mathbb{L}^1$, then since $\tilde{g}$ is continuous and vanishes at 0, the result follows. Thus Lemma 5 is proved. $\diamond$

The asymptotic behavior of ϕ_n will be that of ψ_n and we identify now its limit with the characteristic function of the linear combination of $\left(\Delta_t, B_t\right)_{t\geq 0}$ in the following:

Lemma 6. *For any distinct $t_1, \ldots, t_k \geq 0$ and $\theta_1, \ldots, \theta_k \in \mathbb{R}^2$, the distribution of*

$$\left(n^{-\frac{3}{2}} \sum_{x \in \mathbb{Z}} \Big(\sum_{j=1}^k \theta_j^{(1)} N_{[nt_j]}(x) \Big)^2, \; n^{-\frac{1}{2}} \sum_{j=1}^k \theta_j^{(2)} Y_{[nt_j]} \right)_{j=1\ldots k}$$

converges, as $n \to \infty$, to the distribution of

$$\left(\int_{-\infty}^{\infty} \Big(\sum_{j=1}^k \theta_j^{(1)} L_{t_j}(x) \Big)^2 dx, \sum_{j=1}^k \theta_j^{(2)} B_{t_j} \right)_{j=1\ldots k}.$$

Proof: We proceed like in [6] where a similar result is proved for the horizontal component; although the convergence holds for each component, their possible non-independence prevents to get the convergence for the couple directly and we have to proceed carefully using similar steps and Lemma 2. We decompose the set of all possible indices into small slices where sharp estimates can be made, and proceed on them of two different limits on their sizes afterwards. Let $\tau > 0$ and $a(l,n) = l\tau\sqrt{n}$, $l \in \mathbb{Z}$. Define, in the slice $[a(l,n), a(l+1,n)[$, an average occupation time by

$$T(l,n) = \sum_{j=1}^k \theta_j^{(1)} T_{t_j}^{(n)}(l\tau, (l+1)\tau) = \frac{1}{n} \sum_{j=1}^k \theta_j^{(1)} \sum_{a(l,n) \leq y < a(l+1,n)} N_{[nt_j]}(y).$$

Define also $\quad U(\tau, M, n) = n^{-\frac{3}{2}} \sum_{\substack{x < -M\tau\sqrt{n} \\ \text{or } x \geq M\tau\sqrt{n}}} (\sum_{j=1}^k \theta_j^{(1)} N_{[nt_j]}(x))^2 \quad$ and

$$V(\tau, M, n) = \frac{1}{\tau} \sum_{-M \leq l < M} (T(l,n))^2 + n^{-\frac{1}{2}} \sum_{j=1}^k \theta_j^{(2)} Y_{[nt_j]}.$$

Consider $\delta(\tau, n) = \#\{l \in \mathbb{Z} \mid a(l,n) \leq l < a(l+1,n)\}$ and write

$$\begin{aligned} A(\tau, M, n) \quad &:= \quad n^{-\frac{1}{2}} \sum_{j=1}^k \theta_j^{(2)} Y_{[nt_j]} + n^{-\frac{3}{2}} \sum_{x \in \mathbb{Z}} \Big(\sum_{j=1}^k \theta_j^{(1)} N_{[nt_j]}(x) \Big)^2 - U(\tau, M, n) - V(\tau, M, n) \\ &= \quad n^{-\frac{3}{2}} \sum_{-M \leq l < M} \sum_{a(l,n) \leq x < a(l+1,n)} \left(\Big(\sum_{j=1}^k \theta_j^{(1)} N_{[nt_j]}(x) \Big)^2 - \frac{n^2 \times (T(l,n))^2}{\tau\sqrt{n}\ \delta(\tau,n)} \right). \end{aligned}$$

First step: We first show that the $\mathbb{L}^1$-norm of $A(\tau, M, n)$ is uniformly bounded in n. In the following, C denotes some constant which may vary from line to line. Fix M and n and write

$$\begin{aligned} &\mathbb{E}\Big[\Big| \Big(\sum_{j=1}^k \theta_j^{(1)} N_{[nt_j]}(x) \Big)^2 - \frac{n^2 \times (T(l,n))^2}{\tau\sqrt{n}\ \delta(\tau,n)} \Big| \Big] \\ = \quad &\mathbb{E}\Big[\Big| \sum_{j=1}^k \theta_j^{(1)} N_{[nt_j]}(x) - \frac{n \times T(l,n)}{(\tau\sqrt{n}\ \delta(\tau,n))^{1/2}} \Big| \times \Big| \sum_{j=1}^k \theta_j^{(1)} N_{[nt_j]}(x) + \frac{n \times T(l,n)}{(\tau\sqrt{n}\ \delta(\tau,n))^{1/2}} \Big| \Big] \\ \leq \quad &\mathbb{E}\Big[\Big| \sum_{j=1}^k \theta_j^{(1)} N_{[nt_j]}(x) - \frac{n \times T(l,n)}{(\tau\sqrt{n}\ \delta(\tau,n))^{1/2}} \Big|^2 \Big]^{\frac{1}{2}} \times \ \mathbb{E}\Big[\Big| \sum_{j=1}^k \theta_j^{(1)} N_{[nt_j]}(x) + \frac{n \times T(l,n)}{(\tau\sqrt{n}\ \delta(\tau,n))^{1/2}} \Big|^2 \Big]^{\frac{1}{2}}. \end{aligned}$$

Firstly, $\mathbb{E}\Big[\Big|\sum_{j=1}^k \theta_j^{(1)} N_{[nt_j]}(x) + \frac{n\times T(l,n)}{(\tau\sqrt{n}\ \delta(\tau,n))^{1/2}}\Big|^2\Big]$

$$\begin{aligned}
&\leq C(\delta(\tau,n))^{-2}\mathbb{E}\Big[\Big(\sum_{j=1}^k \sum_{a(l,n)\leq y<a(l+1,n)} |\theta_j^{(1)}|(N_{[nt_j]}(x)+N_{[nt_j]}(y))\Big)^2\Big] \\
&\leq C(\delta(\tau,n))^{-1}\Big(\sum_{j=1}^k |\theta_j^{(1)}|^2\Big) \sum_{j=1}^k \sum_{a(l,n)\leq y<a(l+1,n)} \mathbb{E}\Big[(N_{[nt_j]}(x)+N_{[nt_j]}(y))^2\Big] \\
&\leq C\Big(\sum_{j=1}^k |\theta_j^{(1)}|^2\Big) \sum_{j=1}^k \max_{a(l,n)\leq y<a(l+1,n)} \mathbb{E}\Big[(N_{[nt_j]}(x)+N_{[nt_j]}(y))^2\Big] \\
&\leq C\Big(\sum_{j=1}^k |\theta_j^{(1)}|^2\Big) \sum_{j=1}^k \max_{a(l,n)\leq y<a(l+1,n)} \big\{\mathbb{E}[N_{[nt_j]}(x)^2]+\mathbb{E}[N_{[nt_j]}(y)^2]\big\} \\
&\leq C\Big(\sum_{j=1}^k |\theta_j^{(1)}|^2\Big) \sum_{j=1}^k \max_{a(l,n)\leq y<a(l+1,n)} \Big\{\mathbb{E}[N_{[nt_j]}(x)^3]^{2/3}+\mathbb{E}[N_{[nt_j]}(y)^3]^{2/3}\Big\}
\end{aligned}$$

and similarly,

$$\mathbb{E}\Big[\Big|\sum_{j=1}^k \theta_j^{(1)} N_{[nt_j]}(x) \quad - \quad \frac{n\times T(l,n)}{\delta(\tau,n)}\Big|^2\Big] \leq C\Big(\sum_{i=1}^k |\theta_j^{(1)}|^2\Big) \sum_{j=1}^k \max_{a(l,n)\leq y<a(l+1,n)} \mathbb{E}\Big[\big(N_{[nt_j]}(x)-N_{[nt_j]}(y)\big)^2\Big].$$

Thus, using Lemma 1 and 3 from [6], we have for all n,

$$\mathbb{E}\Big[\Big|A(\tau,M,n)\Big|\Big] \leq C(2M+1)\tau^{3/2}.$$

We will afterwards consider the limit as $M\tau^{3/2}$ goes to zero to approximate the stochastic integral of the local time L_t, and this term will then go to zero. Moreover, we have

$$\begin{aligned}
\mathbb{P}[U(\tau,M,n)\neq 0] &\leq \mathbb{P}[N_{[nt_j]}(x)>0 \text{ for some } x \text{ such that } |x|>M\tau\sqrt{n} \text{ and } 1\leq j\leq k] \\
&\leq \mathbb{P}\Big[N_{\max([nt_j])}(x)>0 \text{ for some } x \text{ such that } |x|>\frac{M\tau}{\sqrt{\max(t_j)}}\sqrt{\max([nt_j])}\ \Big].
\end{aligned}$$

From item b) of Lemma 1 in [6] , we can choose $M\tau$ so large that $\mathbb{P}\big[U(\tau,M,n)\neq 0\big]$ is small. Then, we have proved that for each $\eta>0$, we can choose τ, M and large n such that

$$\mathbb{P}\Big[\Big|n^{-\frac{1}{2}}\sum_{j=1}^k \theta_j^{(2)} Y_{[nt_j]} + n^{-\frac{3}{2}}\sum_{x\in\mathbb{Z}}(\sum_{j=1}^k \theta_j^{(1)} N_{[nt_j]}(x))^2 - V(\tau,M,n)\Big| > \eta\Big] \leq 2\eta.$$

Second step: From Lemma 2, $V(\tau,M,n)$ converges in distribution, when $n\to\infty$, to

$$\frac{1}{\tau}\sum_{-M\leq l<M}\Big(\sum_{j=1}^k \theta_j^{(1)}\int_{l\tau}^{(l+1)\tau} L_{t_j}(x)dx\Big)^2 + \sum_{j=1}^k \theta_j^{(2)} B_{t_j}.$$

The function $x\to L_t(x)$ being continuous and having a.s compact support,

$$\frac{1}{\tau}\sum_{-M\leq l<M}\Big(\sum_{j=1}^k \theta_j^{(1)}\int_{l\tau}^{(l+1)\tau} L_{t_j}(x)dx\Big)^2 + \sum_{j=1}^k \theta_j^{(2)} B_{t_j}$$

converges, as $\tau \to 0, M\tau \to \infty$, to

$$\int_{-\infty}^{\infty} (\sum_{j=1}^{k} \theta_j^{(1)} L_{t_j}(x))^2 \, dx + \sum_{j=1}^{k} \theta_j^{(2)} B_{t_j}. \diamond$$

Putting together Lemma 4, 5 and 6 gives Proposition 1, that proves Theorem 4. $\diamond$

4.3 Proof of Theorem 2

We get the convergence of Theorem 2 from Theorem 4 and Lemma 3 and focus first on the embedded process (X, Y):

Lemma 7.

$$\Big(\frac{1}{n^{3/4}} X_{[nt]}, \frac{1}{n^{1/2}} Y_{[nt]}\Big)_{t\geq 0} \overset{\mathcal{D}}{\Longrightarrow} \Big(m \cdot \Delta_t, B_t\Big)_{t\geq 0}.$$

Proof: We first prove the tightness of the family. The second component is tight in $\mathcal{D}$ (see Donsker's theorem in [1]), so to prove the proposition we only have to prove the tightness of the first one in $\mathcal{D}$. By Theorem 13.5 of Billingsley [1], it suffices to prove that there exists $K > 0$ such that for all $t, t_1, t_2 \in [0, T], T < \infty$, s.t. $t_1 \leq t \leq t_2$, for all $n \geq 1$,

$$\mathbb{E}\Big[|X_{[nt]} - X_{[nt_1]}| \cdot \ |X_{[nt_2]} - X_{[nt]}|\Big] \leq K n^{3/2} |t_2 - t_1|^{\frac{3}{2}}. \tag{4.10}$$

Using Cauchy-Schwarz inequality, it is enough to prove that there exists $K > 0$ such that for all $t_1 \leq t$, for all $n \geq 1$,

$$\mathbb{E}\Big[|X_{[nt]} - X_{[nt_1]}|^2\Big] \leq K n^{3/2} |t - t_1|^{\frac{3}{2}}. \tag{4.11}$$

Since the ϵ's are independent and centered, we have

$$\mathbb{E}\Big[|X_{[nt]} - X_{[nt_1]}|^2\Big] = \sum_{x\in\mathbb{Z}} \mathbb{E}\Big[\sum_{i=N_{[nt_1]-1}(x)+1}^{N_{[nt]-1}(x)} \sum_{j=N_{[nt_1]-1}(x)+1}^{N_{[nt]-1}(x)} \mathbb{E}[\xi_i^{(x)} \xi_j^{(x)} | Y_k, k \geq 0]\Big].$$

From the inequality

$$0 \leq \mathbb{E}[\xi_i^{(x)} \xi_j^{(x)}] \leq m^2 + \mathrm{Var}(\xi_i^{(x)}) = C,$$

we deduce that

$$\mathbb{E}\Big[|X_{[nt]} - X_{[nt_1]}|^2\Big] \quad \leq \quad C \sum_{x\in\mathbb{Z}} \mathbb{E}\Big[(N_{[nt]-1}(x) - N_{[nt_1]-1}(x))^2\Big] = C \sum_{x\in\mathbb{Z}} \mathbb{E}\Big[(N_{[nt]-[nt_1]-1}(x))^2\Big].$$

From item d) of Lemma 1 in [6], as n tends to infinity,

$$\mathbb{E}\big[\sum_x N_n^2(x)\big] \sim C n^{3/2},$$

and there exists some constant $K > 0$ such that

$$\mathbb{E}\Big[|X_{[nt]} - X_{[nt_1]}|^2\Big] \leq K \Big([nt] - [nt_1] - 1\Big)^{\frac{3}{2}} \leq K n^{\frac{3}{2}} \Big(t - t_1\Big)^{\frac{3}{2}}.$$

We get the tightness of the first component by dividing X_n by $n^{3/4}$, and eventually the tightness of the properly normalized embedded process.

To deal with finite dimensional distributions, we rewrite $X_n = X_n^{(1)} + mZ_{n-1}$ with

$$X_n^{(1)} = \sum_{y\in\mathbb{Z}} \epsilon_y \sum_{i=1}^{N_{n-1}(y)} \big(\xi_i^{(y)} - m\big).$$

Using the $\mathbb{L}^2-$convergence proved in the proof of Proposition 2 in [5],

$$\frac{X_n^{(1)}}{n^{3/4}} \overset{n\to\infty}{\longrightarrow} 0, \text{ in Probability}$$

one gets that the finite dimensional distributions of $\left(\frac{X_{[nt]}}{n^{3/4}}, \frac{Y_{[nt]}}{n^{1/2}}\right)_{t\geq 0}$ are asymptotically equivalent to those of $\left(m\cdot\frac{Z_{[nt]}}{n^{3/4}}, \frac{Y_{[nt]}}{n^{1/2}}\right)_{t\geq 0}$. One concludes then using Theorem 4. $\diamond$

In the second step of the proof of Theorem 2, we use Lemma 3 of [5] and that $M_{T_n} = \big(M_{T_n}^{(1)}, M_{T_n}^{(2)}\big) = (X_n, Y_n)$ for any n with

$$\frac{T_n}{n} \overset{n\to\infty}{\longrightarrow} 1+m\ , \ \ \mathbb{P}-\text{almost surely}$$

and the self-similarity of the limit process Δ (index $3/4$) and of the Brownian motion B (index $1/2$). Using the fact that $(T_n)_{n\in\mathbb{N}}$ is strictly increasing, there exists a sequence of integers $(U_n)_n$ which tends to infinity and such that $T_{U_n} \leq n < T_{U_n+1}$. More formally, for any $n\geq 0$, $U_n = \sup\{k\geq 0; T_k\leq n\}$, $(U_{[nt]}/n)_{n\geq 1}$ converges a.s. to the continuous function $\phi(t) := t/(1+m)$, so from Theorem 14.4 from [1],

$$\Big(\frac{1}{n^{3/4}}M^{(1)}_{T_{U_{[nt]}}}, \frac{1}{n^{1/2}}M^{(2)}_{T_{U_{[nt]}}}\Big)_{t\geq 0} \overset{\mathcal{D}}{\Longrightarrow} \big(m\Delta_{\phi(t)}, B_{\phi(t)}\big)_{t\geq 0}.$$

Using Lemma 7, the processes $(m\Delta_{\phi(t)}, B_{\phi(t)})_t$ and $\left(\frac{m}{(1+m)^{3/4}}\Delta_t, \frac{1}{(1+m)^{1/2}}B_t\right)_t$ have the same law, so

$$\Big(\frac{1}{n^{3/4}}M^{(1)}_{T_{U_{[nt]}}}, \frac{1}{n^{1/2}}M^{(2)}_{T_{U_{[nt]}}}\Big)_{t\geq 0} \overset{\mathcal{D}}{\Longrightarrow} \big(\Delta_t^{(m)}, B_t^{(m)}\big)_{t\geq 0}$$

with $\Delta_t^{(m)} = \frac{m}{(1+m)^{3/4}} \cdot \Delta_t$ and $B_t^{(m)} = \frac{1}{\sqrt{1+m}}\cdot B_t$ for all $t\geq 0$. Now, $M^{(2)}_{[nt]} = M^{(2)}_{T_{U_{[nt]}}}$ and $M^{(1)}_{[nt]} = M^{(1)}_{T_{U_{[nt]}}} + \big(M^{(1)}_{[nt]} - M^{(1)}_{T_{U_{[nt]}}}\big)$, so

$$\Big|M^{(1)}_{[nt]} - M^{(1)}_{T_{U_{[nt]}}}\Big| \leq \Big|M^{(1)}_{T_{U_{[nt]}+1}} - M^{(1)}_{T_{U_{[nt]}}}\Big| = \xi^{(Y_{U_{[nt]}})}_{N_{U_{[nt]}}(Y_{U_{[nt]}})}.$$

By remarking that for every $R>0$,

$$\mathbb{P}\Big[\sup_{t\in[0,R]} \frac{1}{n^{3/4}}\xi^{(Y_{U_{[nt]}})}_{N_{U_{[nt]}}(Y_{U_{[nt]}})} \geq \epsilon\Big] \leq [nR]\cdot\mathbb{P}[\xi_1^{(1)}\geq \epsilon n^{3/4}] \leq \frac{[nR]\mathbb{E}[|\xi_1^{(1)}|^2]}{\epsilon^2 n^{3/2}} = o(1),$$

we deduce that for any $R > 0$, $\left(\frac{M^{(1)}_{[nt]} - M^{(1)}_{T_{U_{[nt]}}}}{n^{3/4}}, 0\right)_{t\in[0,R]}$ converges as an element of $\mathcal{D}$ in $\mathbb{P}$-probability to 0. Finally, we get the result:

$$\left(\frac{1}{n^{3/4}} M^{(1)}_{[nt]}, \frac{1}{n^{1/2}} M^{(2)}_{[nt]}\right)_{t\geq 0} \overset{\mathcal{D}}{\Longrightarrow} \left(\Delta^{(m)}_t, B^{(m)}_t\right)_{t\geq 0}.$$

Let us prove now that we could not deduce this result from the convergence of the components because in the joint limiting distribution the two components are not independent. It is enough to prove that Δ_1 and B_1 are not independent and we use that conditionnally to $(B_t)_{0\leq t\leq 1}$, the random variable Δ_1 is the sum of the stochastic integrals

$$\int_0^{+\infty} L_1(x)\, dW_+(x) \quad \text{and} \quad \int_0^{+\infty} L_1(-x)\, dW_-(x)$$

which are independent Gaussian random variables, centered, with variance

$$\int_0^{+\infty} L_1(x)^2\, dx \quad \text{and} \quad \int_0^{+\infty} L_1(-x)^2\, dx.$$

Denote by $V_1 := \int_{\mathbb{R}} L_1^2(x)dx$ the self-intersection time of the Brownian motion $(B_t)_{t\geq 0}$ during the time interval $[0, 1]$.

Lemma 8. *For $n \in \mathbb{N}$ even, there exists a positive real $C(n)$ depending on n s.t.*

$$\mathbb{E}\big[V_1 \cdot B_1^n\big] = C(n) \cdot \mathbb{E}[B_1^n].$$

In particular, V_1 and B_1 are not independent.

Proof: For every $x \in \mathbb{R}$ and $\epsilon \in (0, 1)$, define $J_\varepsilon(x) = \frac{1}{2\varepsilon}\int_0^1 1_{\{|B_s - x|\leq\varepsilon\}}\, ds$. Then, $L_1^2(x)$ is the almost sure limit of $\big(J_\varepsilon(x)\big)^2$ as $\varepsilon \to 0$ so that

$$V_1 \cdot B_1^n = \int_{\mathbb{R}} \Big(\lim_{\varepsilon\to 0} J_\varepsilon(x)^2 B_1^n\Big) dx$$

and by Fubini's theorem for $n \in \mathbb{N}$ even,

$$\mathbb{E}\big[V_1 \cdot B_1^n\big] = \int_{\mathbb{R}} \mathbb{E}\left[\lim_{\varepsilon\to 0} J_\varepsilon(x)^2 B_1^n\right] dx.$$

From the occupation times formula, for every $x \in \mathbb{R}$, for every $\varepsilon > 0$,

$$J_\varepsilon(x) \leq L_1^* := \sup_{x\in\mathbb{R}} L_1(x).$$

So, for every $x \in \mathbb{R}$, for every $\varepsilon > 0$, $J_\varepsilon(x)^2 B_1^n$ is dominated by $(L_1^*)^2\ B_1^n$ which belongs to $\mathbb{L}^1$ since L_1^* and B_1 have moments of any order (see [8] for instance). By dominated convergence theorem, we get

$$\mathbb{E}\big[V_1 \cdot B_1^n\big] = \int_{\mathbb{R}} \lim_{\varepsilon\to 0} \mathbb{E}\left[J_\varepsilon(x)^2 B_1^n\right] dx.$$

But, when $(p_t)_t$ is the Markov transition kernel of the Brownian motion B,

$$\begin{aligned}\mathbb{E}\left[J_\varepsilon(x)^2 B_1^n\right] &= \frac{1}{2\varepsilon^2}\mathbb{E}\left[\int_{0<s<t\le 1} 1_{\{|B_s-x|\le\varepsilon\}}1_{\{|B_t-x|\le\varepsilon\}}\, B_1^n dsdt\right] \\ &= \frac{1}{2\varepsilon^2}\int_{\mathbb{R}^3}\int_{0<s<t\le 1} 1_{\{|y-x|\le\varepsilon\}}1_{\{|z-x|\le\varepsilon\}} p_s(0,y)p_{t-s}(y,z)p_{1-t}(z,u)u^n\, dsdtdydzdu \\ &= 2\int_{\mathbb{R}} du \int_{0<s<t\le 1} dsdt\left[\frac{1}{4\varepsilon^2}\int_{x-\varepsilon}^{x+\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} p_s(0,y)p_{t-s}(y,z)p_{1-t}(z,u)u^n\, dydz\right]\end{aligned}$$

The quantity inside the bracket in the r.h.s. of the last equation converges as $\varepsilon\to 0$ to

$$p_s(0,x)p_{t-s}(x,x)p_{1-t}(x,u)u^n$$

and is bounded by

$$f_{n,x}(u,s,t) := \frac{u^n}{(2\pi)^{3/2}\sqrt{s}\sqrt{t-s}\sqrt{1-t}}\sup_{|z-x|\le 1} e^{-\frac{(z-u)^2}{2(1-t)}}.$$

Let us prove that this function is integrable on $\mathbb{R}\times\{(s,t)\in[0,1]^2; s<t\}$. On the set $[x+1,+\infty[\times\{(s,t)\in[0,1]^2; s<t\}$, $f_{n,x}(u,s,t)$ is bounded by

$$\frac{u^n}{(2\pi)^{3/2}\sqrt{s}\sqrt{t-s}\sqrt{1-t}}e^{-\frac{[u-(x+1)]^2}{2(1-t)}}$$

and

$$\begin{aligned}&\int_{0<s<t\le 1}\frac{1}{\sqrt{s}\sqrt{t-s}\sqrt{1-t}}\left[\int_{x+1}^{\infty} u^n e^{-\frac{[u-(x+1)]^2}{2(1-t)}}\, du\right] dsdt \\ &\le \left(\int_{0<s<t\le 1}\frac{1}{\sqrt{s}\sqrt{t-s}}ds\, dt\right)\left(\int_{\mathbb{R}}[v+|x|+1]^n\ e^{-v^2/2}\, dv\right) < +\infty.\end{aligned}$$

The same type of arguments can be used on the sets $]-\infty,x-1]\times\{(s,t)\in[0,1]^2; s<t\}$ and $]x-1,x+1[$. By dominated convergence theorem we deduce that

$$\begin{aligned}\mathbb{E}[V_1\cdot B_1^n] &= 2\int_{0<s<t\le 1} p_{t-s}(0,0)dsdt\int_{\mathbb{R}}\left[\int_{\mathbb{R}} p_s(0,x)p_{1-t}(x,u)\, dx\right] u^n du \\ &= 2\int_{0<s<t\le 1} p_{t-s}(0,0)\left[\int_{\mathbb{R}} p_{1-t+s}(0,u)u^n\, du\right] dsdt.\end{aligned}$$

Now, by the scaling property of the Brownian motion,

$$\int_{\mathbb{R}} p_{1-t+s}(0,u)u^n\, du = \mathbb{E}[B_{1-t+s}^n] = (1-t+s)^{n/2}\mathbb{E}[B_1^n].$$

Therefore, $\mathbb{E}[V_1\cdot B_1^n] = C(n)\cdot\mathbb{E}[B_1^n]$ where

$$C(n) = 2\int_{0<s<t\le 1}\frac{(1-t+s)^{n/2}}{\sqrt{2\pi(t-s)}}\, dsdt.\ \diamond$$

To get the non-independence, one computes then for n even

$$\begin{aligned}\mathbb{E}\big[\Delta_1^2\cdot B_1^n\big] &= \mathbb{E}\big[\mathbb{E}\big[\Delta_1^2|B_s, 0\le s\le 1\big]\cdot B_1^n\big] = \mathbb{E}\big[V_1\cdot B_1^n\big] \\ &\ne \mathbb{E}\big[V_1\big]\cdot\mathbb{E}\big[B_1^n\big] = \mathbb{E}\big[\Delta_1^2\big]\cdot\mathbb{E}\big[B_1^n\big]\end{aligned}$$

leading to the non-independence of Δ_1 and B_1. $\diamond$.

5 Conclusions and open questions

The functional limit theorem we have proved here, with an horizontal component normalized in $n^{3/4}$ with a non-Gaussian behavior and a more standard vertical one normalized in $n^{1/2}$, strongly indicates the possibility of a local limit theorem where the probability for the walk to come back at the origin would be of order $n^{-5/4}$, in complete coherence with the transience result of [3]. This result is not straightforward and an extra work is needed to get it; this is a work in progress. Other interesting questions could concern different lattices, with e.g. also vertical orientations, but the peculiarities of these studies on oriented lattices is that the methods used are not robust for the moment. Getting robust methods for general oriented lattices seems to be a nice challenge.

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